Robust control for constant thrust rendezvous under thrust failure

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Abstract A robust constant thrust rendezvous approach under thrust failure is proposed based on the relative motion dynamic model. Firstly, the design problem is cast into a convex optimization problem by introducing a Lyapunov function subject to linear matrix inequalities. Secondly, the robust controllers satisfying the requirements can be designed by solving this optimization problem. Then, a new algorithm of constant thrust fitting is proposed through the impulse compensation and the fuel consumption under the theoretical continuous thrust and the actual constant thrust is calculated and compared by using the method proposed in this paper. Finally, the proposed method having the advantage of saving fuel is proved and the actual constant thrust switch control laws are obtained through the isochronous interpolation method, meanwhile, an illustrative example is provided to show the effectiveness of the proposed control design method.

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1. Introduction

As well-known, there are many tasks to be conducted in space such as building and operation of the international space station, inspection and repair of orbiting satellites and conducting lunar/planetary explorations. Some of these tasks are currently conducted by astronauts. However, most of these tasks are highly risky and expensive. Therefore space robot is an indispensable tool for future space activities. 1,2 Therefore, space robot’s autonomous rendezvous is a crucial phase for many important astronomic missions such as intercepting, repairing, saving, docking, large-scale structure assembling and satellite networking.

During the last few decades, the problem of autonomous rendezvous has been extensively studied and many results have been reported. 3–5 For example, the optimal impulsive control method for spacecraft rendezvous is studied in Ref. 6; adaptive control theory is applied to the rendezvous and docking problem in Ref. 7; an annealing algorithm method for rendezvous orbital control is proposed in Ref. 8; a new rendezvous
guidance method based on sliding-mode control theory can be found in Ref. 9. Although there have been many results in this field, the rendezvous orbital control problem has not been fully investigated and still remains challenging. Both impulsive thrust and the continuous thrust assumptions in these results have been exploited through the Pontryagin’s maximum principle respectively.10–12 In actual practice, however, the thrusts of the spacecraft are constant thrusts, therefore, maneuver during rendezvous and docking operations cannot be normally considered as continuous thrust maneuver or impulsive maneuver.13–15 In our previous studies,16 constant thrust fuel-optimal control for spacecraft rendezvous was studied according to Clohessy–Wiltshire (C–W) equation and the analytical solutions. But the traditional open-loop control method used in our previous studies is not applicable while they are often utilized during the long-distance navigation process. To overcome this problem, a robust closed-loop control laws for constant thrust rendezvous to enhance the orbital control accuracy is proposed in this paper. And the fuel consumption of constant thrust is less than that of the continuous thrust by using the method proposed in this paper.

The purpose of this paper is to study the constant thrust rendezvous under thrust failure. In order to compare the fuel consumption under the theoretical continuous thrust and the actual constant thrust, a new algorithm of constant thrust fitting is proposed by using the impulse compensation method. The optimal fuel consumption and the actual working times of the thrusters in three axes can be respectively calculated by using the time series analysis method.

2. Multi-objective robust controller design

There are ten thrusters installed on the space robot as shown in Fig. 1, where thruster 9 and thruster 10 are symmetric, and the thrust of the \( i \)th thrust is defined as \( F_i \) (\( i = 1, 2, \ldots, 10 \)).

The relative motion coordinate system can be established as follows. Firstly, the target spacecraft is assumed as a rigid body and in a circular orbit, and the relative motion can be described by C–W equation. Then, the centroid of the target spacecraft \( C_i \) is selected as the origin of coordinate, the \( x \)-axis is opposite to the target spacecraft motion, the \( y \)-axis is from the center of the Earth to the target spacecraft and the \( z \)-axis is determined by the right-handed rule. Suppose that thrust failure in the \( y \)-axis is shown in Eq. (1):

\[
\begin{align*}
\dot{x} - 2\omega \dot{y} &= \frac{F_x + \eta_x}{m} \\
\dot{y} + 2\omega \dot{x} - 3\omega^2 y &= \frac{\eta_y}{m} \\
\dot{z} + \omega^2 z &= \frac{F_z + \eta_z}{m}
\end{align*}
\]

where \( x, y, \) and \( z \) are the components of the relative position in corresponding axes, \( \omega \) the angular velocity, \( F_x \) and \( F_z \) the vacuum thrust of the space robot, \( \eta_x, \eta_y, \eta_z \) the sum of the perturbation and nonlinear factors in the \( x \)-axis, \( y \)-axis and \( z \)-axis, respectively, and \( m \) the mass of the space robot at the beginning of the rendezvous. The state variable is defined as \( x(t) = [x, y, z, \dot{x}, \dot{y}, \dot{z}]^T \), the control input vector is \( u(t) = [F_x, 0, F_z]^T \), and the vector of the perturbation and nonlinear factors is defined as \( \eta(t) = [\eta_x, \eta_y, \eta_z]^T \), then the state equation can be transformed as

\[
\dot{x}(t) = A_0 x(t) + B_0 (u(t) + \eta(t))
\]

where

\[
A_0 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 2\omega \\
0 & 3\omega^2 & 0 & -2\omega & 0 & 1 \\
0 & 0 & -\omega^2 & 0 & 0 & 1
\end{bmatrix}
\]

\[
B_0 = \frac{1}{m} \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Suppose that the time of rendezvous maneuver is \( T \); and the shortest switching time interval of thrust is \( \Delta T \), there are \( M \) shortest switching time and \( N \) target maneuver positions, and \( T_{ij} \) (\( i = 1, 2, \ldots, N \)) represents the time of the \( i \)-th thrust arc, \( M_i \) (\( i = 1, 2, \ldots, N \)) represents the number of the shortest switching time \( T_i \) in the \( i \)-th thrust arc:

\[
T_i = M_i \Delta T
\]

with

\[
\begin{align*}
T_{ij} &= M_{ij} \Delta T \\
M &= M_{i1} + M_{i2} + \cdots + M_{iN} (i = 1, 2, \ldots, N)
\end{align*}
\]

