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Existence results for the three-point impulsive boundary value problem involving fractional differential equations☆

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ABSTRACT

In this paper, we consider the existence of solutions for a class of three-point boundary value problems involving nonlinear impulsive fractional differential equations. By use of Banach's fixed point theorem and Schauder's fixed point theorem, some existence results are obtained.

1. Introduction

This paper is concerned with the existence of solutions for the three-point impulsive boundary value problem involving nonlinear fractional differential equations:

$$\begin{cases} {}^{C}D^{q}u(t) = f(t, u(t)), \quad 0 < t < 1, \ t \neq t_k, \ k = 1, 2, \dots, p, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \qquad \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), \quad k = 1, 2, \dots, p, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(\xi) = 0, \end{cases} \tag{1.1}$$

where ${}^{C}D^{q}$ is the Caputo fractional derivative, $q \in \mathbb{R}, 1 < q \leq 2, f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, \bar{I}_k : \mathbb{R} \to \mathbb{R}, \xi \in (0, 1), \xi \neq t_k, k = 1, 2, \dots, p$ and $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), \Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and the left-hand limit of the function $u(t)$ at $t = t_k$, and the sequences $\{t_k\}$ satisfy that $0 = t_0 < t_1 < t_2 < \cdots < t_p < t_{p+1} = 1, p \in \mathbb{N}$.

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc. involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. In consequence, fractional differential equations have been of great interest. For details, see [1–9] and the references therein.

Integer-order impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially

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in the area of impulsive differential equations with fixed moments; see for instance [10,11]. Recently, the boundary value problems of impulsive differential equations of integer order have been studied extensively in the literatures (see [12–15]).

To the best of our knowledge, there are few papers that consider the impulsive boundary value problem involving nonlinear differential equations of fractional order. In this paper, we study the existence of solutions for three-point impulsive boundary value problem (1.1); by use of Banach's fixed point theorem and Schauder's fixed point theorem, some existence results are obtained.

The organization of this paper is as follows. In Section 2, we present some necessary definitions and preliminary results that will be used to prove our main results. The proofs of our main results are given in Section 3. In Section 4, we will give an examples to ensure our main result.

2. Preliminaries and lemmas

The material in this section is basic in some sense. For the reader's convenience, we present some necessary definitions from fractional calculus theory and preliminary results.

Definition 2.1. For a function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^{C}D^{q}f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q-n+1}} \mathrm{d}s, \quad n = [q] + 1, $$

where $[q]$ denotes the integer part of real number q.

Definition 2.2. The Riemann–Liouville fractional integral of order q is defined as

$$ I^{q}f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) \mathrm{d}s, \quad q > 0, $$

provided the integral exists.

Definition 2.3. The Riemann–Liouville fractional derivative of order q for a function $f(t)$ is defined by

$$ D^{q}f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_0^t \frac{f(s)}{(t-s)^{q-n+1}} \mathrm{d}s, \quad n = [q] + 1, $$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 ([9]). *Let $q > 0$; then the differential equation*

$$ {}^{C}D^{q}h(t) = 0 $$

has solution $h(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [q] + 1$.

Lemma 2.2 ([9]). *Let $q > 0$; then*

$$ I^{q}\,{}^{C}D^{q}h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1} $$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$, $n = [q] + 1$.

For the sake of convenience, we introduce the following notations.
Let $J = [0, 1], J_0 = [0, t_1], J_1 = (t_1, t_2], \ldots, J_{p-1} = (t_{p-1}, t_p], J_p = (t_p, 1], J' = J \setminus \{t_1, t_2, \ldots, t_p\}$, and
$PC(J) = \{u : [0, 1] \to \mathbb{R} \mid u \in C(J'), u(t_k^+)$ and $u(t_k^-)$ exist, and $u(t_k^-) = u(t_k), 1 \le k \le p\}$.
Obviously, $PC(J)$ is a Banach space with the norm $\|u\| = \sup_{0 \le t \le 1} |u(t)|$.

