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On statistical exhaustiveness

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ABSTRACT

We study statistical versions of several types of convergence of sequences of functions between two metric spaces. Special attention is devoted to statistical versions of recently introduced notions of exhaustiveness (Gregoriades and Papanastassiou (2008) [4]) and strong uniform convergence on a bornology (Beer and Levi (2009) [3]). We obtain a few results about the continuity of the statistical pointwise limit of a sequence of functions.

1. Introduction

Our notation and terminology are standard, as in [1,2]. Throughout this work, $X = (X, d)$ and $Y = (Y, \rho)$ will always denote metric spaces.

If $x \in X$, $A \subset X$ and $\varepsilon > 0$ is a real number, we write

$$S(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\},$$
$$A^\varepsilon := \bigcup_{a \in A} S(a, \varepsilon),$$

to denote the open ε-ball with center x and the ε-enlargement of A.

Given spaces X and Y we denote by Y^X (resp. $C(X, Y)$) the set of all functions (resp. all continuous functions) from X into Y. The topology of pointwise convergence on these function spaces is denoted by τ_p, and the topology of uniform convergence on compacta by τ_{uc}.

In this work we also consider a new topology on function spaces introduced recently in [3] and called the topology of strong uniform convergence. In fact, we study statistical versions of this convergence and convergence with respect to the aforementioned classical topologies and their relationships with statistical versions of two other recently introduced notions, exhaustiveness and weak exhaustiveness [4]. Also, we give some results concerning continuity of the statistical pointwise limit of a sequence of functions which are far-reaching generalizations and extensions of some results in [4].

2. Preliminaries

In this section we familiarize the reader with the basic notions concerning statistical convergence and bornology.

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2.1. Statistical convergence

The idea of statistical convergence appeared, under the name of almost convergence, in the first edition (Warsaw, 1935) of the celebrated monograph [5] of Zygmund. Explicitly, the notion of statistical convergence of sequences of real numbers was introduced by Fast in [6] and Steinhaus in [7] and is based on the notion of asymptotic density of a set $A \subset \mathbb{N}$. Statistical convergence has many applications in different fields of mathematics: number theory, summability theory, trigonometric series, probability theory, measure theory, optimization, approximation theory and so on. For more information see [8] (where statistical convergence was generalized to sequences in topological and uniform spaces) and references therein. We hope that the results in our article will find applications in the aforementioned areas of mathematics.

Let $A \subset \mathbb{N}$ and $n \in \mathbb{N}$. Put $A(n) := \{k \in A : k \le n\}$. Then one defines

$$\underline{\partial}(A) := \liminf_{n\to\infty} \frac{|A(n)|}{n},$$
$$\overline{\partial}(A) := \limsup_{n\to\infty} \frac{|A(n)|}{n}$$

called the *lower asymptotic density* and *upper asymptotic density* of A, respectively. If $\underline{\partial}(A) = \overline{\partial}(A)$, then

$$\partial(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$$

is called the *asymptotic* (or *natural*) *density* of A.

All three densities, if they exist, are in [0, 1]. We recall also that $\partial(\mathbb{N} \setminus A) = 1 - \partial(A)$ for $A \subset \mathbb{N}$. A set $A \subset X$ is said to be *statistically dense* if $\partial(A) = 1$. Let us mention that the union and intersection of two statistically dense sets in $\mathbb{N}$ are also statistically dense.

A sequence $(x_n)_{n\in\mathbb{N}}$ in a topological space X is said to *converge statistically* (or for short, st-*converge*) to $x \in X$ if for every neighborhood U of x, $\partial(\{n \in \mathbb{N} : x_n \notin U\}) = 0$ [8]. This will be denoted by $(x_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau} x$, where τ is a topology on X.

It was shown in [8, Theorem 2.2] (see [9,10] for $X = \mathbb{R}$) that for the first countable spaces this definition is equivalent to the following statement: there exists a subset A of $\mathbb{N}$ with $\partial(A) = 1$ such that the sequence $(x_n)_{n\in A}$ converges to x.

