Variations on a theme of Frobenius about almost commuting unitary matrices

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Abstract

Let $A$ and $B$ be two rotations in $\mathbb{R}^n$ so that $B$ does not rotate any vector by an angle of $\frac{\pi}{2}$ radians. F.G. Frobenius proved that if $A$ commutes with the commutator $[A, B] = A^{-1}B^{-1}AB$, then $A$ and $B$ are commuting rotations. We prove a generalisation for almost commuting unitary matrices, giving explicit estimations of the error terms, which can be useful in numerical matrix computations.

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1. Notation

We begin by defining the basic terms and notation used throughout this paper. Readers interested in the more important ideas of this work should begin directly at Section 2.

For any matrix $A \in \text{Mat}_n(\mathbb{C})$, we write $A^* = \overline{A}^T$, where $\overline{\cdot}$ and $^T$ denote the operations of complex conjugation and matrix transposition, respectively, and define the norm
\[ \|A\| = \max\{|Az|; \ z \in \mathbb{C}^n \text{ and } |z| = 1\}, \]

where \(|z|\) denotes the Euclidean length of the vector \(z \in \mathbb{C}^n\).

Denote by \(U(n)\) the group of unitary \(n \times n\) complex matrices and by \(I\) or \(I_n\) its identity, i.e.,

\[ U(n) = \{A \in \text{Mat}_n(\mathbb{C}); \ AA^* = I\}. \]

The commutator of two matrices \(A, B \in U(n)\) is defined as

\[ [A, B] = A^{-1} B^{-1} AB = A^* B^* AB. \]

The standard Hermitian product \(\langle \cdot, \cdot \rangle\) on \(\mathbb{C}^n\) is taken to be linear in the first argument. On several occasions in the proofs, we use the notation \(\text{span}_c \{z_1, \ldots, z_k\}\) for the \(c\)-vector space generated by \(z_1, \ldots, z_k \in \mathbb{C}^n\) and \(\text{span}_c^\perp \{z_1, \ldots, z_k\}\) for its orthogonal complement.

2. Preliminaries

A discrete group of Euclidean motions with compact fundamental domain is called a crystallographic group. Any crystallographic group \(G\) acting on an \(n\)-dimensional Euclidean space contains \(n\) linearly independent translations which generate an Abelian subgroup of finite index. This is the first Bieberbach Theorem (see [1,2]).

The difficult part of its proof is to show that \(G\) contains one pure translation. The Euclidean motions in \(G\), which rotate any vector by an angle less than \(\pi/2\), generate a nilpotent subgroup of finite index in \(G\). The following theorem due to F.G. Frobenius (see [6] or [8], Hilfssatz 4.2) is used to show that this nilpotent subgroup is in fact Abelian. Frobenius’ method to prove the first Bieberbach Theorem has, in one form or another, become standard in the literature (cf. [5]).

**Theorem 2.1 (Frobenius).** Let \(A, B \in U(n)\), where \(\|B - I\| < \sqrt{2}\). Then

\( [A, [A, B]] = I \) implies \( [A, B] = I \).

The first Bieberbach Theorem can also be formulated in the language of Riemannian geometry as follows: every compact flat \(n\)-dimensional manifold is a quotient of \(\mathbb{R}^n\) by a torsion-free crystallographic group. More generally, if one considers Gromov’s Almost Flat Manifolds, then one is naturally led to a crystallographic pseudogroup (see [4]). Imitating Gromov’s approach, the author has developed the concept of an essential crystallographic set of isometries \(I\). This is a finite set of Euclidean motions with certain properties characterising a bounded, not perfectly regular crystal (see [9,10]). The author showed that an essential crystallographic set of isometries \(I\) defines a crystallographic group \(G\) containing a slightly perturbed part of \(I\). In the proof of this result, the following generalisation of Frobenius’
Theorem for almost commuting unitary matrices, which is also of independent interest, plays a crucial role:

**Theorem 2.2.** Let \( A, B \in U(n) \), where \( \| B - I \| \leq 1 \) and \( \varepsilon \in \left[ 0, \frac{1}{cn} \right] \). Then

\[
\| [A, [A, B]] - I \| \leq \varepsilon \implies \| [A, B] - I \| \leq c_n \sqrt{\varepsilon}.
\]

The constant \( c_n = (3n)^3 \) depends only on the dimension \( n \).

The proof of this theorem follows the lines of the proof of Theorem 2.1 and is based on the fact that if two unitary matrices \( A \) and \( C \) almost commute, then there exists an explicit construction of a unitary change of basis \( V \) so that \( V^*AV \) and \( V^*CV \) are simultaneously almost diagonal (see Definition 3.1 and Lemma 4.3).

