Pergamon

Appl. Math. Lett. Vol. 10, No. 6, pp. 101-106, 1997

0893-9659/97 $17.00 + 0.00

PII: S0893-9659(97)00112-2

Oscillation Criteria for First-Order Neutral Nonlinear Difference Equations with Variable Coefficients

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(Received December 1996; accepted January 1997)

Communicated by D. G. M. Anderson

Abstract—Consider the first-order neutral nonlinear difference equation of the form

$$\Delta\left(y_n - p_n y_{n-\tau}\right) + q_n \prod_{i=1}^{m} \left|y_{n-\sigma_i}\right|^{\alpha_i} \operatorname{sgn} y_{n-\sigma_i} = 0, \qquad n = 0, 1, \dots,$$

where $\tau > 0$, $\sigma_i \geq 0$ $(i = 1, 2, \dots, m)$ are integers, $\{p_n\}$ and $\{q_n\}$ are nonnegative sequences. We obtain new criteria for the oscillation of the above equation without the restrictions $\sum_{n=0}^{\infty} q_n = \infty$ or $\sum_{n=0}^{\infty} nq_n \sum_{j=n}^{\infty} q_j = \infty$ commonly used in the literature.

Keywords—Neutral difference equation, Oscillation criteria, Variable coefficients.

1. INTRODUCTION

This work is motivated by recent investigations [1–4], in which oscillation criteria are given for first-order linear or nonlinear neutral difference equations under assumption either

$$\sum_{n=0}^{\infty} q_n = \infty \tag{1}$$

or

$$\sum_{n=0}^{\infty} nq_n \sum_{j=n}^{\infty} q_j = \infty. \tag{2}$$

In these papers, the hypothesis (1) or (2) plays an essential role. It is then interesting to ask if this condition can be replaced by others. Our aim in this paper is to derive several new criteria for the oscillation of all solutions of an equation of the form

$$\Delta\left(y_n - p_n y_{n-\tau}\right) + q_n \prod_{i=1}^{m} \left|y_{n-\sigma_i}\right|^{\alpha_i} \operatorname{sgn} y_{n-\sigma_i} = 0, \qquad n = 0, 1, 2, \dots, \tag{3}$$

Supported by the Science Foundation of Gansu Province for the Young.

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where τ is a positive integer and $\sigma_1, \sigma_2, \ldots, \sigma_m$ are nonnegative integers, $p_n \geq 0$ and $q_n \geq 0$ for $n \geq 0$ such that q_n is not identically zero for all large n, and each α_i is a positive number for $i = 1, 2, \ldots, m$ such that $\alpha_1 + \cdots + \alpha_m = 1$. The forward difference operator Δ is defined as usual; i.e., $\Delta x_n = x_{n+1} - x_n$.

Let $\mu = \max\{\sigma_1, \ldots, \sigma_m, \tau\}$. Then by a solution of equation (3), we mean a real sequence $\{y_n\}$ that is defined for $n \geq -\mu$ such that (3) is satisfied. By writing (3) in the form of a recurrence relation, it is clear that if $\{y_n\}_{n=-\mu}^{0}$ is given, then equation (3) has a unique solution satisfying these initial values. A solution $\{y_n\}$ of (3) is said to be eventually positive if $y_n > 0$ for all large n, and eventually negative if $y_n < 0$ for all large n. It is said to be oscillatory if it is neither eventually positive nor eventually negative. Equation (3) is said to be oscillatory if each of its solutions is oscillatory. Since $\{y_n\}$ is an eventually positive solution of (3) if and only if $\{-y_n\}$ is an eventually negative solution of (3), equation (3) is oscillatory if and only if it does not have any eventually positive solutions.

2. PREPARATORY LEMMAS

We first quote several preparatory results which will be useful. For the sake of convenience, let $\{y_n\}_{n=a-\tau}^{\infty}$ be a real sequence, then the sequence $\{z_n\}$ defined by

$$z_n = y_n - p_n y_{n-\tau}, \qquad n = a, a+1, \ldots, \tag{4}$$

will be called its associated or comparison sequence (relative to the sequence $\{p_n\}_{n=a}^{\infty}$ and the integer τ).

The first preparatory result is known [2,3], but for the sake of completeness, the proof will be included here.

