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The convolution operation on the spectra of algebras of symmetric analytic functions

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ABSTRACT

We show that the spectrum of the algebra of bounded symmetric analytic functions on ℓ_p, $1 \leq p < +\infty$ with the symmetric convolution operation is a commutative semigroup with the cancellation law for which we discuss the existence of inverses. For $p = 1$, a representation of the spectrum in terms of entire functions of exponential type is obtained which allows us to determine the invertible elements.

0. Introduction and preliminaries

The question of the description of the invariants of a linear transformations group on $\mathbb{C}^n$ which naturally acts on the algebra of polynomials is a typical problem of the classical Invariant Theory. Such invariants form algebras of symmetric polynomials with respect to given groups and have been investigated in the classical cases (see e.g. [1,2]). It is very important for these studies to describe the spectra of the algebras of invariants. The cases when a group (or even a semigroup) of symmetry acts on infinite-dimensional Banach spaces were considered in [3–6]. For the infinite-dimensional case we need to work with a natural completion of the algebra of continuous polynomials, that is, the algebra of analytic functions of bounded type. In this case, we can use some methods and ideas developed in [7,8].

Aron et al. introduced in [7] a convolution operation in the spectrum of the algebra $H_b(X)$ of analytic functions of bounded type defined on a complex Banach space X. This convolution is defined relying on translations on X. Later Aron et al. [8] discussed the commutativity of that convolution and proved that for $X = \ell_p$, it is not commutative.

By a *symmetric* function on ℓ_p we mean a function which is invariant under any reordering of the sequence in ℓ_p. The algebra of symmetric analytic functions of bounded type with the topology of the uniform convergence on bounded sets will be denoted by $\mathcal{H}_{bs}(\ell_p)$. We denote by $\mathcal{M}_{bs}(\ell_p)$ its spectrum, that is the set of all continuous scalar valued homomorphisms.

When dealing with symmetric analytic functions the translation operators are not well defined anymore. This is why in [6] the authors introduced the so-called "intertwining" operators that lead them to define a "symmetric" convolution operation as is described in the next section. We prove that an endomorphism of $\mathcal{H}_{bs}(\ell_p)$ commutes with all intertwining operators if and only if it is a convolution operator. The results in this paper show that, contrary to the non-symmetric case, the symmetric convolution is indeed commutative. Also a representation of $\mathcal{M}_{bs}(\ell_1)$ in terms of entire functions

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of exponential type is obtained. Such representation allows us to determine the invertible elements in $\mathcal{M}_{bs}(\ell_1)$ for such symmetric convolution. Finally we present a description of the elements in the spectrum through certain points in ℓ_1^+.

In [3] it is proved that, similarly to the classical finite dimensional case, the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \quad k = \lceil p \rceil, \lceil p \rceil + 1 \cdots \tag{0.1}$$

form an *algebraic basis* – named *the power series* basis – in the algebra of all symmetric polynomials on ℓ_p (here $\lceil p \rceil$ is the smallest integer that is greater than or equal to p). This means that for every symmetric polynomial P of degree $\lceil p \rceil + n - 1, n \geq 1$ there is a polynomial q on $\mathbb{C}^n$ such that $P(x) = q(F_{\lceil p \rceil}(x), \ldots, F_{\lceil p \rceil + n-1}(x))$. Actually, q is unique as pointed out in [5].

For background on analytic functions on infinite-dimensional spaces, we refer the reader to [9] or to [10].

1. The symmetric convolution

Remark 1.1. There is no $w \in \ell_p$, $w \neq 0$, such that $g(x) = f(x + w)$ is symmetric for every symmetric $f \in \mathcal{H}_{bs}(\ell_p)$.

Proof. There is $i_0 \in \mathbb{N}$, such that $|w_n| < \frac{1}{3}$ if $n \geq i_0$. Assume that $f(\cdot + w)$ belongs to $\mathcal{H}_{bs}(\ell_p)$ for every symmetric $f \in \mathcal{H}_{bs}(\ell_p)$. Then for every fixed permutation σ and each element in the basis of ℓ_p, $f(e_{\sigma(i)} + w) = g(e_{\sigma(i)}) = g(e_i) = f(e_i + w)$, $\forall f \in \mathcal{H}_{bs}(\ell_p)$. Thus $e_{\sigma(i)} + w$ is a permutation of $e_i + w$, that is, $1 + w_{\sigma(i)} = w_{j_i}$ for some index $j_i \in \mathbb{N}$.

Since σ is a bijection, the set $\{\sigma(i) > i_0\}$ is infinite, so there are infinite terms w_{j_i} with absolute value greater that $\frac{2}{3}$. Impossible. □

Next we recall some definitions.

Definition 1.2 (*[6]*). Let $x, y \in \ell_p$, $x = (x_1, x_2, \ldots,)$ and $y = (y_1, y_2, \ldots,)$. We define the *intertwining* $x \bullet y \in \ell_p$ according to

$$x \bullet y = (x_1, y_1, x_2, y_2, \ldots,).$$

The mapping $f \mapsto T_y^s(f)$ where $T_y^s(f)(x) = f(x \bullet y)$ will be referred as to the *intertwining operator*. Observe that $T_x^s \circ T_y^s = T_{x \bullet y}^s = T_y^s \circ T_x^s$: Indeed, $[T_x^s \circ T_y^s](f)(z) = T_x^s[T_y^s(f)](z) = T_y^s(f)(z \bullet x) = f((z \bullet x) \bullet y) = f(z \bullet (x \bullet y))$, since f is symmetric.