Lemma 2.3. *Let $y \in C[0, 1]$ and $\xi \in (t_l, t_{l+1})$; l is a nonnegative integer, $0 \le l \le p$, $1 < q \le 2$. A function $u \in PC(J)$ is a solution of the boundary value problem*

$$ \begin{cases} {}^{C}D^{q}u(t) = y(t), \quad 0 < t < 1,\ t \ne t_k,\ k = 1, 2, \ldots, p, \\ \Delta u|_{t=t_k} = I_k(u(t_k)), \qquad \Delta u'|_{t=t_k} = \bar{I}_k(u(t_k)), \quad k = 1, 2, \ldots, p, \\ u(0) + u'(0) = 0, \qquad u(1) + u'(\xi) = 0, \end{cases} \tag{2.1} $$

if and only if u is a solution of the integral equation

$$u(t) = \begin{cases} \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\mathrm{d}s + M(1-t), \quad t \in J_0; \\ \frac{1}{\Gamma(q)}\int_{t_k}^t (t-s)^{q-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)\mathrm{d}s \\ \quad + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s + \sum_{i=1}^{k}(t-t_i)\bar{I}_i(u(t_i)) \\ \quad + \sum_{i=1}^{k} I_i(u(t_i)) + M(1-t), \quad t \in J_k,\ k = 1, 2, \ldots, p, \end{cases} \tag{2.2}$$

where

$$\begin{aligned} M = {} & \frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s \\ & + \frac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s \\ & + \sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i)) + \sum_{i=1}^{l}\bar{I}_i(u(t_i)) + \sum_{i=1}^{p} I_i(u(t_i)). \end{aligned}$$

Proof. Suppose that u is a solution of (2.1). By applying Lemma 2.2, we have

$$u(t) = I^q y(t) - c_1 - c_2 t = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)\mathrm{d}s - c_1 - c_2 t, \quad t \in J_0, \tag{2.3}$$

for some $c_1, c_2 \in \mathbb{R}$. Then, we have

$$u'(t) = \frac{1}{\Gamma(q-1)}\int_0^t (t-s)^{q-2}y(s)\mathrm{d}s - c_2, \quad t \in J_0. \tag{2.4}$$

If $t \in J_1$, then, we have

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}y(s)\mathrm{d}s - d_1 - d_2(t-t_1), \\ u'(t) &= \frac{1}{\Gamma(q-1)}\int_{t_1}^t (t-s)^{q-2}y(s)\mathrm{d}s - d_2, \end{aligned}$$

for some $d_1, d_2 \in \mathbb{R}$. Thus,

$$\begin{aligned} u(t_1^-) &= \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)\mathrm{d}s - c_1 - c_2 t_1, \\ u(t_1^+) &= -d_1, \\ u'(t_1^-) &= \frac{1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)\mathrm{d}s - c_2, \\ u'(t_1^+) &= -d_2. \end{aligned}$$

In view of $\Delta u|_{t=t_1} = u(t_1^+) - u(t_1^-) = I_1(u(t_1))$ and $\Delta u'|_{t=t_1} = u'(t_1^+) - u'(t_1^-) = \bar{I}_1(u(t_1))$, we have

$$\begin{aligned} -d_1 &= \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)\mathrm{d}s + I_1(u(t_1)) - c_1 - c_2 t_1 \\ -d_2 &= \frac{1}{\Gamma(q-1)}\int_0^{t_1}(t_1-s)^{q-2}y(s)\mathrm{d}s + \bar{I}_1(u(t_1)) - c_2. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(q)}\int_{t_1}^t (t-s)^{q-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(q)}\int_0^{t_1}(t_1-s)^{q-1}y(s)\mathrm{d}s + \frac{t-t_1}{\Gamma(q-1)} \\ & \times \int_0^{t_1}(t_1-s)^{q-2}y(s)\mathrm{d}s + (t-t_1)\bar{I}_1(u(t_1)) + I_1(u(t_1)) - c_1 - c_2 t, \quad t \in J_1. \end{aligned}$$

In a similar way, we can obtain

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(q)}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)\mathrm{d}s \\ & + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{k}(t-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s + \sum_{i=1}^{k}(t-t_i)\bar{I}_i(u(t_i)) \\ & + \sum_{i=1}^{k}I_i(u(t_i)) - c_1 - c_2 t, \quad t\in J_k,\ k=1,2,\ldots,p. \end{aligned} \tag{2.5}$$

By (2.3), (2.4) and the boundary condition $u(0)+u'(0)=0$, we can obtain $c_1+c_2=0$.