LEMMA 1. *(See [2,3].) Suppose that there is an integer $N > 0$ such that*

$$p_{N+k\tau} \leq 1, \qquad k \geq 0. \tag{5}$$

Then for any eventually positive solution $\{y_n\}$ of (3), the sequence $\{z_n\}$ defined by (4) will satisfy $z_n > 0$ and $\Delta z_n \leq 0$ for all large n.

PROOF. It is clear from (3) that $\Delta z_n \leq 0$ and is not identically zero for all large n. Thus, the sequence $\{z_n\}$ is of constant positive or constant negative sign eventually. Suppose that $y_n > 0$, $z_n < 0$ for $n \geq T$, then $z_n \leq z_T < 0$ for $n > T$. By choosing k^* so large that $N + k^*\tau \geq T$, we see from (4) that

$$\begin{aligned} y_{N+k^*\tau+j\tau} &= p_{N+k^*\tau+j\tau} y_{N+k^*\tau+(j-1)\tau} \\ &\leq z_T + y_{N+k^*\tau+(j-1)\tau} = z_T + z_{N+k^*\tau+(j-1)\tau} + p_{N+k^*\tau+(j-1)\tau} y_{N+k^*\tau+(j-2)\tau} \\ &\leq \cdots \leq y_{T+k^*\tau} + (j+1) z_T, \qquad j \geq 0. \end{aligned}$$

By letting $j \to \infty$, we see that the right-hand side diverges to $-\infty$, which is contrary to our assumption that $y_n > 0$ for $n \geq T$. The proof is complete.

The following result due to Zhang and Cheng [3] is an extension and improvement of Theorem 2.1 in [4].

LEMMA 2. *(See [3, Corollary 1].) Suppose there is an integer $N > 0$ such that (5) holds. Suppose further that either (H_1) $p_n + q_n \min\{\sigma_1, \ldots, \sigma_m\} > 0$ or $(H_2) \min\{\sigma_1, \ldots, \sigma_m\} > 0$, and q_n does not vanish identically over sets of consecutive integers of the form $\{a, a+1, \ldots, a + \min\{\sigma_1, \ldots, \sigma_m\}$. Then every solution of (3) is oscillatory if and only if*

$$\Delta(y_n - p_n y_{n-\tau}) + q_n \prod_{i=1}^{m} |y_{n-\sigma_i}|^{\alpha_i} \operatorname{sgn} y_{n-\sigma_i} \leq 0, \qquad n = 0, 1, 2, \ldots, \tag{6}$$

does not have an eventually positive solution.

3. OSCILLATION CRITERIA

In what follows, we will derive some sufficient conditions for the oscillation of all solutions of equation (3). The following lemma plays an important role.

LEMMA 3. *Suppose* $p_n \geq 1$, $q_n \geq 0$, *for* $n \geq 0$ *and*

$$\sum_{n=0}^{\infty} q_n \prod_{j=0}^{n-1} \left(1 + \frac{jq_j}{\tau}\right) = +\infty. \tag{7}$$

Then for every eventually positive solution $\{y_n\}$ *of (6), the sequence* $\{z_n\}$ *defined by (4) will satisfy* $z_n < 0$ *and* $\Delta z_n \leq 0$ *for all large* n.

PROOF. It is clear that (6) and (7), that $\Delta z_n \leq 0$, and is not identically zero for all large n. Consequently, $\{z_n\}$ is eventually positive or eventually negative. Suppose to the contrary that $y_n > 0, \Delta z_n \leq 0$, and $z_n > 0$ for $n \geq T$; then in view of (4), we see that $y_n > p_n y_{n-\tau} \geq y_{n-\tau} > 0$ for $n \geq T + \tau$. Thus,

$$y_n \geq \min\{y_{T-\tau}, y_{T-\tau+1}, \dots, y_{T-1}\} \equiv M > 0, \qquad n \geq T + 2\tau = T_1. \tag{8}$$

For convenience, we denote

$$N(n) = \left[\frac{n - T_1}{\tau}\right],$$

where $[n - T_1/\tau]$ is the integer part of $(n - T_1)/\tau$. Then

$$y_n \geq z_n + y_{n-\tau} \geq z_n + z_{n-\tau} + \dots + z_{n-(N(n)-1)\tau} + y_{n-N(n)\tau}, \qquad n \geq T_1. \tag{9}$$