The process of collision avoidance maneuver can be considered as the system state variables changing from a non-zero initial state \( x(0) \) to a desired state \( x(T) = 0 \), where \( T \) is the time required for collision avoidance maneuver. To make the controller complies with engineering practice, we should consider the following two important questions:

(1) Parameter uncertainty. Due to measurement errors and the complex interactions between celestial bodies, it is difficult to accurately obtain the angular velocity of the target spacecraft. \( \omega_0 \) is defined as the theoretical angular velocity of the target spacecraft, and \( \Delta \omega \) the uncertainty of the parameters, then the actual angular velocity of the target spacecraft is
\( \omega = \omega_0 + \Delta \omega \)  

(2) Control input constraints. To make the size of thrust vector comply with engineering practice, the control input vector \( u(t) \) must satisfy the following condition:  

\[
\|u(t)\|_2 \leq u_{\text{max}}
\]  

where \( u_{\text{max}} \) is the maximum thrust that thrusters can be output; \( \| \cdot \|_2 \) denotes the 2-norm. The uncertainty of the above parameters are taken into account and a theoretical state feedback controller \( K \) is also introduced, then the state equation can be transformed as  

\[
\begin{align*}
\dot{x}(t) &= A_1 x(t) + B_1 u(t) \\
U(t) &= K x(t)
\end{align*}
\]  

where \( A_1 = A_0 + \Delta A \), with \( \Delta A \) a uncertainty matrix constituted by \( \omega_0 \) and \( \Delta \omega \); \( B_1 = B_0 + \Delta B \), with \( \Delta B \) a uncertainty matrix constituted by \( \eta(t) \). The norm of \( \Delta A \) and \( \Delta B \) is bounded as  

\[
\|\Delta A\|_2 \leq \varsigma, \quad \|\Delta B\|_2 \leq \beta
\]  

Then the state equation of the closed-loop system can be described  

\[
\dot{x}(t) = [A_0 + \Delta A + (B_0 + \Delta B)K]x(t)
\]  

Define the Lyapunov function as  

\[
V(t) = x(t)P x(t)
\]  

where \( P \) is a positive definite symmetric matrix.  

According to the system stability theory, the necessary and sufficient conditions for robust stability of the system Eq. (10) are  

\[
PA + A^T P < 0 \quad (\forall A)
\]  

In this section, a multi-objective controller design strategy will be proposed by translating a multi-objective controller design problem into a convex optimization problem. And the conditions of closed-loop system stability and control input constraints can be met simultaneously. First, two lemmas are introduced which will be used in the following derivation.  

**Lemma 1.** (Schur complement lemma). For a given matrix  

\[
S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad \text{the following three conditions are equivalent:}
\]

\[
\begin{align*}
1) & \quad S < 0 \\
2) & \quad S_{11} < 0, \quad S_{22} - S_{12}^T S_{21}^{-1} S_{12} < 0 \\
3) & \quad S_{22} < 0, \quad S_{11} - S_{12} S_{22}^{-1} S_{21} < 0
\end{align*}
\]  

**Lemma 2.** Given matrices with appropriate dimensions \( H \) and \( E \). For any matrix \( F \) which satisfies the condition \( F^T F \leq I \), there must be a parameter \( \omega > 0 \) that makes the following inequality holds.  

\[
H F^T + E^T F^T H^T \leq \omega^{-1} H H^T + \epsilon E^T E
\]  

For the control input constraint Eq. (7), assume the initial conditions satisfy:  

\[
x^T(0)Px(0) \leq \rho
\]  

where \( \rho \) is a given positive constant.  

Then assume that the following conditions can be satisfied:  

\[
x^T(t)Px(t) \leq x^T(0)Px(0) \leq \rho
\]  

\[
u^T(t)u(t) \leq x^T(t)K^T K x(t) \leq u_{\text{max}}^2
\]  

Then the following results can be obtained according to Lemma 1.  

\[
\begin{bmatrix}
\begin{bmatrix} u_{\text{max}}^2 & K \\ K^T & P \end{bmatrix} \\
\begin{bmatrix} \rho I & x(0) \\ x(0) & P^{-1} \end{bmatrix}
\end{bmatrix} \geq 0
\]  

Using the above conclusion, we can get the following theorem to solve multi-objective controller design problem.  

