A New Approach for Analytical Computation of Hamiltonian of a Satellite Perturbed Motion

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The problem of computation of the Hamiltonian for the perturbed motion of an artificial Earth satellite with large eccentricity is considered. The method of calculation of items of the perturbing function, which is presented in the form of Jacobi nome series, is described. Some advantages of the new approach to the expansion of the Hamiltonian are shown by computations within computer algebra systems. Mathematical techniques for computations with elliptic functions, which are used in the suggested method of Hamiltonian expansion, are considered. Numerical verification and accuracy checking of analytical results is investigated by means of computer algebra system MAPLE.

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1. Introduction

The problem of constructing theories for the motion of artificial Earth satellites with large eccentricities is important in modern celestial mechanics and there are many difficulties to overcome. These difficulties are connected mainly with the slow convergence of the series of the satellite’s perturbing function (Gaposhkin, 1973). Such a slow convergence is explained as follows. We cannot consider the eccentricity as a convenient small parameter in the case when the orbit is essentially different from a circular one. Moreover the series for elliptic motion cannot converge when the eccentricity is larger than the Laplace limit. This fact in its turn gives certain limitations on the class of orbits for which the classical methods for expanding the perturbing function can be used. The classical approach is not even convenient for moderately large eccentricities near to the Laplace limit, as the slow convergence of these series means that we must take into account many thousands (and sometimes many millions) of terms in the expansion of the perturbing function and Hamiltonian.

At the same time the accuracy in computing the coefficients of the expansion in terms of the perturbing function decreases with an increase in the eccentricity, because the Hansen coefficients (Giacaglia, 1976) can only be determined within Bessel or hypergeometric function series; and these series also converge very slowly in the domain of large eccentricities.

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Therefore choosing a convenient small parameter for the expansion of the perturbing function in the case when the eccentricity is not small is difficult.

In this paper we consider an application of Brumberg’s (1992) method for expanding a perturbing function for the analytical computation of the Hamiltonian of the perturbed motion of a satellite. We will describe some algorithms and mathematical techniques for the computations and expanding the Hamiltonian, in the case of large eccentricities, using elliptic functions.

2. Expansion of the Hamiltonian into Jacobi Nome Series

We consider the Hamiltonian of the perturbed motion of an artificial Earth satellite, taking into account some perturbations from the gravitational field of Earth, lunar–solar gravitational perturbations and perturbations from light pressure without the shade effect. One can see that the polar coordinates of the satellite (radius-vector \( r \) and true anomaly \( v \)) are included in the Hamiltonian only in the form \((r/a)^n \exp(jmv)\), where \( a \) is the semi-major axis, \( n, m \) are integers, \( j = \sqrt{-1} \).

The classical expansion (Gaposhkin, 1973) in multiples of the mean anomaly \( l \)

\[
(r/a)^n \exp(jmv) = \sum_{s=-\infty}^{\infty} X^{n,m}_s(e) \exp(jsl),
\]  

where \( e \) is the eccentricity and \( X^{n,m}_s(e) \) are Hansen coefficients, converges very slowly for large eccentricities.

Brumberg (1992) suggested that it would be more convenient to expand the series using the Jacobi nome \( q \) (instead of the eccentricity \( e \)) introducing a new independent variable, the elliptic anomaly \( w \):

\[
w = \pi/2 \left( \frac{F(E + \pi/2, e)}{K(e)} - 1 \right),
\]

where \( F \) and \( K \) are elliptic and complete elliptic integrals of the first kind respectively, \( E \) is the eccentric anomaly.

The Jacobi nome

\[
q = \exp(-\pi K(\sqrt{1-e^2})/K(e))
\]

remains small even for large eccentricities (for example, \( e = 0.74 \) corresponds to \( q = 0.4941 \ldots \)) and expansions by Jacobi nome \( q \) converge rapidly even in the domain where the eccentricity is greater than the Laplace limit.

In accordance with Brumberg’s (1992) paper there is the following expansion for the polar coordinates of the satellite

\[
(r/a)^n \exp(jmv) = \sum_{s=-\infty}^{\infty} B^{n,m}_s(q) \exp(jsw),
\]

where the coefficients \( B^{n,m}_s(q) \) (called elliptic Hansen coefficients) can be determined (in contrast to classical Hansen coefficients) in closed form as functions of complete elliptic integrals of the first and second kind with the eccentricity as a modulus. The closed form for coefficients \( B(q) \) enables them to be computed without loss of accuracy using symbolic computation of the Hamiltonian in computer algebra systems.