The above remark explains why we are led to use the intertwining operators to define the convolution in $\mathcal{M}_{bs}(\ell_p)$.

Definition 1.3 (*[6]*). Given $f \in \mathcal{H}_{bs}(\ell_p)$ and $\theta \in \mathcal{H}_{bs}(\ell_p)'$, its symmetric convolution $\theta \star f$ is defined by $(\theta \star f)(x) = \theta[T_x^s(f)]$.

As pointed out in [6], it turns out that $\theta \star f \in \mathcal{H}_{bs}(\ell_p)$.

Definition 1.4 (*[6]*). For any ϕ and θ in $\mathcal{H}_{bs}(\ell_p)'$, its *symmetric convolution* is defined according to

$$(\phi \star \theta)(f) = \phi(\theta \star f) = \phi(y \mapsto \theta(T_y^s f)).$$

Corollary 1.5 (*[6]*). *If $\phi, \theta \in \mathcal{M}_{bs}(\ell_p)$, then $\phi \star \theta \in \mathcal{M}_{bs}(\ell_p)$.*

Theorem 1.6. (a) *For every $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$ the following holds:*

$$(\varphi \star \theta)(F_k) = \varphi(F_k) + \theta(F_k). \tag{1.1}$$

(b) *The semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$ is commutative, the evaluation at 0, δ_0, is its identity and the cancellation law holds.*

Proof. Observe that for each element F_k in the algebraic basis of polynomials, $\{F_k\}$, we have

$$(\theta \star F_k)(x) = \theta(T_x^s(F_k)) = \theta(F_k(x) + F_k) = F_k(x) + \theta(F_k).$$

Therefore,

$$(\varphi \star \theta)(F_k) = \varphi(F_k + \theta(F_k)) = \varphi(F_k) + \theta(F_k).$$

To check that the convolution is commutative, that is, $\phi \star \theta = \theta \star \phi$, it suffices to prove it for symmetric polynomials, hence for the basis $\{F_k\}$. Bearing in mind (1.1) and also by exchanging parameters $(\theta \star \varphi)(F_k) = \theta(F_k) + \varphi(F_k) = (\varphi \star \theta)(F_k)$ as we wanted.

It also follows from (1.1) that the cancellation rule is valid for this convolution: If $\varphi \star \theta = \psi \star \theta$, then $\varphi(F_k) + \theta(F_k) = \psi(F_k) + \theta(F_k)$, hence $\varphi(F_k) = \psi(F_k)$, and thus, $\varphi = \psi$. □

Example 1.7. There exist nontrivial invertible elements in the semigroup $(\mathcal{M}_{bs}(\ell_p), \star)$:

In [5, Example 3.1] it was constructed a continuous homomorphism $\varphi = \Psi_1$ on the uniform algebra $A_{us}(B_{\ell_p})$ such that $\varphi(F_p) = 1$ and $\varphi(F_i) = 0$ for all $i > p$. In a similar way, given $\lambda \in \mathbb{C}$ we can construct a continuous homomorphism Ψ_λ on the uniform algebra $A_{us}(|\lambda|B_{\ell_p})$ such that $\Psi_\lambda(F_p) = \lambda$ and $\Psi_\lambda(F_i) = 0$ for all $i > p$: It suffices to consider for each $n \in \mathbb{N}$, the element $v_n = \left(\frac{\lambda}{n}\right)^{1/p}(e_1 + \cdots + e_n)$ for which $F_p(v_n) = \lambda$, and $\lim_n F_j(v_n) = 0$. Now, the sequence $\{\delta_{v_n}\}$ has an accumulation point Ψ_λ in the spectrum of $A_{us}(|\lambda|B_{\ell_p})$. We use the notation ψ_λ for the restriction of Ψ_λ to the subalgebra $\mathcal{H}_{bs}(\ell_p)$ of $A_{us}(|\lambda|B_{\ell_p})$. It turns out that $\psi_\lambda \star \psi_{-\lambda} = \delta_0$ since for all elements F_j in the algebraic basis, $(\psi_\lambda \star \psi_{-\lambda})(F_j) = \psi_\lambda(F_j) + \psi_{-\lambda}(F_j) = 0 = \delta_0(F_j)$.

Therefore, we obtain a complex line of invertible elements $\{\psi_\lambda : \lambda \in \mathbb{C}\}$.

As in the non-symmetric case [7, Theorem 5.5], the following holds:

Proposition 1.8. *Every $\varphi \in \mathcal{M}_{bs}(\ell_p)$ lies in a schlicht complex line through δ_0.*

Proof. For every $z \in \mathbb{C}$, consider the composition operator $L_z : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ defined according to $L_z(f)((x_n)) := f((zx_n))$, and then, the restriction L_z^* to $\mathcal{M}_{bs}(\ell_p)$ of its transpose map. Now put $\varphi^z := L_z^*(\varphi) = \varphi \circ L_z$. Observe that $\varphi^z(F_k) = \varphi \circ L_z(F_k) = \varphi((F_k(z\cdot))) = z^k\varphi(F_k)$. Also, $\varphi^0 = \delta_0$.