Note that $\{z_n\}$ is nonincreasing and $y_{n-N(n)\tau} \geq M$ for $n \geq T_1$. Thus, by (6) we obtain that

$$y_n \geq N(n) z_n + M, \qquad n \geq T_1. \tag{10}$$

Substituting this into (6), we have

$$\Delta z_n + q_n \prod_{i=1}^{m} \left[N(n - \sigma_i) z_{n-\sigma_i} + M\right]^{\alpha_i} \leq 0, \qquad n \geq T_1 + \tau = T_2. \tag{11}$$

By Holder's inequality [5, p. 20], we have

$$\prod_{i=1}^{m} [N(n - \sigma_i) z_{n-\sigma_i} + M]^{\alpha_i} \geq \prod_{i=1}^{m} [N(n - \sigma_i)]^{\alpha_i} \prod_{i=1}^{m} [z_{n-\sigma_i}]^{\alpha_i} + M.$$

Furthermore,

$$\Delta z_n + q_n \prod_{i=1}^{m} [N(n - \sigma_i)]^{\alpha_i} z_n + q_n M \leq 0, \qquad n \geq T_2. \tag{12}$$

Then

$$\Delta \left\{ z_n \prod_{j=T_2}^{n-1} \left[1 + q_j \prod_{i=1}^{m} [N(i - \sigma_i)]^{\alpha_i}\right] \right\} + M q_n \prod_{j=T_2}^{n-1} \left[1 + q_j \prod_{i=1}^{m} [N(i - \sigma_i)]^{\alpha_i}\right] \leq 0, \qquad n \geq T_2. \tag{13}$$

Summing (13) from T_2 to $n \geq T_2$, we have

$$z_{n+1} \prod_{j=T_2}^{n} \left[1 + q_j \prod_{i=1}^{m} [N(i-\sigma_i)]^{\alpha_i}\right] - z_{T_2} + M \sum_{k=T_2}^{n} q_k \prod_{j=T_2}^{k-1} \left[1 + q_j \prod_{i=1}^{m} [N(i-\sigma_i)]^{\alpha_i}\right] \leq 0, \qquad n \geq T_2. \tag{14}$$

If the condition

$$\sum_{n=0}^{\infty} q_n = \infty \tag{15}$$

is satisfied, then it is easy to see from (15), and the fact that $y_n \geq M$ for $n \geq T_1$, that $\lim_{n\to\infty} z_n = -\infty$, which is a contradiction. Hence, we assume that

$$\sum_{n=0}^{\infty} q_n < \infty. \tag{16}$$

Noting that

$$\frac{\prod_{i=1}^{m} [N(n-\sigma_i)]^{\alpha_i}}{n} \to \frac{1}{\tau}, \qquad (n \to \infty),$$

it is easy to see that

$$\sum_{n=T_2}^{\infty} q_n \left[\frac{n}{\tau} - \prod_{i=1}^{m} [N(n-\sigma_i)]^{\alpha_i}\right]$$

is absolutely convergent and

$$\lim_{n\to\infty} \frac{\prod_{j=T_2}^{n-1} \left[1 + q_j \prod_{i=1}^{m} [N(i-\sigma_i)]^{\alpha_i}\right]}{\prod_{j=T_2}^{n-1} [1 + q_j (i/\tau)]}$$

exists. By condition (7), we obtain

$$\sum_{n=T_2}^{\infty} q_n \prod_{j=T_2}^{n-1} \left[1 + q_j \frac{i}{\tau}\right] = \infty.$$

Letting $n \to \infty$ in (14), we obtain a contradiction and the proof is complete.

As an immediate consequence of Lemmas 1 and 3, we obtain the following result which improves [1,2 Theorem 1] and [3, Theorem 2].

THEOREM 1. *Suppose $p_n = 1$ and $q_n \geq 0$ for $n \geq 0$ and (7) holds. Then every solution of (3) is oscillatory.*

EXAMPLE 1. Consider the neutral delay difference equation

$$\Delta (y_n - y_{n-\tau}) + n^{-\beta} y_{n-\sigma} = 0, \qquad n = 0, 1, 2, \ldots, \tag{17}$$

where $0 < \tau \leq 1$, $\sigma > 0$, $1 < \beta \leq 2$. This equation satisfies all the conditions of Theorem 1. Hence, all solutions of (17) oscillate. On the other hand, by [6, Theorem 5.6], we see that equation (17) has a bounded nonoscillatory solution if and only if $\beta > 2$. But the results in [1–3] are not applicable to this equation when $3/2 < \beta \leq 2$.