The author believes that Theorem 2.2 could be applied to obtain an improvement of the pinching constant in Gromov’s Almost Flat Manifold Theorem (see [4]), since it is precisely the result that, according to Buser and Karcher (see [3]), is required to adapt Bieberbach’s original proof to prove the Almost Flat Manifold Theorem. He also hopes that the results stated in this paper, especially Lemmas 4.1–4.3, could be of interest in numerical matrix computations.

### 3. Almost diagonal matrices

For the proof of Theorem 2.2 we need some special definitions and lemmas.

**Definition 3.1.** Let \( \varepsilon \in [0, 1] \). The matrix \( A = (a_{ij}) \in U(n) \) is called

(I) \( \varepsilon \)-almost the identity if \( \Re(a_{ii}) \geq 1 - \frac{1}{2} \varepsilon^2 \) for all \( i \in \{1, \ldots, n\} \), and

(II) \( \varepsilon \)-almost diagonal if \( |a_{ii}| \geq 1 - \varepsilon^2 \) for all \( i \in \{1, \ldots, n\} \).

If \( A \) is \( \varepsilon \)-almost the identity and \( z \in \mathbb{C} \) with \( |z| = 1 \), then the matrices \( A \) and \( zA \) are \( \varepsilon \)-almost diagonal. If \( A \) is \( \varepsilon \)-almost the identity or an \( \varepsilon \)-almost diagonal matrix, then \( |a_{ij}| \leq \varepsilon \) for all \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \).

**Lemma 3.2.** Let \( A \in U(n) \). If \( \| A - I \| < \sqrt{2} \), then the diagonal elements \( a_{ii} \) of \( A \) satisfy

\[
|a_{ii}| \geq 1 - \frac{1}{2} \| A - I \|^2 > 0 \quad \text{for all} \ i \in \{1, \ldots, n\}.
\]

**Proof.** Let \( z \in \mathbb{C}^n \) with \( |z| = 1 \). We expand the square of the assumed inequality to get

\[
2 > \| A - I \|^2 \geq |Az - z|^2 = 2 - 2 \Re(Az, z),
\]

and therefore \( \Re(Az, z) \geq 1 - \frac{1}{2} \| A - I \|^2 \). If we take \( z \) to be one of the standard basis vectors \( e_1, \ldots, e_n \) of \( \mathbb{C}^n \), then...
|aii| \geq \Re(a_{ii}) = \Re(Ae_i, e_i) \geq 1 - \frac{1}{2} \|A - I\|^2 > 0,

as required. □

**Corollary 3.3.** Let $E \in U(n)$ and $\epsilon \in [0, 1]$. If $\|E - I\| \leq \epsilon$, then $E$ is $\epsilon$-almost the identity.

The converse of Corollary 3.3 does not hold, but we can show the following weaker result:

**Lemma 3.4.** Let $E \in U(n)$ and $\epsilon \in [0, 1]$. If $E$ is $\epsilon$-almost the identity, then $\|E - I\| \leq \sqrt{n\epsilon}$.

**Proof.** For any $z \in \mathbb{C}^n$ with $|z| = 1$ and for any $i \in \{1, \ldots, n\}$, the Cauchy–Schwartz inequality implies

\[
|(E - I)z_i|^2 = \left|\sum_{k=1}^{n} (e_{ik} - \delta_{ik})z_k\right|^2 \leq \left(\sum_{k=1}^{n} |e_{ik} - \delta_{ik}|^2\right) \cdot \left(\sum_{k=1}^{n} |z_k|^2\right)
\]

\[
= 2 - 2\Re(e_{ii}),
\]

where $\delta_{ij}$ is the Kronecker symbol. From the assumption about $E$ we conclude

\[
\|E - I\| \leq \left(\sum_{k=1}^{n} |(E - I)z_i|^2\right)^{\frac{1}{2}} \leq \sqrt{n\epsilon},
\]

as required. □

**4. Proof of Theorem 2.2**

To prove Theorem 2.2, we follow the lines of the proof of Frobenius’ Theorem, which we quickly recall below (see [8], Hilfssatz 4.2):

**Proof of Theorem 2.1.** Since $A$ and $[A, B]$ commute, we can assume, using a unitary change of basis if necessary, that $A$ and $[A, B]$ are simultaneously diagonal. If we set $C = [A, B]$, then $AB = BAC$, where $A$ and $C$ diagonal. If we compare the diagonal entries, then

\[
a_{ii}b_{ii} = b_{ii}a_{ii}c_{ii} \quad \text{for all } i \in \{1, \ldots, n\}.
\]

We have $|a_{ii}| = 1$ since $A \in U(n)$ is diagonal, and $b_{ii} \neq 0$ since $\|B - I\| < \sqrt{2}$ (see Lemma 3.2). Therefore $c_{ii} = 1$ for all $i \in \{1, \ldots, n\}$. Hence $[A, B] = I$. □
Lemma 4.1. Let $A, B, C \in U(n)$, where $\|B - I\| \leq 1$ and $A, C$ are $\varepsilon$-almost diagonal with $\varepsilon \in [0, \frac{1}{2}]$. If $AB = BAC$, then $C$ is $(7\sqrt{n}\varepsilon)$-almost the identity.