For each $f \in \mathcal{H}_{bs}(\ell_p)$ the self-map of $\mathbb{C}$ defined according to $z \rightsquigarrow \varphi^z(f)$ is entire by Aron et al. [7, Lemma 5.4(i)]. Therefore, the mapping $z \in \mathbb{C} \rightsquigarrow \varphi^z \in \mathcal{M}_{bs}(\ell_p)$ is analytic.

Since $\varphi \neq \delta_0$, the set $\Sigma := \{k \in \mathbb{N} : \varphi(F_k) \neq 0\}$ is non-empty. Let m be the first element of Σ, so that $\varphi(F_m) \neq 0$. Then if $\varphi^z = \varphi^w$, one has $z^m\varphi(F_m) = w^m\varphi(F_m)$, hence $z^m = w^m$. Taking the principal branch of the mth root, the map $\xi \rightsquigarrow \varphi^{\sqrt[m]{\xi}}$ is one-to-one. □

Recall that a linear operator $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is said to be a *convolution operator* if there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Let us denote $H_{conv}(\ell_p) := \{T \in L(\mathcal{H}_{bs}(\ell_p)) : T \text{ is a convolution operator}\}$.

Proposition 1.9. *A continuous homomorphism $T : \mathcal{H}_{bs}(\ell_p) \to \mathcal{H}_{bs}(\ell_p)$ is a convolution operator if and only if it commutes with all intertwining operators T_y^s, $y \in \ell_p$.*

Proof. Assume there is $\theta \in \mathcal{M}_{bs}(\ell_p)$ such that $Tf = \theta \star f$. Fix $y \in \ell_p$. Then $[T \circ T_y^s](f)(x) = [T(T_y^s(f))](x) = [\theta \star T_y^s(f)](x) = \theta[T_x^s(T_y^s(f))] = \theta[T_{x\bullet y}^s(f)]$. On the other hand, $[T_y^s \circ T](f)(x) = [T_y^s(Tf)](x) = Tf(x \bullet y) = (\theta \star f)(x \bullet y) = \theta[T_{x\bullet y}^s(f)]$.

Conversely, set $\theta = \delta_0 \circ T$. Clearly, $\theta \in \mathcal{M}_{bs}(\ell_p)$. Let us check that $Tf = \theta \star f$: Indeed, $(\theta \star f)(x) = \theta[T_x^s(f)] = [T(T_x^s(f))](0) = [T_x^s(T(f))](0) = Tf(0 \bullet x) = Tf(x)$. □

Consider the mapping Λ defined by $\Lambda(\theta)(f) = \theta \star f$, that is,

$$\Lambda : \mathcal{M}_{bs}(\ell_p) \to H_{conv}(\ell_p)$$
$$\theta \mapsto f \rightsquigarrow \theta \star f \equiv \Lambda(\theta)(f).$$

It is, clearly, bijective. Moreover we obtain a representation of the convolution semigroup

Proposition 1.10. *The mapping Λ is an isomorphism from $(\mathcal{M}_{bs}(\ell_p), \star)$ into $(H_{conv}(\ell_p), \circ)$ where $\circ$ denotes the usual composition operation.*

Proof. First, notice that using the above proposition,

$$\begin{aligned}\Lambda(\varphi \star \theta)(f)(x) &= [(\varphi \star \theta) \star f](x) = (\varphi \star \theta)(T_x^s f) = \varphi(\theta \star T_x^s f)\\ &= \varphi[\Lambda(\theta)(T_x^s f)] = \varphi[(\Lambda(\theta) \circ T_x^s)(f)] = \varphi[(T_x^s \circ \Lambda(\theta))(f)].\end{aligned}$$

On the other hand,

$$[\Lambda(\varphi) \circ \Lambda(\theta)](f)(x) = \Lambda(\varphi)[\Lambda(\theta)(f)](x) = [\varphi \star \Lambda(\theta)(f)](x) = \varphi[T_x^s(\Lambda(\theta)(f))].$$

Thus the statement follows. □

As a consequence, the homomorphism θ is invertible in $(\mathcal{M}_{bs}(\ell_p), \star)$, if and only if the convolution operator $\Lambda(\theta)$ is an algebraic isomorphism. Observe also that for $\psi \in \mathcal{M}_{bs}(\ell_p)$, one has

$$\psi \circ \Lambda(\theta) = \psi \star \theta,$$

because $[\psi \circ \Lambda(\theta)](f) = \psi[\Lambda(\theta)(f)] = \psi(\theta \star f) = (\psi \star \theta)(f)$.

Next we address the question of solving the equation $\varphi = \psi \star \theta$ for given $\varphi, \theta \in \mathcal{M}_{bs}(\ell_p)$. We begin with a general lemma.

Lemma 1.11. *Let A, B be Fréchet algebras and $T : A \to B$ an onto homomorphism. Then T maps (closed) maximal ideals onto (closed) maximal ideals.*

Proof. Since T is onto, it maps ideals in A onto ideals in B. Let $\mathcal{J} \subset A$ be a maximal ideal. We prove that $T(\mathcal{J})$ is a maximal ideal in B: If $\mathcal{I}$ is another ideal with $T(\mathcal{J}) \subset \mathcal{I} \subset B$, it turns out that for the ideal $T^{-1}(\mathcal{I})$, $\mathcal{J} \subset T^{-1}(T(\mathcal{J})) \subset T^{-1}(\mathcal{I})$, hence either $\mathcal{J} = T^{-1}(\mathcal{I})$, or $A = T^{-1}(\mathcal{I})$. That is, either $T(\mathcal{J}) = \mathcal{I}$, or $B = \mathcal{I}$.

