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Classifying tame blocks and related algebras up to stable equivalences of Morita type

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ABSTRACT

We contribute to the classification of finite dimensional algebras under stable equivalence of Morita type. More precisely we give a classification of Erdmann's algebras of dihedral, semi-dihedral and quaternion type and obtain as byproduct the validity of the Auslander–Reiten conjecture for stable equivalences of Morita type between two algebras, one of which is of dihedral, semi-dihedral or quaternion type.

Introduction

Stable categories were introduced very early in the representation theory of algebras and played a major rôle in the development of Auslander–Reiten theory. Nevertheless, already in the 1970s Auslander and Reiten knew that equivalences of stable categories can behave very badly. For example, there are indecomposable finite dimensional algebras which are stably equivalent to a direct product of two algebras none of which is separable [1, Example 3.5].

Around 1990 the concept of derived categories became popular in the representation theory of groups and algebras by mainly two developments. First Happel interpreted successfully tilting theory in the framework of derived categories and secondly Broué formulated his famous abelian defect group conjecture in this framework. Many homological constructions are more natural in the language of derived categories than in module categories. Work of Rickard [22] and Keller–Vossieck [14] show that an equivalence between derived categories of self-injective algebras implies an equivalence of a very particular shape between the stable categories of these algebras. The stable equivalences coming from derived equivalences are induced by tensoring with bimodules which are invertible almost as for Morita equivalences. This discovery in mind, Broué defined two algebras A and B to be stably equivalent of Morita type if there is an A-B-bimodule M and a B-A-bimodule N, which are projective considered as modules on either side only and so that there are isomorphisms of bimodules $M \otimes_B N \simeq A \oplus P$ for a projective A-A-bimodule P and $N \otimes_A M \simeq B \oplus Q$ for a projective B-B-bimodule Q.

It soon became clear that stable equivalences of Morita type are much better behaved than arbitrary stable equivalences. Nevertheless, classes of algebras which are classified up to stable equivalence of Morita type are rare. In recent joint work with Yuming Liu [19] we gave several invariants which we shall apply in the present paper to classify algebras up to stable equivalences of Morita type. These invariants were used in [24] to classify symmetric algebras of polynomial growth up to stable equivalence of Morita type. The main additional problem with respect to derived equivalences is that the number of non-projective simple modules is not proven to be an invariant under stable equivalences of Morita type. This fact is the long-standing open Auslander–Reiten conjecture.

Erdmann gave a list [6] of algebras which are defined by properties of their Auslander–Reiten quiver, the Cartan matrix and the representation type and which include all blocks of finite groups of tame representation type. These algebras are

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given by a finite number of quivers and relations which depend on various parameters. The so-defined algebras fall into three classes, called of dihedral, of semi-dihedral and of quaternion type. In each class there are algebras with one, with two and with three simple modules. Moreover, within each class and with a fixed number of simple modules there are a number of finite so-called families of algebras. Each family is given by a fixed quiver with relations depending on several parameters. Erdmann's classification is up to Morita equivalences. Holm pursued further this approach and classified the algebras in Erdmann's list up to derived equivalences [9,10]. Various algebras in different families are proved to be derived equivalent. However the three classes of the algebras are closed under derived equivalences. We shall give an account of his results in Section 2.

In the present work we classify the algebras of dihedral, semi-dihedral and quaternion type up to stable equivalences of Morita type. Our classification is almost as complete as for derived equivalences and the classification coincides roughly with the derived equivalence classification. In particular we show the Auslander–Reiten conjecture for stable equivalences of Morita type between these classes of algebras and we find that the classes of algebras are also closed under stable equivalences of Morita type. Within each class of algebras with a fixed number of simple modules we are able to distinguish in most cases two families up to stable equivalences whenever it is known to distinguish the families up to derived equivalences. However, in a few cases (see Theorem 7.1 for the technical details) we are not able to distinguish the families up to stable equivalences of Morita type even though the families can be distinguished up to derived equivalences.

The paper is organised as follows. In Section 1 we recall some of the invariants under stable equivalence of Morita type we use in the sequel. In Section 2 we recall Holm's derived equivalence classification of algebras of dihedral, semi-dihedral and quaternion type. In Section 3 we present an independent classification for the case of tame blocks of group rings. The proof is much simpler than the general case, and hence we decided to present the arguments separately, though, of course, the general theorem includes this case as well. Moreover, a short summary of Holm's result on Hochschild cohomology of tame blocks is given there. In Section 4 we classify dihedral type algebras up to stable equivalences of Morita type. The main tool is a result of Pogorzały [21, Theorem 7.3]. This section is the first technical core of the paper. In Section 5 we compute the centres of the algebras of semi-dihedral and of quaternion type. This section prepares the classification result for these classes of algebras. In Section 6 we show how one can distinguish stable equivalence classes of Morita type using basically invariants derived from the centre. This part is the second technical core of the paper. In Section 7 we finally summarise large parts of what was proved before. This section contains the main result Theorem 7.1 of the paper. Moreover, in this section we recall the results of an earlier paper of Holm and the second author [12] which distinguishes derived equivalence classes of algebras corresponding to one family and two different parameters in a very subtle situation. The method of proof uses a derived invariant [26] which was shown [19] to be an invariant under stable equivalences of Morita type.

Throughout this paper K denotes a field. Mostly we assume K to be algebraically closed. A K-algebra is assumed to be finite dimensional associative algebra with unit over K and modules over a K-algebra are always assumed to be finite dimensional.

1. Stable invariants

In this section we shall explain and state most of the various properties of algebras invariant under stable equivalences of Morita type used in what follows.

The stable category A-$\underline{mod}$ of a finite dimensional K-algebra A has the same objects as the category of A-modules and morphisms, denoted by $\underline{Hom}_A(M, N)$ from M to N, are equivalence classes of morphisms of A-modules modulo those factoring through projective A-modules.

The first reduction is a result of Keller–Vossieck and Rickard.

Theorem 1.1 (*Keller–Vossieck [14] and Rickard [22]*)**.** *Let K be a field and let A and B be two self-injective K-algebras. If the bounded derived categories $D^b(A)$ and $D^b(B)$ of A and B are equivalent as triangulated categories, then the algebras A and B are stably equivalent of Morita type.*

Hence in order to give a classification of a class of self-injective algebras up to stable equivalences of Morita type we can start from a classification up to derived equivalences and decide for two representatives of the derived equivalence classes whether they are stably equivalent of Morita type.

In order to do so we use several criteria, some linked to invariants around the centres of the algebras.

