Coeffective and de Rham cohomologies on almost contact manifolds

Marisa Fernández¹ and Raúl Ibáñez²

Departamento de Matemáticas, Facultad de Ciencias, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain

Manuel de León³
Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, Spain

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Abstract: We discuss the relation of the coeffective cohomology of some classes of almost contact manifolds with the topology of the manifold. A topological bound for the coeffective numbers is obtained. The lower bound is attained for compact cosymplectic manifolds, and the upper one for contact and (non-compact) exact almost cosymplectic manifolds. Finally, the behaviour of the coeffective cohomology under deformations is studied and an almost cosymplectic version of the Moser Stability Theorem is given.

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1. Introduction

The coeffective cohomology was introduced by T. Bouché in 1990 [6] for symplectic manifolds. In general, given a closed 2-form \( \omega \) on a manifold \( M \) we may consider the mapping \( L : \Lambda^k(M) \rightarrow \Lambda^{k+2}(M) \) defined by \( L\alpha = \alpha \wedge \omega \), where \( \Lambda^k(M) \) is the space of \( k \)-forms on \( M \). The space \( \mathcal{A}^k(M) \) of coeffective \( k \)-forms is defined to be \( \mathcal{A}^k(M) = \text{Ker } L = \{ \alpha \in \Lambda^k(M) \mid \alpha \wedge \omega = 0 \} \), and since \( \omega \) is closed, the complex

\[
\cdots \rightarrow \mathcal{A}^{k-1}(M) \xrightarrow{d} \mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M) \xrightarrow{d} \cdots
\]

is a differential subcomplex of the de Rham complex, thus, it defines the so-called coeffective cohomology of \( M \) with respect to \( \omega \).

Our interest on the coeffective cohomology is to obtain some information on the topology of the manifold by considering geometric structures related to the 2-form \( \omega \). This is, for instance, the case of symplectic and Kähler structures discussed by T. Bouché [6] (see also [2, 12, 13]).

¹ E-mail: mtpferol@lg.ehu.es.
² E-mail: mtpibtor@lg.ehu.es.
³ Corresponding author. E-mail: mdeleon@fresno.csic.es.
The purpose of this paper is to study the coeffective cohomology for almost contact metric structures such that the fundamental 2-form $\Phi$ is closed, in particular, almost cosymplectic and contact structures. For almost cosymplectic manifolds the coeffective cohomology was first considered in [10], where it was proved that for compact cosymplectic manifolds

$$H^k(\mathcal{A}(M)) \cong \tilde{H}^k(M), \quad \forall k \geq n + 2, \quad (1)$$

where $H^k(\mathcal{A}(M))$ denotes the coeffective cohomology group of degree $k$, $\tilde{H}^k(M)$ the de Rham cohomology group of degree $k$ truncated by the class of the fundamental 2-form $\Phi$ and $\dim M = 2n + 1$. But, in general, this isomorphism is not satisfied for arbitrary compact almost cosymplectic manifolds as it was shown in [10, 12].

The paper is structured as follows. In Section 2, we recall the main definitions and results on almost contact metric structures, and we introduce the coeffective cohomology for such kind of manifolds. In Section 3, we relate the coeffective and de Rham cohomologies by means of a long exact sequence in cohomology. From there, we obtain that the coeffective cohomology groups of a manifold of finite type have finite dimension (called the coeffective numbers and denoted by $c_k(M)$). Also, we prove that the coeffective numbers are bounded by topological numbers depending on the Betti numbers $b_k(M)$ of the manifold:

$$b_k(M) - b_{k+2}(M) \leq c_k(M) \leq b_k(M) + b_{k+1}(M), \quad k \geq n + 2. \quad (2)$$

The main problem to exhibit counterexamples of almost contact metric manifolds with closed fundamental 2-form and not satisfying (1) is to compute the coeffective cohomology groups; however, for a compact nilmanifold (resp. a completely solvable manifold) $\Gamma \backslash G$, Nomizu’s Theorem permits us to calculate the de Rham cohomology at the Lie algebra level, and a similar result for the coeffective cohomology has been proved by the authors in [12]. In Section 4 we exhibit a new proof by using the long exact sequence in cohomology described in Section 3.

Section 5 is devoted to discuss the coeffective cohomology of compact cosymplectic and contact manifolds. We prove that the lower bound for the coeffective numbers is obtained for compact cosymplectic manifolds:

$$c_k(M) = b_k(M) - b_{k+2}(M), \quad k \geq n + 2; \quad (3)$$

that is, they are topological invariants that measure the jumps on the Betti numbers. But, in the contrary, we have that the upper bound is overtaken for (compact or not) contact manifolds. Both, compact cosymplectic and Sasakian manifolds have global properties similar to the compact Kähler manifolds [4, 5, 9, 16], but the above results show that their behaviour with respect to the coeffective cohomology dramatically differs. Apart from the above results on the coeffective numbers, the isomorphism (1) is satisfied for compact cosymplectic manifolds, but this is not the case for compact Sasakian manifolds, as we have proved in Section 5 by constructing some counterexamples. Also in that section we exhibit examples of compact almost cosymplectic manifolds not satisfying the isomorphism (1).

In Section 6 we will be concerned with non-compact exact almost cosymplectic manifolds, for which we obtain that

$$c_k(M) = b_k(M) + b_{k+1}(M), \quad k \geq n + 2. \quad (4)$$
Using this formula, we construct examples of non-compact exact cosymplectic manifolds not satisfying the isomorphism (1), in contrast to the case of compact cosymplectic manifolds.

Finally, in Section 7, we study the variation of the coeffective cohomology under deformations of the almost cosymplectic structure on compact manifolds. First, we prove an almost cosymplectic version of the Moser Stability Theorem for compact symplectic manifolds, and then we show that the coeffective cohomology is invariant by strong isotopies, but not by pseudo-isotopies (even if they preserve the Reeb vector field $\xi$ associated to the almost cosymplectic structures).

2. Definitions and basic facts

Let $M$ be a real $(2n + 1)$-dimensional smooth manifold, $\mathfrak{X}(M)$ the Lie algebra of vector fields on $M$ and $\Lambda^k(M)$ the space of $k$-forms on $M$. Suppose that $\Phi \in \Lambda^2(M)$ and $\eta \in \Lambda^1(M)$ are such that $\Phi^\flat \wedge \eta \neq 0$.

(i) The pair $(\Phi, \eta)$ is said to be an \textit{almost cosymplectic structure} on $M$ if $\Phi$ and $\eta$ are closed (i.e., $d\Phi = d\eta = 0$), and $(M, \Phi, \eta)$ is called an \textit{almost cosymplectic manifold}.

(ii) If $\Phi = d\eta$, then $\eta$ is said to be a \textit{contact structure} and $(M, \eta)$ is called a \textit{contact manifold}.

Now, an almost contact metric structure $(\phi, \xi, \eta, g)$ on a $(2n + 1)$-dimensional manifold $M$ is given by a Riemannian metric $g$ and an almost contact structure $(\phi, \xi, \eta)$ such that they are compatible:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \mathfrak{X}(M).$$

Then, the associated fundamental 2-form $\Phi$ can be defined:

$$\Phi(X, Y) = g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

It is not hard to see that $\Phi^\flat \wedge \eta \neq 0$. Then, if $d\Phi = d\eta = 0$, it defines an almost cosymplectic structure, and if $\Phi = d\eta$, it defines a contact structure. Conversely, it is known [4] that given an almost cosymplectic or a contact structure, there exists an associated almost contact metric structure (this is not unique).