The new independent variable (the elliptic anomaly \( w \)) can be defined as a function of
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the mean anomaly \( l \) by means of generalized Kepler equation

\[
\begin{align*}
  w &= l + \sum_{f=1}^{\infty} C_f \sin fl, \\
  \text{where coefficients } C_f \text{ can be obtained in an analytical form by means of the Deprit algorithm. These symbolic computations were executed by Klioner (1992) within the computer algebra system MATHEMATICA (Wolfram, 1991).}
\end{align*}
\]

Using some of the advantages of the described method for expanding the perturbing function a package of algorithms and programs was elaborated by the authors for analytical computation of the special functions in the Hamiltonian and for computing the Hamiltonian of the perturbed motion of an artificial satellite in analytical form within computer algebra.

3. Computation of Special Functions in the Hamiltonian

If we construct a semianalytical theory for the satellite motion using an expansion in multiples of the elliptic anomaly \( w \), we face the problem of computing some special functions in the Hamiltonian. The computation of elliptic Hansen coefficients and their derivatives is part of this problem. For the solution of this problem we need to elaborate some efficient algorithms for the computations using elliptic integrals. It is important that our theory is intended first of all for satellites with large eccentricities. Therefore the Taylor expansion for elliptic integrals is not convenient and we can only use the Fukushima (1991) algorithm, which enables the elliptic integrals with arbitrary modulus to be calculated very efficiently. By means of this algorithm we obtained analytical expressions for the derivatives with respect to the eccentricity from Jacobi nome \( q \) and from complete elliptic integrals of the first kind \( K \) and the second kind \( I \) with the eccentricity as a modulus in the computer algebra system MAPLE (Char et al., 1993).

Hence, we can obtain (neglecting terms of order \( q^{16} \) and more) the following expressions:

\[
\frac{dq}{de} = e (1 - e^2)^{-3/4}(1 + \alpha) \times \left( 0.250 00 + 0.156 250 \alpha^4 + 0.131 836 \alpha^8 + 0.119 011 \alpha^{12} + \cdots \right),
\]

where

\[
\alpha = \frac{1 - (1 - e^2)^{1/4}}{1 + (1 - e^2)^{1/4}},
\]

\[
\frac{dK}{de} = \frac{\pi(1 + 2q^4)}{(1 + (1 - e^2)^{1/4})^2} \left( \frac{32q^3 \frac{dq}{de} + 2(1 + 2q^4)e}{(1 + (1 - e^2)^{1/4})(1 - e^2)^{3/4}} \right),
\]

\[
\frac{dI}{de} = \frac{2\pi^2 dq}{K \frac{dq}{de}} \left( \frac{(q - 4q^4 + 9q^9)(-2 + 8q^3 - 18q^6)}{(1 - 2q + 2q^4 - 2q^9)^2} - \frac{1 - 16q^3 + 81q^8}{1 - 2q + 2q^4 - 2q^9} \right) + \frac{dK}{de} \left( \frac{2\pi^2(q - 4q^4 + 9q^9)}{K^2(1 - 2q + 2q^4 - 2q^9)} + 1 \right).
\]

Then by means of the recurrence relations for elliptic Hansen coefficients (Brumberg...
and the initial values for the recurrences using the MAPLE function of differentiation we obtained the recurrence relations for the first and second derivatives of the elliptic Hansen coefficients and initial values for these recurrences. The analytical expressions (simultaneously with the numerical values) for these coefficients and their derivatives were calculated for all harmonics of the Earth potential from the 2nd to 12th and for all lunar-solar harmonics from the 2nd to 6th inclusive. These analytical expressions, are too cumbersome to be given here, but they are quite convenient for calculations on computers.

Using the numerical computation on MAPLE we can calculate the limits for $s$ in expansion (4) to obtain the necessary accuracy by evaluating the series in multiples of the elliptic anomaly.

For example, for the eccentricity $e = 0.74$ we can demonstrate the following result. The errors of approximation neglecting small terms of the series (4) are not greater than $10^{-5}$, if we take into account maximal 25 (by their absolute values) terms for every $m$ by computation of perturbations from the second harmonics of the Earth's gravitational potential. For the third harmonics of the Earth's potential this quantity is equal to 29, and for the second lunar-solar harmonics this quantity is equal to 21. At the same time for errors of the order $10^{-7}$ we must take into account 37, 38 and 25 terms respectively.

This way we can check numerically the convergence of the analytical expansions and therefore we can maintain the necessary accuracy in the analytical theory. This is very important for the computation of derivatives of elliptic Hansen coefficients, because their numerical values are often much greater than the numerical values of the corresponding coefficients.

Such research was also executed for coefficients $C_f$ (these coefficients also contain the elliptic integrals) of the solution of the generalized Kepler equation (5) and for the derivatives of these coefficients with respect to the eccentricity.