We first recall a construction due to Broué. Let A and B be K-algebras. If A is stably equivalent of Morita type to B, then the subcategory of the stable category of bimodules generated by $A \otimes_K A^{op}$-modules which are projective on either side is equivalent to the analogous category of $B \otimes_K B^{op}$-modules. The $A \otimes_K A^{op}$-module A is mapped to the $B \otimes_K B^{op}$-module B under this equivalence. Therefore

$$\underline{End}_{A\otimes_K A^{op}}(A) \simeq \underline{End}_{B\otimes_K B^{op}}(B).$$

Broué denotes by $Z^{st}(A) := \underline{End}_{A\otimes_K A^{op}}(A)$ the stable centre and by

$$Z^{pr}(A) := \ker\left(End_{A\otimes_K A^{op}}(A) \longrightarrow \underline{End}_{A\otimes_K A^{op}}(A)\right)$$

the projective centre of A, where $Z(A) \simeq End_{A\otimes_K A^{op}}(A)$ denotes the centre of A.

Theorem 1.2 (*Broué [4, Proposition 5.4]*). *Let K be a field and let A and B be two K-algebras which are stably equivalent of Morita type. Then $Z^{st}(A) \simeq Z^{st}(B)$.*

The centre is usually not known to be an invariant under stable equivalences of Morita type. One of the main results of [19] deals with degree 0 Hochschild homology.

Theorem 1.3 (*Liu et al.[19, Theorem 1.1 and Corollary 1.2]*). *Let K be an algebraically closed field and let A and B be two indecomposable K-algebras which are stably equivalent of Morita type. Then $\dim_K(HH_0(A)) = \dim_K(HH_0(B))$ if and only if the number of isomorphism classes of simple A-modules equals the number of isomorphism classes of simple B-modules. If the algebras are symmetric, then the invariance of the number of simple modules is equivalent to $\dim Z(A) = \dim Z(B)$.*

The second point of the above theorem follows from the first one, because for a symmetric algebra A, we have $Hom_K(HH_0(A), K) \simeq Z(A)$ as vector spaces, and hence we get a partial answer to the invariance of the centre under stable equivalences of Morita type.

Moreover, a very useful criterion was given in [19] as well in order to estimate the dimension of the projective centre.

Proposition 1.4 (*Liu et al.[19, Corollary 2.7]*). *Let K be an algebraically closed field of any characteristic and let A be an indecomposable symmetric K-algebra with n isomorphism classes of simple modules. Then the dimension of the projective centre equals the rank of the Cartan matrix seen as linear mapping $K^n \longrightarrow K^n$.*

If K is a field of characteristic $p \geq 0$ then we shall denote in the sequel by p-rank of the Cartan matrix the rank of the Cartan matrix seen as linear mapping $K^n \longrightarrow K^n$. Recall that for every $A \otimes_K A^{op}$-module V the algebra $Ext^*_{A\otimes_K A^{op}}(A, V)$ is naturally a module over the centre $Z(A)$. Define the Higman ideal $H(A)$ to be the intersection of the annihilators of the $Z(A)$-module $Ext^*_{A\otimes_K A^{op}}(A, V)$ for all $A \otimes_K A^{op}$-modules V (cf. [8]).

Proposition 1.5 (*[19, Proposition 2.3] and [7, Lemma 4.1]*). *The projective centre of an algebra equals the Higman ideal of A. For symmetric algebras the Higman ideal is in the socle of the algebra.*

A classical invariant, popularised by Külshammer [17], is the Reynolds ideal defined for any K-algebra A as $R(A) := Z(A) \cap \mathrm{Soc}(A)$. For a perfect field K of characteristic $p > 0$ and a symmetric K-algebra A define $T_n(A) := \{x \in A \mid x^{p^n} \in [A, A]\}$ where $[A, A]$ is the K-subspace of A generated by all expressions $ab - ba$ for $a, b \in A$. Then $T_n(A)^{\perp}$ is the orthogonal complement of $T_n(A)$ in A with respect to the symmetrising bilinear form of A. These spaces $T_n^{\perp}(A)$ form a decreasing sequence of ideals of the centre $Z(A)$ with intersection $R(A)$, that is,

$$Z^{pr}(A) \subseteq R(A) = \bigcap_n T_n^{\perp}(A) \subseteq \cdots \subseteq T_1^{\perp}(A) \subseteq T_0^{\perp}(A) = Z(A).$$

Proposition 1.6 (*[19, Proposition 6.10] and [15, Proposition 5.8]*). *Let K be an algebraically closed field of characteristic $p > 0$ and let A be a symmetric algebra. Then the ideals $T_n^{\perp}(A)/Z^{pr}(A)$ of $Z^{st}(A)$ are an invariant, as ideals, under stable equivalences of Morita type via the isomorphism in Theorem 1.2. In particular $R(A)/Z^{pr}(A)$ is an invariant under stable equivalences of Morita type.*

The following result will be crucial in what follows.

Theorem 1.7. *Let K be an algebraically closed field and let A and B be two finite dimensional symmetric indecomposable K-algebras which are stably equivalent of Morita type. If K is of positive characteristic or the Cartan matrix of A is non-singular considered as a matrix over K, then we have an isomorphism of algebras $Z(A)/R(A) \simeq Z(B)/R(B)$.*

Proof. The case of positive characteristic is contained in Proposition 1.6. In case of non-singular Cartan matrix over K, by [23, Proposition 5.1] (or see Proposition 1.8 and the discussions afterwards), B has also non-singular Cartan matrix over K. So the rank of the Cartan matrix is equal to the number of isomorphism classes of simple modules and thus so is $\dim(Z^{pr}(A))$. Brauer shows [3, Statement (3A)] that the number of simple A-modules equals the dimension of $A/(J(A)+[A, A])$, where $J(A)$ denotes the Jacobson radical. For a symmetric algebra A the orthogonal space of $J(A)+[A, A]$ with respect to the symmetrising form is $R(A)$, as $J(A)^{\perp} = \mathrm{Soc}(A)$ and $[A, A]^{\perp} = Z(A)$. We obtain thus that the number of simple A-modules is the dimension of the Reynolds ideal. By [7, Lemma 4.1] and [19, Proposition 2.4] we get $Z^{pr}(A) \subseteq R(A)$. The dimensions coincide (cf. Proposition 1.4) and so we have $Z^{pr}(A) = R(A)$ and $Z^{st}(A) = Z(A)/R(A)$. Now we use Theorem 1.2. □

Let A be an indecomposable finite dimensional algebra and let C_A be its Cartan matrix. The Cartan matrix induces in a natural way a mapping of the Grothendieck group $G_0(A)$ of abelian groups (the Grothendieck group taken in the sense of A-modules modulo exact sequences). The stable Grothendieck group $G_0^{st}(A)$ is defined as the cokernel of C_A:

$$G_0(A) \xrightarrow{C_A} G_0(A) \longrightarrow G_0^{st}(A) \longrightarrow 0.$$

Proposition 1.8 (*Xi [23, Section 5]*). *Let A and B be finite dimensional indecomposable K-algebras and suppose that A and B are stably equivalent of Morita type. Then $G_0^{st}(A) \simeq G_0^{st}(B)$. In particular the absolute value of the Cartan determinant is preserved.*

It is clear by this statement that a stable equivalence of Morita preserves those elementary divisors of the Cartan matrix which are different from 1, including their multiplicity. Note that in order to avoid ambiguities the elementary divisors are supposed to be non-negative.