**Theorem 1.** For the uncertain dynamic system of Eq. (8) with the parameter uncertainty of Eq. (9) and the control input constraints of Eq. (7) as well as the given positive constant \( \rho \), if there exists a corresponding dimension of the matrix \( L \), a symmetric positive definite matrix \( X \) and two parameters \( \epsilon_1 < 0 \) and \( \epsilon_2 > 0 \), then the sufficient condition for robust stability is that there exists a state feedback controller \( K \) which can meet the conditions of Eqs. (19) and (20) simultaneously.  

\[
\begin{bmatrix}
\Sigma & X & L \\
L^T & -\epsilon_2 & 0 \\
X^T & 0 & -\epsilon_1
\end{bmatrix} < 0
\]  

\[
\begin{bmatrix}
\begin{bmatrix} u_{\text{max}}^2 I & L \\ L^T & \frac{x}{\rho} \end{bmatrix} \geq 0 \\
\begin{bmatrix} \rho I & x^T(0) \\ x(0) & X \end{bmatrix} \geq 0
\end{bmatrix}
\]  

where the block matrix in Eq. (19) are  

\[
\Sigma = XX^T + A_0 X + L^T B_0 + B_0 L + \epsilon_1 x^T I + \epsilon_2 \beta^T I, \quad \text{then the theoretical state feedback controller} \quad K \quad \text{can be calculated as}
\]

\[
K = LX^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\
k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\
k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \end{bmatrix}
\]  

**Proof.** The derivation of the Lyapunov function Eq. (9) can be written as  

\[
\dot{V}(t) = \dot{x}^T(t)Px(t) + x^T(t)Px(t) = x^T(t)(A^T P + PA)x(t)
\]  

Eq. (21) can be written in the form of Eq. (23) when \( A = A_0 + \Delta A + (B_0 + \Delta B)K \) is substituted into it.  

\[
A^T P + PA_0 + PB_0 K + K^T B_0^T P + P \Delta A + \Delta A^T P + K^T \Delta B^T P + \Delta P \beta K < 0
\]  

Define \( X = P^{-1}, L = KX \), then the following results can be obtained with each side of Eq. (22) multiplied by \( X \).
$$\begin{align*}
X \Delta A^T & + A X + L^T B_k^T + B_k \mathbf{L} \\
+ X \Delta A^T + \Delta AX + L^T \Delta B^T + \Delta BL & < 0
\end{align*}$$

(24)

According to Lemma 2, there exist two parameters $\epsilon_1 > 0$ and $\epsilon_2 > 0$ which can make Eq. (25) holds.

$$\begin{align*}
X \Delta A^T + \Delta AX & \leq \epsilon_1 X \Delta A^T + \frac{1}{\epsilon_1} X^2 \\
+ L^T \Delta B^T + \Delta BL & \leq \epsilon_2 L^T \Delta B^T + \frac{1}{\epsilon_2} L^2 \mathbf{L}
\end{align*}$$

(25)

According to the condition of Eq. (22) and Lemma 1, Eqs. (23) and (25) can be converted into the form of Eq. (19) which is a sufficient condition for system robust stability. And Eq. (20) which is a sufficient condition to meet the system input constraints can be obtained by using the variable substitution $\mathbf{X} = P^{-1}$ into Eq. (18).  \(\square\)

3. Constant thrust fitting through impulse compensation

The theoretical state feedback controller $K$ can be calculated by Theorem 1. In actual practice, however, the thrusts of the space robot are constant thrusts, therefore, maneuver during rendezvous and docking operations cannot be normally considered as continuous thrust maneuver or impulsive maneuver. To overcome this problem, based on our previous work, a constant thrust fitting by using the isochronous interpolation method is proposed.

Suppose that the theoretical state feedback controller as shown in Eq. (21) and the theoretical continuous thrusts are $F_x$, $F_y$, $F_z$ in three axes, respectively. Then the results of Eq. (26) can be obtained.

$$\begin{align*}
F_x &= k_{11} x + k_{12} y + k_{13} + k_{14} V_x + k_{15} V_y + k_{16} V_z \\
0 &= k_{21} x + k_{22} y + k_{23} z + k_{24} V_x + k_{25} V_y + k_{26} V_z \\
F_z &= k_{31} x + k_{32} y + k_{33} + k_{34} V_x + k_{35} V_y + k_{36} V_z
\end{align*}$$

(26)

where $V_x, V_y, V_z$ are the components of the relative velocity in x-axis, y-axis and z-axis, respectively. According to Eq. (1), we can obtain

$$\begin{align*}
x_{i+1} &= -\frac{1}{m_0} [k_{11} x_i + (6 m_0 \omega + k_{12}) y_i + k_{13} z_i] \\
+ (k_{14} - 4 m_0 \omega) V_{x_i} + k_{15} V_{y_i} + k_{16} V_{z_i} \\
+ \frac{4}{m_0} (k_{11} x_i + k_{12} y_i + k_{13} z_i + k_{14} V_{x_i}) \\
+ (k_{15} + 0.5 m_0 \omega) V_{y_i} + k_{16} V_{z_i} & \cos \omega t_i \\
&+ \frac{4}{m_0} (k_{11} x_i + k_{12} y_i + k_{13} z_i + k_{14} V_{x_i}) \\
+ k_{14} V_{x_i} + (k_{15} + 0.5 m_0 \omega) V_{y_i} + k_{16} V_{z_i} \\
&+ \frac{2 k_{11} + (6 m_0 \omega^2 + k_{12}) y_i + k_{13} z_i] \\
+ (k_{14} - 3 m_0 \omega) V_{x_i} + k_{15} V_{y_i} + k_{16} V_{z_i} & t_i \\
&- \frac{3}{2 m_0} [k_{11} x_i + k_{12} y_i + k_{13} z_i + k_{14} V_{x_i} + k_{15} V_{y_i} + k_{16} V_{z_i}] (t_i)^2
\end{align*}$$