2.2. Bornology

Recall that a *bornology* on a metric space (X, d) is a family $\mathcal{B}$ of nonempty subsets of X which is closed under finite unions, is hereditary (i.e. closed under taking nonempty subsets) and forms a cover of X [11,12]. A *base* for a bornology $\mathcal{B}$ on (X, d) is a subfamily $\mathfrak{B}_0$ of $\mathfrak{B}$ which is cofinal in $\mathcal{B}$ with respect to the inclusion, i.e. for each $B \in \mathcal{B}$ there is $B_0 \in \mathcal{B}_0$ such that $B \subset B_0$. A base is called *closed* (*compact*) if all its members are closed (compact) subsets of X. For example, the family $\mathcal{F}$ of all nonempty finite subsets of X is a bornology on X; it is the smallest bornology on X and has a closed (in fact a compact) base. Another bornology that will be used in this article is the collection $\mathcal{K}_r$ of all nonempty relatively compact subsets (i.e. subsets with compact closure).

In [3], Beer and Levi defined a new topology $\tau^s_{\mathcal{B}}$ on the set Y^X, named the *topology of strong uniform convergence on a bornology* $\mathcal{B}$ on X, and initiated the study of function spaces Y^X and $\mathrm{C}(X, Y)$ with this new topology. This study was continued further in [13,14]. For a bornology $\mathcal{B}$ on X with *closed base* and for a function $f \in (Y^X, \tau^s_{\mathcal{B}})$, the standard local base at f is the collection of sets

$$[B, \varepsilon]^s(f) = \{g \in Y^X : \exists\delta > 0,\, \rho(g(x), f(x)) < \varepsilon,\ \forall x \in B^\delta\} \quad (B \in \mathfrak{B},\ \varepsilon > 0).$$

The topology $\tau^s_{\mathcal{B}}$ is stronger than the topology of uniform convergence on elements of $\mathcal{B}$.

3. Statistical exhaustiveness

Recently Gregoriades and Papanastassiou [4, Def. 2.1] defined the notion of exhaustiveness (and its relatives) and studied, among other things, its relations with certain types of convergence. The following definition is a statistical version of this notion.

Definition 3.1. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to be *statistically exhaustive* (or for short, st-*exhaustive*) at a point $x \in X$ if for each $\varepsilon > 0$ there are $\delta > 0$ and a statistically dense set $M \subset \mathbb{N}$ such that for each $y \in S(x, \delta)$ we have $\rho(f_n(y), f_n(x)) < \varepsilon$ for each $n \in M$. The sequence $(f_n)_{n\in\mathbb{N}}$ is st-*exhaustive* if it is st-exhaustive at every $x \in X$.

Every exhaustive sequence $(f_n)_{n\in\mathbb{N}}$ is st-exhaustive. The converse need not be true as the following example shows.

Example 3.2. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions in $\mathbb{R}^{\mathbb{R}}$ defined in this way:

$$f_n(x) = \begin{cases} -1 & \text{if } x \le 0,\ n \text{ is prime},\\ 1/n & \text{if } x \le 0,\ n \text{ is not prime},\\ 1 & \text{if } x > 0,\ n \text{ is prime},\\ 1/2n & \text{if } x > 0,\ n \text{ is not prime}. \end{cases}$$

This sequence is st-exhaustive at 0. Indeed, the set P of prime natural numbers has asymptotic density 0, and so $\partial(\mathbb{N}\setminus P) = 1$. Take $\varepsilon > 0$ and $n_0 \in \mathbb{N}\setminus P$ such that $1/2n_0 < \varepsilon$. For each $n \in (\mathbb{N}\setminus P)\cap\{n \in \mathbb{N} : n > n_0\}$ and each $y \in (-1/2, 1/2)$ we have $|f_n(y) - f_n(0)| \leq 1/2n < 1/2n_0 < \varepsilon$.