We shall use frequently the following result of Pogorzały. For basic facts about special biserial algebras see [5].

Theorem 1.9. *[21, Theorem 0.1 and Theorem 7.3] Let A be a special biserial algebra over an algebraically closed field K. If a K-algebra B is stably equivalent to A, then A and B have the same number of non-isomorphic non-projective simple modules. If furthermore, A is self-injective and is not a Nakayama algebra, then B is also a self-injective special biserial algebra.*

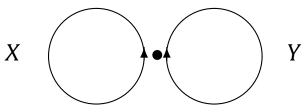
2. Algebras of dihedral, semi-dihedral and quaternion type

In this section we shall give Erdmann's list of algebras of dihedral, semi-dihedral and quaternion type.

By Theorem 1.1 of Keller–Vossieck and Rickard, for two self-injective algebras A and B, an equivalence $D^b(A) \simeq D^b(B)$ of the bounded derived categories implies that A and B are stably equivalent of Morita type. Hence, as basis of our discussion we shall use the list of Holm [10] of algebras of dihedral, semi-dihedral and quaternion type up to derived equivalences. There are three families: the algebras of dihedral type, the algebras of semi-dihedral type, the algebras of quaternion type. Each family is subdivided into three subclasses: algebras with one simple module, algebras with two simple modules and algebras with three simple modules. Each subfamily contains algebras given by quivers and relations, depending on parameters.

	Dihedral	Semi-dihedral	Quaternion
1 simple	$K[X,Y]/(XY, X^m - Y^n)$, $m \geq n \geq 2, m+n > 4$; $D(1\mathcal{A})_1^1 = K[X,Y]/(X^2,Y^2)$; $(charK = 2)$ $K[X,Y]/(X^2, YX - Y^2)$; $D(1\mathcal{A})_1^k, k \geq 2$; $(charK = 2)$ $D(1\mathcal{A})_2^k(d)$, $k \geq 2, d \in \{0,\ 1\}$;	$SD(1\mathcal{A})_1^k, k \geq 2$; $(char(K) = 2)$ $SD(1\mathcal{A})_2^k(c,d)$ $k \geq 2, (c,d) \neq (0,0)$;	$Q(1\mathcal{A})_1^k, k \geq 2$; $(charK = 2)$ $Q(1\mathcal{A})_2^k(c,d)$, $k \geq 2, (c,d) \neq (0,0)$;
2 simples	$D(2\mathcal{B})^{k,s}(c)$, $k \geq s \geq 1, c \in \{0,\ 1\}$	$SD(2\mathcal{B})_1^{k,t}(c)$ $k \geq 1, t \geq 2, c \in \{0,\ 1\}$; $SD(2\mathcal{B})_2^{k,t}(c)$ $k \geq 1, t \geq 2$, $k+t \geq 4, c \in \{0,\ 1\}$;	$Q(2\mathcal{B})_1^{k,s}(a,c)$ $k \geq 1, s \geq 3, a \neq 0$;
3 simples	$D(3\mathcal{K})^{a,b,c}$, $a \geq b \geq c \geq 1$; $D(3\mathcal{R})^{k,s,t,u}$, $s \geq t \geq u \geq k \geq 1, t \geq 2$	$SD(3\mathcal{K})^{a,b,c}$ $a \geq b \geq c \geq 1, a \geq 2$;	$Q(3\mathcal{K})^{a,b,c}$ $a \geq b \geq c \geq 1, b \geq 2$, $(a,b,c) \neq (2,2,1)$; $Q(3\mathcal{A})_1^{2,2}(d)$ $d \notin \{0,1\}$

All algebras with one simple module in the above list have the quiver of type $1\mathcal{A}$



and the relations are given as follows.

$D(1\mathcal{A})_1^k : X^2, Y^2, (XY)^k - (YX)^k$;

$D(1\mathcal{A})_2^k(d) : X^2 - (XY)^k, Y^2 - d \cdot (XY)^k, (XY)^k - (YX)^k, (XY)^k X, (YX)^k Y$;

$SD(1\mathcal{A})_1^k : (XY)^k - (YX)^k, (XY)^k X, Y^2, X^2 - (YX)^{k-1} Y$;

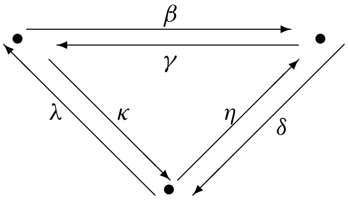
$SD(1\mathcal{A})_2^k(c,d) : (XY)^k - (YX)^k, (XY)^k X, Y^2 - d(XY)^k, X^2 - (YX)^{k-1} Y + c(XY)^k$;

$Q(1\mathscr{A})_1^k : (XY)^k - (YX)^k, (XY)^k X, Y^2 - (XY)^{k-1}X, X^2 - (YX)^{k-1}Y;$

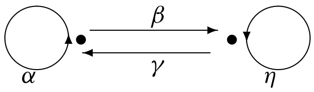
$Q(1\mathscr{A})_2^k(c, d) : X^2 - (YX)^{k-1}Y - c(XY)^k, Y^2 - (XY)^{k-1}X - d(XY)^k, (XY)^k - (YX)^k, (XY)^k X, (YX)^k Y.$

The quivers of the algebras of type $2\mathscr{B}$, $3\mathscr{K}$, $3\mathscr{A}$ and $3\mathscr{R}$ are respectively:

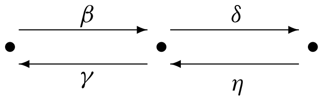
type $3\mathscr{K}$



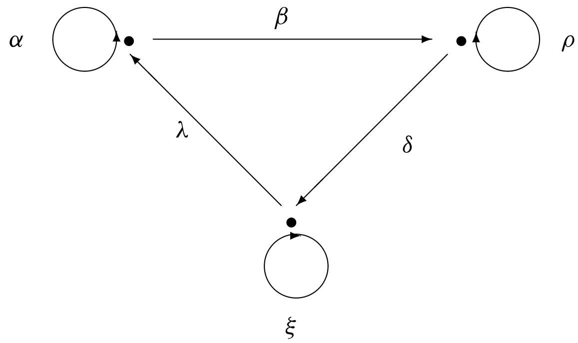
type $2\mathscr{B}$



type $3\mathscr{A}$



type $3\mathscr{R}$



The relations are respectively

$D(2\mathscr{B})^{k,s}(c) : \beta\eta, \eta\gamma, \gamma\beta, \alpha^2 - c(\alpha\beta\gamma)^k, (\alpha\beta\gamma)^k - (\beta\gamma\alpha)^k, \eta^s - (\gamma\alpha\beta)^k;$