On the other hand, by (2.5), we have

$$\begin{aligned} u(1) = {} & \frac{1}{\Gamma(q)}\int_{t_p}^{1}(1-s)^{q-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(q)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i) \\ & \times \int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s + \sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i)) + \sum_{i=1}^{p}I_i(u(t_i)) - c_1 - c_2, \\ u'(\xi) = {} & \frac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s + \sum_{i=1}^{l}\bar{I}_i(u(t_i)) - c_2. \end{aligned}$$

By the boundary condition $u(1)+u'(\xi)=0$ and $c_1+c_2=0$, we obtain

$$\begin{aligned} c_1 = {} & -\frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)\mathrm{d}s - \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s \\ & - \frac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)\mathrm{d}s - \frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s \\ & - \sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i)) - \sum_{i=1}^{l}\bar{I}_i(u(t_i)) - \sum_{i=1}^{p}I_i(u(t_i)), \end{aligned} \tag{2.6}$$

$$\begin{aligned} c_2 = {} & \frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}y(s)\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s \\ & + \frac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}y(s)\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}y(s)\mathrm{d}s \\ & + \sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i)) + + \sum_{i=1}^{l}\bar{I}_i(u(t_i)) + \sum_{i=1}^{p}I_i(u(t_i)). \end{aligned} \tag{2.7}$$

Substituting (2.6) and (2.7) into (2.3), (2.5) respectively, and letting $M=c_2$, we get (2.2).

Conversely, we assume that u is a solution of the integral equation (2.2). In view of the relations ${}^{C}D^{p}I^{p}y(t)=y(t)$ for $p>0$, we get

$${}^{C}D^{q}u(t)=y(t), \quad 0<t<1, t\neq t_k\, k=1,2,\ldots,p,\ 1<q\leq 2.$$

Moreover, it can easily be shown that $\Delta u|_{t=t_k}=I_i(u(t_k))$, $\Delta u'|_{t=t_k}=\bar{I}_i(u(t_k))$, $k=1,2,\ldots,p$. Also it can easily be verified that the boundary conditions $u(0)+u'(0)=0$, $u(1)+u'(\xi)=0$ are satisfied. The proof is completed. □

3. Existence of solutions

Let $\xi\in(t_l,t_{l+1})$; l is a nonnegative integer, $0\leq l\leq p$. Define an operator $T: PC(J)\to PC(J)$ by

$$\begin{aligned} (Tu)(t) = {} & \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}f(s,u(s))\mathrm{d}s + \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}f(s,u(s))\mathrm{d}s \\ & + \frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}f(s,u(s))\mathrm{d}s + \sum_{0<t_k<t}(t-t_k)\bar{I}_k(u(t_k)) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{0<t_k<t} I_k(u(t_k)) + (1-t)\Bigg\{\frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}f(s,u(s))\mathrm{d}s + \frac{1}{\Gamma(q-1)}\\
&\times\int_{t_l}^{\xi}(\xi-s)^{q-2}f(s,u(s))\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f(s,u(s))\mathrm{d}s\\
&+\frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}f(s,u(s))\mathrm{d}s + \sum_{i=1}^{p}(1-t_i)\bar{I}_i(u(t_i)) + \sum_{i=1}^{l}\bar{I}_i(u(t_i)) + \sum_{i=1}^{p}I_i(u(t_i))\Bigg\}.
\end{aligned}
$$

Clearly, the fixed points of the operator T are solutions of problem (1.1). Our first result is based on Banach's fixed point theorem.

Theorem 3.1. *Assume that:*

(C_1) *There exists a constant $L_1 > 0$ such that $|f(t,x) - f(t,y)| \le L_1|x-y|$, for each $t \in J$ and all $x, y \in \mathbb{R}$.*
(C_2) *There exist constants $L_2, L_3 > 0$ such that $|I_k(x) - I_k(y)| \le L_2|x-y|$, $|\bar{I}_k(x) - \bar{I}_k(y)| \le L_3|x-y|$, for each $t \in J$ and all $x, y \in \mathbb{R},\ k = 1, 2, \ldots, p$.*

If

$$L_1\left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)}\right) + p(2L_2+3L_3) < 1,$$

then problem (1.1) has a unique solution.