On the other hand, this sequence is not exhaustive at 0 because for every $\delta > 0$ and every $y \in (-\delta, \delta)$ we have $|f_n(y) - f_n(0)| = 1$ for infinitely many n.

Notice that the subsequence $(f_m)_{m\in P}$ of the sequence $(f_n)_{n\in\mathbb{N}}$ in the previous example is not st-exhaustive. However, the following is true.

Lemma 3.3. *A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is* st*-exhaustive if and only if each of its statistically dense subsequence is* st*-exhaustive.*

Proof. We have only to prove that a statistically dense subsequence $(f_{n_k})_{k\in\mathbb{N}}$ (i.e. the set $M = \{n_k : k \in \mathbb{N}\}$ has density 1) of the st-exhaustive sequence $(f_n)_{n\in\mathbb{N}}$ is also st-exhaustive. Suppose not, and let $x \in X$ and $\varepsilon > 0$ witness this fact. This means that:

$(*)$ For each $\delta' > 0$ and each statistically dense subset T of $\mathbb{N}$ there exist $y \in S(x, \delta')$ and $t \in T$ such that $\rho(f_t(x), f_t(y)) \geq \varepsilon$.

Since $(f_n)_{n\in\mathbb{N}}$ is st-exhaustive there are $\delta > 0$ and a statistically dense $K \subset \mathbb{N}$ such that $\rho(f_k(x), f_k(y)) < \varepsilon$ for each $y \in S(x, \delta)$ and each $k \in K$. On the other hand, by applying $(*)$ to δ and the statistically dense set $K \cap M$, we have that for some $y \in S(x, \delta)$ and some $k_0 \in K \cap M$ it holds that $\rho(f_{k_0}(x), f_{k_0}(y)) \geq \varepsilon$. This is a contradiction. □

The next definition is a statistical version of the classical notion of α-convergence (known also as continuous convergence [2, Chapt. 7]).

Definition 3.4. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X *statistically α-converges* to $f \in Y^X$, denoted as $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\alpha} f$, if for every $x \in X$ and every sequence $(x_n)_{n\in\mathbb{N}}$ in X converging to x, the sequence $(f_n(x_n))_{n\in\mathbb{N}}$ st-converges to $f(x)$.

The following theorem describes relations of st-exhaustiveness to other types of st-convergence.

Theorem 3.5. *For a sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X and a function $f \in Y^X$ the following are equivalent:*

(1) $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\alpha} f$*;*
(2) $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} f$ *and $(f_n)_{n\in\mathbb{N}}$ is* st*-exhaustive;*
(3) *f is continuous and* $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_{uc}} f$.

If X is locally compact, then (1)–(3) are equivalent also to:

(4) *f is continuous and* $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau^s_{\mathcal{K}r}} f$.

Proof. (1) ⇒ (2): It is obvious that (1) implies $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} f$, so we have to prove that $(f_n)_{n\in\mathbb{N}}$ is st-exhaustive. Suppose not. Then there are $x \in X$ and $\varepsilon > 0$ such that:

(✠) For each $n \in \mathbb{N}$ and each statistically dense set $T \subset \mathbf{N}$ there exists $x_n \in S(x, 1/n)$ and $t \in T$ such that $\rho(f_t(x_n), f_t(x)) \geq \varepsilon$.

Since $(f_n)_{n\in\mathbb{N}}$ st $-\alpha$-converges to f and $(x_n)_{n\in\mathbb{N}}$ converges to x, it follows that $(f_n(x_n))_{n\in\mathbb{N}}$ st-converges to $f(x)$. So, there is a statistically dense set $M_1 \subset \mathbb{N}$ such that $\rho(f_m(x_m), f(x)) < \varepsilon/2$ for all $m \in M_1$. On the other hand, since $(f_n)_{n\in\mathbb{N}}$ statistically converges to f at x, there is $M_2 \subset \mathbf{N}$ with $\partial(M_2) = 1$ such that $\rho(f_m(x), f(x)) < \varepsilon/2$ for all $m \in M_2$. Thus for each $m \in M_1\cap M_2$ we have

$$\rho(f_m(x_m), f_m(x)) \leq \rho(f_m(x_m), f(x)) + \rho(f(x), f_m(x)) < \varepsilon,$$

which contradicts (✠) because $M_1 \cap M_2$ has density 1.