$SD(2\mathscr{B})_1^{k,t}(c) : \gamma\beta, \eta\gamma, \beta\eta, \alpha^2 - (\beta\gamma\alpha)^{k-1}\beta\gamma - c(\alpha\beta\gamma)^k, \eta^t - (\gamma\alpha\beta)^k, (\alpha\beta\gamma)^k - (\beta\gamma\alpha)^k;$

$SD(2\mathscr{B})_2^{k,t}(c) : \beta\eta - (\alpha\beta\gamma)^{k-1}\alpha\beta, \eta\gamma - (\gamma\alpha\beta)^{k-1}\gamma\alpha, \gamma\beta - \eta^{t-1}, \alpha^2 - c(\alpha\beta\gamma)^k, \beta\eta^2, \eta^2\gamma;$

$Q(2\mathscr{B})_1^{k,s}(a, c) : \gamma\beta - \eta^{s-1}, \beta\eta - (\alpha\beta\gamma)^{k-1}\alpha\beta, \eta\gamma - (\gamma\alpha\beta)^{k-1}\gamma\alpha, \alpha^2 - a(\beta\gamma\alpha)^{k-1}\beta\gamma - c(\beta\gamma\alpha)^k, \alpha^2\beta, \gamma\alpha^2;$

$D(3\mathscr{K})^{a,b,c} : \beta\delta, \delta\lambda, \lambda\beta, \gamma\kappa, \kappa\eta, \eta\gamma, (\beta\gamma)^a - (\kappa\lambda)^b, (\lambda\kappa)^b - (\eta\delta)^c, (\delta\eta)^c - (\gamma\beta)^a;$

$D(3\mathscr{R})^{k,s,t,u} : \alpha\beta, \beta\rho, \rho\delta, \delta\xi, \xi\lambda, \lambda\alpha, \alpha^s - (\beta\delta\lambda)^k, \rho^t - (\delta\gamma\beta)^k, \xi^u - (\lambda\beta\delta)^k;$

$SD(3\mathscr{K})^{a,b,c} : \kappa\eta, \eta\gamma, \gamma\kappa, \delta\lambda - (\gamma\beta)^{a-1}\gamma, \beta\delta - (\kappa\lambda)^{b-1}\kappa, \lambda\beta - (\eta\delta)^{c-1}\eta;$

$Q(3\mathscr{K})^{a,b,c} : \beta\delta - (\kappa\lambda)^{a-1}\kappa, \eta\gamma - (\lambda\kappa)^{a-1}\lambda, \delta\lambda - (\gamma\beta)^{b-1}\gamma, \kappa\eta - (\beta\gamma)^{b-1}\beta, \lambda\beta - (\eta\delta)^{c-1}\eta,$
$\gamma\kappa - (\delta\eta)^{c-1}\delta, \gamma\beta\delta, \delta\eta\gamma, \lambda\kappa\eta;$

$Q(3\mathscr{A})_1^{2,2}(d) : \beta\delta\eta - \beta\gamma\beta, \delta\eta\gamma - \gamma\beta\gamma, \eta\gamma\beta - d\eta\delta\eta, \gamma\beta\delta - d\delta\eta\delta, \beta\delta\eta\delta, \eta\gamma\beta\gamma.$

The following result suggests that we only need to consider internally these three classes of algebras in order to classify them up to stable equivalences of Morita type.

Proposition 2.1. *If two indecomposable algebras A and B are stably equivalent of Morita type and A is of dihedral (resp. semi-dihedral, quaternion) type, then so is B.*

Proof. These classes of algebras are defined in terms of the nature of their Auslander–Reiten quiver and the Cartan matrix. An algebra A is of one of these types if

- A is symmetric, indecomposable and tame;
- the Cartan matrix of A is non-singular;
- the stable Auslander–Reiten quiver of A has the following components

	Dihedral type	Semi-dihedral type	Quaternion type
Tubes	Rank 1 and 3 At most two 3-tubes	Rank at most 3 At most one 3-tube	Rank at most 2
Others	Non-periodic components of tree class A_∞^∞ or $\tilde{A}_{1,2}$	$\mathbb{Z}A_\infty^\infty$ and $\mathbb{Z}D_\infty$	

By a result of Yuming Liu ([18, Corollary 2.4]) (resp. Krause ([16, last corollary of the article])), if two algebras are stably equivalent of Morita type and one of them is symmetric (resp. tame), so is the other. If two algebras are stably equivalent of Morita type, they are stably equivalent. By [2, Chapter X, Corollary 1.9], two stably equivalent indecomposable self-injective algebras have isomorphic stable Auslander–Reiten quivers, if they are of Loewy length at least three. Note that all algebras of dihedral, semi-dihedral, quaternion type are of Loewy length at least three, so if two algebras within these classes are stably equivalent, they have isomorphic stable Auslander–Reiten quivers. By Xi's result Proposition 1.8 if two algebras are stably equivalent of Morita type, the fact that the Cartan matrix of one algebra is non-singular implies that the Cartan matrix of the other algebra is non-singular as well. Therefore, the defining properties are preserved by a stable equivalence of Morita type between two indecomposable algebras. □

3. Tame blocks

3.1. Derived classification

The following is a classification of algebras of dihedral, semi-dihedral and quaternion type up to derived equivalence, as given by Holm [9], which could occur as blocks of group algebras. For some cases the question if there is a block of a group algebra with this derived equivalence type is not clear yet. We include in this case the algebra as well. Now let K be an algebraically closed field of characteristic two. Let A be a tame block of defect $n \geq 2$. Then A is derived equivalent to one of the following algebras.

	Dihedral	Semi-dihedral	Quaternion
1 simple	$D(1\mathcal{A})_1^{2^{n-2}}$, $n \geq 2$;	$SD(1\mathcal{A})_1^{2^{n-2}}$, $n \geq 4$;	$Q(1\mathcal{A})_1^{2^{n-2}}$, $n \geq 3$;
2 simples	$D(2\mathcal{B})^{1,2^{n-2}}(c)$, $c \in \{0, 1\}$, $n \geq 3$;	$SD(2\mathcal{B})_1^{1,2^{n-2}}(c)$, $c \in \{0, 1\}$, $n \geq 4$; $SD(2\mathcal{B})_2^{2,2^{n-2}}(c)$, $c \in \{0, 1\}$, $n \geq 4$;	$Q(2\mathcal{B})_1^{2,2^{n-2}}(a, c)$, $n \geq 3$, $a \in K^*$, $c \in K$;
3 simples	$D(3\mathcal{K})^{2^{n-2},1,1}$, $n \geq 2$;	$SD(3\mathcal{K})^{2^{n-2},2,1}$, $n \geq 4$;	$Q(3\mathcal{K})^{2^{n-2},2,2}$, $n \geq 3$.