Proof. We set $A = (a_{ij}), B = (b_{ij})$ and $C = (c_{ij})$ and compare the diagonal entries of both sides of the equation $AB = BAC$:

$$
\sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k,l=1}^{n} b_{ik} a_{kl} c_{li} = \sum_{k,l=1}^{n} b_{ik} a_{kl} c_{li} + \sum_{k=1}^{n} b_{ik} a_{ki} c_{ii}.
$$

$$
a_{ii} b_{ii} (1 - c_{ii}) = \sum_{k,l=1}^{n} b_{ik} a_{kl} c_{li} - \sum_{k=1}^{n} a_{ik} b_{ki} c_{ii} + \sum_{k=1}^{n} b_{ik} a_{ki} c_{ii}, \quad (1)
$$

$$
|a_{ii}| |b_{ii}| (1 - c_{ii}) \leq \sum_{k,l=1}^{n} |b_{ik}| |a_{kl}| |c_{li}| + \sum_{k=1}^{n} |a_{ik}| |b_{ki}| + |c_{ii}| \sum_{k=1}^{n} |b_{ik}| |a_{ki}|. \quad (2)
$$

Notice that if $A$ and $C$ were diagonal, then the right-hand side of Eq. (1) would be zero. Using the fact that $A$ and $C$ are $\varepsilon$-almost diagonal, we can bound the right-hand side of Inequality (2) by the following expression:

$$
\left( \sum_{k=1}^{n} |b_{ik}|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{l=1}^{n} \left( \sum_{j=1}^{n} |a_{kl}| |c_{lj}| \right)^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{n} |a_{ik}|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{k=1}^{n} |b_{ki}|^2 \right)^{\frac{1}{2}}
$$

$$
+ |c_{ii}| \left( \sum_{k=1}^{n} |b_{ik}|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{k=1}^{n} |a_{ki}|^2 \right)^{\frac{1}{2}}
$$

$$
\leq \left[ \sum_{k=1}^{n} \left( \sum_{l=1}^{n} |a_{kl}|^2 \right) \cdot \left( \sum_{j=1}^{n} |c_{lj}|^2 \right) \right]^{\frac{1}{2}} + \left( \sum_{k=1}^{n} |a_{ik}|^2 \right)^{\frac{1}{2}}
$$

$$
+ |c_{ii}| \left( \sum_{k=1}^{n} |a_{ki}|^2 \right)^{\frac{1}{2}}
$$

$$
\leq \left( \sum_{k=1}^{n} (1 - |c_{ii}|^2) \right)^{\frac{1}{2}} + 2 \left( 1 - |a_{ii}|^2 \right)^{\frac{1}{2}}
$$

$$
\leq (\sqrt{n} + 2) \varepsilon.
$$
The assumption about $B$ implies $|b_{ii}| \geq \frac{1}{2}$ (see Lemma 3.2). Since $A$ is $\varepsilon$-almost diagonal with $\varepsilon \in [0, \frac{1}{2}]$, we obtain $|a_{ii}| \geq \frac{\sqrt{3}}{2}$ by Definition 3.1. Therefore

$$|1 - c_{ii}| \leq \frac{1}{|a_{ii}||b_{ii}|} (\sqrt{n} + 2) \varepsilon \leq \frac{4}{\sqrt{3}} (\sqrt{n} + 2) \varepsilon.$$

Since $C$ is $\varepsilon$-almost diagonal, $|c_{ii}|^2 \geq 1 - \varepsilon^2$, which implies that

$$\frac{16}{\sqrt{3}^2} (\sqrt{n} + 2)^2 \varepsilon^2 \geq |1 - c_{ii}|^2 = (1 - c_{ii})(1 - \bar{c}_{ii}) = 1 - 2 \text{Re}(c_{ii}) + |c_{ii}|^2 \geq 2 - 2 \text{Re}(c_{ii}) - \varepsilon^2.$$ 

Consequently, $\text{Re}(c_{ii}) \geq 1 - \frac{\sqrt{3}}{49} n \varepsilon^2$ and so $C$ is $(7\sqrt{n}\varepsilon)$-almost the identity. \(\square\)

**Lemma 4.2.** Let $C$ be a given complex $n \times n$ unitary matrix with column vectors $c_1, \ldots, c_n$ satisfying $|\langle c_i, c_j \rangle| \leq \varsigma$ and $1 \geq |c_i|^2 \geq 1 - \varsigma$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$.