Let now $\varphi \in M(A)$ and $\mathcal{J} = Ker(\varphi)$, be a closed maximal ideal. Then $T(\mathcal{J})$ is a maximal ideal in B, so there is a character ψ on B such that $Ker(\psi) = T(\mathcal{J})$. Then $Ker(\varphi) \subset Ker(\psi \circ T)$, because if $\varphi(a) = 0$, that is, $a \in \mathcal{J}$, we have $T(a) \in Ker(\psi)$. By the maximality, either $\varphi = \psi \circ T$, or $\psi \circ T = 0$, hence $\psi = 0$. In the former case, ψ is also continuous since being T an open mapping, if (b_n) is a null sequence in B, there is a null sequence $(a_n) \subset A$ such that $T(a_n) = b_n$; thus $\lim_n \psi(b_n) = \lim_n \psi \circ T(a_n) = \lim_n \varphi(a_n) = 0$. □

Remark 1.12. Let A, B be Fréchet algebras and $T: A \to B$ be an onto homomorphism. If $T(Ker(\varphi))$ is a proper ideal, then there is a unique $\psi \in M(B)$ such that $\varphi = \psi \circ T$.

Corollary 1.13. *Let $\theta \in \mathcal{M}_{bs}(\ell_p)$. Assume that $\Lambda(\theta)$ is onto. If $\Lambda(\theta)(Ker\varphi)$ is a proper ideal, then the equation $\varphi = \psi \star \theta$ has a unique solution. In case $\Lambda(\theta)(Ker\varphi) = \mathcal{H}_{bs}(\ell_p)$, then the equation $\varphi = \psi \star \theta$ has no solution.*

Proof. The first statement is just an application of the remark, since $\psi \star \theta = \psi \circ \Lambda(\theta) = \varphi$. For the second statement, if some solution ψ exists, then again $\psi \circ \Lambda(\theta) = \psi \star \theta = \varphi$, so $\psi(\mathcal{H}_{bs}(\ell_p)) = (\psi \circ \Lambda(\theta))((Ker\varphi)) = \varphi(Ker\varphi) = 0$. Therefore, then also $\varphi = 0$. □

2. A weak polynomial topology on $\mathcal{M}_{bs}(\ell_p)$

Let us denote by w_p the topology in $\mathcal{M}_{bs}(\ell_p)$ generated by the following neighborhood basis:

$$U_{\varepsilon,k_1,\ldots,k_n}(\psi) = \{\psi \star \varphi : \varphi \in \mathcal{M}_{bs}(\ell_p) \text{ and } |\varphi(F_{k_j})| < \varepsilon, j = 1, \ldots, n\}.$$

It is easy to check that the convolution operation is continuous for the w_p topology, since thanks to (1.1),

$$U_{\varepsilon/2,k_1,\ldots,k_n}(\theta) \star U_{\varepsilon/2,k_1,\ldots,k_n}(\psi) \subset U_{\varepsilon,k_1,\ldots,k_n}(\theta \star \psi).$$

We say that a function $f \in \mathcal{H}_{bs}(\ell_p)$ is *finitely generated* if there are a finite number of the basis functions $\{F_k\}$ and an entire function q such that $f = q(F_1, \ldots, F_j)$.

Theorem 2.1. *A function $f \in \mathcal{H}_{bs}(\ell_p)$ is w_p-continuous if and only if it is finitely generated.*

Proof. Clearly, every finitely generated function is w_p-continuous. Let us denote by V_n the finite dimensional subspace in ℓ_p spanned by the basis vectors $\{e_1, \ldots, e_n\}$. First we observe that if there is a positive integer m such that the restriction $f_{|V_n}$ of f to V_n is generated by the restrictions of $F_1, \ldots, F_m$ to V_n for every $n \geq m$, then f is finitely generated. Indeed, for given $n \geq k \geq m$ we can write

$$f_{|V_k}(x) = q_1(F_1(x), \ldots, F_m(x)) \quad \text{and} \quad f_{|V_n}(x) = q_2(F_1(x), \ldots, F_m(x))$$

for some entire functions q_1 and q_2 on $\mathbb{C}^n$. Since

$$\{(F_1(x), \ldots, F_m(x)) : x \in V_k\} = \mathbb{C}^m$$

(see e.g. [5]) and $f|_{V_n}$ is an extension of $f|_{V_k}$ we have $q_1(t_1, \ldots, t_n) = q_2(t_1, \ldots, t_n)$. Hence $f(x) = q_1(F_1(x), \ldots, F_m(x))$ on ℓ_p because $f(x)$ coincides with $q_1(F_1(x), \ldots, F_m(x))$ on the dense subset $\bigcup_n V_n$.

