Discrete Mathematics 27 (1979) 111–116

ON THE CYCLOMATIC NUMBER OF A HYPERGRAPH

B.D. ACHARYA

The Mehta Research Institute of Mathematics and Mathematical Physics, 26, Dilkusha, New Katra, Allahabad UP 211 002, India

Received 25 January 1978
Revised 6 March 1979

This note generalizes the notion of cyclomatic number (or cycle rank) from Graph Theory to Hypergraph Theory and links it up with the concept of planarity in hypergraphs which was recently introduced by R.P. Jones. Sharp bounds are obtained for the cyclomatic number of the planar hypergraphs and, further, it is shown that the upper bound is attainable if, and only if the hypergraph satisfies Krewera's condition.

Introduction

By a *hypergraph* H we mean an ordered pair $(X, \mathscr{E})$, where X is a finite set whose elements are called *vertices* and $\mathscr{E}$ is a collection of nonempty subsets E of X called *edges*; we then write $H=(X, \mathscr{E})$, $X=X(H)$ and $\mathscr{E}=\mathscr{E}(H)$ for the hypergraph H, its vertex set and its edge set, respectively. The vertices in the set $Y(H)=X(H)-\bigcup_{E\in\mathscr{E}(H)} E$ are called the *isolates* of H. All the hypergraphs H treated in [3] are *isolate-free* in the sense that $Y(H)=\emptyset$.

Multiplicity of a set S of vertices of a hypergraph H, denoted $m(S, H)$, is the number of times S appears as an edge of H. Clearly, for any edge E of H one has $m(E, H)\geq 1$; E is said to be *simple* (*multiple*) if $m(E, H)=1$ ($m(E, H)>1$). A *simple hypergraph* is one in which every edge is simple. A *multigraph* (*graph*) is a hypergraph $H=(X, \mathscr{E})$ such that $|E|=2$ and $m(E, H)\geq 1$ ($m(E, H)=1$) for every edge E in H.

For terms in Hypergraph Theory and Graph Theory, not specifically defined here, we refer the reader to Berge [3] and Harary [7], respectively.

This note is mainly concerned with investigations on a new invariant of a hypergraph, called its *cyclomatic number*, and its links with the notion of planarity in hypergraphs which was recently introduced in [8].

The cyclomatic number of a hypergraph

Let $H=(X, \mathscr{E})$ be a hypergraph. The *intersection multigraph* (or shortly, *IM-graph*) $G(H)$ of H is defined (see [9]) as follows: The vertex set of $G(H)$ is the edge set $\mathscr{E}(H)$ of H and $x(E, E')$ is defined to be an edge of $G(H)$, joining the vertices representing the edges E and E' of H if, and only if $x\in E\cap E'$ in H.

Clearly, every edge of $G(H)$ may be regarded as labeled by the vertex of H it is associated with. Further, H and $G(H)$ have the same number of components and the *representative graph* $L(H)$ of H, as defined by Berge (see [3, p. 400]), is a spanning subgraph of $G(H)$. In particular, if H is an isolate-free graph then $G(H)$ is isomorphic to the well known *line-graph* $L(H)$ of H (i.e. $G(H)=L(H)$). Some special classes of IM-graphs, viz. that of designs, are recently treated in [4, 5].

A cycle $C=(x_1E_1x_2E_2\cdots x_qE_qx_1)$, of length q, in a hypergraph H is said to be *significant* if $q\geq 3$ (see [9]). A *multi-forest* is a multigraph without significant cycles and a *multi-tree* is a connected multiforest. A *maximal spanning multiforest* (*multitree*) *of a multigraph* is a spanning submultiforest (multitree) of that graph having maximum number of edges.

Now, given a simple hypergraph $H=(X,\mathscr{E})$ its *cyclomatic number* $\gamma(H)$ is defined as

$$\gamma(H)=|Y(H)|+\sum_{E\in\mathscr{E}(H)}|E|-|X|-e(T) \tag{1}$$

where $e(T)$ denotes the number of edges in a maximal spanning multiforest of $G(H)$. Clearly, if H is isolate-free then we have