3.2. Hochschild cohomology of tame blocks

If A and B are two algebras which are stably equivalent of Morita type, then Xi shows [23, Theorem 4.2] that the Hochschild cohomology groups $HH^m(A)$ and $HH^m(B)$ are isomorphic for any $m \geq 1$. Furthermore, in a recent paper of the first author with Shengyong Pan [20], we proved that a stable equivalence of Morita type preserves the algebra structure of the stable Hochschild cohomology, that is, the Hochschild cohomology modulo the projective centre.

For the sake of completeness we recall results of Holm [11] which allow to distinguish a certain number of pairs of algebras up to stable equivalences of Morita type, although we could avoid using these results in what follows, mainly because they only deal with blocks of group rings with one or three simple modules.

3.2.1. Dihedral type

By [11, Theorem 2.2] the Hochschild cohomology groups of a block with dihedral defect group of order 2^n with $n \geq 2$ and one simple module has dimension $\dim(HH^i(B)) = 2^{n-2} + 3 + 4i$ for $i \geq 0$.

By [11, Theorem 2.8] the Hochschild cohomology groups of a block with dihedral defect group of order 2^n with $n \geq 2$ and three simple modules has dimension $2^{n-2} + 3$ in degree 0, and dimension $2^{n-2} + 1$ in degree 1. Further, for all $i \geq 1$, $\dim(HH^{3i-1}(B)) = 2^{n-2} - 1 + 4i$ and $\dim(HH^{3i}(B)) = dim(HH^{3i+1}(B)) = 2^{n-2} + 1 + 4i$.

3.2.2. Semi-dihedral type

By [11, Theorem 3.2] the Hochschild cohomology groups of a block with semi-dihedral defect group of order 2^n with $n \geq 4$ and one simple module has dimension $2^{n-2}+3$ in degree 0, dimension $2^{n-2}+6$ in degree 1, dimension $2^{n-2}+7$ in degree 2, and dimension $2^{n-2}+8$ in degree 3. Further, $\dim(HH^{i+4}(B)) = dim(HH^i(B)) + 8$ for all $i \geq 0$.

By [11, Theorem 3.3] the Hochschild cohomology groups of a block with semi-dihedral defect group of order 2^n with $n \geq 4$ and three simple modules has dimension $2^{n-2}+4$ in degrees 0 and 3, dimension $2^{n-2}+2$ in degrees 1 and 2, and dimension $2^{n-2}+5$ in degree 4. Further, for all $i \geq 1$ we get $\dim(HH^{i+4}(B)) = dim(HH^i(B)) + 2 + x(i)$, where $x(i)$ is 0 if 3 divides i, and $x(i) = 1$ else.

3.2.3. Quaternion type

By [11, Theorem 4.2] a block with one simple module and quaternion defect group of order 2^n with $n \geq 3$ has periodic Hochschild cohomology groups with period 4 and dimension $2^{n-2}+3$ in degrees congruent 0 or 3 mod 4 and of dimension $2^{n-2}+5$ in degrees congruent 1 or 2 mod 4.

By [11, Theorem 4.6] a block with three simple modules and quaternion defect group of order 2^n with $n \geq 3$ has periodic Hochschild cohomology groups with period 4 and dimension $2^{n-2}+5$ in degrees congruent 0 or 3 mod 4 and of dimension $2^{n-2}+3$ in degrees congruent 1 or 2 mod 4.

3.3. Blocks of dihedral defect groups

Proposition 3.1. *Let K be an algebraically closed field of characteristic 2 and let A be a dihedral block of defect $n \geq 2$. Then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(1\mathcal{A})_1^{2^{n-2}}$; $D(2\mathcal{B})^{1,2^{n-2}}(c)$ (for $n \geq 3$) with $c = 0$ or $c = 1$; $D(3\mathcal{K})^{2^{n-2},1,1}$. As a consequence, the derived classification coincides with the classification up to stable equivalences of Morita type.*

Remark 3.2. Before giving the proof, we remark that for a dihedral block with two simple modules, we do not know whether the case $c = 1$ really occurs. All known examples have zero as the value of this scalar. But this does not influence our result, since $D(2\mathcal{B})^{k,s}(0)$ is NOT derived equivalent to $D(2\mathcal{B})^{k,s}(1)$. There are several proofs of this fact (cf. [13, Corollary 5.3] [12, Theorem 1.1]). One can also use the second part of Theorem 1.9, which says that an algebra stably equivalent to a self-injective special biserial algebra which is not a Nakayama algebra is itself a self-injective special biserial algebra. Notice that $D(2\mathcal{B})^{k,s}(0)$ is a symmetric special biserial algebra, but $D(2\mathcal{B})^{k,s}(1)$ is not. As a consequence the algebras $D(2\mathcal{B})^{k,s}(1)$ cannot be stably equivalent to any algebra of the other classes.

Proof. Since Holm's result [10] implies that any algebra of dihedral type is derived equivalent to one in the list we gave, we just need to show that any two algebras in the list are not stably equivalent of Morita type.

We prove that for different parameter $s \neq t$, $D(2\mathcal{B})^{1,s}(1)$ is NOT stably equivalent of Morita type to $D(2\mathcal{B})^{1,t}(1)$. To this end, one computes the dimension of the stable centre, that is, the quotient of the centre by the projective centre. By Proposition 1.4, for a symmetric algebra, the dimension of the projective centre is the 2-rank of the Cartan matrix, that is the rank of the Cartan matrix seen as linear mapping over the field K of characteristic $p = 2$. We thus have that for $A = D(2\mathcal{B})^{1,2^{n-2}}(1)$, $\dim(Z^{st}(A)) = 2^{n-2}+2$ for $n \geq 3$ (for the dimension of the centre and the Cartan matrix, see the appendix of [6]). Since $n \geq 3$ this dimension distinguishes two algebras with different parameters in this class. Another way to see this is to use the absolute value of the determinant of the Cartan matrix, which is invariant under stable equivalences of Morita type, by Proposition 1.8. In fact, the absolute value of the determinant of the Cartan matrix of $D(2\mathcal{B})^{1,s}(1)$ is $4s$.