Let f be a w_p-continuous function in $\mathcal{H}_{bs}(\ell_p)$. Then f is bounded on a neighborhood $U_{\varepsilon,1,\ldots,m} = \{x \in \ell_p : |F_1(x)| < \varepsilon, \ldots, |F_m(x)| < \varepsilon\}$. For a given $n \geq m$ let

$$f|_{V_n}(x) = q(F_1(x), \ldots, F_m(x))$$

be the representation of $f|_{V_n}(x)$ for some entire function q on $\mathbb{C}^n$. Since $\{(F_1(x), \ldots, F_m(x)) : x \in V_n\} = \mathbb{C}^m$, $q(t_1, \ldots, t_n)$ must be bounded on the set $\{|t_1| < \varepsilon, \ldots, |t_m| < \varepsilon\}$. The Liouville Theorem implies $q(t_1, \ldots, t_n) = q(t_1, \ldots, t_m, 0 \ldots, 0)$, that is, $f|_{V_n}$ is generated by $F_1, \ldots, F_m$. Since it is true for every n, f is finitely generated. □

For example $f(x) = \sum_{n=1}^{\infty} \frac{F_n(x)}{n!}$ is not w_p-continuous.

Proposition 2.2. *The topology w_p is Hausdorff.*

Proof. If $\varphi \neq \psi$, then there is a number k such that

$$|\varphi(F_k) - \psi(F_k)| = \rho > 0.$$

Let $\varepsilon = \rho/3$. Then for every θ_1 and θ_2 in $U_{\varepsilon,k}(0)$,

$$|(\varphi \star \theta_1)(F_k) - (\varphi \star \theta_2)(F_k)| = |(\varphi(F_k) - \psi(F_k)) - (\theta_2(F_k)) - \theta_1(F_k)| \geq \rho/3. \quad \square$$

Proposition 2.3. *On bounded sets of $\mathcal{M}_{bs}(\ell_p)$ the topology w_p is finer than the weak-star topology $w(\mathcal{M}_{bs}(\ell_p), \mathcal{H}_{bs}(\ell_p))$.*

Proof. Since $(\mathcal{M}_{bs}(\ell_p), w_p)$ is a first-countable space, it suffices to verify that for a bounded sequence $(\varphi_i)_i$ which is w_p convergent to some ψ, we have $\lim_i \varphi_i(f) = \psi(f)$ for each $f \in \mathcal{H}_{bs}(\ell_p)$: Indeed, by the Banach–Steinhaus theorem, it is enough to see that $\lim_i \varphi_i(P) = \psi(P)$ for each symmetric polynomial P. Being $\{F_k\}$ an algebraic basis for the symmetric polynomials, this will follow once we check that $\lim_i \varphi_i(F_k) = \psi(F_k)$ for each F_k. To see this, notice that given $\varepsilon > 0$, $\varphi_i \in U_{\varepsilon,k}$ for i large enough, that is, there is θ_i such that $\varphi_i = \psi \star \theta_i$ with $|\theta_i(F_k)| < \varepsilon$. Then, $|\varphi_i(F_k) - \psi(F_k)| = |\theta_i(F_K)| < \varepsilon$ for i large enough. □

Proposition 2.4. *If $(\mathcal{M}_{bs}(\ell_p), \star)$ is a group, then w_p coincides with the weakest topology on $\mathcal{M}_{bs}(\ell_p)$ such that for every polynomial $P \in \mathcal{H}_{bs}(\ell_p)$ the Gelfand extension $\widehat{P}$ is continuous on $\mathcal{M}_{bs}(\ell_p)$.*

Proof. The sets $F_k^{-1}(B(F_k(\psi), \varepsilon))$ generate the weakest topology such that all $\widehat{P}$ are continuous. Let $\theta \in \mathcal{M}_{bs}(\ell_p)$ be such that $|F_k(\theta) - F_k(\psi)| < \varepsilon$. Set $\varphi = \theta \star \psi^{-1}$. Then $|F_k(\varphi)| = |F_k(\theta) - F_k(\psi)| < \varepsilon$ and $\theta = \psi \star \varphi$. □

3. Representations of the convolution semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$

In this section we consider the case $\mathcal{H}_{bs}(\ell_1)$. This algebra admits besides the power series basis $\{F_n\}$, another natural basis that is useful for us: It is given by the sequence $\{G_n\}$ defined by $G_0 = 1$, and

$$G_n(x) = \sum_{k_1 < \cdots < k_n}^{\infty} x_{k_1} \cdots x_{k_n},$$

and we refer to it as the *basis of elementary* symmetric polynomials.

Lemma 3.1. *We have that $\|G_n\| = 1/n!$*

Proof. To calculate the norm, it is enough to deal with vectors in the unit ball of ℓ_1 whose components are non-negative. And we may restrict ourselves to calculate it on L_m the linear span of $\{e_1, \ldots, e_m\}$ for $m \geq n$. We do the calculation in an inductive way over m.

Since $G_{n|_{L_m}}$ is homogeneous, its norm is achieved at points of norm 1. If $m = n$, then G_n is the product $x_1 \cdots x_n$. By using the Lagrange multipliers rule, we deduce that the maximum is attained at points with equal coordinates, that is at $\frac{1}{n}(e_1 + \cdots + e_n)$. Thus $|G_n(\frac{1}{n}, \overset{n}{\ldots}, \frac{1}{n}, 0, \ldots)| = 1/n^n \leq \frac{1}{n!}$.

Now for $m > n$, and $x \in L_m$, we have $G_n(x) = \sum_{k_1 < \cdots < k_n \leq m} x_{k_1} \cdots x_{k_n}$. Again the Lagrange multipliers rule leads to either some of the coordinates vanish or they are all equal, hence they have the same value $\frac{1}{m}$. In the first case, we are led back to some the previous inductive steps, with L_k with $k < m$, so the aimed inequality holds. While in the second one, we have $G_n(\frac{1}{m}, \overset{m}{\ldots}, \frac{1}{m}, 0, \ldots) = \binom{m}{n} \frac{1}{m^n} \leq \frac{1}{n!}$.