(2) ⇒ (3): First we prove that f is continuous. Let $x \in X$ and $\varepsilon > 0$ be fixed. As $(f_n)_{n\in\mathbb{N}}$ is st-exhaustive at x, there is $\delta > 0$ and a set $M_1 \subset \mathbb{N}$ with $\partial(M_1) = 1$ such that for every $y \in S(x, \delta)$ we have $\rho(f_n(x), f_n(y)) < \varepsilon/3$ for all $n \in M_1$. Fix $z \in S(x, \delta)$. Since $(f_n(x))_{n\in\mathbb{N}}$ and $(f_n(z))_{n\in\mathbb{N}}$ st-converge to $f(x)$ and $f(z)$, respectively, there are statistically dense subsets M_2 and M_3 of $\mathbb{N}$ such that $\rho(f_n(x), f(x)) < \varepsilon/3$ for every $n \in M_2$ and $\rho(f_n(z), f(z)) < \varepsilon/3$ for each $n \in M_3$. Pick an arbitrary $m \in M_1 \cap M_2 \cap M_3$. Then

$$\rho(f(x), f(z)) \leq \rho(f(x), f_m(x)) + \rho(f_m(x), f_m(z)) + \rho(f_m(z), f(z)) < \varepsilon,$$

i.e. f is continuous at x.

Let now $\varepsilon > 0$ and let K be a compact subset of X. By continuity of f for every $x \in K$ there is μ_x such that $d(x, y) < \mu_x$ implies $\rho(f(x), f(y)) < \varepsilon/3$. Since $(f_n)_{n\in\mathbb{N}}$ is st-exhaustive at each $x \in K$, there exist $\delta_x > \mu_x$, $x \in K$, and sets $P_x \subset \mathbf{N}$ of asymptotic density 1 such that for each $y \in S(x, \delta_x)$ and each $n \in P_x$ one has $\rho(f_n(y), f_n(x)) < \varepsilon/3$. By compactness of K there are finitely many $x_1, \ldots, x_k$ such that $K \subset \bigcup_{1=1}^{k} S(x_i, \delta_{x_i})$.

As, by (2), $(f_n(x_i))_{n\in\mathbb{N}}$ st-converges to $f(x_i)$ for every $i \le k$, there are sets $Q_i \subset \mathbb{N}, i \le k$, such that $\partial(Q_i) = 1$ and $\rho(f_n(x_i), f(x_i)) < \varepsilon/3$ for every $n \in Q_i$. By continuity of f at every x_i, there are $\delta_i > 0$ such that for every $y \in S(x_i, \delta_i)$, $\rho(f(x_i), f(y)) < \varepsilon/3$. Let $M = \bigcap_{i=1}^{k}(P_{x_i} \cap Q_i)$ and $\delta = \min\{\delta_{x_1}, \dots, \delta_{x_k}, \delta_1, \dots, \delta_k\}$. Let $z \in K$ be arbitrary. Then $z \in S(x_i, \delta_{x_i})$ for some $i \le k$ and thus for every $m \in M$ we have

$$\rho(f_m(z), f(z)) \le \rho(f_m(z), f_m(x_i)) + \rho(f_m(x_i), f(x_i)) + \rho(f(x_i), f(z)) < \varepsilon.$$

Thus $(f_n)_{n\in\mathbb{N}}$ uniformly converges to f on K.