Let $\varsigma \in [0, \frac{1}{2n}]$. Then there exists a unitary matrix $C' \in U(n)$ so that

$$\|C - C'\| \leq 4n^2 \varsigma.$$

**Proof.** In a first step we show that the given matrix $C$ is non-singular; in a second step we apply a Gram–Schmidt procedure to its columns to obtain the required unitary matrix $C'$.

(a) We show that $\{c_1, \ldots, c_n\}$ is a basis for $\mathbb{C}^n$.

Using Gershgorin’s Circle Theorem (see [7]), we conclude that every eigenvalue $\lambda$ of the Gram matrix $\langle c_i, c_j \rangle$ satisfies

$$\left| \frac{|c_i|^2 - \lambda}{n-1} \right| \leq \sum_{i=1, i \neq j}^{n} |\langle c_i, c_j \rangle| \leq (n-1) \varsigma.$$

Since $\varsigma < \frac{1}{n}$, all eigenvalues are non-zero. Thus, the Gram matrix has rank $n$. The vectors $c_1, \ldots, c_n$ are therefore linearly independent.

(b) We apply the procedure of Gram–Schmidt\(^1\) inductively to find the orthonormal vectors $c'_1, \ldots, c'_n$.

Normalising all $c_i$, we obtain $c^{(1)}_i = \frac{c_i}{|c_i|}$ and

$$\left| \langle c^{(1)}_i, c^{(1)}_j \rangle \right| = \frac{|\langle c_i, c_j \rangle|}{|c_i||c_j|} \leq \frac{\varsigma}{1 - \varsigma}.$$

\(^1\) As one of the referees pointed out, we could also prove step (b) by replacing the matrix $C$ by the matrix $Q$ of the QR-factorisation $C = QR$ and then estimating the distance $\|C - Q\| = \|QR - Q\| = \|R - I\|$. We do not follow this idea, since the explicit estimation of $\|R - I\|$ uses estimations of the coefficients in the Gram–Schmidt procedure which are as tedious as ours.
We choose $c_1^{(1)}$ as the first unit vector. Then we project all $c_1^{(1)}, \ldots, c_n^{(1)}$ orthogonally onto $\mathrm{span}_C\{c_1^{(1)}\}$ and normalise these projections to get

$$c_i^{(2)} = \frac{c_i^{(1)} - \langle c_i^{(1)}, c^{(1)}_1 \rangle c_1^{(1)}}{|c_i^{(1)} - \langle c_i^{(1)}, c^{(1)}_1 \rangle c_1^{(1)}|}.$$

The following estimates of the resulting change of basis will be needed later:

$$|c_i^{(2)} - c_i^{(1)}|^2 = 2 - 2\sqrt{1 - |\langle c_i^{(1)}, c_1^{(1)} \rangle|^2} \leq 2|\langle c_i^{(1)}, c_1^{(1)} \rangle|^2 \leq 2\left(\frac{\varsigma}{1 - \varsigma}\right)^2,$$

$$||\langle c_i^{(2)}, c_j^{(2)} \rangle|| \leq \sqrt{|\langle c_i^{(1)}, c_j^{(1)} \rangle|^2 - |\langle c_i^{(1)}, c_1^{(1)} \rangle|^2} \cdot \sqrt{|\langle c_j^{(1)}, c_1^{(1)} \rangle|^2 - |\langle c_j^{(1)}, c_1^{(1)} \rangle|^2} \leq \frac{\varsigma}{1 - 2\varsigma}$$

for all $i, j \in \{2, \ldots, n\}$ where $i \neq j$.

Now suppose that the vectors $c_1^{(1)}, \ldots, c_{k-1}^{(1)}$ are orthonormal and $c_k^{(k)}, \ldots, c_n^{(k)} \in \mathrm{span}_C\{c_1^{(1)}, \ldots, c_{k-1}^{(1)}\}$ pairwise satisfy $|\langle c_k^{(k)}, c_j^{(k)} \rangle| \leq \frac{\varsigma}{1-k\varsigma}$ for all $i, j \in \{k, \ldots, n\}$ with $i \neq j$. The next unit vector is $c_k^{(k)}$. We project all $c_k^{(k)}, \ldots, c_n^{(k)}$ orthogonally onto $\mathrm{span}_C\{c_1^{(1)}, \ldots, c_k^{(k)}\}$ and normalise them. We obtain

$$c_i^{(k+1)} = \frac{c_i^{(k)} - \langle c_i^{(k)}, c_k^{(k)} \rangle c_k^{(k)}}{|c_i^{(k)} - \langle c_i^{(k)}, c_k^{(k)} \rangle c_k^{(k)}|}$$

and conclude as above that

$$|c_i^{(k+1)} - c_i^{(k)}| \leq \frac{\sqrt{2}\varsigma}{1-k\varsigma} \quad \text{and} \quad ||\langle c_i^{(k+1)}, c_j^{(k+1)} \rangle|| \leq \frac{\varsigma}{1-(k+1)\varsigma}$$

for all $i, j \in \{k+1, \ldots, n\}$ with $i \neq j$.