Proof. Let $x, y \in PC(J)$. Then, for each $t \in J$, we have

$$
\begin{aligned}
|(Tx)(t)-(Ty)(t)| \le{}& \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\
&+\frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\
&+\frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\
&+\sum_{0<t_k<t}(t-t_k)|\bar{I}_k(x(t_k))-\bar{I}_k(y(t_k))| + \sum_{0<t_k<t}|I_k(x(t_k))-I_k(y(t_k))|\\
&+\frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\
&+\frac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\
&+\frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\
&+\frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,x(s))-f(s,y(s))|\mathrm{d}s\\
&+\sum_{i=1}^{p}(1-t_i)|\bar{I}_i(x(t_i))-\bar{I}_i(y(t_i))| + \sum_{i=1}^{l}|\bar{I}_i(x(t_i))-\bar{I}_i(y(t_i))| + \sum_{i=1}^{p}|I_i(x(t_i))-I_i(y(t_i))|\\
\le{}& \frac{L_1\|x-y\|}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\mathrm{d}s + \frac{2L_1\|x-y\|}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\mathrm{d}s\\
&+\frac{3L_1\|x-y\|}{\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\mathrm{d}s + \frac{L_1\|x-y\|}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}\mathrm{d}s\\
&+2pL_2\|x-y\| + 3pL_3\|x-y\|\\
\le{}& \left[L_1\left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)}\right) + p(2L_2+3L_3)\right]\|x-y\|.
\end{aligned}
$$

Thus,

$$\|Tx - Ty\| \le \left[L_1\left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)}\right) + p(2L_2 + 3L_3)\right]\|x - y\|.$$

Since

$$L_1\left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)}\right) + p(2L_2 + 3L_3) < 1,$$

consequently T is a contraction. As a consequence of Banach's fixed point theorem, we deduce that T has a fixed point which is a solution of problem (1.1). □

Theorem 3.2. *Assume that:*

(C_3) *The function $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and there exists a constant $N_1 > 0$ such that*

$$|f(t, u)| \le N_1, \quad \text{for each } t \in J \text{ and all } u \in \mathbb{R}.$$

(C_4) *The functions $I_k, \bar{I}_k : \mathbb{R} \to \mathbb{R}$ are continuous, and there exist constants $N_2, N_3 > 0$ such that*

$$|I_k(u)| \le N_2, \qquad |\bar{I}_k(u)| \le N_3, \quad \text{for all } u \in \mathbb{R},\ k = 1, 2, \ldots, p.$$

Then problem (1.1) has at least one solution.

Proof. We shall use Schauder's fixed point theorem to prove that T has a fixed point. The proof will be given in four steps.

Step 1: T is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $PC(J)$.

$$\begin{aligned}
|(Tu_n)(t) - (Tu)(t)| \le{}& \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \sum_{0<t_k<t}(t-t_k)|\bar{I}_k(u_n(t_k)) - \bar{I}_k(u(t_k))| + \sum_{0<t_k<t}|I_k(u_n(t_k)) - I_k(u(t_k))| \\
&+ \frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \sum_{i=1}^{p}(1-t_i)|\bar{I}_i(u_n(t_i)) - \bar{I}_i(u(t_i))| + \sum_{i=1}^{l}|\bar{I}_i(u_n(t_i)) - \bar{I}_i(u(t_i))| + \sum_{i=1}^{p}|I_i(u_n(t_i)) - I_i(u(t_i))| \\
\le{}& \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{2}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{3}{\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,u_n(s)) - f(s,u(s))|\mathrm{d}s \\
&+ 3\sum_{i=1}^{p}|\bar{I}_i(u_n(t_i)) - \bar{I}_i(u(t_i))| + 2\sum_{i=1}^{p}|I_i(u_n(t_i)) - I_i(u(t_i))|.
\end{aligned}$$

Since f, I and $\bar{I}$ are continuous functions, then we have

$$\|Tu_n - Tu\| \to 0 \quad \text{as } n \to \infty.$$

Step 2. T maps bounded sets into bounded sets.

Indeed, it is enough to show that, for any $\rho > 0$, there exists a positive constant L such that, for each $u \in \Omega_\rho = \{u \in PC(J) \mid \|u\| \le \rho\}$, we have $\|Tu\| \le L$. By (C_3) and (C_4), we have, for each $t \in J$,

$$\begin{aligned}
|(Tu)(t)| \le{}& \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}|f(s,u(s))|\mathrm{d}s + \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}|f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}(t-t_k)\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}|f(s,u(s))|\mathrm{d}s + \sum_{0<t_k<t}(t-t_k)|\bar{I}_k(u(t_k))| \\
&+ \sum_{0<t_k<t}|I_k(u(t_k))| + \frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}|f(s,u(s))|\mathrm{d}s + \frac{1}{\Gamma(q-1)} \\
&\times \int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,u(s))|\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u(s))|\mathrm{d}s + \sum_{i=1}^{p}(1-t_i)|\bar{I}_i(u(t_i))| + \sum_{i=1}^{l}|\bar{I}_i(u(t_i))| + \sum_{i=1}^{p}|I_i(u(t_i))| \\
\le{}& \frac{N_1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}\mathrm{d}s + \frac{2N_1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\mathrm{d}s \\
&+ \frac{3N_1}{\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\mathrm{d}s + \frac{N_1}{\Gamma(q-1)}\int_{t_l}^{\xi}(\xi-s)^{q-2}\mathrm{d}s + 2pN_2 + 3pN_3 \\
\le{}& N_1\left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)}\right) + p(2N_2+3N_3).
\end{aligned}$$