Let $(\phi, \xi, \eta, g)$ be an almost contact metric structure on a $(2n + 1)$-dimensional manifold $M$. It is called [4, 20]:

(i) \textit{normal} if $[\phi, \phi] + 2d\eta \otimes \xi = 0$.

(ii) \textit{cosymplectic} if it is normal and almost cosymplectic,

(iii) \textit{quasi-Sasakian} if it is normal and $\Phi$ is closed,

(iv) \textit{Sasakian} if it is contact and normal.

From now on we shall consider that $M$ is a $(2n + 1)$-dimensional almost contact metric manifold with closed fundamental 2-form $\Phi$. If $d$ denotes the exterior derivative on $M$, then we have the de Rham differential complex

$$\cdots \longrightarrow \Lambda^{k-1}(M) \xrightarrow{d} \Lambda^k(M) \xrightarrow{d} \Lambda^{k+1}(M) \longrightarrow \cdots$$

whose cohomology $H^*(M)$ is the de Rham cohomology of $M$. Also, let

$$\mathcal{A}^k(M) = \{ \alpha \in \Lambda^k(M) \mid \alpha \wedge \Phi = 0 \}$$
be the subspace of $\Lambda^k(M)$ of **coeffective forms** on $M$ (in case that we are considering more than one structure on the manifold we shall add a reference to the structure on the notation). Alternatively, we can introduce the linear mapping $L : \Lambda^k(M) \rightarrow \Lambda^{k+2}(M)$ defined by $L(\alpha) = \alpha \wedge \Phi$ and hence

$$A^k(M) = \ker\{L : \Lambda^k(M) \rightarrow \Lambda^{k+2}(M)\}.$$  

Since $\Phi$ is closed, $L$ and $d$ commute which implies that

$$\cdots \rightarrow A^{k-1}(M) \xrightarrow{d} A^k(M) \xrightarrow{d} A^{k+1}(M) \rightarrow \cdots$$

is a differential subcomplex of the de Rham complex. Its cohomology is called **coeffective cohomology** of the manifold and it is denoted by $H^*(\mathcal{A}(M))$.

**Proposition 2.1.** [9] Let $M$ be a $(2n + 1)$-dimensional almost contact metric manifold, then $L$ is injective for $k \leq n - 1$, and surjective for $k \geq n$.

**Corollary 2.2.** $A^k(M) = \{0\}$, for $k \leq n - 1$. And as a consequence, $H^k(\mathcal{A}(M)) = \{0\}$, for $k \leq n - 1$.

On the other hand, since the fundamental 2-form $\Phi$ is closed, it defines a de Rham cohomology class $[\Phi] \in H^2(M)$. Then, we can consider the de Rham cohomology groups truncated by the class of the fundamental 2-form $\Phi$, that is,

$$\tilde{H}^k(M) = \{\alpha \in H^k(M) \mid \alpha \wedge [\Phi] = 0\}.$$  

Notice that in the contact case, $\tilde{H}^k(M) = H^k(M)$.

The relation between the coeffective cohomology and the de Rham cohomology truncated by $[\Phi]$ has been discussed in [10, 12] for compact almost cosymplectic manifolds. In fact, Chinea, de León and Marrero have proved that for a compact cosymplectic manifold $M$ we have

$$H^k(\mathcal{A}(M)) \cong \tilde{H}^k(M), \quad \forall k \neq n, n + 1,$$

where $\dim M = 2n + 1$. This result does not hold for arbitrary almost cosymplectic manifolds as we have seen in [10, 12].

Moreover, from Proposition 2.1 it can be easily seen that for any almost contact metric manifold of dimension $2n + 1$, with $d\Phi = 0$, the natural mapping

$$\psi_k : H^k(\mathcal{A}(M)) \rightarrow \tilde{H}^k(M)$$

defined by

$$\psi(\alpha) = [\alpha]$$

is surjective for $k \geq n + 1$.

(Note that we have denoted by $[\cdot]$ the de Rham cohomology classes and by $\{\cdot\}$ the coeffective cohomology classes.)
3. Coeffective and de Rham cohomologies related by exact sequences

The aim of this section is to relate the coeffective cohomology with the de Rham cohomology by means of a long exact sequence in cohomology. As in the previous section, let \( M \) be an almost contact metric manifold of dimension \( 2n + 1 \) with closed fundamental 2-form \( \Phi \). We consider the following short exact sequence for any degree \( k \):

\[
0 \longrightarrow \text{Ker } I = \mathcal{A}^k(M) \overset{i}{\longrightarrow} \Lambda^k(M) \overset{L}{\longrightarrow} \text{Im}^k+2 I \longrightarrow 0
\]

Since \( L \) and \( d \) commute, then (6) becomes a short exact sequence of differential complexes:

\[
\begin{array}{ccccccc}
0 & 0 & 0 &  &  &  \\
 & L & & L & & L \\
\cdots \longrightarrow & \text{Im}^k+1 L & \overset{d}{\longrightarrow} & \text{Im}^k+2 I. & \overset{d}{\longrightarrow} & \text{Im}^k+3 I. & \longrightarrow \cdots \\
\uparrow & & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \Lambda^{k-1}(M) & \overset{d}{\longrightarrow} & \Lambda^k(M) & \overset{d}{\longrightarrow} & \Lambda^{k+1}(M) & \longrightarrow \cdots \\
\uparrow & & & \uparrow & & \uparrow \\
\cdots \longrightarrow & \mathcal{A}^{k-1}(M) & \overset{d}{\longrightarrow} & \mathcal{A}^k(M) & \overset{d}{\longrightarrow} & \mathcal{A}^{k+1}(M) & \longrightarrow \cdots \\
\uparrow & & & \uparrow & & \uparrow \\
0 & 0 & 0 & & & & \\
\end{array}
\]

Therefore, we have the associated long exact sequence in cohomology [14]:

\[
\cdots \longrightarrow H^k(\mathcal{A}(M)) \overset{H(i)}{\longrightarrow} H^k(M) \overset{H(L)}{\longrightarrow} H^{k+2}(\text{Im } L) \overset{C_{k+2}}{\longrightarrow} H^{k+1}(\mathcal{A}(M)) \longrightarrow \cdots 
\]

where \( H(i) \) and \( H(L) \) are the induced homomorphisms in cohomology by \( i \) and \( L \), respectively, and \( C_{k+2} \) is the connecting homomorphism defined in the following way: for \([\alpha] \in H^{k+2}(\text{Im } L)\), then \( C_{k+2}[\alpha] = [d\beta] \), for \( \beta \in \Lambda^k(M) \) such that \( L\beta = \alpha \).