Now we proceed inductively to find $c_1^{(1)}, \ldots, c_n^{(n)}$. The maximal change $|c_i - c_i^{(i)}|$ for $i \in \{1, \ldots, n\}$ can be estimated as follows:

$$|c_i - c_i^{(i)}| \leq |c_i - c_i^{(1)}| + \cdots + |c_i^{(i-1)} - c_i^{(i-1)}| \leq \varsigma + \sum_{j=1}^{i} \frac{\sqrt{2}\varsigma}{1-j\varsigma} \leq (1 + 2\sqrt{2}/i)\varsigma \leq 4\varsigma,$$

using the assumption $\varsigma \leq \frac{1}{2\varsigma}$.

We define the unitary matrix $C' \in U(n)$ with column vectors

$$c'_1 = c_1^{(1)}, \ldots, c'_n = c_n^{(n)}.$$

By construction we obtain $|c_{ij} - c'_{ij}| \leq 4\varsigma$ for all $i, j \in \{1, \ldots, n\}$. For any vector $z \in C^n$ with $|z| = 1$, we have
\[(C - C')z_i \leq 4n\sqrt{n}\varsigma \text{ for all } i \in \{1, \ldots, n\},\]
which implies that
\[\|C - C'\| = \max \{ |(C - C')z| : z \in \mathbb{C}^n \text{ and } |z| = 1 \} \leq 4n^2\varsigma,\]
as required. □

**Lemma 4.3.** Let \(A, C \in U(n)\) and \(\epsilon \in [0, \frac{1}{2n}]\). If \(\|A - I\| \leq \epsilon\), then there exists a unitary matrix \(V \in U(n)\) so that \(V^*AV\) and \(V^*CV\) are simultaneously \((3n^{3/2}\sqrt{\epsilon})\)-almost diagonal.

**Proof.** The following proof follows the idea of simultaneous diagonalisation of commuting unitary matrices. It is split up into several shorter parts:

(a) **Partitioning the spectrum of \(A\):**

Let \(\{a_1, \ldots, a_r\}\) be the set of eigenvalues of \(A\). We divide \(\{a_1, \ldots, a_r\}\) into subsets \(P_1, \ldots, P_s\) of nearby eigenvalues as follows: the eigenvalues \(a_i\) and \(a'_i\) belong to the same class \(P\) if and only if there exist indices \(i_1, \ldots, i_k\) so that \(a_{i_1} = a_i\) and \(a_{i_k} = a'_i\) and \(|a_{i_j} - a_{i_{j-1}}| \leq \sqrt{\epsilon}\) for all \(j \in [2, \ldots, k]\). The diameter of a class \(P\) is 
\[\max \{|a_i - a_j| : a_i, a_j \in P\} \leq n\sqrt{\epsilon},\]
since there are at most \(n\) different eigenvalues. The distance between two different classes \(P\) and \(P'\) is
\[\min \{|a - a'| : a \in P, a' \in P', P \neq P'\} > \sqrt{\epsilon}.\]

Let \(\mathbb{C}^n = E_{a_1} \oplus \cdots \oplus E_{a_r}\) be the orthogonal decomposition of \(\mathbb{C}^n\) into eigenspaces of \(A\). For each class \(P\) we define the corresponding subspace
\[E_P = \bigoplus_{a \in P} E_a.\]

For each \(i \in \{1, \ldots, r\}\), let \(p_i : \mathbb{C}^n \to E_{a_i}\) be the orthogonal projection onto the eigenspace \(E_{a_i}\); for each \(j \in \{1, \ldots, s\}\), let \(q_j : \mathbb{C}^n \to E_{P_j}\) be the orthogonal projection onto \(E_{P_j}\).