Now consider other classes of algebras. Pogorzały proved the Auslander–Reiten conjecture for self-injective special biserial algebras (see the first part of Theorem 1.9). Thus two indecomposable non-simple self-injective special biserial algebras with different numbers of simple modules cannot be stably equivalent. By Theorem 1.3, we know that for symmetric algebras, this is equivalent to saying that their centres have the same dimension. Now by computing the dimension of the centre, we obtain easily that the number of simple modules and the defect n characterise equivalence classes under stable equivalences of Morita type of dihedral blocks which are special biserial. On can also use the computations of Holm about Hochschild cohomology of dihedral blocks resumed in Section 3.2 to distinguish dihedral blocks with one simple module from those with three simple modules. □

3.4. Blocks with semi-dihedral defect groups

Proposition 3.3. *Let K be an algebraically closed field of characteristic 2 and let A be a semi-dihedral block of defect $n \geq 4$. Then A is stably equivalent of Morita type to an algebra in the following families (1), (2), (3):*

(1) $SD(1\mathcal{A})^{2^{n-2}}$ *with* $n \geq 4$;

(2) (a) $SD(2\mathcal{B})_1^{1,2^{n-2}}(c)$ *with* $n \geq 4$, $c \in \{0, 1\}$;

 (b) $SD(2\mathcal{B})_2^{2,2^{n-2}}(c)$ *with* $n \geq 4$, $c \in \{0, 1\}$;

(3) $SD(3\mathcal{K})^{2^{n-2},2,1}$, $n \geq 4$.

Two algebras with different number of simple modules in the above list are not stably equivalent of Morita type. Two algebras in the above list with the same number of isomorphism classes of simple modules and different parameter n are not stably equivalent of Morita type.

Remark 3.4. In the above classification we still have an unsolved scalar problem, that is, as in the case of derived equivalence classification, we cannot determine whether $SD(2\mathcal{B})_1^{1,2^{n-2}}(0)$ (resp. $SD(2\mathcal{B})_2^{2,2^{n-2}}(0)$) is stably equivalent of Morita type to $SD(2\mathcal{B})_1^{1,2^{n-2}}(1)$ (resp. $SD(2\mathcal{B})_2^{2,2^{n-2}}(1)$) or not. Therefore, up to these problems, the derived classification coincides with the classification up to stable equivalences of Morita type.

Proof. Since a derived equivalence between self-injective algebras induces a stable equivalence of Morita type, the first part of the statement of the proposition is true simply by the derived equivalence classification of Holm. We now prove that the classification is complete up to the problems cited above.

By the result of Holm on Hochschild cohomology of semi-dihedral blocks, a semi-dihedral block with one simple module cannot be stably equivalent of Morita type to a semi-dihedral block with three simple modules. The dimension of the centre and the Cartan matrix can be obtained from the appendix of [6]. The dimension of the stable centre of $SD(1\mathcal{A})^{2^{n-2}}$ with $n \geq 4$ is $2^{n-2}+3$, it is $2^{n-2}+2$ for $SD(2\mathcal{B})_1^{1,2^{n-2}}(c)$ and is $2^{n-2}+4$ for $SD(2\mathcal{B})_2^{2,2^{n-2}}(c)$, while for $SD(3\mathcal{K})^{2^{n-2},2,1}$, it is $2^{n-2}+3$. This invariant distinguishes semi-dihedral blocks with two simple modules from those with one or three simple modules and it also distinguishes $SD(2\mathcal{B})_1^{1,2^{n-2}}(c)$ from $SD(2\mathcal{B})_2^{2,2^{n-2}}(c)$. $\square$

3.5. Blocks with quaternion defect groups

Proposition 3.5. *Let K be an algebraically closed field of characteristic* 2 *and let A be a block with generalised quaternion defect groups of defect* $n \geq 3$. *Then A is stably equivalent of Morita type to one of the following algebras:*

(1) $Q(1\mathcal{A})^{2^{n-2}}$ *with* $n \geq 3$;
(2) $Q(2\mathcal{B})_1^{2,2^{n-2}}(a, c)$ *with* $n \geq 3$, $a \in K^*$, $c \in K$;
(3) $Q(3K)^{2^{n-2},2,2}$ *with* $n \geq 3$.

Two algebras of quaternion type with different number of simple modules in the above list are not stably equivalent of Morita type. Two algebras of quaternion type with the same number of simple modules and different parameters n in the above list are not stably equivalent of Morita type.

Remark 3.6. The above classification is complete up to some scalar problem, that is, as in the case of derived equivalence classification, we cannot determine whether $Q(2\mathcal{B})_1^{2,2^{n-2}}(a, c)$ is not stably equivalent of Morita type to $Q(2\mathcal{B})_1^{2,2^{n-2}}(a', c')$ for $(a, c) \neq (a', c')$. Therefore, up to these scalar problems, the derived classification coincides with the classification up to stable equivalences of Morita type.

Proof. Since a derived equivalence between self-injective algebras induces a stable equivalence of Morita type, the statement of the proposition is true simply by the derived equivalence classification of Holm. We now prove that the classification is complete up to the scalar problem.

The dimension of the stable centre is $2^{n-2}+3$ for $Q(1\mathcal{A})^{2^{n-2}}$, is $2^{n-2}+4$ for $Q(2\mathcal{B})_1^{2,2^{n-2}}(a, c)$ and is $2^{n-2}+5$ for $Q(3\mathcal{K})^{2^{n-2},2,2}$ (for the dimension of the centre and the Cartan matrix, see the appendix of [6]). This invariant thus distinguishes these algebras up to stable equivalences of Morita type up to the scalar problem. One can also use the result of Holm on Hochschild cohomology of blocks with generalised quaternion defect groups to distinguish blocks with generalised quaternion defect groups having one simple module from those having three simple modules. $\square$

4. Algebras of dihedral type

We classify algebras of dihedral type up to stable equivalences of Morita type in this section. Notice that all algebras of dihedral type except $B_1 = K[X, Y]/(X^2, Y^2 - XY)$ and $D(1\mathcal{A})_2^k(d)$ are special biserial. As B_1 and $D(1\mathcal{A})_2^k(d)$ are local algebras and by the result Theorem 1.9 of Pogorzały, one only needs to consider separately dihedral algebras with one, two or three simple modules.

4.1. One simple module

The Morita equivalence classification of algebras of dihedral type with one simple module is displayed in Section 2.

Proposition 4.1. *Let K be an algebraically closed field of characteristic* $p \geq 0$ *and let A be an algebra of dihedral type with one simple module. Then A is stably equivalent of Morita type to an algebra in one and only one of the families* (1), (2), (3), (4) *or* (5) *in the following list:*

(1) *$A_1(m,n) := K[X,Y]/(XY, X^m - Y^n)$ with $m \geq n \geq 2$ and $m+n > 4$;*
(2) *$D(1\mathcal{A})_1^1$;*
(3) *$D(1\mathcal{A})_1^k$ with $k \geq 2$;*
(4) *if $p = 2$, B_1*
(5) *if $p = 2$, $D(1\mathcal{A})_2^k(d)$ with $k \geq 2$ and $d \in \{0, 1\}$.*

If $A_1(m,n)$ is stably equivalent of Morita type to $A_1(m',n')$ then $(m,n) = (m',n')$.
If $D(1\mathcal{A})_1^k$ is stably equivalent of Morita type to $D(1\mathcal{A})_1^{k'}$ then $k = k'$.
If $D(1\mathcal{A})_2^k(d)$ is stably equivalent of Morita type to $D(1\mathcal{A})_2^{k'}(d')$ then $k = k'$.