From Proposition 2.1 we know that \( \text{Im}^{k+2} L = \Lambda^{k+2}(M) \), for \( k \geq n \). As a consequence, we have

\[
H^{k+2}(\text{Im } L) = H^{k+2}(M),
\]

for \( k \geq n + 1 \). Furthermore, the long exact sequence in cohomology (7) may be expressed as

\[
\cdots \longrightarrow H^k(\mathcal{A}(M)) \overset{H(i)}{\longrightarrow} H^k(M) \overset{H(L)}{\longrightarrow} H^{k+2}(M) \overset{C_{k+2}}{\longrightarrow} H^{k+1}(\mathcal{A}(M)) \longrightarrow \cdots 
\]

for such degrees. Now, we shall decompose the long exact sequence (8) in 5-terms exact se-
sequences:
\[ 0 \to \text{Im} C_{k+1} \xrightarrow{i} H^k(\mathcal{A}(M)) \xrightarrow{H(i)} H^k(M) \xrightarrow{H(L)} H^{k+2}(M) \xrightarrow{C_{k+2}} \text{Im} C_{k+2} \to 0. \]  
(9)

If \( M \) is of finite type, the de Rham cohomology groups have finite dimension. Denote by \( b_k(M) = \dim H^k(M) \) the \( k \)th Betti number of \( M \). Since \( 0 \leq \dim(\text{Im} C_k) \leq b_k(M) \), for \( k \geq n + 3 \) we have the following result:

**Proposition 3.1.** Let \( M \) be a \((2n + 1)\)-dimensional almost contact metric manifold with closed fundamental 2-form \( \Phi \) and of finite type, then the coeffective cohomology group \( H^k(\mathcal{A}(M)) \) has finite dimension, for \( k \geq n + 2 \).

Thus, we define the coeffective numbers \( c_k(M) = \dim H^k(\mathcal{A}(M)) \). (Remember that \( c_k(M) = 0 \) for \( k \leq n - 1 \).)

**Remark 3.2.** Note that because the de Rham cohomology groups of a compact manifold have finite dimension (or of manifolds that can be continuously deformed onto a compact subset of itself), we immediately deduce from Proposition 3.1 that the coeffective cohomology groups of a compact almost cosymplectic or contact \((2n + 1)\)-dimensional manifold have also finite dimension for \( k \neq n, n + 1 \). This result was already obtained for compact almost cosymplectic manifolds in [10] by computing the symbol of the coeffective complex.

From (9), we have
\[ \dim(\text{Im} C_{k+1}) - \dim H^k(\mathcal{A}(M)) + \dim H^k(M) - \dim H^{k+2}(M) + \dim(\text{Im} C_{k+2}) = 0, \]
for \( k \geq n + 2 \), from which we deduce
\[ \dim(\text{Im} C_{k+1}) - c_k(M) + b_k(M) - b_{k+2}(M) + \dim(\text{Im} C_{k+2}) = 0. \]  
(10)

Now, as a consequence of (10), we obtain that the coeffective numbers are bounded by upper and lower limits depending on the Betti numbers of the manifold.

**Theorem 3.3.** For \( k \geq n + 2 \), we have
\[ b_k(M) - b_{k+2}(M) \leq c_k(M) \leq b_k(M) + b_{k+1}(M). \]  
(11)

**4. Coeffective cohomology on nilmanifolds and solvmanifolds**

The main problem to construct examples of compact almost cosymplectic, more generally almost contact metric with closed fundamental 2-form, manifolds not satisfying the isomorphism (5) is the difficulty to compute the coeffective cohomology. In [12] we have proved a Nomizu’s type theorem for compact almost cosymplectic nilmanifolds and completely solvable
manifolds. In this section we shall prove such result by a more simple method using the long exact sequence in cohomology (8).

Let \( M = \Gamma \setminus G \) be a compact nilmanifold (resp. solvmanifold), that is, \( G \) is a simply connected nilpotent (resp. solvable) Lie group with discrete subgroup \( \Gamma \) such that the space of right cosets \( \Gamma \setminus G \) is compact. Let

\[
\cdots \longrightarrow \Lambda^{k-1}(g^*) \xrightarrow{d} \Lambda^k(g^*) \xrightarrow{d} \Lambda^{k+1}(g^*) \longrightarrow \cdots
\]

be the differential complex where \( \Lambda^k(g^*) \) denotes the space of left invariant \( k \)-forms on \( G \). Its cohomology \( H^*(g^*) \) is the Chevalley–Eilenberg cohomology of the Lie algebra \( g \) of \( G \). We have

**Theorem 4.1.** (Nomizu [19]) Let \( M = \Gamma \setminus G \) be a compact nilmanifold, then there exists an isomorphism of cohomology groups

\[ H^k(g^*) \cong H^k(M). \]

(notice that the natural map \( m_{\text{dR}} : H^k(g^*) \longrightarrow H^k(M) \) defined by \( m_{\text{dR}}[\alpha^*] = [\alpha] \), where \( \alpha^* \in \Lambda^k(g^*) \) and \( \alpha \) is the projected \( k \)-form on \( M \), is a linear isomorphism.)

From now on, let \((\phi, \xi, \eta, g)\) be an almost contact metric structure on \( M = \Gamma \setminus G \), such that the fundamental 2-form \( \Phi \) and the 1-form \( \eta \) come from left invariant forms \( \Phi^*, \eta^* \) on \( G \), and \( d\Phi^* = d\Phi = 0 \). Consider now the differential subcomplex of coeffective left invariant forms

\[
\cdots \longrightarrow \mathcal{A}^{k-1}(g^*) \xrightarrow{d} \mathcal{A}^k(g^*) \xrightarrow{d} \mathcal{A}^{k+1}(g^*) \longrightarrow \cdots
\]

whose cohomology is denoted by \( H^*(\mathcal{A}(g^*)) \).

By similar arguments that those used in Section 3, if we define the mapping \( L^* : \Lambda^k(g^*) \longrightarrow \Lambda^{k+2}(g^*) \) by \( L\alpha^* = \alpha^* \wedge \Phi^* \), then for any degree we get the short exact sequence

\[
0 \longrightarrow \text{Ker} L^* = \mathcal{A}^k(g^*) \xrightarrow{i^*} \Lambda^k(g^*) \xrightarrow{L^*} \text{Im}^k L^* \longrightarrow 0
\]

and, since \( L^* \) and \( d \) commutes, the associated long exact sequence in cohomology similar to (7) for left invariant forms. As \( L^* \) is surjective in the same degrees as \( L \) (see [12]), then for \( k \geq n + 1 \) we deduce that

\[
\cdots \longrightarrow H^k(\mathcal{A}(g^*)) \xrightarrow{H(i^*)} H^k(g^*) \xrightarrow{H(L^*)} H^{k+2}(g^*) \xrightarrow{C_{k+2}^*} H^{k+1}(\mathcal{A}(g^*)) \longrightarrow \cdots \quad (12)
\]

**Theorem 4.2.** The natural mapping \( m_{\epsilon} : H^k(\mathcal{A}(g^*)) \longrightarrow H^k(\mathcal{A}(M)) \), defined by \( m_{\epsilon}[\alpha^*] = [\alpha] \), is an isomorphism of cohomology groups, for \( k \geq n + 2 \).