An oscillation criterion can now be derived as a consequence of Theorem 1, which improves [2, Theorem 2] and [3, Theorem 3].

THEOREM 2. *Suppose there is an integer $N > 0$ such that (5) for $k \geq 0$, and suppose p_n, $q_n \geq 0$ for $n \geq 0$ and (7) holds. Suppose further that*

$$q_n \prod_{i=1}^{m} p_{n-\sigma_i}^{\alpha_i} \geq q_{n-\tau}, \tag{18}$$

for all large n. Then every solution of (3) is oscillatory.

PROOF. Suppose to the contrary that $\{y_n\}$ is an eventually positive solution of (3). Then by means of Lemma 1, the sequence $\{z_n\}$ defined by $z_n = y_n - p_n y_{n-\tau}$ will satisfy $z_n > 0$ for all large n. In view of (4), we have

$$\begin{aligned}\Delta z_n &= -q_n \prod_{i=1}^{m} y_{n-\sigma_i}^{\alpha_i} \\ &= -q_n \prod_{i=1}^{m} \left(z_{n-\sigma_i} + p_{n-\sigma_i} y_{n-\tau-\sigma_i}\right)^{\alpha_i} \\ &\leq -q_n \left\{ \prod_{i=1}^{m} z_{n-\sigma_i}^{\alpha_i} + \prod_{i=1}^{m} p_{n-\sigma_i}^{\alpha_i} \prod_{i=1}^{m} y_{n-\tau-\sigma_i}^{\alpha_i} \right\},\end{aligned}$$

for all large n, where we have used the Holder's inequality [5, p. 20] to obtain our last inequality. Since (3) implies

$$\Delta z_{n-\tau} + q_{n-\tau} \prod_{i=1}^{m} y_{n-\tau-\sigma_i}^{\alpha_i} = 0,$$

we have

$$\Delta z_n - \Delta z_{n-\tau} + q_n \prod_{i=1}^{m} z_{n-\sigma_i}^{\alpha_i} \leq \left(q_{n-\tau} - q_n \prod_{i=1}^{m} p_{n-\sigma_i}^{\alpha_i} \right) \prod_{i=1}^{m} y_{n-\tau-\sigma_i}^{\alpha_i}.$$

In view of our hypothesis,

$$\Delta \left(z_n - z_{n-\tau}\right) + q_n \prod_{i=1}^{m} z_{n-\sigma_i}^{\alpha_i} \leq 0,$$

for all large n. This is contrary to Theorem 1 and the proof is complete.

The following result is an extension of [3, Theorem 4], which does not require the assumption (2).

THEOREM 3. *Suppose that $p_n, q_n \geq 0$ for $n \geq 0$ such that (5) and (7) hold. Suppose further that there is some number $r \in (0,1)$ such that*

$$q_n \prod_{i=1}^{m} p_{n-\sigma_i}^{\alpha_i} \geq r q_{n-\tau}, \tag{19}$$

for all large n. Then equation (3) is oscillatory provided that the following recurrence relation:

$$\Delta w_n + \frac{r}{1-r} q_n w_{n-\tau-\sigma} \leq 0, \qquad \sigma = \min\left\{\sigma_1, \ldots, \sigma_m\right\}, \quad n \geq 0,$$

does not have an eventually positive solution.

By Lemma 2, the proof of Theorem 3 is obvious, we omit it here.

Our final result deals with the case $p_n \geq 1$ and $m = 1$, which improves [2, Theorem 6] by dropping the condition (2).

THEOREM 4. *Suppose $m = 1$, $p_n \geq 1$, $q_n \geq 0$ for $n \geq 0$ such that (7) holds. Suppose further that there is a number $\alpha \geq 1$ such that $p_{n-\sigma_1} q_n \leq \alpha q_{n-\tau}$ for all large n; then every solution of (3) is oscillatory.*

EXAMPLE 2. The neutral difference equation

$$\Delta\left(y_n - \frac{n+1}{n} y_{n-1}\right) + n^{-\beta} y_{n-2} = 0, \qquad n = 0, 1, \ldots, \tag{20}$$

where $3/2 < \beta < 2$, satisfies all the conditions of Theorem 4. Hence, all solutions of (20) oscillate. But [2, Theorem 6] is not applicable to the equation since $3/2 < \beta < 2$.

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