(27)

$$\begin{align*}
y_{i+1} &= \left( \frac{2}{\epsilon_1} V_{x_i} - 3 y_i \right) \cos (\omega t_i) + \frac{1}{\epsilon_1} V_{y_i} \sin (\omega t_i) + 4 y_i - \frac{1}{\epsilon_1} V_{z_i}
\end{align*}$$

(28)

$$\begin{align*}
z_{i+1} &= \frac{1}{m_0 \omega^2} (k_{31} x_i + k_{32} y_i + k_{33} z_i + k_{34} V_{x_i} + k_{35} V_{y_i} + k_{36} V_{z_i}) \\
&+ \frac{V_{x_i}}{m_0 \omega} \sin (\omega t_i) - \frac{1}{m_0 \omega^2} (k_{31} x_i + k_{32} y_i + (k_{33} - m_0 \omega^2) z_i + k_{34} V_{x_i} \\
&+ k_{35} V_{y_i} + k_{36} V_{z_i}) \cos (\omega t_i)
\end{align*}$$

(29)

$$\begin{align*}
V_{x(i+1)} &= -\frac{1}{m_i} [k_{11} x_i + (6 m_0 \omega + k_{12}) y_i + k_{13} z_i] \\
&+ (k_{14} - 4 m_0 \omega) V_{x_i} + k_{15} V_{y_i} + k_{16} V_{z_i} \cos \omega t_i \\
&+ \frac{4}{m_0} (k_{11} x_i + k_{12} y_i + k_{13} z_i + k_{14} V_{x_i}) \\
&+ (k_{15} + 0.5 m_0 \omega) V_{y_i} + k_{16} V_{z_i} & \sin \omega t_i \\
&+ \frac{1}{m_0} (2 k_{11} + (6 m_0 \omega^2 + k_{12}) y_i + k_{13} z_i] \\
&+ (k_{14} - 3 m_0 \omega) V_{x_i} + k_{15} V_{y_i} + k_{16} V_{z_i}] \\
&- \frac{3}{m_i} (k_{11} x_i + k_{12} y_i + k_{13} z_i + k_{14} V_{x_i} + k_{15} V_{y_i} + k_{16} V_{z_i}) t_i
\end{align*}$$

(30)

$$\begin{align*}
V_{y(i+1)} &= (3 \omega y_i - 2 V_{x_i}) \sin \omega t_i + V_{x_i} \cos \omega t_i + 4 y_i
\end{align*}$$

(31)

$$\begin{align*}
V_{z(i+1)} &= V_{z_i} \cos \omega t_i + \frac{1}{m_0} (k_{31} x_i + k_{32} y_i + (k_{33} - m_0 \omega^2) z_i) \\
&+ k_{34} V_{x_i} + k_{35} V_{y_i} + k_{36} V_{z_i} \sin \omega t_i
\end{align*}$$

(32)

where $m_i (i = 1, 2, \ldots, N)$ is the initial mass of the space robot at the beginning of the $i$th thrust arc; $t_i, t'_i, t''_i$ are the theoretical working time of thrusters in the $i$th thrust arc, respectively; $x_i, y_i, z_i$ and $V_{x_i}, V_{y_i}, V_{z_i}$ ($i = 1, 2, \ldots, N$) are the initial position and the initial relative velocity of the space robot at the beginning of the $i$th thrust arc; $x_{i+1}, y_{i+1}, z_{i+1}$ and $V_{x(i+1)}, V_{y(i+1)}, V_{z(i+1)}$ ($i = 1, 2, \ldots, N-1$) are the target maneuver position and the target relative velocity of the space robot at the end of the $i$th thrust arc. According to Eqs. (27)–(32), the theoretical working time of thrusters in the $i$th thrust arc are $t_i, t'_i, t''_i$ can be calculated, respectively.

Then a new algorithm of constant thrust fitting is proposed by using the impulse compensation method as follows. Define that the actual constant thrusts of the space robot are $F_x, F_y, F_z$ and the maximum thrusts are $F'_x, F'_y, F'_z$ in the x-axis, y-axis and z-axis, respectively. Suppose that the thrusters in these three axes can provide different sizes of constant thrust to meet different thrust requirements. Taking the thruster in x-axis as an example, suppose that there are $N_x$ (which is the number of the thrust levels) different sizes of constant thrust denoted as $F_{x_1}, F_{x_2}, F_{x_3}, F_{x_4}, N_{(x-1)} F_{x_1}, \ldots, N_{(x)} F_{x_1}, F_x$.

The physical meaning of the thrust $F_x > 0$ is $F_x$ along the positive x-axis and the first thruster are open, and that of the thrust $F_x < 0$ is $F_x$ along the negative x-axis and the third thruster are open, and the thrust $F_x = 0$ means no thrusters along the x-axis are open. The similar defining method can be applied to the y-axis and the z-axis.
Therefore the constant thrust fitting should be discussed in several categories. For convenience, take the x-axis thrust $F_x$ for example.