(b) **Let \(A \in U(n)\) and \(z \in \mathbb{C}^n\) with \(|z| = 1\). Let \(q(z)\) be the orthogonal projection of \(z\) onto the eigenspace \(E_P\).** If \(Az = az + h\) for some eigenvalue \(a \in P\) of \(A\) and for some \(h \in \mathbb{C}^n\), then
\[|z - q(z)| \leq \frac{|h|}{\sqrt{\epsilon}}.\]

Using the notation of part (a), we can write \(A : \mathbb{C}^n \to \mathbb{C}^n\) as
\[Az = a_1p_1(z) + \cdots + a_rp_r(z),\]
where \(z = p_1(z) + \cdots + p_r(z)\) and \(h = p_1(h) + \cdots + p_r(h)\). Comparing components of the equation \(Az = az + h\) gives \(a_i p_i(z) = ap_i(z) + p_i(h)\), so
\[(a_i - a)p_i(z) = p_i(h) \text{ for all } i \in \{1, \ldots, r\} \text{ and } a \in P.\]
The orthogonal projection \( q(z) \) of \( z \) onto \( E_P \) is \( q(z) = p_1(z) + \cdots + p_{i+|P|-1}(z) \), where \( \{a_i, \ldots, a_{i+|P|-1}\} = P \), and \(|P|\) is the cardinality of the set \( P \). Now we calculate

\[
|z - q(z)|^2 = \sum_{k=1}^{r} |p_k(z)|^2 = \sum_{k=1}^{r} |p_k(h)|^2 \frac{|p_k(h)|^2}{|a_k - a|^2}.
\]

Since the eigenvalues \( a \in P \) and \( a_k \) for all \( k \in \{1, \ldots, i - 1, i + |P|, \ldots, r\} \) are in different classes, they satisfy \( |a_k - a| \geq \sqrt{\varepsilon} \). Thus

\[
|z - q(z)|^2 \leq \sum_{k \in \{1, \ldots, i - 1, i + |P|, \ldots, r\}} \frac{|p_k(h)|^2}{\varepsilon} \leq \frac{1}{\varepsilon} \sum_{k=1}^{r} |p_k(h)|^2 = \frac{1}{\varepsilon} |h|^2.
\]

(c) Let \( u_i \in E_{\tilde{a}_i} \) be a unit-length eigenvector of \( A^* \) corresponding to the eigenvalue \( \tilde{a}_i \). The hypothesis \( \| [A, C] - I \| \leq \varepsilon \) implies that

\[
[A, C] = A^*C^*AC \in U(n)
\]

is \( \varepsilon \)-almost the identity (see Corollary 3.3). We set \( E = A^*C^*AC \). Since the matrices \( E^*A^* = C^*A^*C \) and \( A^* \) have the same spectrum, we obtain

\[
A^*C^*u_i = E C^*A^*u_i = \tilde{a}_i E C^*u_i.
\]

Therefore \( C^*u_i \) is a unit-length eigenvector of \( E^*A^* \) corresponding to the eigenvalue \( \tilde{a}_i \); in other words, \( C^*u_i \) is almost an eigenvector of \( A^* \), in the sense that

\[
A^*C^*u_i = \tilde{a}_i C^*u_i + \tilde{a}_i (E - I) C^*u_i
\]

and

\[
|\tilde{a}_i (E - I) C^*u_i| \leq \|E - I\| \cdot |C^*u_i| \leq \varepsilon.
\]

Now we use the result of part (b) with the following settings: replace \( A \) by \( A^* \), \( z \) by \( C^*u_i \) and \( h \) by \( \tilde{a}_i (E - I) C^*u_i \), and let \( P \) be the class containing \( \tilde{a}_i \). Then the projection \( q(C^*u_i) \) of \( C^*u_i \) onto the space \( E_P \) differs from \( C^*u_i \) by

\[
|z - q(z)| = |C^*u_i - q(C^*u_i)| \leq \frac{|h|}{\sqrt{\varepsilon}} = \sqrt{\varepsilon}.
\]

The vector \( C^*u_i \) can be written in the unitary basis \( \{u_1, \ldots, u_n\} \) of eigenvectors of \( A^* \) as \( C^*u_i = \sum_{k=1}^{n} c_{ik} u_k \), where \( C^* = (c_{ik}) \). Let \( U \in U(n) \) be the unitary matrix with column vectors \( u_1, \ldots, u_n \). Since \( U \) is a unitary matrix and \( \{u_1, \ldots, u_i+|P|-1\} \) is a basis for \( E_P \), we immediately get \( |c_{ik}| \leq \sqrt{\varepsilon} \) for all \( k \in \{1, \ldots, i - 1, i + |P|, \ldots, n\} \). We obtain

\[
U^*C^*U = \begin{pmatrix} C_{P_i} & f_{ij} \\ f_{ij}^* & C_{P_i} \end{pmatrix} \in U(n),
\]
i.e., $U^*C^*U$ is almost a block matrix, in the sense that all the matrix entries $f_{ij}$ and $f'_{ij}$ outside the blocks $C_{P_1}, \ldots, C_{P_s}$ satisfy $|f_{ij}| \leq \sqrt{\varepsilon}$ and $|f'_{ij}| \leq \sqrt{\varepsilon}$ for all meaningful $(i, j)$-combinations.