**Proof.** Consider the long exact sequences in cohomology (8) and (12), and the mappings \( m_{\text{dR}} \).
$m_c$, that is,

\[ \cdots \to H^k(A(M)) \overset{H(i)}{\to} H^k(M) \overset{H(L)}{\to} H^{k+2}(M) \overset{C_{k+2}}{\to} H^{k+1}(A(M)) \to \cdots \]

\[ \cdots \to H^k(A(g^*)) \overset{H(i^*)}{\to} H^k(g^*) \overset{H(L^*)}{\to} H^{k+2}(g^*) \overset{C_{k+2}}{\to} H^{k+1}(A(g^*)) \to \cdots \]

for $k \geq n + 2$. Since all the diagrams commute and the mappings $m_{dR}$ are linear isomorphisms from Nomizu’s Theorem (Theorem 4.1), then $m_c$ are also linear isomorphisms for $k \geq n + 2$.

(As above, we have denoted by $[\cdot]$ the de Rham cohomology classes and by $\{\cdot\}$ the coeffective cohomology classes.)

**Remark 4.3.** Taking into account that the isomorphism of Theorem 4.1 still holds for completely solvable manifolds (see [15]), and for $G$ a simply connected and solvable Lie group and $\Gamma \subset G$ a lattice such that $\text{Ad} \Gamma$ and $\text{Ad} G$ have the same Zariski closures $\text{Aut}_\Gamma(g)$ (see [21]), then the result of Theorem 4.2 is still true for these cases.

### 5. Compact almost contact metric manifolds

In this section, we shall use the long exact sequence in cohomology (8) and Hodge Theorem [23] to obtain some results on the coeffective cohomology for compact manifolds. In particular, we shall prove that the coeffective cohomology is a topological invariant for compact cosymplectic and contact manifolds. The last part of the section is devoted to discuss the behaviour of the coeffective cohomology for compact quasi-Sasakian manifolds.

#### 5.1. Compact cosymplectic manifolds

We shall study the coeffective cohomology for compact cosymplectic manifolds, which are almost contact metric manifolds enjoying similar properties to the compact Kähler manifolds [4, 9].

**Theorem 5.1.** Let $M$ be a compact cosymplectic manifold of dimension $2n + 1$, then

\[ c_k(M) = b_k(M) - b_{k+2}(M), \quad k \geq n + 2. \]  

**Proof.** To prove this result we shall show that the mapping $C_{k+2}$ identically vanishes, then from (10) we obtain the relation (13).

Let $\alpha \in H^{k+2}(M)$. Taking into account the Hodge theorem [23], we may consider the unique harmonic representative $\alpha$ of the de Rham cohomology class. Since the map $L : \Lambda^k_H(M) \to \Lambda^{k+2}_H(M)$ (where $\Lambda^k_H(M)$ is the space of harmonic $k$-forms) is surjective for $k \geq n$ [9], then there exist a harmonic $k$-form $\beta$ such that $L \beta = \alpha$. From the definition of the connecting homomorphism, $C_{k+2} \alpha = [d \beta] = 0$.  $$\square$$
Remark 5.2. Since the Betti numbers are topological invariants for differentiable manifolds [23], Theorem 5.1 implies that the coeffective cohomology groups are topological invariants for compact cosymplectic manifolds. Moreover, since
\[ b_k(M) \geq b_{k+2}(M), \quad k \geq n + 1, \]
for compact cosymplectic manifolds [9], then the coeffective numbers measure these jumps on the Betti numbers.

Now, we shall show some examples of compact almost cosymplectic (non-cosymplectic) manifolds and their behaviour with respect to the isomorphism (5) and the inequalities (11).

Example 5.3. Let \((M, \omega)\) be a compact symplectic manifold of dimension \(2n\), then the product manifold \(M \times S^1\), with the closed 2-form \(\omega\) and the closed 1-form \(\theta\) (the standard volume form on \(S^1\)), is a compact almost cosymplectic manifold of dimension \(2n + 1\).

Now, we consider the 6-dimensional compact nilmanifold \(R^6 = \Gamma \backslash G\), where \(G\) is a simply connected nilpotent Lie group of dimension 6 defined by the left invariant 1-forms \(\{\alpha_i, 1 \leq i \leq 6\}\) such that
\[
\begin{align*}
d\alpha_1 &= 0, \quad 1 \leq i \leq 3, \\
d\alpha_4 &= -\alpha_1 \wedge \alpha_2, \\
d\alpha_5 &= -\alpha_1 \wedge \alpha_3, \\
d\alpha_6 &= -\alpha_1 \wedge \alpha_4,
\end{align*}
\]
and \(\Gamma\) is a discrete and uniform subgroup of \(G\).

It will be convenient to introduce an abbreviated notation for wedge products; we write \(\alpha_{ij} = \alpha_i \wedge \alpha_j\), \(\alpha_{ijk} = \alpha_i \wedge \alpha_j \wedge \alpha_k\), and so forth.

An easy computation, using Nomizu’s theorem (see Section 4), shows that the de Rham cohomology of \(R^6\) is:
\[
\begin{align*}
H^0(R^6) &= \{1\}, \\
H^1(R^6) &= \{[\alpha_1], [\alpha_2], [\alpha_3]\}, \\
H^2(R^6) &= \{[\alpha_{15}], [\alpha_{16}], [\alpha_{23}], [\alpha_{24}], [\alpha_{35}], [\alpha_{25} + \alpha_{34}]\}, \\
H^3(R^6) &= \{[\alpha_{135}], [\alpha_{145}], [\alpha_{146}], [\alpha_{156}], [\alpha_{234}], [\alpha_{235}], [\alpha_{246}], [\alpha_{236} + \alpha_{245}]\}, \\
H^4(R^6) &= \{[\alpha_{1246}], [\alpha_{1256}], [\alpha_{1356}], [\alpha_{1456}], [\alpha_{2345}], [\alpha_{2346}]\}, \\
H^5(R^6) &= \{[\alpha_{12456}], [\alpha_{13456}], [\alpha_{23456}]\}, \\
H^6(R^6) &= \{[\alpha_{123456}]\}.
\end{align*}
\]

\(R^6\) does not admit Kähler structures because the first Betti number is odd, \(b_1(R^6) = 3\), but it admits symplectic structures, for instance,
\[
\omega = \alpha_{15} + \alpha_{16} + \alpha_{25} + \alpha_{34}.
\]
Therefore, $\tilde{R}^6 = R^6 \times S^1$ is an almost cosymplectic nilmanifold of dimension 7 (notice that it does not admit cosymplectic structures because its first Betti number is even). Its de Rham cohomology is obtained from (14) and the Künneth theorem [14].

Now, Theorem 4.2 permits us to calculate the coeffective cohomology for $\tilde{R}^6$:

$$H^5(\mathcal{A}(\tilde{R}^6)) = \{[\alpha_{12457}], [\alpha_{12467}], [\alpha_{13567}], [\alpha_{12567} - \alpha_{23457}],$$

$$[\alpha_{12567} + \alpha_{23467}], [\alpha_{12456}], [\alpha_{13456}], [\alpha_{23456}]\},$$

$$H^i(\mathcal{A}(\tilde{R}^6)) = H^i(\tilde{R}^6), i = 6, 7,$$

where $\alpha_7 = \theta$.