Remark 4.2. We do not know whether $D(1A)_2^k(0)$ and $D(1A)_2^k(1)$ are stably equivalent of Morita type or not.

The proof combines the following five claims below using some invariants of these algebras shown in the following tables. The tables can be obtained as follows: The dimensions of the centres and the Cartan matrices can be obtained from the list at the end of Erdmann's book [6]. The dimension of the projective centre is the rank of the Cartan matrix, seen as linear map of a K-vector space, the so-called p-rank (Proposition 1.4). Further, by definition $\dim Z^{st}(A) + \dim Z^{pr}(A) = \dim Z(A)$. The stable Grothendieck group is the cokernel of the Cartan mapping, seen as endomorphism of the ordinary Grothendieck group (cf. e.g. [19, Section 4]).

Characteristic zero case

Algebra A	$A_1(m,n)$	$D(1\mathcal{A})_1^1$	$D(1\mathcal{A})_1^k$
$\dim Z(A)$	$n+m$	4	$k+3$
$\dim Z^{pr}(A)$	1	1	1
$\dim Z^{st}(A)$	$n+m-1$	3	$k+2$
C_A	$[n+m]$	$[4]$	$[4k]$
G_0^{st}	$\mathbb{Z}/(n+m)$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$

Characteristic two case

Algebra A	$A_1(m,n)$	$D(1\mathcal{A})_1^1$	$D(1\mathcal{A})_1^k$	B_1	$D(1\mathcal{A})_2^k(d)$
$\dim Z(A)$	$n+m$	4	$k+3$	4	$k+3$
$\dim Z^{pr}(A)$	0 or 1	0	0	0	0
$\dim Z^{st}(A)$	$n+m$ or $n+m-1$	4	$k+3$	4	$k+3$
C_A	$[n+m]$	$[4]$	$[4k]$	$[4]$	$[4k]$
G_0^{st}	$\mathbb{Z}/(n+m)$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$

Characteristic $p > 2$ case

Algebra A	$A_1(m,n)$	$D(1\mathcal{A})_1^1$	$D(1\mathcal{A})_1^k$
$\dim Z(A)$	$n+m$	4	$k+3$
$\dim Z^{pr}(A)$	0 or 1	1	0 or 1
$\dim Z^{st}(A)$	$n+m$ or $n+m-1$	3	$k+3$ or $k+2$
C_A	$[n+m]$	$[4]$	$[4k]$
G_0^{st}	$\mathbb{Z}/(n+m)$	$\mathbb{Z}/4$	$\mathbb{Z}/4k$

By Pogorzały's result (Theorem 1.9), if two finite dimensional self-injective algebras A and B over an algebraically closed field are stably equivalent, and if A is special biserial and not a Nakayama algebra, then B is special biserial as well. Hence we only need to compare $A_1(m,n)$ and $D(1\mathcal{A})_1^1$ with $D(1\mathcal{A})_1^k$, since they are special biserial, and compare B_1 with $D(1A)_2^k(d)$, since they are not special biserial.

Claim 1. *$D(1\mathcal{A})_1^1$ cannot be stably equivalent of Morita type to $A_1(m,n)$ or $D(1\mathcal{A})_1^k$.*
Indeed, the stable Grothendieck groups differ, because $m+n > 4$ and $k \geq 2$.

Similarly one proves

Claim 1′. *B_1 cannot be stably equivalent of Morita type to $D(1\mathcal{A})_2^k(d)$.*

Claim 2. *$A_1(m,n)$ cannot be stably equivalent of Morita type to $D(1\mathcal{A})_1^k$.*
Compare their stable centres and their stable Grothendieck groups.

Claim 3. *$A = A_1(m,n)$ is not stably equivalent of Morita type to $A' = A_1(m',n')$ for $(m,n) \neq (m',n')$.*
Now suppose that $A = A_1(m,n)$ is stably equivalent of Morita type to $A' = A(m',n')$, then by comparing their stable Grothendieck groups, $n+m = n'+m'$. The algebra A is commutative and rather easy to describe. A basis of the algebra is given by $\{X^u, Y^v \mid 0 \leq u \leq m;\ 1 \leq v \leq n-1\}$. Hence the Loewy length of $A = Z(A)$ is $m+1$ (recalling that $m \geq n$ by

hypothesis). The projective centre is $\{0\}$ if and only if the characteristic divides $m+n$ and equals the 1-dimensional socle otherwise. Hence the Loewy length of the stable centre of $A(m,n)$ is m if the characteristic p divides $m+n$ and is $m+1$, otherwise. Hence $m=m'$ and this implies $n=n'$.

Claim 4. $D(1\mathcal{A})_1^k$ *cannot be stably equivalent of Morita type to* $D(1\mathcal{A})_1^l$ *for* $k \neq l$
Comparing the orders of the stable Grothendieck groups gives the result.

Claim 5. $D(1\mathcal{A})_2^k(d)$ *cannot be stably equivalent of Morita type to* $D(1\mathcal{A})_2^l(d')$ *for* $k \neq l$.
Consider the stable Grothendieck groups or the stable centres.

4.2. Two simple modules

For algebras of dihedral type with two simple modules, we have the following result of Holm.

Proposition 4.3 (*[10, Proposition 3.1]*)**.** *Let K be an algebraically closed field of characteristic* $p \geq 0$ *and let A be an algebra of dihedral type with two simple module. Then A is derived equivalent to* $D(2\mathcal{B})^{k,s}(0)$ *with* $k \geq s \geq 1$ *or (when* $p=2$*)* $D(2\mathcal{B})^{k,s}(1)$ *with* $k \geq s \geq 1$.

Proposition 4.4. *Let K be an algebraically closed field of characteristic* $p \geq 0$ *and let A be an algebra of dihedral type with two simple module. Then A is stably equivalent of Morita type to one and exactly one of the following algebras:* $D(2\mathcal{B})^{k,s}(0)$ *with* $k \geq s \geq 1$ *or if* $p=2$, $D(2\mathcal{B})^{k,s}(1)$ *with* $k \geq s \geq 1$.

Proof. By the result of Pogorzały (Theorem 1.9), in case of characteristic two, the algebras $D(2\mathcal{B})^{k,s}(0)$ and $D(2\mathcal{B})^{k,s}(1)$ are not stably equivalent of Morita type.