Case 1. If the theoretical working time of x-axis thruster in the $i$th thrust arc is $t_i' = 0$, then the actual constant thrust of the space robot in the x-axis is $F_x = F_x' = 0$.

Case 2. If the theoretical working time of x-axis thruster in the $i$th thrust arc is $\Delta T \leq t_i' \leq T_0 = M_0 \Delta T$, then the constant thrust fitting should be discussed in several subcategories.

Case 3. If the theoretical working time of x-axis thruster in the $i$th thrust arc is $t_i' = \Delta T < T_0$ and $t_i'$ can be anyone of $M_0$ shortest switching time interval in the $i$th thrust arc. Without loss of generality, suppose that $t_i'$ is the first shortest switching time interval and the impulse error of the x-axis in the $i$th thrust arc $\Delta I_{ix}$ can be calculated as follows:

Step 1. Choose the size of the constant thrust in Case 1.

There are $N_x + 1$ thrust levels can be selected: $\frac{\lambda F_x}{N_x}$ ($j = 0, 1, \ldots, N_x$) and the level of the constant thrust in Case 3 can be calculated as

$$\lambda = \left[ N_x \int_{T_i}^{T_i + \Delta T} \frac{|F_x'(t)|}{F_x} \right]$$

(34)

where $\lfloor \cdot \rfloor$ means the bracket function; $|F_x'(t)|$ means the absolute value of $F_x'$; $\lambda$ is the maximum constant thrust. Then $\frac{\lambda F_x}{N_x}$ ($\lambda = 0, 1, \ldots, N_x$) can be calculated by the Eq. (34).

Step 2. Calculate the impulse error.

$$\Delta I_{ix} = \text{sgn}(F_x'(t)) \left| \int_{T_i}^{T_i + \Delta T} \frac{\lambda F_x \Delta T}{N_x} \right|$$

(35)

Step 3. Determine the value of the impulse compensation threshold.

Suppose that the value of the impulse compensation threshold is a positive constant $\gamma > 0$.

(1) If the impulse error $\Delta I_{ix}$ satisfies the condition of Eq. (36),

$$\left| \int_{T_i}^{T_i + \Delta T} |F_x'(t)|dt - \frac{\lambda F_x \Delta T}{N_x} \right| \leq \gamma$$

(36)

the actual constant thrust of the space robot in the x-axis can be calculated as

$$F_x = \text{sgn}(F_x'(t)) \frac{\lambda F_x \Delta T}{N_x}$$

(37)

then the space robot will not carry out impulse compensation,

(2) If the impulse error $\Delta I_{ix}$ satisfies the condition of Eq. (38),

$$\gamma < \left| \int_{T_i}^{T_i + \Delta T} |F_x'(t)|dt - \frac{\lambda F_x \Delta T}{N_x} \right| \leq \hat{F}_x \Delta T$$

(38)

then the space robot should carry out impulse compensation.

The size of the impulse error in the x-axis of the space robot (means the constant thrust impulse compensation of the space robot) can be calculated as

$$\left\{ \begin{array}{ll} \Delta I_{ix} = F_x \Delta T = \frac{\hat{F}_x \Delta T}{N_x} & (F_x'(t) < 0) \\ \Delta I_{ix} = F_x \Delta T = -\frac{\hat{F}_x \Delta T}{N_x} & (F_x'(t) > 0) \end{array} \right.$$ 

(39)

where $F_1$ and $F_3$ represent the size of the first thrust and the third thrust of the space robot. The detailed derivation process of the Case 4 and Case 5 are given in Appendix A.

4. Comparison of fuel consumption and calculation of switch control laws

4.1. Comparison of fuel consumption

We could get the theoretical continuous thrust $F_x$, $F_0 = 0$, $F_x'$ through the robust controller design as described in Section 3, and then the fuel consumption of the space robot under the theoretical continuous thrust can be calculated. In this section, the fuel consumptions under the theoretical continuous thrust and under the actual constant thrust are compared from the perspective of impulse compensation. We have already calculated different impulse compensation according to different conditions from Section 3, and then the fuel savings under the actual constant thrust can be calculated as follows. Without loss of generality, take the fuel savings in Case 5 (in the x-axis) as an example. Suppose that the mass-flow-rate of propellant of the space robot’s thruster is assumed to be $m_0$ in the x-axis. Because the impulse error $\Delta I_{ix}$ is

$$\Delta I_{ix} = \sum_{j=0}^{M_0} \int_{T_i}^{T_i + (j+1)\Delta T} \left( F_x'(t) - \text{sgn}(F_x'(t)) \frac{\hat{F}_x}{N_x} \right) dt$$

(40)

the fuel savings in the x-axis in the $i$th thrust arc $\Delta P_{xi}$ can be calculated as

$$\Delta P_{xi} = \sum_{j=0}^{M_0} \int_{T_i}^{T_i + (j+1)\Delta T} \left( m_0 N_x |F_x'(t)| - \lambda \text{sgn}(F_x'(t)) m_0 \right) dt$$

(41)