Therefore, with respect to the inequalities (11), we have:

$$b_5(\tilde{R}^6) - b_7(\tilde{R}^6) = 8 < c_5(\tilde{R}^6) - 9 < b_5(\tilde{R}^6) + b_6(\tilde{R}^6) = 13,$$

$$b_6(\tilde{R}^6) = c_6(\tilde{R}^6) = 4 < b_6(\tilde{R}^6) + b_7(\tilde{R}^6) = 5.$$

And, as it was shown in [12], the isomorphism (5) is not satisfied for $\tilde{R}^6$ in degree 5, because

$$\tilde{H}^5(\tilde{R}^6) = \{[\alpha_{12457}], [\alpha_{13567}], [\alpha_{14567}], [\alpha_{12567} + \alpha_{23457}],$$

$$[\alpha_{23457} + \alpha_{23467}], [\alpha_{12456}], [\alpha_{13456}], [\alpha_{23456}]\},$$

that is, dim $\tilde{H}^5(\tilde{R}^6) = 8$, but $c_5(\tilde{R}^6) = 9$.

**Example 5.4.** Let $(M, \omega)$ be a compact symplectic manifold of dimension $2n$, and let $(N, \Phi, \eta)$ be a compact almost cosymplectic manifold of dimension $2m + 1$. Then the product manifold $M \times N$, with the closed 2-form $\omega + \Phi$ and the closed 1-form $\eta$, is a compact almost cosymplectic manifold of dimension $2(n + m) + 1$.

First, we consider the compact symplectic manifold $(R^6, \omega)$ described in Example 5.3. On the other hand, we consider the 3-dimensional compact completely solvable manifold $M(k) = S_1 \setminus D_1$ (see [3, 17]), where $D_1$ is a simply connected solvable non-nilpotent Lie group of dimension 3 defined by the left invariant 1-forms $\{\beta_1, \beta_2, \beta_3\}$ such that

$$d\beta_1 = -k\beta_1 \wedge \beta_3, \quad d\beta_2 = k\beta_2 \wedge \beta_3, \quad d\beta_3 = 0.$$

$M(k)$ does not admit cosymplectic structures (because $M(k) \times S^1$ does not admit complex, then Kähler, structures [17]), but it admits almost cosymplectic structures, for example:

$$\Phi = \beta_1 \wedge \beta_2, \quad \text{and} \quad \eta = \beta_3.$$

Therefore, $R^6 \times M(k)$ is a compact almost cosymplectic manifold of dimension 9. We shall see that the isomorphism (5) is not satisfied in degree 7. From Section 2 we know that the natural mapping

$$\psi_7 : H^7(\mathcal{A}(R^6 \times M(k))) \longrightarrow \tilde{H}^7(R^6 \times M(k))$$

is surjective; but, we shall see that $\psi_7$ is not injective and therefore,

$$H^7(\mathcal{A}(R^6 \times M(k))) \not\cong \tilde{H}^7(R^6 \times M(k)).$$
The 7-form $\alpha_{1245} \wedge \beta_{123} \in \Lambda^{7}(R^{6} \times M(k))$ is coeffective and closed, then it defines a cohomology class in $H^{7}(A(R^{6} \times M(k)))$ and this class is non-zero, because at the Lie algebra level
\[ \alpha_{1245} \wedge \beta_{123} = d((\alpha_{256} + \alpha_{346}) \wedge \beta_{123}). \] (15)
and $(\alpha_{256} + \alpha_{346}) \wedge \beta_{123}$ is not coeffective, then from Theorem 4.2, the class $\{\alpha_{1245} \wedge \beta_{123}\}$ is non-zero. However, from (15) we have
\[ \psi_{7}\{\alpha_{1245} \wedge \beta_{123}\} = 0. \]

5.2. Compact Sasakian manifolds

This subsection is devoted to study the coeffective cohomology of compact contact manifolds and, in particular, the case of compact Sasakian manifolds. In fact, compact Sasakian manifolds also have many global properties similar to the compact Kähler manifolds [4], but their behaviour with respect to the coeffective cohomology will be dramatically different.

Theorem 5.5. Let $M$ be a contact manifold of dimension $2n + 1$, then
\[ c_{k}(M) = b_{k}(M) + b_{k+1}(M), \quad k \geq n + 2. \]

Proof. From the exactness of the fundamental 2-form $\Phi = d\eta$, the mapping $H(L)$ identically vanishes and then the result follows from (8). \(\square\)

Now, Theorem 5.5 will give us the relation between the coeffective cohomology and the de Rham cohomology (truncated by the class of $\Phi$) for contact manifolds.

Corollary 5.6. Let $M$ be a compact (or non-compact of finite type) contact manifold with $b_{k}(M) \neq 0$ for some $k$ such that $n + 3 \leq k \leq 2n + 1$, then
\[ H^{k-1}(A(M)) \cong \tilde{H}^{k-1}(M) = H^{k-1}(M). \]

Now, we shall show some examples of compact contact manifolds for which the isomorphism (5) is not satisfied.

Example 5.7. The standard contact structure on $\mathbb{C}^{n+1}$ induces a contact structure on the unit sphere $S^{2n+1}$. Moreover, $S^{2n+1}$ with this structure is a Sasakian space form [4].

It is known that the Betti numbers of the sphere $S^{2n+1}$ are:
\[ b_{0}(S^{2n+1}) = b_{2n+1}(S^{2n+1}) = 1, \quad b_{k}(S^{2n+1}) = 0, \quad k = 1, \ldots, 2n. \]

Therefore, from Corollary 5.6 the isomorphism (5) is not satisfied in degree $2n$.

Example 5.8. Consider the $(2n + 1)$-dimensional compact nilmanifold $M(n, 1) = \Gamma \backslash G$ (see [8]), where $G$ is a simply connected nilpotent Lie group of dimension $(2n + 1)$ defined by the left invariant 1-forms $\{\alpha_{i}, \beta_{i}, \gamma \mid 1 \leq i \leq n\}$ such that
\[ d\alpha_{i} = d\beta_{i} = 0, \quad d\gamma = -\sum_{i=1}^{n} \alpha_{i} \wedge \beta_{i}. \]
and Γ is a discrete and uniform subgroup of G.

In [8] Cordero, Fernández and de León have shown that $\mathbb{M}(r, 1)$ can have no cosymplectic structures, but it always admits a Sasakian structure. Denote by $\{U_i, V_i, T \mid 1 \leq i \leq n\}$ the basis of vector fields on $\mathbb{M}(n, 1)$ dual to $\{a_i, \beta_i, \gamma \mid 1 \leq i \leq n\}$, then the almost contact metric structure $(\phi, \eta, \xi, g)$ given by

$$\phi = \sum_{i=1}^{n} \alpha_i \otimes V_i - \sum_{i=1}^{n} \beta_i \otimes U_i, \quad \xi = T, \quad \eta = \gamma,$$

$$g = \sum_{i=1}^{n} \alpha_i \otimes \alpha_i + \sum_{i=1}^{n} \beta_i \otimes \beta_i + \gamma \otimes \gamma,$$

with fundamental form

$$\Phi = d\eta = -\sum_{i=1}^{n} \alpha_i \wedge \beta_i,$$

is a normal contact structure, that is, $\mathbb{M}(n, 1)$ is a compact Sasakian manifold.