Thus,

$$\|Tu\| \le N_1\left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)}\right) + p(2N_2+3N_3) := L.$$

Step 3. T maps bounded sets into equicontinuous sets.

Let Ω_ρ be a bounded set of $PC(J)$ as in Step 2, and let $u \in \Omega_\rho$. For each $t \in J_k$, $0 \le k \le p$, we have

$$\begin{aligned}
|(Tu)'(t)| \le{}& \frac{1}{\Gamma(q-1)}\int_{t_k}^{t}(t-s)^{q-2}|f(s,u(s))|\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-2}|f(s,u(s))|\mathrm{d}s \\
&+ \sum_{0<t_k<t}|\bar{I}_k(u(t_k))| + \frac{1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}|f(s,u(s))|\mathrm{d}s + \frac{1}{\Gamma(q-1)} \\
&\times \int_{t_l}^{\xi}(\xi-s)^{q-2}|f(s,u(s))|\mathrm{d}s + \frac{1}{\Gamma(q-1)}\sum_{i=1}^{p}(1-t_i)\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u(s))|\mathrm{d}s \\
&+ \frac{1}{\Gamma(q-1)}\sum_{i=1}^{l}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}|f(s,u(s))|\mathrm{d}s + \sum_{i=1}^{p}(1-t_i)|\bar{I}_i(u(t_i))| + \sum_{i=1}^{l}|\bar{I}_i(u(t_i))| + \sum_{i=1}^{p}|I_i(u(t_i))| \\
\le{}& \frac{N_1}{\Gamma(q-1)}\left(\int_{t_k}^{t}(t-s)^{q-2}\mathrm{d}s + \int_{t_l}^{\xi}(\xi-s)^{q-2}\mathrm{d}s\right) + \frac{3N_1}{\Gamma(q-1)}\sum_{i=1}^{p}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-2}\mathrm{d}s \\
&+ \frac{N_1}{\Gamma(q)}\sum_{i=1}^{p+1}\int_{t_{i-1}}^{t_i}(t_i-s)^{q-1}\mathrm{d}s + 2pN_2 + 3pN_3 \\
\le{}& \left(\frac{3p+2}{\Gamma(q)} + \frac{p+1}{\Gamma(q+1)}\right)N_1 + pN_2 + 3pN_3 := M.
\end{aligned}$$

Hence, letting $t'', t' \in J_k$, $t' < t''$, $0 \leq k \leq p$, we have

$$|(Tu)(t'') - (Tu)(t')| \leq \int_{t'}^{t''} |(Tu)'(s)|\mathrm{d}s \leq M(t'' - t').$$

So, $T(\Omega_\rho)$ is equicontinuous on all $J_k (k = 0, 1, 2, \ldots, p)$. We can conclude that $T : PC(J) \to PC(J)$ is completely continuous.

Step 4. A priori bounds

Now it remains to show that the set

$$\Omega = \{u \in PC(J) \mid u = \lambda Tu \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Let $u \in \Omega$; then $u = \lambda Tu$ for some $0 < \lambda < 1$. Thus, for each $t \in J$, we have

$$\begin{aligned} u(t) = {} & \frac{\lambda}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, u(s))\mathrm{d}s + \frac{\lambda}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, u(s))\mathrm{d}s \\ & + \frac{\lambda}{\Gamma(q-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{q-2} f(s, u(s))\mathrm{d}s + \lambda \sum_{0<t_k<t} (t - t_k) \bar{I}_k(u(t_k)) \\ & + \lambda \sum_{0<t_k<t} I_k(u(t_k)) + (1-t)\lambda \left\{ \frac{1}{\Gamma(q)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} f(s, u(s))\mathrm{d}s \right. \\ & + \frac{1}{\Gamma(q-1)} \int_{t_l}^{\xi} (\xi - s)^{q-2} f(s, u(s))\mathrm{d}s + \frac{1}{\Gamma(q-1)} \sum_{i=1}^{p} (1 - t_i) \\ & \times \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} f(s, u(s))\mathrm{d}s + \frac{1}{\Gamma(q-1)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} f(s, u(s))\mathrm{d}s \\ & \left. + \sum_{i=1}^{p} (1 - t_i) \bar{I}_i(u(t_i)) + \sum_{i=1}^{l} \bar{I}_i(u(t_i)) + \sum_{i=1}^{p} I_i(u(t_i)) \right\}. \end{aligned}$$