4.2. Calculation of switch control laws

There are three types of time intervals in each thrust arc: the accelerating time intervals, the zero-thrust time intervals and the decelerating time intervals. The task of space robot rendezvous maneuver is converted into the calculation of the number and sequence of three types of time intervals in three axes, respectively. In this section, the fuel consumptions under the theoretical continuous thrust and the actual constant thrust are calculated and compared by using the method proposed in this paper. At last, the actual constant thrust switch control laws are obtained through the isochronous interpolation method. Without loss of generality, take the Case 5 in the x-axis as an example. If the impulse error $\Delta I_{ix}$ satisfies the condition of Eq. (42),

$$\left| \sum_{j=0}^{M_0} \int_{T_i}^{T_i + (j+1)\Delta T} \left( F_x'(t) - \text{sgn}(F_x'(t)) \frac{\hat{F}_x}{N_x} \right) dt \right| \leq \gamma$$

(42)

the total number of the accelerating time intervals and the decelerating time intervals in the 1st thrust arc is $M_1$ and the number of zero-thrust time intervals is $M_0 - M_1$. The position of the three types of time intervals is decided by the curve of the theoretical continuous thrust $F_x'$. 
If the impulse error $\Delta I_{ij}$ satisfies the condition of Eq. (43),
\[
\left| \int_{T_i}^{T_i+M_{ij}T} \left( F_s(t) - \int_{T_i}^{T_i+M_{ij}T} \delta \text{sgn}(F_s(t)) \frac{\dot{F}_s}{N_x} \right) dt \right| \leq \gamma \tag{43}
\]
the total number of the accelerating time intervals and the decelerating time intervals is $M_1 - m_1$ and the total number of zero-thrust time intervals is $M_{ij} - M_1 + m_1$, where $m_1$ is the number of zero-thrust time intervals for impulse compensation. The position of the three types of time intervals is decided by the curve of the theoretical continuous thrust $F_s$.

If the impulse error $\Delta I_{ij}$ satisfies the condition of Eq. (44),
\[
\left| \int_{T_i}^{T_i+M_{ij}T} \left( F_s(t) - \int_{T_i}^{T_i+M_{ij}T} \delta \text{sgn}(F_s(t)) \frac{\dot{F}_s}{N_x} \right) dt \right| > \gamma \tag{44}
\]
the total number of the accelerating time intervals and the decelerating time intervals is $M_1 - m_1 + m_2$ and the number of zero-thrust time intervals is $M_{ij} - M_1 + m_1 - m_2$, where $m_1$ is the number of zero-thrust time intervals for impulse error. The position of the three types of time intervals is decided by the curve of the theoretical continuous thrust $F_s$.

At last, the switch control laws for the rendezvous maneuver can be given in three axes. For convenience, take the time intervals in the $i$th thrust arc in the $x$-axis for example:
\[
S_{ci} = \left\{ \left( T_i + j\Delta T, \delta \text{sgn}(F_s(t)) \frac{\dot{F}_s}{N_x} \right) \right\} \quad j = 1, 2, \ldots, M_{ij} \tag{45}
\]
where $S_{ci}$ is the switch control laws in the $x$-axis in the $i$th thrust arc.

5. Simulation example

The height of target spacecraft is assumed to be 356 km in a circular orbit, then the mean angular velocity is $\Delta \omega = 0.0654 \times 10^{-3}$ rad/s and the uncertainty parameters is assumed as $\Delta \omega = \pm 1 \times 10^{-3}$ rad/s. The initial mass of the space robot is assumed to be 180 kg at the beginning of rendezvous maneuver. The size of thrusters is assumed to be $\pm 1200$ N in three axes and the shortest switching time is $\Delta T = 1$ s. The initial position and velocity of the space robot are assumed to be [1000 m, 500 m, $-200$ m] and [$-10$ m/s, $-5$ m/s, 2 m/s]. Suppose that the thrusters in three axes can provide 12 different sizes of constant thrust. Suppose that the value of the impulse compensation threshold is a positive constant $\gamma = 300$ N s. The norms of the uncertainty matrix $\Delta A$ and $\Delta B$ are bounded:
\[
\|\Delta A\|_2 \leq \alpha = 0.005, \quad \|\Delta B\|_2 \leq \beta = 0.005 \tag{46}
\]

Then the following results can be obtained by solving Eqs. (19) and (20).

\[
X = 10^3 \times \begin{bmatrix}
1.9148 & 0.0042 & -0.0004 & -0.0928 & -0.0002 & 0 \\
0.0042 & 1.9647 & -0.0102 & 0 & -0.0984 & 0 \\
-0.0004 & -0.0102 & 0.1029 & 0 & 0.0011 & -0.0003 \\
-0.0928 & 0 & 0 & 0.0093 & 0 & 0 \\
-0.0002 & -0.0984 & 0.0011 & 0 & 0.0102 & 0 \\
0 & 0 & -0.0003 & 0 & 0 & 0
\end{bmatrix}
\]

\[
L = 10^3 \times \begin{bmatrix}
-5.6641 & -0.1481 & -0.0005 & -5.4300 & -0.0034 & -0.0004 \\
-0.3833 & -5.6643 & 5.9611 & 0.0121 & -5.4400 & 0.0131 \\
-0.3043 & 0.0307 & 6.5771 & 0.0131 & 0.5805 & -0.2148
\end{bmatrix}
\]

Then the theoretical state feedback controller $K$ can be calculated and the three types of the thrusters working time: the accelerating time intervals, the zero-thrust time intervals and the decelerating time intervals can be calculated. By calculation, $T = 60$ s is the total time of rendezvous and is divided into 20 shortest switching time intervals.

\[
K = \begin{bmatrix}
-0.6036 & 0.0008 & -0.0036 & -11.8482 & 0.0052 & -0.4016 \\
-0.0010 & -0.5783 & 0.6166 & 0.0191 & -11.0384 & -6.2805 \\
0.0010 & -0.0442 & -0.3822 & 0.0059 & -0.9476 & -315.9693
\end{bmatrix}
\]

Fig. 2 shows the changes of $x, y, z$ during rendezvous maneuver. Fig. 3 shows the changes of $V_x, V_y, V_z$ during rendezvous maneuver. By calculation and simulation, $V_x$ changes from $-10$ m/s to 0 m/s, $V_y$ from $-5$ m/s to 0 m/s, and $V_z$ from 2 m/s to 0 m/s.