If all eigenvalues of $A^*$ have pairwise distance bigger than $\sqrt{\varepsilon}$, then $|P_j| = \cdots = |P_n| = 1$ and $C_{P_j} \in \text{Mat}_1(C)$ for all $j \in \{1, \ldots, n\}$, and so the proof of Lemma 4.3 is complete. Otherwise we suppose without loss of generality that $|P_1| = m/\varepsilon = 1$ and

$$U^*C^*U = \begin{pmatrix} C_{P_1} \ F' \\ F \end{pmatrix},$$

where $F$ is a complex $(n-m) \times m$-matrix with $|f_{ij}| \leq \sqrt{\varepsilon}$, and $F'$ is a complex $m \times (n-m)$-matrix with $|f'_{ij}| \leq \sqrt{\varepsilon}$ for all meaningful $(i, j)$-combinations.\(^2\)

Since $C_{P_1}$ is in general not a unitary matrix it is not certain that we can diagonalise $C_{P_1}$ with a unitary change of basis. Therefore we apply Lemma 4.2 to find a unitary matrix $C_{P_1}$ “close” to $C_{P_1}$ which is diagonalisable.

Note that the matrix $U^*A^*U$ on the other hand is diagonal by construction.

(d) For each block $C_{P_j}$ there exists a unitary matrix $S_{P_j} \in U(|P_j|)$ so that $S_{P_j}C_{P_j}S_{P_j}$ is almost diagonal. We can therefore replace each $C_{P_j}$ by a nearby diagonalisable matrix of $U(|P_j|)$ using Lemma 4.2. We then apply this new change of basis $S$ to the matrix $C_{P_j}$ and conclude that $S^*U^*C^*US$ is $(\sqrt{8n^2} \sqrt{\varepsilon})$-almost diagonal.

Indeed, $C \in U(n)$ implies $U^*C^*U \in U(n)$, thus we can again suppose without loss of generality that

$$\begin{pmatrix} C_{P_1} \ F' \\ F \end{pmatrix} = \begin{pmatrix} C_{P_1} & F' \\ F & \end{pmatrix} = \begin{pmatrix} C_{P_1}C_{P_1} + F'^*F' & 0 \\ 0 & \end{pmatrix} = \begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix},$$

hence

$$C_{P_1}C_{P_1} = I_m - F'^*F' \quad \text{where } |(F'^*F')_{ij}| \leq n\varepsilon.$$

Let $c_1, \ldots, c_m$ be the column vectors of the matrix $C_{P_1}$. Then

$|\langle c_i, c_j \rangle| \leq n\varepsilon$ and $1 \geq |\langle c_i, c_j \rangle| \geq 1 - n\varepsilon$

for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$.

Applying Lemma 4.2 with $\varsigma = n\varepsilon \leq \frac{1}{2n}$, we obtain a matrix $C'_{P_1} \in U(m)$ close to $C_{P_1}$, i.e.,

$$|C_{P_1} - C'_{P_1}| \leq 4n^3\varepsilon.$$

The unitary matrix $C'_{P_1}$ is diagonalisable, i.e., there exists $S_{P_1} \in U(m)$ with column vectors $s_1, \ldots, s_m$ so that $S_{P_1}C'_{P_1}S_{P_1} = \text{diag}(\lambda_1, \ldots, \lambda_m)$, where $\lambda_1, \ldots, \lambda_m$ is the spectrum of $C'_{P_1}$. Using the decomposition

$$C_{P_1}s_i = C'_{P_1}s_i + (C_{P_1} - C'_{P_1})s_i,$$

\(^2\) The symbol $\star$ denotes matrix entries which are not of interest.
we can estimate the diagonal entries of the matrix $S_{P_i} C_{P_i} S_{P_i}$ as follows:

$$\begin{align*}
|\langle s_i, C_{P_i} s_i \rangle| &\geq |\bar{s}_i| - |s_i| \cdot |(C_{P_i} - C'_{P_i}) s_i| \geq 1 - 4n^3 \varepsilon, \\
|\langle s_i, C_{P_i} s_i \rangle|^2 &\geq 1 - 8n^5 \varepsilon
\end{align*}$$

for all $i \in \{1, \ldots, m\}$. For each block $C_{P_i}$, we can use the same procedure to find a unitary matrix $S_{P_j} \in U(|P_j|)$ with analogous properties. We then set

$$S = \begin{pmatrix} S_{P_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_{P_m} \end{pmatrix} \in U(n).$$

By the definition of almost diagonal matrices, the proof of part (d) is complete.

(e) Investigation of the matrix $S^* U^* A^* U S$.