This implies by (C_3) and (C_4) that, for each $t \in J$, we have

$$\begin{aligned} |u(t)| \leq {} & \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} |f(s, u(s))|\mathrm{d}s + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} |f(s, u(s))|\mathrm{d}s \\ & + \frac{1}{\Gamma(q-1)} \sum_{0<t_k<t} (t - t_k) \int_{t_{k-1}}^{t_k} (t_k - s)^{q-2} |f(s, u(s))|\mathrm{d}s + \sum_{0<t_k<t} (t - t_k) |\bar{I}_k(u(t_k))| \\ & + \sum_{0<t_k<t} |I_k(u(t_k))| + \frac{1}{\Gamma(q)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1} |f(s, u(s))|\mathrm{d}s + \frac{1}{\Gamma(q-1)} \\ & \times \int_{t_l}^{\xi} (\xi - s)^{q-2} |f(s, u(s))|\mathrm{d}s + \frac{1}{\Gamma(q-1)} \sum_{i=1}^{p} (1 - t_i) \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} |f(s, u(s))|\mathrm{d}s \\ & + \frac{1}{\Gamma(q-1)} \sum_{i=1}^{l} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2} |f(s, u(s))|\mathrm{d}s + \sum_{i=1}^{p} (1 - t_i) |\bar{I}_i(u(t_i))| + \sum_{i=1}^{l} |\bar{I}_i(u(t_i))| + \sum_{i=1}^{p} |I_i(u(t_i))| \\ \leq {} & \frac{N_1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1}\mathrm{d}s + \frac{2N_1}{\Gamma(q)} \sum_{i=1}^{p+1} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-1}\mathrm{d}s + \frac{3N_1}{\Gamma(q-1)} \sum_{i=1}^{p} \int_{t_{i-1}}^{t_i} (t_i - s)^{q-2}\mathrm{d}s \\ & + \frac{N_1}{\Gamma(q-1)} \int_{t_l}^{\xi} (\xi - s)^{q-2}\mathrm{d}s + 2pN_2 + 3pN_3 \\ \leq {} & N_1 \left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)} \right) + p(2N_2 + 3N_3). \end{aligned}$$

Thus, for every $t \in J$, we have

$$\|u\| \leq N_1 \left(\frac{2p+3}{\Gamma(q+1)} + \frac{3p+1}{\Gamma(q)} \right) + p(2N_2 + 3N_3).$$

This shows that the set Ω is bounded. As a consequence of Schauder's fixed point theorem, we deduce that T has a fixed point which is a solution of problem (1.1). □

4. Example

Let $q = \frac{3}{2},\ \xi = \frac{1}{2}$, $p = 1$. We consider the following boundary value problem:

$$\begin{cases} {}^{C}D^{\frac{3}{2}}u(t) = f(t,u), \quad t \neq \dfrac{1}{3},\ 0 < t < 1, \\ \Delta u|_{t=\frac{1}{3}} = I\left(u\left(\dfrac{1}{3}\right)\right), \qquad \Delta u'|_{t=\frac{1}{3}} = \bar{I}\left(u\left(\dfrac{1}{3}\right)\right), \\ u(0) + u'(0) = 0, \qquad u(1) + u'\left(\dfrac{1}{2}\right) = 0, \end{cases} \tag{4.1}$$

where

$$f(t,u) = \frac{1 + tu^2 \sin^4 u}{1 + t^2 + u^2}, \qquad I(u) = \frac{3 + 2u^2}{1 + u^2}, \qquad \bar{I}(u) = \frac{5u^2}{1 + u^2}.$$

Obviously, f, I and $\bar{I}$ are continuous functions, and

(1) $|f(t,u)| \leq 1$, for each $t \in J$ and all $u \in \mathbb{R}$.
(2) $|I(u)| \leq 3,\ |\bar{I}(u)| \leq 5$, for all $u \in \mathbb{R}$.

So conditions (C_3) and (C_4) hold; by Theorem 3.2, problem (4.1) has at least one solution.

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