Figs. 4 and 5 show the changes of theoretical thrust $F_x, F_z$ during rendezvous maneuver and the constant thrust fitting of $F_x$. According to the proposed criterion of this article, the space robot should carry out impulse compensation and the size of the constant thrust impulse compensation is the same but the time of the constant thrust impulse compensation in the $x$-axis is different. Fig. 6 shows the trajectory of space
robot during rendezvous maneuver. It can be seen that with the switch control, the space robot can get to 20 target positions smoothly.

The fuel consumption under the theoretical continuous thrust and under the actual constant thrust can be compared from the perspective of impulse compensation. Without loss of generality, take the fuel savings in the \(x\)-axis as an example.

The theoretical continuous thrust \(F_{x}/C_3\) can be obtained through the robust controller design as mentioned in Section 3, and then the fuel consumption of the space robot under the theoretical continuous thrust can be calculated. Suppose that the mass-flow-rate of propellant of the space robot’s thruster is assumed to be 12 g/s in the \(x\)-axis, then fuel savings satisfy the Case 3. Therefore the fuel savings in the \(x\)-axis can be calculated as

\[
\Delta P_{x,t} = \sum_{i=0}^{20} \Delta P_{x,i} = \sum_{i=1}^{20} \sum_{j=1}^{3} \int_{T_{i}}^{T_{i+j}\Delta T} \left( \frac{12[|F_{x}(t)|]}{100} - 12\lambda m_0 \text{sgn}(F_{x}(t)) \right) dt
\]

\[
= 29.5266 \text{ g}
\]

The switch control laws can be given according to the sizes and the directions of the thrust of the space robot. Take the switch control law in the \(x\)-axis as an example:

\[
S_x = \{1000 \text{ m}; -10 \text{ m/s}; (4\Delta T, 1200 \text{ N}); \ldots; (4\Delta T, -600 \text{ N}); \ldots; (29\Delta T, 0 \text{ N}); (30\Delta T, 600 \text{ N})\}
\]

Finally, the switch sequences and the switch control laws for the entire process of rendezvous maneuver can be obtained according to the thrusters’ working times in three axes and the goal of rendezvous can be achieved.

6. Conclusions

A robust control method for the space robot rendezvous maneuver based on C–W equation is proposed in this paper. In our next study, variable thrust angle constant thrust rendezvous will be studied. In particular, the rendezvous process will be divided into in-plane motion and out-plane motion based on the relative motion dynamic model. For the in-plane motion, the calculation of thrust angle control laws is cast into a convex optimization problem by introducing a Lyapunov function subject to linear matrix inequalities, and the robust controllers satisfying the requirements can be designed by solving this optimization problem. For the out-plane motion, a new algorithm of variable thrust angle constant thrust fitting will be proposed.

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Appendix A.

Case 4: If the theoretical working time of \(x\)-axis thruster in the \(i\)th thrust arc \(T_i = M_i \Delta T\), the impulse error in the \(x\)-axis in the \(i\)th thrust arc \(\Delta I_{x_i}\) can be calculated as

\[
\Delta I_{x_i} = \sum_{j=0}^{M_i} \int_{T_{i+j}\Delta T}^{T_{i+(j+1)\Delta T}} \left( F_{x}(t) - \text{sgn}(F_{x}(t)) \frac{\dot{x}}{N_x} \right) dt
\]  

(A1)

Furthermore, if there exist \(n_1\) shortest switching time intervals satisfying the following conditions, without loss of generality, we suppose that these time intervals are the first
\( n_1 \) shortest switching time interval. Take the \( j \)th shortest switching time interval as an example,
\[
\int_{T_{j+1}}^{T_{j+1} + \Delta T} F_x(t) \, dt \leq \gamma \quad (A2)
\]
then the size of the impulse compensation can be calculated as follows.

(1) If the impulse error \( \Delta I_{ai} \) satisfies the condition of Eq. (A3),
\[
\sum_{j=0}^{M_1} \int_{T_{j+1}}^{T_{j+1} + \Delta T} \left[ F_x(t) - \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \right] \, dt \leq \gamma \quad (A3)
\]
the actual constant thrust of the space robot in the \( x \)-axis can be calculated. Taking the \( j \)th shortest switching time interval as an example,
\[
F_x = \int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt \quad (j = 0, 1, \ldots, M_1)
\]
(A4)

then the space robot will not carry out impulse compensation.