Since the Betti numbers of $\mathbb{M}(n, 1)$ are all non-zero for $0 \leq k \leq 2n + 1$, then Corollary 5.6 says us that the isomorphism (5) fails for some degrees.

We shall end this section by relating coeffective forms and harmonicity in Sasakian manifolds.

**Theorem 5.9.** [22] Let $M$ be a $(2n + 1)$-dimensional compact Sasakian manifold with contact form $\eta$. For any harmonic $k$-form $\alpha$ with $k \leq n$, then $i(\xi)\alpha = 0$.

**Corollary 5.10.** Let $M$ be a $(2n + 1)$-dimensional compact Sasakian manifold with contact form $\eta$. For any harmonic $k$-form $\alpha$ with $k \geq n + 1$, then $\alpha \wedge \eta = 0$.

**Theorem 5.11.** Let $M$ be a $(2n + 1)$-dimensional compact Sasakian manifold with contact form $\eta$, then every harmonic $k$-form is coeffective for $k \geq n + 1$.

**Proof.** Let $\alpha$ be an harmonic $k$-form with $k \geq n + 1$. From Corollary 5.10, $\alpha \wedge \eta = 0$ and then $\alpha = i(\xi)(\alpha) \wedge \eta$. Now,

$$d(i(\xi)(\alpha)) = L_{\xi}\alpha - i(\xi)(d\alpha) = 0,$$

since $\xi$ is a Killing vector field and $\alpha$ is harmonic. Therefore,

$$0 = d\alpha = d(i(\xi)(\alpha)) \wedge \eta + (-1)^{k-1}i(\xi)(\alpha) \wedge d\eta = (-1)^{k-1}i(\xi)(\alpha) \wedge \Phi,$$

that is, $i(\xi)\alpha$ is a coeffective $(k - 1)$-form and then $\alpha$ is a coeffective harmonic $k$-form. QED

### 5.3. Compact quasi-Sasakian manifolds

Although quasi-Sasakian structures are apparently close to cosymplectic or Sasakian structures (see Section 2), we shall show, by constructing some examples, that for compact quasi-Sasakian manifolds the isomorphism (5) is not satisfied and the coeffective cohomology does not live in the bounds of (11).
Example 5.12. Consider the compact nilmanifold $M(n, 1)$ studied in Example 5.8, but with different almost contact metric structures. We distinguish the even and odd cases:

Case 1. $n = 2s + 1$ ($s \geq 1$): As we have seen, in $M(n, 1)$ there exists a basis of left invariant 1-forms $\{\alpha_i, \beta_i, \gamma \mid 1 \leq i \leq n\}$ such that

$$d\alpha_i = d\beta_i = 0, \quad d\gamma = -\sum_{i=1}^{n} \alpha_i \wedge \beta_i,$$

and the dual basis of vector fields $\{U_i, V_i, T \mid 1 \leq i \leq n\}$. Then, we take the almost contact metric structure $(\phi, \eta, \xi, g)$ given by

$$\phi = \sum_{i=1}^{s} (\alpha_{2i-1} \otimes U_{2i} - \alpha_{2i} \otimes U_{2i-1} + \beta_{2i-1} \otimes V_{2i} - \beta_{2i} \otimes V_{2i-1})$$

$$+ \alpha_{2s+1} \otimes V_{2s+1} - \beta_{2s+1} \otimes U_{2s+1},$$

$$\xi = T, \quad \eta = \gamma.$$

$$g = \sum_{i=1}^{n} \alpha_i \otimes \alpha_i + \sum_{i=1}^{n} \beta_i \otimes \beta_i + \gamma \otimes \gamma.$$

Then, the associated fundamental 2-form is

$$\Phi = \sum_{i=1}^{s} (\alpha_{2i-1} \wedge \alpha_{2i} + \beta_{2i-1} \wedge \beta_{2i}) + \beta_{2s+1} \wedge \alpha_{2s+1},$$

and hence, $d\Phi = 0, \quad d\eta = -\sum_{i=1}^{n} \alpha_i \wedge \beta_i$. It is no hard to check that $[\phi, \phi] + 2d\eta \otimes \xi = 0$, that is, the almost contact metric structure is normal. Therefore, $(\phi, \eta, \xi, g)$ is a quasi-Sasakian structure on $M(n, 1)$, but it is neither cosymplectic ($d\eta \neq 0$) nor Sasakian ($d\eta \neq \Phi$).

In order to simplify the calculations we shall consider $s = 1$, that is, $\dim M(n, 1) = 7$. First, we consider Nomizu’s Theorem to calculate the de Rham cohomology groups, in particular, we obtain the following Betti numbers:

$$b_5(M(n, 1)) = 14, \quad b_6(M(n, 1)) = 6, \quad b_7(M(n, 1)) = 1,$$

and therefore, if we truncate by the class of the fundamental 2-form $\Phi$:

$$\tilde{b}_5(M(n, 1)) = 13, \quad \tilde{b}_6(M(n, 1)) = 6, \quad \tilde{b}_7(M(n, 1)) = 1.$$

Now, from Theorem 4.2 we compute the coeffective cohomology groups, in particular,

$$c_5(M(n, 1)) = 15, \quad c_6(M(n, 1)) = 6, \quad c_7(M(n, 1)) = 1.$$

Therefore,

(i) $H^5(\mathcal{A}(M(n, 1))) \not\cong \tilde{H}^5(M(n, 1))$,

(ii) $b_5(M(n, 1)) - b_7(M(n, 1)) = 13 < c_5(M(n, 1)) = 15$

$$< b_5(M(n, 1)) + b_6(M(n, 1)) = 20.$$
Case 2. $n = 2s$ ($s \geq 1$): We consider the almost contact metric structure $(\phi, \eta, \xi, g)$ given by

$$\phi = \sum_{i=1}^{s} (\alpha_{2i-1} \otimes U_{2i} - \alpha_{2i} \otimes U_{2i-1} + \beta_{2i-1} \otimes V_{2i} - \beta_{2i} \otimes V_{2i-1}),$$

$$\xi = T, \quad \eta = \gamma,$$

$$g = \sum_{i=1}^{n} \alpha_i \otimes \alpha_i + \sum_{i=1}^{n} \beta_i \otimes \beta_i + \gamma \otimes \gamma.$$

Then, the associated fundamental 2-form is

$$\Phi = \sum_{i=1}^{s} (\alpha_{2i-1} \wedge \alpha_{2i} + \beta_{2i-1} \wedge \beta_{2i}),$$

and hence, $d\Phi = 0$, $d\eta = -\sum_{i=1}^{n} \alpha_i \wedge \beta_i$ and $[\phi, \phi] + 2d\eta \otimes \xi = 0$. Therefore, $(\phi, \eta, \xi, g)$ is a quasi-Sasakian structure on $M(n, 1)$, but it is neither cosymplectic nor Sasakian.