(3) ⇒ (1): Let $\varepsilon > 0$ and $x \in X$ be given. Suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in X converging to x. Since $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact set in X, by (3) there exists a set $M_0 \subset \mathbb{N}$ such that $\partial(M_0) = 1$ and for every $z \in K$ and every $m \in M_0$, $\rho(f_m(z), f(z)) < \varepsilon/2$. Since f is continuous at x, there is $\delta > 0$ such that $\rho(f(x), f(y)) < \varepsilon/2$ for every $y \in S(x, \delta)$. Also, $(x_n)_{n\in\mathbb{N}}$ converges to x and thus there is $n_0 \in \mathbb{N}$ such that $x_n \in S(x, \delta)$ for every $n \ge n_0$. The set $M = M_0 \cap \{n \in \mathbb{N} : n \ge n_0\}$ is statistically dense in $\mathbb{N}$ and for each $m \in M$ we have

$$\rho(f_m(x_m), f(x)) \le \rho(f_m(x_m), f(x_m)) + \rho(f(x_m), f(x)) < \varepsilon.$$

(3) ⇔ (4): It follows from the known fact of [3, Theorem 6.2] that for a locally compact space X, $\tau^s_{\mathcal{K}_r}$ coincides with the compact–open topology on Y^X. □

4. Weak statistical exhaustiveness

In [4], the notion of weak exhaustiveness of a sequence of functions was introduced, which turns out to be equivalent to the continuity of the pointwise limit of such a sequence [4, Theorem 4.2.3]. In this section we give a statistical version of this notion which is unexpectedly also equivalent to the continuity of the statistical pointwise limit of the sequence.

Definition 4.1. A sequence $(f_n)_{n\in\mathbb{N}}$ in Y^X is said to be st-*weakly exhaustive at* $x \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that for each $y \in S(x, \delta)$ there exists a statistically dense subset M_y of $\mathbb{N}$, depending on y, such that for all $n \in M_y$ we have $\rho(f_n(y), f_n(x)) < \varepsilon$. The sequence $(f_n)_{n\in\mathbb{N}}$ is st-*weakly exhaustive* if it is st-weakly exhaustive at every $x \in X$.

Lemma 4.2. *Let* $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} f$ *in* Y^X. *Then* $(f_n)_{n\in\mathbb{N}}$ *is* st-*weakly exhaustive if and only if* f *is continuous.*

Proof. (⇒): Let $x \in X$ and $\varepsilon > 0$. Since $(f_n)_{n\in\mathbb{N}}$ is st-weakly exhaustive at x, there is $\delta > 0$ such that for every $y \in S(x, \delta)$ there is a set $M_y \subset \mathbb{N}$ such that $\partial(M_y) = 1$ and for every $n \in M_y$, $\rho(f_n(y), f_n(x)) < \varepsilon/3$. Also, since $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} f$, there is $M_0 \subset \mathbb{N}$ with $\partial(M_0) = 1$ such that $\rho(f_n(y), f(y)) < \varepsilon/3$ and $\rho(f_n(x), f(x)) < \varepsilon/3$ for every $n \in M_0$. Pick an arbitrary $z \in S(x, \delta)$ and any $m \in M_0 \cap M_z$. Then

$$\rho(f(z), f(x)) \le \rho(f(z), f_m(z)) + \rho(f_m(z), f_m(x)) + \rho(f_m(x), f(x)) < \varepsilon.$$

(⇐): Let $x \in X$ and $\varepsilon > 0$. Since f is continuous at x there is $\delta > 0$ such that for every $y \in S(x, \delta)$, $\rho(f(y), f(x)) < \varepsilon/2$. The st-pointwise convergence of $(f_n)_{n\in\mathbb{N}}$ to f implies the existence of a set $A \subset \mathbb{N}$ with $\partial(A) = 1$ such that for every $n \in A$ we have $\rho(f_n(x), f(x)) < \varepsilon/4$, $\rho(f_n(y), f(y)) < \varepsilon/4$. Thus for every $n \in A$ and every $y \in S(x, \delta)$ we have

$$\rho(f_n(x), f_n(y)) \le \rho(f_n(x), f(x)) + \rho(f(x), f(y)) + \rho(f(y), f_n(y)) < \varepsilon. \quad \square$$

Theorem 4.3. *For a sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\mathrm{C}(X, Y)\mathrm{st} - \tau_p$*-converging to a function* $f \in Y^X$ *the following are equivalent:*

(1) $(f_n)_{n\in\mathbb{N}}$ *is* st-*weakly exhaustive;*
(2) $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau^s_{\mathcal{F}}} f$.
(3) f *is continuous.*

In order to display what we exactly prove we divide the proof of this theorem into the next two propositions.