Here we would like to pay attention to the following fact. The decrease in the derivatives with an increase in $f$ is slower than the decrease in the coefficients themselves. However this is correct for large eccentricities only. When the eccentricity is close to zero, the situation is reversed. Investigation by means of computer algebra shows that

$$\forall f \lim_{e \to 0} C_f(e) = 0,$$

and

$$\lim_{e \to 0} \left(\frac{dC_f}{de}\right) = \begin{cases} 1, & \text{if } f = 1, \\ 0, & \text{in other cases}. \end{cases}$$

In such a way we are able:

- to compute in analytical form the new special functions of celestial mechanics with elliptic integrals,
- to investigate the convergence of the expansions of these functions,
- to evaluate the accuracy of constructing a theory of satellite motion.

We also executed an analogous symbolic computation with numerical verification for the inclination functions (Gooding, 1971) and for other standard functions. The presence of such functions in the Hamiltonian is not connected with the method of expansion of satellite polar coordinates.
4. MAPLE package for Analytical Computation of the Hamiltonian

The package for analytical computation of the Hamiltonian by means of computer algebra is elaborated on the basis of programs for the computation of some special functions of celestial mechanics. In this package we take as a basis the expansion of coordinates into Jacobi nome series (4), because the series in multiples of the elliptic anomaly converges more rapidly and can be used for a wider class of orbits than the classical expansion (1) in multiples of the mean anomaly.

However, to preserve some of the advantages that the Hamiltonian formalism gives us, the Hamiltonian is presented in this package as a function of canonical Delaunay variables \( L, G, \Theta, l, g, \theta \) in the following form

\[
H = \sum_{s=-\infty}^{\infty} \sum_{t=0}^{\infty} \Phi_{st}(L, G, \Theta, g, \theta) \left[ \sum_{f=-\infty}^{\infty} C_{\lfloor f \rfloor} \exp(j\varphi_{stf}) \right]^t, \tag{6}
\]

where

\[
\varphi_{stf} = \begin{cases} 
(s/t + f) \operatorname{sgn}(f), & \text{if } t \neq 0, \\
 s, & \text{if } t = 0.
\end{cases}
\]

To obtain expansion (6) we used the generalized Kepler equation (5) for the elliptic anomaly and then we expand the multiplier

\[
\exp\left(js\left( l + \sum_{f=1}^{\infty} C_f \sin fl \right) \right)
\]

into a Taylor series in the neighbourhood of the point \( jsl \).

In expression (6) \( \Phi_{st} \) is a function of slow Delaunay variables \( L, G, \Theta, g, \theta \). It contains some elliptic Hansen coefficients (Brumberg, 1992), Gooding (1971) inclination functions and some standard functions and coefficients, which appear in the expansion of the Earth potential into a series by spherical functions (Gaposhkin, 1973) and in expansion of lunar–solar coordinates into a series by orbit elements (Cok, 1977).

In the input file of the main program of the package we must introduce the number of maximum Earth potential harmonics and the number of maximum lunar–solar harmonics that we take into consideration. In the output we obtain an analytical expression for the Hamiltonian of the corresponding perturbed motion. The limits of the summation for \( s, t \) and \( f \) in formula (6) can be defined via numerical estimation of terms, the numerical values of which are smaller than the parameter \( \varepsilon \) (this small parameter we must determine for the accuracy checking). However these limits can be calculated beforehand. There is a special numerical support program for symbolic computations in the package for this purpose. The preliminary definition of the limits of summing in series enables essential economies in computer memory and computing time of the main MAPLE program for analytical computations to be made. It is important that within the time in which we construct the satellite motion theory these limits cannot change essentially. This means that numerical calculation of the limits should only be made once.

As a result, we obtain an analytical expression for the Hamiltonian of the satellite’s perturbed motion. Simultaneously the accuracy is checked neglecting small terms of the series in formula (6).

The MAPLE package is realized on an IBM PC AT 486 in the MS-DOS operating system.
5. Conclusion

Computations using this MAPLE package on computers show the possibility of using the analytical expansion of the Hamiltonian of the satellite motion to construct an analytical theory by means of computer algebra. Of course, the analytical expressions for the Hamiltonian items are cumbersome. But they are quite suitable to be used in the theory of motion. Simultaneously the possibility of checking the accuracy is demonstrated by symbolic computations within the computer algebra systems. Our computer experiments also show that the approach, based on the expansion of the Hamiltonian by Jacobi nome, is most convenient for computations using computer algebra in the case of large eccentricity. This is connected with the closed form of the coefficients of the expansion and with fast convergence of the series in multiples of the elliptic anomaly.

In the future we propose using the advantages of the described approach and to apply our MAPLE package for the construction of a semianalytical theory of motion for an artificial Earth satellite with large eccentricity.

A similar approach as well as classical methods of the perturbation theory such as those described in Laskar (1985, 1990) and Bretagnon (1983) may also be used in constructing a general planetary theory (Brumberg, 1992).

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References


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