(2) Suppose that
\[
\left[ \int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt - \int_{T_{j+1} + n_1 \Delta T}^{T_{j+1} + M_1 \Delta T} \frac{\dot{\text{sgn}}(F_x(t))}{\Delta T} \, dt \right] = n_2
\]
(A5)

Furthermore, if the impulse error \( \Delta I_{ai} \) satisfies the following condition,
\[
\int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt - \int_{T_{j+1} + n_1 \Delta T}^{T_{j+1} + M_1 \Delta T} \frac{\dot{\text{sgn}}(F_x(t))}{\Delta T} \, dt \leq \gamma
\]
(A6)

the actual constant thrust of the space robot in the \( x \)-axis can be calculated as
\[
F_x = \int_{T_{j+1} + n_1 \Delta T}^{T_{j+1} + M_1 \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt \quad (j = n_1 + 1, n_1 + 2, \ldots, M_1)
\]
(A7)
then the space robot will not carry out impulse compensation.

(3) If the impulse error \( \Delta I_{ai} \) satisfies the condition of Eq. (A8),
\[
\int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt - \int_{T_{j+1} + n_1 \Delta T}^{T_{j+1} + M_1 \Delta T} \frac{\dot{\text{sgn}}(F_x(t))}{\Delta T} \, dt > \gamma
\]
(A8)

then the space robot should carry out impulse compensation and the size of the constant thrust impulse compensation in the \( x \)-axis can be calculated as
\[
\Delta I_{ai} = F_x n_2 \Delta T = \frac{n_2 \dot{F}_x}{N_x} \Delta T \quad (F_x(t) < 0)
\]
\[
\Delta I_{ai} = F_x n_2 \Delta T = -\frac{n_2 \dot{F}_x}{N_x} \Delta T \quad (F_x(t) > 0)
\]
(A9)

**Case 5:** If the theoretical working time of \( x \)-axis thruster in the \( i \)th thrust arc is \( t'_i = H_i \Delta T \), where \( H_i \) (\( i = 1, 2, \ldots, N_i \)) is the \( i \)th time interval; \( 1 < H_i < M_i \), and \( t'_i \) can be any \( H_i \) shortest switching time interval in the \( i \)th thrust arc. Without loss of generality, we suppose that \( t'_i \) is the first \( h_i \) shortest switching time interval and the impulse error in the \( x \)-axis in the \( i \)th thrust arc \( \Delta I_{ai} \) can be calculated as
\[
\Delta I_{ai} = \sum_{j=0}^{H_i} \int_{T_{j+1}}^{T_{j+1} + \Delta T} \left( F_x(t) - \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \right) \, dt \quad (A10)
\]
Furthermore, if there exist \( h_i \) shortest switching time intervals satisfying the following conditions, without loss of generality, we suppose that these time intervals are the first \( h_i \) shortest switching time interval. Taking the \( j \)th shortest switching time interval as an example,
\[
\int_{T_{j+1}}^{T_{j+1} + \Delta T} F_x(t) \, dt \leq \gamma \quad (A11)
\]

Then the size of the impulse compensation can be calculated.

(1) If the impulse error \( \Delta I_{ai} \) satisfies the condition of Eq. (A12),
\[
\sum_{j=0}^{H_i} \int_{T_{j+1}}^{T_{j+1} + \Delta T} \left| F_x(t) - \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \right| \, dt \leq \gamma
\]
(A12)

the actual constant thrust of the space robot in the \( x \)-axis can be calculated. Take the \( j \)th shortest switching time interval as an example,
\[
F_x = \int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt \quad (j = 0, 1, \ldots, H_i)
\]
(A13)
then the space robot will not carry out impulse compensation.

(2) Suppose that
\[
\left[ \int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt - \int_{T_{j+1} + h_i \Delta T}^{T_{j+1} + H_i \Delta T} \frac{\dot{\text{sgn}}(F_x(t))}{\Delta T} \, dt \right] = h_2
\]
(A14)

Furthermore, if the impulse error \( \Delta I_{ai} \) satisfies the condition of Eq. (A15),
\[
\int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt - \int_{T_{j+1} + h_i \Delta T}^{T_{j+1} + H_i \Delta T} \frac{\dot{\text{sgn}}(F_x(t))}{\Delta T} \, dt \leq \gamma
\]
(A15)

the actual constant thrust of the space robot in the \( x \)-axis can be calculated as
\[
F_x = \int_{T_{j+1} + h_i \Delta T}^{T_{j+1} + H_i \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt \quad (j = h_1 + 1, h_1 + 2, \ldots, H_i)
\]
(A16)
them the space robot will not carry out impulse compensation.

(3) If the impulse error \( \Delta I_{ai} \) satisfies the condition of Eq. (A17),
\[
\int_{T_{j+1}}^{T_{j+1} + \Delta T} \text{sgn}(F_x(t)) \frac{\dot{F}_x}{N_x} \, dt - \int_{T_{j+1} + h_i \Delta T}^{T_{j+1} + H_i \Delta T} \frac{\dot{\text{sgn}}(F_x(t))}{\Delta T} \, dt > \gamma
\]
(A17)
then the space robot should carry out impulse compensation and the size of the constant thrust impulse compensation in the x-axis can be calculated as
\[
\Delta I_{ix} = F \frac{h \Delta T}{N_x} \quad (F_x(t) < 0) \tag{A18}
\]
\[
\Delta I_{ix} = F \frac{h \Delta T}{N_x} \quad (F_x(t) > 0)
\]

References


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