To simplify calculations, take $s = 2$, and so, $\dim M(n, 1) = 9$. From Nomizu’s Theorem we obtain:

$$b_6(M(n, 1)) = 48, \quad b_7(M(n, 1)) = 27, \quad b_8(M(n, 1)) = 8, \quad b_9(M(n, 1)) = 1,$$

and then,

$$\tilde{b}_6(M(n, 1)) = 32, \quad \tilde{b}_7(M(n, 1)) = 26, \quad \tilde{b}_8(M(n, 1)) = 8, \quad \tilde{b}_9(M(n, 1)) = 1.$$

And, from Theorem 4.2 we obtain:

$$c_6(M(n, 1)) = 40, \quad c_7(M(n, 1)) = 26, \quad c_8(M(n, 1)) = 8, \quad c_9(M(n, 1)) = 1.$$

Therefore,

(i) $H^6(A(M(n, 1))) \not\cong \tilde{H}^6(M(n, 1)),$

(ii) $b_6(M(n, 1)) - b_8(M(n, 1)) = 40 < c_6(M(n, 1)) = 42$

$$< b_6(M(n, 1)) + b_7(M(n, 1)) = 75,$$

$$b_7(M(n, 1)) - b_9(M(n, 1)) = c_7(M(n, 1)) = 26$$

$$< b_7(M(n, 1)) + b_8(M(n, 1)) = 35,$$

$$b_8(M(n, 1)) = c_8(M(n, 1)) = 8 < b_8(M(n, 1)) + b_9(M(n, 1)) = 9.$$

6. Non-compact almost contact metric manifolds

In differential geometry and physics there exists important examples of non-compact almost cosymplectic and contact manifolds. This section is devoted to study the coeffective cohomology in the non-compact case.
The first example of non-compact almost cosymplectic manifold is $\mathbb{R}^{2n+1}$ with the standard structure:

$$\Phi_0 = dx_1 \wedge dx_{n+1} + \cdots + dx_n \wedge dx_{2n}, \quad \eta = dx_{2n+1},$$

where $(x_1, \ldots, x_{2n+1})$ are the natural coordinates on $\mathbb{R}^{2n+1}$. Moreover, the standard contact structure on $\mathbb{R}^{2n+1}$ is given by the standard 1-form:

$$\eta = dx_{2n+1} - \sum_{i=1}^{n} x_{n+i} dx_i.$$

The Betti numbers of $\mathbb{R}^{2n+1}$ are well-known:

$$b_0(\mathbb{R}^{2n+1}) = 1, \quad b_k(\mathbb{R}^{2n+1}) = 0, \quad \text{for } k \geq 1. \quad (16)$$

**Proposition 6.1.** Let $(\Phi, \eta)$ be any almost cosymplectic (resp. contact) structure on $\mathbb{R}^{2n+1}$ or on any smooth manifold homeomorphic to $\mathbb{R}^{2n+1}$, then

$$c_k(M) = c_k(\mathbb{R}^{2n+1}) = 0, \quad \text{for } k \neq n, n + 1.$$

Thus, the isomorphism (5) between the coeffective cohomology and the de Rham cohomology truncated by $[\Phi]$ is satisfied for $k \neq n, n + 1$.

**Proof.** If $M$ is homeomorphic to $\mathbb{R}^{2n+1}$, then $b_k(M) = b_k(\mathbb{R}^{2n+1})$. Therefore the result follows from (16) and (8). \(\square\)

**Theorem 6.2.** Let $M$ be a (non-compact) exact almost cosymplectic manifold of dimension $2n + 1$, then

$$c_k(M) = b_k(M) + b_{k+1}(M), \quad \text{for } k \geq n + 2.$$

**Proof.** As in Theorem 5.5. \(\square\)

**Corollary 6.3.** Let $M$ be a (non-compact) exact almost cosymplectic manifold of dimension $2n + 1$, which is of finite type and with $b_k(M) \neq 0$ for some $k$ such that $n + 3 \leq k \leq 2n + 1$, then

$$H^{k-1}(\mathbb{A}(M)) \cong \tilde{H}^{k-1}(M) = H^{k-1}(M).$$

**Example 6.4.** Let $M$ be a compact contact manifold of dimension $2n + 1$ with contact structure $\eta$. The product manifold $\tilde{M} = M \times \mathbb{R}^2$ of dimension $2n + 3$ is an exact almost cosymplectic manifold, with the fundamental 2-form $\Phi = d(e^t \eta) = e^t dt \wedge \eta + e^t d\eta$ (notice that it is exact) and the 1-form $ds$, where $t, s$ are the standard coordinates on $\mathbb{R}^2$. Moreover, if $M$ is Sasakian, then $\tilde{M}$ is cosymplectic.

By the Künneth formula $b_k(\tilde{M}) = b_k(M)$, then if $b_k(\tilde{M}) \neq 0$ for some $k$ such that $k \geq n + 5$, then $\tilde{M}$ satisfies the conditions of the Corollary 5.6 and the isomorphism (5) is not satisfied.

Two particular examples are the compact Sasakian manifolds $S^{2n+1}$ and $M(n, 1)$, for sufficiently large $n$. 
Remark 6.5. The results for non-compact contact manifolds were obtained in Theorem 5.5 and Corollary 5.6.

7. Deformation of almost cosymplectic structures

The purpose of this section is to study the variation of the coeffective cohomology if we perform a deformation of the almost cosymplectic or contact structure on a compact manifold. In Section 5, we have already obtained a partial answer to this question, in fact, we have shown that for a compact cosymplectic or a contact (compact or not) manifold, the coeffective cohomology is a topological invariant.

We shall prove an analogous result for almost cosymplectic structures to the known Moser Stability Theorem [18].

First of all, let us recall the definition of the basic cohomology of an almost contact metric manifold.

For any integer \( k \), we shall denote by \( \Lambda^k_B(M) \) the subspace of \( \text{basic} \ k\text{-forms} \), that is,

\[
\Lambda^k_B(M) = \{ \alpha \in \Lambda^k(M) \mid i(\xi)\alpha = 0 = L_\xi\alpha \}
\]

\[
= \{ \alpha \in \Lambda^k(M) \mid i(\xi)\alpha = 0 = i(\xi)da \}.
\]

If \( C^\infty_B(M) \) is the space of basic functions on \( M \) then \( \Lambda^k_B(M) \) is a \( C^\infty_B(M) \)-module. Also, we can consider the subcomplex of the de Rham complex:

\[
\cdots \to \Lambda^{k-1}_B(M) \xrightarrow{d} \Lambda^k_B(M) \xrightarrow{d} \Lambda^{k+1}_B(M) \to \cdots
\]

The cohomology of this complex is denoted by \( H^*_B(M) \) and called the \textit{basic de Rham cohomology} associated to \( \xi \) [11].

Remark 7.1. Moreover, if \( \xi \) is a Killing vector field, then the foliation \( F \) on \( M \) defined by the vector field \( \xi \) is a transversally oriented Riemannian foliation and the space of leaves \( \overline{M} = M/F \) satisfies that \( H^*_B(M) = H^k(\overline{M}) \).