$$\gamma(H)=\sum_{E\in\mathscr{E}(H)}|E|-|X|-e(T) \tag{2}$$

and, further, if H has $k=k(H)$ components $H_1, H_2, \ldots, H_k$ then

$$\gamma(H)=\sum_{i=1}^{k}\gamma(H_i). \tag{3}$$

The reason for calling $\gamma(H)$, defined as in (1), the 'cyclomatic number of H' is the fact that when H is given to be a graph, the expression for $\gamma(H)$ given in (1) reduces to $|\mathscr{E}(H)|-|X(H)|+k(H)$ which is the usual cyclomatic number (or *cycle rank*) defined for the graph (see [3, Chapter 2]; and [7, p. 39]). For a graph H, since $\gamma(H)=|\mathscr{E}(H)|-|X(H)|+k(H)$ gives the number of cycles in a basis for the cycle space of H (see [7, p. 39]) one is naturally led to expect such an interpretation for $\gamma(H)$ when H is given to be a simple hypergraph in general. Though such an interpretation for $\gamma(H)$ in the case of simple hypergraphs which are not graphs is not known so far, some progress has already been made in this direction by determining simple hypergraphs H for which $\gamma(H)=0$ (see [1]).

Cyclomatic number versus planarity in hypergraphs

In this section, we shall link up the notions of planarity and cyclomatic number for the simple hypergraphs. In Graph Theory such a relation already exists. For a planar graph H with V vertices, E edges, and k components it is well known that $\gamma(H)=F-1$, where F is given by the famous Euler Polyhedron Formula $V-E+F=k+1$ and is nothing but the number of faces in a plane imbedding of H (see [7, p. 104]).

The idea of planarity in hypergraphs was recently introduced by R.P. Jones in [8] as follows: Represent the vertices of a simple hypergraph each by its own distinct point in the plane and then represent each edge by a subset of the plane homeomorphic to a closed disc containing all those points representing vertices contained in that edge. If in such a representation the subsets representing any two edges intersect only in points representing vertices common to both the edges, then the representation is called a *plane imbedding* of the hypergraph. Further, a simple hypergraph is said to be *planar* if it has a plane imbedding. For instance, the hypergraph

$$H=(\{1,2,3,4\}, \{\{1,2,3\}, \{2,3,4\}\})$$

has a plane imbedding which is shown in Fig. 1 and therefore H is a planar hypergraph by definition.

In a plane imbedding of a planar hypergraph H, we shall regard the portions of the plane that represent the edges of H as shaded (as shown in Fig. 1) with vertices of H in a given edge appearing as black nodes on the boundary of the region representing that edge. Then the unshaded regions of the plane, including the infinite portion of the plane (also unshaded as our hypergraphs are finite by definition), are called the *faces* of H. Note that the infinite unshaded portion of the plane in a plane imbedding of H is also counted as a face of H because we may regard this portion as having been 'folded over' homeomorphically into a finite closed disc. Thus, for example, there are two faces for the planar hypergraph H shown in Fig. 1.

Not every hypergraph is planar. For example, the famous 'Fano plane' (see [3, p. 428]) is nonplanar.

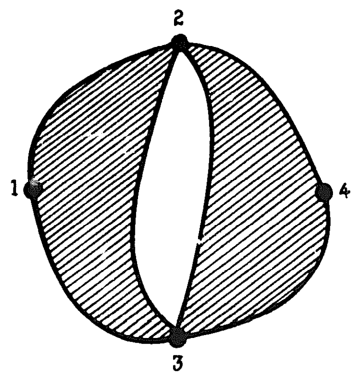

Fig. 1. A planar hypergraph H with $F=2$.

Several characterizations of planar hypergraphs were obtained in [8]. Among many interesting results established therein the following is useful to us here.

Theorem 1. (R.P. Jones). *Let $H=(X,\mathscr{E})$ be a planar simple hypergraph with $F=F(H)$ faces and $k=k(H)$ components. Then*

$$F+|X|+|\mathscr{E}|=1+k+\sum_{E\in\mathscr{E}}|E|. \tag{4}$$

Using this theorem, we can get an expression for the cyclomatic number $\gamma(H)$ of a planar simple hypergraph H in general as follows.

Theorem 2. *For any planar simple hypergraph H,*

$$\gamma(H)=|\mathscr{E}(H)|+F(H)-(e(T)+k(H)+1) \tag{5}$$

where $e(T)$ is as defined in (1).

Proof. This follows from (1) and (4).

Theorems 1 and 2 show how the concepts of cyclomatic number and planarity in hypergraphs are interrelated.

We now establish bounds for the cyclomatic number $\gamma(H)$ of a planar simple hypergraph H in terms of basic parameters of H. In what follows we shall say that a given hypergraph H satisfies *Krewera's condition* whenever any two edges of H have at most one vertex in common (see [6]). We shall need the following lemma whose proof is quite straightforward and, therefore, will be omitted.