The matrix $U \in U(n)$ was chosen so that $U^* A^* U$ is diagonal with eigenvalues partitioned into classes $P_1, \ldots, P_m$. Without loss of generality we can again restrict our investigation to the first class $P_1 = \{\bar{a}_1, \ldots, \bar{a}_m\}$:

$$S_{P_1}^* \begin{pmatrix} \bar{a}_1 \\ \vdots \\ \bar{a}_m \end{pmatrix} S_{P_1} = \bar{a}_1 S_{P_1}^* \begin{pmatrix} 1 & 0 \\ 0 & a_1 \bar{a}_2 \\ \vdots \\ 0 & 0 & \cdots & a_1 \bar{a}_m \end{pmatrix} S_{P_1},$$

where $|1 - a_1 \bar{a}_k| = |a_1 - a_k| \leq n \sqrt{\varepsilon}$, so $\text{Re}(a_1 \bar{a}_k) \geq 1 - \frac{1}{2} n^2 \varepsilon$ for all $k \in \{1, \ldots, m\}$. Let

$$Q_{P_i} = \begin{pmatrix} 1 & 0 \\ 0 & a_1 \bar{a}_2 \\ \vdots \\ 0 & 0 & \cdots & a_1 \bar{a}_m \end{pmatrix} \text{ and } S_{P_1}^* \begin{pmatrix} 1 & 0 \\ 0 & a_1 \bar{a}_2 \\ \vdots \\ 0 & 0 & \cdots & a_1 \bar{a}_m \end{pmatrix} S_{P_1}.$$

Note that the above argument implies that the matrix $Q_{P_1}$ is $(n \sqrt{\varepsilon})$-almost the identity. Using Lemma 3.4, we obtain

$$\left\| S_{P_1}^* Q_{P_1} S_{P_1} - I \right\| = \left\| Q_{P_1} - I \right\| \leq n^2 \sqrt{\varepsilon},$$

so the matrix $S_{P_1}^* Q_{P_1} S_{P_1}$ is $(n^2 \sqrt{\varepsilon})$-almost the identity and $S^* U^* A^* U S$ is $(n^2 \sqrt{\varepsilon})$-almost diagonal.

Part (d) implies $S^* U^* C^* U S$ is $(\sqrt{\varepsilon} n^2)$-almost diagonal; part (e) implies $S^* U^* A^* U S$ is $(n^2 \sqrt{\varepsilon})$-almost diagonal. Let $V = US \in U(n)$. Since $\max \{n^2, \sqrt{\varepsilon} n^2\} \leq 3n^2$, we have constructed a unitary matrix $V$ so that $V^* AV$ and $V^* CV$ are simultaneously $(3n^2 \sqrt{\varepsilon})$-almost diagonal. $\square$
Now we have all the ingredients necessary to prove Theorem 2.2:

**Proof of Theorem 2.2.** Let \( C = [A, B] \) and \( E = [A, [A, B]] \in U(n) \). We write \( A = (a_{ij}), B = (b_{ij}) \) and \( C = (c_{ij}) \). By the hypothesis, \( \|E - I\| \leq \varepsilon \).

Using Lemma 4.3, we can find a unitary matrix \( V \in U(n) \) so that \( V^* A V \) and \( V^* [A, B] V \) are simultaneously \((3n^2 \sqrt{\varepsilon})\)-almost diagonal. By definition of the norm, \( \|E - I\| = \|V^* E V - I\| \leq \varepsilon \).

Now set
\[
A' = V^* A V, \quad B' = V^* B V, \quad C' = [A', B'] \quad \text{and} \quad E' = V^* E V.
\]
Then \([A', [A', B']] = E'\) and \( A' B' = B' A' C'\), where \( A' \) and \( C' \) are \((3n^2 \sqrt{\varepsilon})\)-almost diagonal. Lemma 4.1 implies that \( C' \) is \((21n^2 \sqrt{\varepsilon})\)-almost the identity. In other words, since \( V[A', B'] V^* = [A, B] \), Lemma 3.4 implies that
\[
\|V[A', B'] V^* - I\| = \|[A, B] - I\| \leq 21n^2 \sqrt{\varepsilon} \leq (3n)^3 \sqrt{\varepsilon},
\]
which completes the proof. \( \square \)

**Remark 4.4.** Numerical simulations in dimensions smaller than 8 confirm that the square-root dependence on \( \varepsilon \) in Theorem 2.2 cannot be improved. Indeed, they show that there is no hope to get a linear dependence.

On the other hand, the coefficient \( c_n = (3n)^3 \) is obviously not optimal, since it was obtained by relaxing several stricter estimates. It is not clear whether the order of \( c_n \) could be improved.

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