Moreover, $\|G_n\| \geq \lim_m \binom{m}{n} \frac{1}{m^n} = \frac{1}{n!}$. This completes the proof. □

Let $\mathbb{C}\{t\}$ be the space of all power series over $\mathbb{C}$. We denote by $\mathcal{F}$ and $\mathcal{G}$ the following maps from $\mathcal{M}_{bs}(\ell_1)$ into $\mathbb{C}\{t\}$

$$\mathcal{F}(\varphi) = \sum_{n=1}^{\infty} t^{n-1} \varphi(F_n) \quad \text{and} \quad \mathcal{G}(\varphi) = \sum_{n=0}^{\infty} t^n \varphi(G_n).$$

Let us recall that every element $\varphi \in \mathcal{M}_{bs}(\ell_1)$ has a radius-function

$$R(\varphi) = \limsup_{n\to\infty} \|\varphi_n\|^{\frac{1}{n}} < \infty,$$

where φ_n is the restriction of φ to the subspace of n-homogeneous polynomials [6].

Proposition 3.2. *The mapping $\varphi \in \mathcal{M}_{bs}(\ell_1) \xrightarrow{\mathcal{G}} \mathcal{G}(\varphi) \in \mathcal{H}(\mathbb{C})$ is one-to-one and ranges into the subspace of entire functions on $\mathbb{C}$ of exponential type. The type of $\mathcal{G}(\varphi)$ is less than or equal to $R(\varphi)$.*

Proof. Using Lemma 3.1,

$$\limsup_{n\to\infty} \sqrt[n]{n!|\varphi_n(G_n)|} \leq \limsup_{n\to\infty} \sqrt[n]{n!\|\varphi_n\| \, \|G_n\|} = \limsup_{n\to\infty} \sqrt[n]{\|\varphi_n\|} = R(\varphi) < \infty,$$

hence $\mathcal{G}(\varphi)$ is entire and of exponential type less than or equal to $R(\varphi)$. That $\mathcal{G}$ is one-to-one follows from the fact $\{G_n\}$ is a basis. □

Theorem 3.3. *The following identities hold:*

(1) $\mathcal{F}(\varphi \star \theta) = \mathcal{F}(\varphi) + \mathcal{F}(\theta)$.
(2) $\mathcal{G}(\varphi \star \theta) = \mathcal{G}(\varphi)\mathcal{G}(\theta)$.

Proof. The first statement is a trivial corollary of the properties of the convolution. To prove the second we observe that

$$G_n(x \bullet y) = \sum_{k=0}^{n} G_k(x) G_{n-k}(y).$$

Thus

$$(\theta \star G_n)(x) = \theta(T_x^s(G_n)) = \theta\left(\sum_{k=0}^{n} G_k(x) G_{n-k}\right) = \sum_{k=0}^{n} G_k(x)\theta(G_{n-k}).$$

Therefore,

$$(\varphi \star \theta)(G_n) = \varphi\left(\sum_{k=0}^{n} G_k(x)\theta(G_{n-k})\right) = \sum_{k=0}^{n} \varphi(G_k)\theta(G_{n-k}).$$

Hence, being the series absolutely convergent,

$$\begin{aligned}\mathcal{G}(\varphi)\mathcal{G}(\theta) &= \sum_{k=0}^{\infty} t^k \varphi(G_k) \sum_{m=0}^{\infty} t^m \theta(G_m) = \sum_{n=0}^{\infty} \sum_{k+m=n} t^n \varphi(G_k)\theta(G_m) \\ &= \sum_{n=0}^{\infty} t^n \sum_{k+m=n} \varphi(G_k)\theta(G_m) = \sum_{n=0}^{\infty} t^n (\varphi \star \theta)(G_n) = \mathcal{G}(\varphi \star \theta). \quad \square\end{aligned}$$

Example 3.4. Let ψ_λ be as defined in Example 1.7. We know that $\mathcal{F}(\psi_\lambda) = \lambda$. To find $\mathcal{G}(\psi_\lambda)$ note that

$$G_k(v_n) = \left(\frac{\lambda}{n}\right)^k \binom{n}{k}, \quad \text{hence } \varphi(G_k) = \lim_n G_k(v_n) = \frac{\lambda^k}{k!}$$

and so

$$\mathcal{G}(\psi_\lambda)(t) = \lim_{n\to\infty} \sum_{k=0}^{n} (\lambda t)^k \psi_\lambda(G_n) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(\lambda t)^k}{k!} = e^{\lambda t}.$$

According to well-known Newton's formula we can write for $x \in \ell_1$,

$$nG_n(x) = F_1(x)G_{n-1}(x) - F_2(x)G_{n-2}(x) + \cdots + (-1)^{n+1} F_n(x). \tag{3.1}$$

Moreover, if ξ is a complex homomorphism (not necessarily continuous) on the space of symmetric polynomials $\mathcal{P}_s(\ell_1)$, then

$$n\xi(G_n) = \xi(F_1)\xi(G_{n-1}) - \xi(F_2)\xi(G_{n-2}) + \cdots + (-1)^{n+1}\xi(F_n). \tag{3.2}$$

Next we point out the limitations of the construction's technique described in 1.7.