Theorem 7.2. If \( M \) is a \((2n+1)\)-dimensional compact manifold and \((\Phi_t, \eta_t)\) is a continuous \( 1 \)-parameter family of almost cosymplectic structures on \( M \) which has the property that the associated Reeb vector fields \( \xi_t \) and both the basic de Rham cohomology classes \([\Phi_t]_B\) and the de Rham cohomology classes \([\eta_t]\) are independent of \( t \), then for each \( t \in [0, 1] \) there exists a diffeomorphism \( \phi_t \) such that \( \phi^*_t(\Phi_t) = \Phi_0 \) and \( \phi^*_t(\eta_t) = \eta_0 \) (that is, \( \phi_t \) is a cosymplectomorphism).

Proof. Let \( \xi \) be the associated Reeb vector field. Since the basic de Rham cohomology classes \([\Phi_t]_B\) and the de Rham cohomology classes \([\eta_t]\) are independent of \( t \), we have

\[
\Phi_t - \Phi_0 = d\alpha_t, \quad \text{and} \quad \eta_t - \eta_0 = d\beta_t,
\]

for a basic \(1\)-form \( \alpha_t \) (that is, \( i(\xi)\alpha_t = 0 = i(\xi)d\alpha_t \)) and a function \( \beta_t \). Then, we have

\[
\dot{\Phi}_t = d\dot{\alpha}_t \quad \text{and} \quad \dot{\eta}_t = d\dot{\beta}_t.
\]
Next, since \( \varphi : \frak{X}(M) \longrightarrow \Lambda^1(M) \), defined by \( \varphi(X) = i(X)\Phi_t + \eta_t(X)\eta_t \) is an isomorphism [1, 7], there exists a unique vector field \( X_t \) solution of the equations:

\[
i(X_t)\Phi_t = -\dot{\alpha}_t, \quad i(X_t)\eta_t = -\dot{\beta}_t.
\]

Now, let \( \phi_t \) be the flow of \( X_t \). Then,

\[
\frac{d}{dt}\phi_t^*\Phi_t = \phi_t^*(\dot{\Phi}_t + d(i(X_t)\Phi_t)) = 0,
\]

\[
\frac{d}{dt}\phi_t^*\eta_t = \phi_t^*(\dot{\eta}_t + d(i(X_t)\eta_t)) = 0,
\]

and, therefore we obtain the result. \( \square \)

**Remark 7.3.** In general, it is not true that for two almost cosymplectic structures \((\Phi_i, \eta_i)\) \((i = 0, 1)\), then \((\Phi_t = \Phi_0 + t(\Phi_1 - \Phi_0), \eta_t = \eta_0 + t(\eta_1 - \eta_0))\) is an almost cosymplectic manifold, but it is true if the structures are sufficiently close.

**Definition 7.4.** Two almost cosymplectic structures \((\Phi_0, \eta_0)\) and \((\Phi_1, \eta_1)\) are said to be deformation equivalent (or pseudo-isotopic) if they can be joined by a continuous 1-parameter family of almost cosymplectic structures \((\Phi_t, \eta_t), t \in [0, 1]\). If, moreover, the fundamental 2-forms \(\Phi_t\) are cohomologous and the 1-forms \(\eta_t\) are cohomologous too, with respect to the de Rham cohomology, they are called isotopic. Furthermore, if the associated Reeb vector field \(\xi_t\) does not depend on \(t\), \(\Phi_t\) are cohomologous with respect to the basic de Rham cohomology and \(\eta_t\) are cohomologous with respect to the de Rham cohomology, they are called strongly isotopic.

**Corollary 7.5.** For a compact almost cosymplectic manifold \(M\) of dimension \(2n + 1\), the coeffective cohomology is invariant by strong isotopies.

But, we shall see by constructing an example, that the coeffective cohomology is not invariant under deformations of the almost cosymplectic structure, even if the associated Reeb vector fields are the same.

**Example 7.6.** We consider the 7-dimensional manifold \(\tilde{R}^6\) described in Section 5 (Example 5.3) and the continuous 1-parameter family of almost cosymplectic structures given by:

\[
(\Phi_t = (2 - t)\alpha_{15} + \alpha_{16} + (2 - t)(\alpha_{25} + \alpha_{34}) + (t - 1)(\alpha_{24} + \alpha_{35}), \quad \eta_t = \theta).
\]

A direct computation shows that

\[
\Phi_t^3 \wedge \theta = 2(3 - 2t)\alpha_{123456} \wedge \theta,
\]

from which we deduce that \((\Phi_t, \eta_t)\) defines a continuous 1-parameter family of almost cosymplectic structures on \(\tilde{R}^6\) for \(t \neq \frac{3}{2}\).
From Theorem 4.2 we compute the coeffective cohomology on \( \tilde{R}^6 \) with respect to \((\Phi_t, \eta_t)\):

\[
\begin{align*}
H^5(\mathcal{A}(\tilde{R}^6)) &= \{\alpha_{14567}, \{\alpha_{12467} - (t-1)\alpha_{23457}\}, \\
&\quad \{\alpha_{13567} + (t-1)\alpha_{23457}\}, \{\alpha_{23467} + (2-t)\alpha_{23457}\}, \{\alpha_{12457}\}, \\
&\quad \{\alpha_{12456}\}, \{\alpha_{13456}\}, \{\alpha_{23456}\}\}, \\
\end{align*}
\]

\( H^k(\mathcal{A}(\tilde{R}^6)) = H^k(\tilde{R}^6), \quad k = 6, 7, \)

where the function \( e(t) \) takes the value 1 for \( t = 1 \) and the value 0 for \( t \neq 1 \).

Then, from (17) we obtain

\[
c_5(\tilde{R}^6, \Phi_t) = 8 + e(t).
\]

Therefore, the almost cosymplectic structures on \( \tilde{R}^6 \),

\[
(\Phi_0 = 2\alpha_{15} + \alpha_{16} + 2\alpha_{25} + 2\alpha_{34} - \alpha_{24} - \alpha_{35}, \quad \eta_0 = \theta)
\]

and

\[
(\Phi_1 = \alpha_{15} + \alpha_{16} + \alpha_{25} + \alpha_{34}, \quad \eta_1 = \theta)
\]

are deformation equivalent but their coeffective cohomology is not the same. Indeed, from (18) we have that \( c_5(\tilde{R}^6, \Phi_0) = 8 \neq 9 = c_5(\tilde{R}^6, \Phi_1) \), that is,

\[
H^5(\mathcal{A}(\tilde{R}^6, \Phi_0)) \neq H^5(\mathcal{A}(\tilde{R}^6, \Phi_1)).
\]

**Theorem 7.7.** The almost cosymplectic structures \((\Phi_0, \eta_0)\) and \((\Phi_1, \eta_1)\) defined on the compact manifold \( \tilde{R}^6 = R^6 \times S^1 \) are deformation equivalent but not cosymplectomorphic.

**Proof.** Taking into account that the almost cosymplectic structures \((\Phi_0, \eta_0)\) and \((\Phi_1, \eta_1)\) have different coeffective cohomology, then they cannot be cosymplectomorphic. \( \square \)

**Remark 7.8.** From this example we may obtain trivially two almost cosymplectic structures deformation equivalent by a continuous 1-parameter family of almost cosymplectic structures that preserves the associated Reeb vector field and the volume form, but with different coeffective cohomology, therefore they can not be cosymplectomorphic.

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**References**

Coeffective and de Rham cohomologies