Lemma. *Let $H=(X,\mathscr{E})$ be a simple hypergraph and T be a maximal spanning multiforest of $G(H)$. Then*

$$e(T)\geq|\mathscr{E}(H)|-k(H), \tag{6}$$

where equality holds if, and only if H satisfies Krewera's condition.

Theorem 3. *Let $H=(X,\mathscr{E})$ be a planar simple hypergraph such that any two intersecting edges have at most t vertices in common. Then*

$$(F-1)-(t-1)(|\mathscr{E}|-k(H))\leq\gamma(H)\leq F-1 \tag{7}$$

where the bounds are attainable. Further, $\gamma(H)=F-1$ if, and only if H satisfies Krewera's condition.

Proof. Let T be any maximal spanning multiforest of $G(H)$. Then by the Lemma we have $e(T)\geq|\mathscr{E}|-k(H)$. Invoking this fact in (5) we get $\gamma(H)\leq F-1$, the right hand inequality in (7). Further, since $e(T)=|\mathscr{E}|-k$ if, and only if H satisfies

Krewera's condition (by the Lemma) it follows from Theorem 2 that $\gamma(H)=F-1$ if, and only if H satisfies Krewera's condition. The existence of such hypergraphs follows from the truth of this relation (viz. $\gamma=F-1$) for all planar graphs and the fact that there exist infinitely many planar graphs. Moreover, there do exist infinitely many planar hypergraphs which are not graphs and which satisfy Krewera's condition (for instance, any acyclic hypergraph would be one such). Thus, it follows that the upper bound for $\gamma(H)$ in (7) is attainable (in fact, we have obtained a characterization of planar hypergraphs satisfying the relation $\gamma=F-1$).

We now turn to the lower bound for $\gamma(H)$ claimed in (7). Again, let T be any maximal spanning multiforest of $G(H)$. Then it is easy to see that $e(T)\leqslant t(|\mathscr{E}|-k)$ and that $e(T)=t(|\mathscr{E}|-k)$ if, and only if any two intersecting edges of H have exactly t vertices in common. Invoking this fact in (5) we obtain the lower bound for γ in (7). It is also clear that

$$\gamma(H)=(F-1)-(t-1)(|\mathscr{E}|-k) \tag{8}$$

if, and only if any two intersecting edges of H have exactly t vertices in common.

It only remains to show the existence of planar hypergraphs satisfying (8). We give below one such infinite family of planar hypergraphs:

Let q and t be integers such that $q\geqslant 2$ and $t\geqslant 1$. Consider the $2t$-uniform *hypercycle* $H=H(q,2t)$ defined as follows.

(i) H has qt vertices and exactly q edges $E_1, E_2, \ldots, E_q$.
(ii) $|E_i|=2t$ for all $i\in\{1,2,\ldots,q\}$.
(iii) $|E_i\cap E_{i+1}|=|E_q\cap E_1|=t$ for all $i\in\{1,2,\ldots,q-1\}$.
(iv) $E_i\cap E_j=\emptyset$ for all $i,j\in\{1,2,\ldots,q\}$ with $|i-j|\geqslant 2$.
(v) Every vertex of H is contained in exactly two edges of H.

It is easy to see that $H(q,2t)$ has a plane imbedding, hence it is a planar hypergraph. It has $q(t-1)+2$ faces. Therefore,

$$(F-1)-(t-1)(|\mathscr{E}(H)|-k(H))=(q(t-1)+2-1)-(t-1)(q-1)=t.$$

On the other hand, since any maximal spanning multiforest T of $G(H)$ is a multigraph in which multiplicity of each edge is t so that $e(T)=t(q-1)$, one has

$$\begin{aligned}\gamma(H)&=\sum_{E\in\mathscr{E}(H)}|E|-|X(H)|-e(T)\\&=q(2t)-qt-t(q-1)=t.\end{aligned}$$

Thus $H(q,2t)$ is a planar hypergraph for which γ attains the lower bound given in (7).

This completes the proof.

Corollary 3.1. *For every nonnegative integer n there exist infinitely many planar simple hypergraphs H such that $\gamma(H)=n$.*

Proof. If n is a positive integer, the result follows from the last part of the proof of Theorem 3. If $n=0$ then one may consider any simple hypergraph H without significant cycles. All such hypergraphs are planar. Moreover, it is known that $\gamma(H)=0$ for these hypergraphs (see [9]).

Also, we mention here that for every positive integer n the existence of *infinitely* many nonplanar simple hypergraphs H for which $\gamma(H)=n$ is known (see [2])

Acknowledgements

I am thankful to the referee for recommending the present revised version of the rather elaborate original version of this note.

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