Remark 3.5. Let ξ be a complex homomorphism on $\mathcal{P}_s(\ell_1)$ such that $\xi(F_m) = c \neq 0$ for some $m \geq 2$ and $\xi(F_n) = 0$ for $n \neq m$. Then ξ is not continuous.

Proof. Using formula (3.2) we can see that

$$\xi(G_{km}) = (-1)^{m+1} \frac{\xi(F_m)\xi(G_{(k-1)m})}{km}$$

and $\xi(G_n) = 0$ if $n \neq km$ for some $k \in \mathbb{N}$. By induction we have

$$\xi(G_{km}) = \frac{\left((-1)^{m+1} c/m\right)^k}{k!}$$

and so

$$\mathcal{G}(\xi)(t) = 1 + \sum_{k=1}^{\infty} \frac{\left((-1)^{m+1}c/m\right)^k}{k!} t^{km} = 1 + \sum_{k=1}^{\infty} \frac{\left((-1)^{m+1}\frac{ct^m}{m}\right)^k}{k!} = e^{\left((-1)^{m+1}\frac{ct^m}{m}\right)}.$$

Hence $\mathcal{G}(\xi)(t) = e^{-\frac{(-ct)^m}{m}} = e^{-\frac{(-c)^m}{m}t^m}$. Since $m \geq 2$, $\mathcal{G}(\xi)$ is not of exponential type. So if ξ were continuous, it could be extended to an element in $\mathcal{M}_{bs}(\ell_1)$, leading to a contradiction with Proposition 3.2. □

According to the Hadamard Factorization Theorem (see [11, p. 27]) the function of exponential type $\mathcal{G}(\varphi)(t)$ is of the form

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\left(1-\frac{t}{a_k}\right)e^{t/a_k}, \tag{3.3}$$

where $\{a_k\}$ are the zeros of $\mathcal{G}(\varphi)(t)$. If $\sum |a_k|^{-1} < \infty$, then this representation can be reduced to

$$\mathcal{G}(\varphi)(t) = e^{\lambda t}\prod_{k=1}^{\infty}\left(1-\frac{t}{a_k}\right). \tag{3.4}$$

Recall how ψ_λ was defined in Example 1.7.

Proposition 3.6. *If $\varphi \in (\mathcal{M}_{bs}(\ell_1), \star)$ is invertible, then $\varphi = \psi_\lambda$ for some λ. In particular, the semigroup $(\mathcal{M}_{bs}(\ell_1), \star)$ is not a group.*

Proof. If φ is invertible then $\mathcal{G}(\varphi)(t)$ is an invertible entire function of exponential type and so has no zeros. By Hadamard's factorization (3.3) we have that $\mathcal{G}(\varphi)(t) = e^{\lambda t}$ for some complex number λ. Hence $\varphi = \psi_\lambda$ by Proposition 3.2.

The evaluation $\delta_{(1,0,\ldots,0,\ldots)}$ does not coincide with any ψ_λ since, for instance, $\psi_\lambda(F_2) = 0 \neq 1 = \delta_{(1,0,\ldots,0,\ldots)}(F_2)$. □

Another consequence of our analysis is the following remark.

Corollary 3.7. *Let Φ be a homomorphism of $\mathcal{P}_s(\ell_1)$ to itself such that $\Phi(F_k) = -F_k$ for every k. Then Φ is discontinuous.*

Proof. If Φ is continuous it may be extended to a continuous homomorphism $\widetilde{\Phi}$ of $\mathcal{H}_{bs}(\ell_1)$. Then for $x = (1, 0, \ldots, 0, \ldots)$, we have $\delta_x \star (\delta_x \circ \widetilde{\Phi}) = \delta_0$. However, this is impossible since δ_x is not invertible. □

We close this section by analyzing further the relationship established by the mapping $\mathcal{G}$.

It is known from Combinatorics (see e.g. [12, pp. 3,4]) that

$$\mathcal{G}(\delta_x)(t) = \prod_{k=1}^{\infty}(1+x_k t) \quad \text{and} \quad \mathcal{F}(\delta_x)(t) = \sum_{k=1}^{\infty}\frac{x_k}{1-x_k t} \tag{3.5}$$

for every $x \in c_{00}$. Formula (3.5) for $\mathcal{G}(\delta_x)$ is true for every $x \in \ell_1$: Indeed, for fixed t, both the infinite product and $\mathcal{G}(\delta_x)(t)$ are analytic functions on ℓ_1.

Taking into account formula (3.5) we can see that the zeros of $\mathcal{G}(\delta_x)(t)$ are $a_k = -1/x_k$ for $x_k \neq 0$. Conversely, if $f(t)$ is an entire function of exponential type which is equal to the right hand side of (3.4) with $\sum |a_k|^{-1} < \infty$, then for $\varphi \in \mathcal{M}_{bs}(\ell_1)$ given by $\varphi = \psi_\lambda \star \delta_x$, where $x \in \ell_1$, $x_k = -1/a_k$ and ψ_λ as defined in Example 1.7, it turns out that $\mathcal{G}(\varphi)(t) = f(t)$. So we just have to examine entire functions of exponential type with Hadamard canonical product

$$f(t) = \prod_{k=1}^{\infty}\left(1-\frac{t}{a_k}\right)e^{t/a_k} \tag{3.6}$$

with $\sum |a_k|^{-1} = \infty$. Note first that the growth order of $f(t)$ is not greater than 1. According to Borel's theorem [11, p. 30] the series