Proposition 4.4. *Let* $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} f$ *in* Y^X *and let* $(f_n)_{n\in\mathbb{N}}$ *be* st-*weakly exhaustive. Then* $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau^s_{\mathcal{F}}} f$.

Proof. Let $F = \{x_1, \dots, x_k\}$ be a finite subset of X and $\varepsilon > 0$. By assumption, $(f_n)_{n\in\mathbb{N}}$ is st-weakly exhaustive at every x_i, , $i \le k$, so for $i \le k$ there is a $\delta_i > 0$ such that for every $y \in S(x_i, \delta_i)$ there is a statistically dense subset M_y of $\mathbb{N}$ such that $\rho(f_n(y), f_n(x_i)) < \varepsilon/3$ for all $n \in M_y$. On the other hand, for every $i \le k$, the sequence $(f_n(x_i))_{n\in\mathbb{N}}$st-converges to $f(x_i)$ which implies the existence of statistically dense sets $M_i \subset \mathbb{N}$, $i \le k$, such that $\rho(f_n(x_i), f(x_i)) < \varepsilon/3$ for all $n \in M_i$, $i \le k$. By Lemma 4.2, f is continuous at every x_i so there are $\delta'_i > 0$, $i \le k$, such that $y \in S(x_i, \delta'_i)$, $i \le k$, implies $\rho(f(x_i), f(y)) < \varepsilon/3$. Put $\delta = \min\{\delta_1, \dots, \delta_k, \delta'_1, \dots, \delta'_k\}$ and let $z \in F^\delta$; hence $z \in S(x_j, \delta)$ for some $j \le k$. The set $M = M_z \cap \bigcap_{i\le k} M_i$ is statistically dense in $\mathbb{N}$ and for every $m \in M$ we have

$$\rho(f_m(z), f(z)) \le \rho(f_m(z), f_m(x_j)) + \rho(f_m(x_j), f(x_j)) + \rho(f(x_j), f(z)) < \varepsilon. \quad \square$$

Proposition 4.5. *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathrm{C}(X, Y)$ *and* $f \in Y^X$ *such that* $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau^s_{\mathcal{F}}} f$. *Then* $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} f$ *and* $(f_n)_{n\in\mathbb{N}}$ *is* st-*weakly exhaustive.*

Proof. By Lemma 4.2 it suffices to prove that f is continuous. Let $x \in X$, $\varepsilon > 0$. By assumption there is $\delta_x > 0$ and a set $A \subset \mathbb{N}$ with $\partial(A) = 1$ such that for every $n \in A$ and every $y \in S(x, \delta_x)$ it holds that $\rho(f_n(y), f(y)) < \varepsilon/3$. Every f_n, $n \in \mathbb{N}$, is continuous and thus for every $n \in A$ there is $\delta_n > 0$ such that for every $y \in S(x, \delta_n)$, $\rho(f_n(x), f_n(y)) < \varepsilon/3$. Let m be an arbitrary element in A and let $\delta = \min\{\delta_m, \delta_x\}$. Then for any $z \in S(x, \delta)$ we have

$$\rho(f(x), f(z)) \le \rho(f(x), f_m(x)) + \rho(f_m(x), f_m(z)) + \rho(f_m(z), f(z)) < \varepsilon,$$

what we wanted to prove. □

Example 4.6. There is an st-weakly exhaustive sequence of functions which is not st-exhaustive.

Let M be a statistically dense subset of $\mathbb{N}$, say $M = \mathbb{N} \setminus P$, where P is the set of prime numbers. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathbb{R}^{\mathbb{R}}$ defined as follows:

If $n \notin M$, then $f_n(x) = 0$ for each $x \in \mathbb{R}$.