$$\sum_{k=1}^{\infty}\frac{1}{|a_k|^{1+d}}$$

converges for every $d > 0$. Let

$$\Delta_f = \limsup_{n\to\infty}\frac{n}{|a_n|}, \qquad \eta_f = \limsup_{r\to\infty}\left|\sum_{|a_n|<r}\frac{1}{a_n}\right|$$

and $\gamma_f = \max(\Delta_f, \eta_f)$. Due to Lindelöf's theorem [11, p. 33] the type σ_f of f and γ_f simultaneously are equal either to zero, or to infinity, or to positive numbers. Hence $f(t)$ of the form (3.6) is a function of exponential type if and only if $\sum |a_k|^{-1-d}$ converges for every $d > 0$ and γ_f is finite.

Corollary 3.8. *If a sequence $(x_n) \notin \ell_p$ for some $p > 1$, then there is no $\varphi \in \mathcal{M}_{bs}(\ell_1)$ such that*

$$\varphi(F_k) = \sum_{n=1}^{\infty}x_n^k$$

for all k.

Let $x = (x_1, \ldots, x_n, \ldots)$ be a sequence of complex numbers such that $x \in \ell_{1+d}$ for every $d > 0$,

$$\limsup_{n\to\infty} n|x_n| < \infty, \qquad \limsup_{r\to 1} \left| \sum_{\frac{1}{|x_n|}<r} x_n \right| < \infty \tag{3.7}$$

and $\lambda \in \mathbb{C}$. Let us denote by $\delta_{(x,\lambda)}$ a homomorphism on the algebra of symmetric polynomials $\mathcal{P}_s(\ell_1)$ of the form

$$\delta_{(x,\lambda)}(F_1) = \lambda, \qquad \delta_{(x,\lambda)}(F_k) = \sum_{n=1}^{\infty} x_n^k, \quad k > 1.$$

Proposition 3.9. *Let $\varphi \in \mathcal{M}_{bs}(\ell_1)$. Then the restriction of φ to $\mathcal{P}_s(\ell_1)$ coincides with $\delta_{(x,\lambda)}$ for some $\lambda \in \mathbb{C}$ and x satisfying (3.7).*

Proof. Consider the exponential type function $\mathcal{G}(\varphi)$ given by (3.3) and the corresponding sequence $x = (\frac{-1}{a_n})$.

If $x \in \ell_1$, then according to (3.4), $\varphi = \psi_\lambda \star \delta_x$. If $x \notin \ell_1$, then $\mathcal{G}(\varphi)(t) = e^{\lambda t} \prod_{n=1}^{\infty}\Big(1 + tx_n\Big)e^{-tx_n}$ and, on the other hand, $\mathcal{G}(\varphi)(t) = \sum_{n=0}^{\infty} \varphi(G_n)t^n$.

We have

$$\begin{aligned}\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n}\right)'_t &= \lambda e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n} \\ &\quad + e^{\lambda t}\left(-tx_1^2 e^{-tx_1}\prod_{n\neq 1}(1+tx_n)\,e^{-tx_n} - tx_2^2 e^{-tx_2}\prod_{n\neq 2}(1+tx_n)\,e^{-tx_n} - \cdots\right) \\ &= \lambda e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n} - te^{\lambda t}\sum_{k=1}^{\infty}x_k^2 e^{-tx_k}\prod_{n\neq k}(1+tx_n)\,e^{-tx_n}\end{aligned}$$

and

$$\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n}\right)'\Bigg|_{t=0} = \lambda.$$

So by the uniqueness of the Taylor coefficients, $\varphi(G_1) = \varphi(F_1) = \lambda$.

Now

$$\begin{aligned}\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n}\right)''_t &= \left(\lambda e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n}\right)'_t - \left(te^{\lambda t}\sum_{k=1}^{\infty}x_k^2 e^{-tx_k}\prod_{n\neq k}(1+tx_n)\,e^{-tx_n}\right)'_t \\ &= \lambda^2 e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n} - \lambda te^{\lambda t}\sum_{k=1}^{\infty}x_k^2 e^{-tx_k}\prod_{n\neq k}(1+tx_n)\,e^{-tx_n} \\ &\quad - e^{\lambda t}\sum_{k=1}^{\infty}x_k^2 e^{-tx_k}\prod_{n\neq k}(1+tx_n)\,e^{-tx_n} - t\left(e^{\lambda t}\sum_{k=1}^{\infty}x_k^2 e^{-tx_k}\prod_{n\neq k}(1+tx_n)\,e^{-tx_n}\right)'_t\end{aligned}$$

and

$$\left(e^{\lambda t}\prod_{n=1}^{\infty}(1+tx_n)\,e^{-tx_n}\right)''\Bigg|_{t=0} = \lambda^2 - \sum_{k=1}^{\infty}x_k^2.$$

Then

$$\varphi(G_2) = \frac{\lambda^2 - F_2(x)}{2} = \frac{(\varphi(F_1))^2 - F_2(x)}{2}.$$

On the other hand,

$$\varphi(G_2) = \frac{\varphi(F_1^2) - \varphi(F_2)}{2}$$

and we have

$$\varphi(F_2) = F_2(x).$$

Now using induction we obtain the required result. $\square$

Question 3.10. Does the map $\mathcal{G}$ act onto the space of entire functions of exponential type?

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