If $n \in M$, then

$$f_n(x) = \begin{cases} 0 & \text{if } x \in (-\infty, -1/n) \cup \{0\} \cup (1/n, \infty), \\ nx + 1 & \text{if } x \in [-1/n, 0), \\ -nx + 1 & \text{if } x \in (0, 1/n]. \end{cases}$$

Then $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} \underline{0}$, where $\underline{0}$ denotes the constantly zero function. By Lemma 4.2, the sequence $(f_n)_{n\in\mathbb{N}}$ is st-weakly exhaustive.

Let us prove that $(f_n)_{n\in\mathbb{N}}$ is not st-exhaustive at $0 \in \mathbb{R}$. Let $\varepsilon \in (0, 1)$ be given, and let K be any statistically dense subset of $\mathbb{N}$ and $\delta > 0$. Pick $k \in K \cap M$. Let $y \in (0, \delta)$ be such that $y < (1 - \varepsilon)/k$. Then $-ky + 1 > \varepsilon$, i.e. $|f_k(y) - f_k(0)| > \varepsilon$. This means that $(f_n)_{n\in\mathbb{N}}$ is not st-exhaustive at 0.

Another statistical convergence that gives continuity for $\mathrm{st} - \tau_p$-convergent sequences of functions in $\mathrm{C}(X, Y)$ is a statistical version of the classical Alexandroff convergence introduced in 1948 in [15] (see [13, Def. 2.8]).

Definition 4.7. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathrm{C}(X, Y)$ is said to be *statistically Alexandroff convergent to* $f \in Y^X$, denoted by $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-Al} f$, provided $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-\tau_p} f$ and for every $\epsilon > 0$ and every statistically dense set $A \subset \mathbb{N}$ there exist an infinite set $M_A = \{n_1 < n_2 < \cdots n_k < \cdots\} \subset A$ and an open cover $\mathcal{U} = \{U_n : n \in A\}$ such that for every $x \in U_k$ we have $\rho(f_{n_k}(x), f(x)) < \epsilon$.

Theorem 4.8. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{C}(X, Y)$ and $f \in Y^X$. If $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-Al} f$, then f is continuous.*

Proof. Let $x \in X$ and let $(x_i)_{i\in\mathbb{N}}$ be a sequence converging to x. We prove that the sequence $(f(x_i))_{i\in\mathbb{N}}$ converges to $f(x)$. Let $\varepsilon > 0$ be given. Since $(f_n(x))_{n\in\mathbb{N}} \xrightarrow{st} f(x)$, there is a statistically dense set $B_x \subset \mathbb{N}$ such that $\rho(f_n(x), f(x)) < \varepsilon/3$ for every $n \in B_x$. By assumption there are an infinite set $M = \{n_1 < n_2 < \cdots < n_k < \cdots\} \subset B_x$ and an open cover $\mathcal{U} = \{U_n : n \in B_x\}$ of X such that for every $z \in U_k$, $\rho(f_{n_k}(z), f(z)) < \varepsilon/3$. Let k be such that $x \in U_k$. Since f_{n_k} is continuous at x and $(x_i)_{i\in\mathbb{N}}$ converges to x, there is $i_0 \in \mathbb{N}$ such that for every $i \ge i_0$, $x_i \in U_k$ and $\rho(f_{n_k}(x_i), f_{n_k}(x)) < \varepsilon/3$. Thus for $i \ge i_0$ we have

$$\rho(f(x_i), f(x)) \le \rho(f(x_i), f_{n_k}(x_i)) + \rho(f_{n_k}(x_i), f_{n_k}(x)) + \rho(f_{n_k}(x), f(x)) < \varepsilon,$$

which means that $(f(x_i))_{i\in\mathbb{N}}$ converges to $f(x)$, i.e. f is continuous at x. □

Problem 4.9. Is the converse of the previous theorem true, i.e. does continuity of f imply $(f_n)_{n\in\mathbb{N}} \xrightarrow{st-Al} f$?

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