Erdmann [6, Tables page 294 ff] shows that the centre of $D(2\mathcal{B})^{k,s}(c)$ is of dimension $k+s+2$. The dimension of the Reynolds ideal is 2, since the algebras have 2 simple modules. Now for any characteristic p and for different parameters $(k,s) \neq (k',s')$ such $k \geq s \geq 1$ and $k' \geq s' \geq 1$, if $D(2\mathcal{B})^{k,s}(c)$ is stably equivalent to $D(2\mathcal{B})^{k',s'}(c)$, then comparing the dimension of the centre modulo the Reynolds ideal gives $k+s=k'+s'$. Erdmann shows (cf. [6, Tables page 294ff]) that the Cartan determinant of the algebra $D(2\mathcal{B})^{k,s}(c)$ is $4ks$. Since the absolute values of the determinants of the Cartan matrices are the same, we get $ks=k's'$. This implies that $k=k'$ and $s=s'$. $\square$

4.3. Three simple modules

Proposition 4.5. *Let K be an algebraically closed field of characteristic* $p \geq 0$ *and let A be an algebra of dihedral type with two simple modules. Then A is stably equivalent of Morita type to one and exactly one of the following algebras:* $D(3\mathcal{K})^{a,b,c}$ *with* $a \geq b \geq c \geq 1$ *or* $D(3\mathcal{R})^{k,s,t,u}$ *with* $s \geq t \geq u \geq k \geq 1$ *and* $t \geq 2$.

Proof. Again, we consider different parameters of type $D(3\mathcal{K})^{a,b,c}$ or of type $D(3\mathcal{R})^{k,s,t,u}$. By Theorem 1.7, one can use the algebra structure of the centre modulo the Reynolds ideal to distinguish stable equivalences classes of Morita type.

Using the explicit basis of the centres [10, Lemma 3.16] allows by a very straightforward identification to determine the quotient

$$Z(D(3\mathcal{K})^{a,b,c})/R(D(3\mathcal{K})^{a,b,c}) \simeq K[A,B,C]/(A^a,B^b,C^c,AB,AC,BC)$$

and hence two algebras of type $D(3\mathcal{K})^{a,b,c}$ can only be stably equivalent of Morita type if the parameters a,b,c coincide (cf. Theorem 1.7).

Using [10, Lemma 3.17], we also get that

$$Z(D(3\mathcal{R})^{k,s,t,u})/R(D(3\mathcal{R})^{k,s,t,u}) \simeq K[U,V,W,T]/(U^s,V^t,W^u,T^k,UV,UW,UT,VW,VT,WT)$$

and again two algebras of type $D(3\mathcal{R})^{k,s,t,u}$ can only be stably equivalent of Morita type if the parameters coincide.

Holm shows [10, Remark after Lemma 3.17] that the stable Auslander–Reiten quivers of algebras of type $D(3\mathcal{K})^{a,b,c}$ and of algebras of type $D(3\mathcal{R})^{k,s,t,u}$ are different. Hence algebras of these two types cannot be stably equivalent of Morita type. Another method is as follows. If $D(3\mathcal{R})^{k,s,t,u}$ is stably equivalent of Morita type to $D(3\mathcal{K})^{a,b,c}$, then the quotients of the algebras modulo the Reynolds ideals coincide, and hence $k=1$, $a=s$, $b=t$ and $c=u$. The Cartan matrix of $D(3\mathcal{R})^{1,a,b,c}$ is displayed in [6, Tables page 294ff] and its determinant is $ab+ac+bc+abc$. Now, since $a \geq b=t \geq 2$ and $c \geq 1$, we get that $abc \geq ab$, $abc > bc$ and $abc > ac$, so that $0 < ab+ac+bc+abc < 4abc$, whereas the Cartan determinant of $D(3\mathcal{K})^{a,b,c}$ is $4abc$. Since the absolute values of the Cartan determinants coincide (cf. Proposition 1.8), we obtain a contradiction. $\square$

Although our results Propositions 4.1, 4.4 and 4.5 are only a complete classification up to a scalar problem in the case of an algebra with one simple module, we can prove nevertheless the following special case of the Auslander–Reiten conjecture.

Corollary 4.6. *Let A be an indecomposable algebra which is stably equivalent of Morita type to an algebra of dihedral type. Then this algebra has the same number of simple modules as the algebra of dihedral type.*

Proof. By Proposition 2.1, A is necessarily of dihedral type. Then apply our classification results above. Notice that although we cannot determine whether $D(1\mathcal{A})^k(0)$ and $D(1\mathcal{A})^k(1)$ are stably equivalent of Morita type or not, they have the same number of simple modules. $\square$

5. Centres of semi-dihedral and quaternion type algebras

We shall study the centres and the stable centres of the involved algebras.

5.1. Semi-dihedral type

An algebra of semi-dihedral type with one simple module is Morita equivalent to $SD(1\mathcal{A})_1^k$ with $k \geq 2$ or to (in case of characteristic 2) $SD(1\mathcal{A})_2^k(c, d)$ with $k \geq 2$ and $(c, d) \neq (0, 0)$. Recall from [6, Corollary III.1.3] that for each of these algebras, the centre has dimension $k + 3$. Indeed, more precisely denote by A one of the above algebras. Then the centre $Z(A)$ has a K-basis given by

$$\{1; (XY)^i + (YX)^i; (XY)^k; X(YX)^{k-1}; (YX)^{k-1}Y \mid 1 \leq i \leq k-1\}.$$

Lemma 5.1. *Let K be an algebraically closed field and let A be one of the algebras $SD(1\mathcal{A})_1^k$ with $k \geq 2$ or (in case of characteristic 2) $SD(1\mathcal{A})_2^k(c, d)$ with $k \geq 2$ and $(c, d) \neq (0, 0)$.*

If K is of characteristic 2, then

$$Z(A) \simeq K[U, T, V, W]/(U^k, T^2, V^2, W^2, UT, UV, UW, TV, TW, VW)$$

and $R(A) = Z(A) \cap \mathrm{Soc}(A) = K \cdot T$.

If K is of characteristic different from 2, then

$$Z(A) \simeq K[U, V, W]/(U^{k+1}, V^2, W^2, UV, UW, VW)$$

and $R(A) = Z(A) \cap \mathrm{Soc}(A) = K \cdot U^k$.

Proof. We need to identify U with $XY + YX$, observe that $((XY) + (YX))^i = (XY)^i + (YX)^i$, and identify T with $(XY)^k$, the element V with $X(YX)^{k-1}$ and the element W with $(YX)^{k-1}Y$. If K is of characteristic 2, then $U^k = (XY)^k + (YX)^k = 0$, and if K is of characteristic different from 2, then $U^k = (XY)^k + (YX)^k = 2(XY)^k \neq 0$. □

The Cartan matrix of the algebra A is the matrix $(4k)$ of size 1×1.

Now we turn to the case of two simple modules. Recall from the table at the beginning of Section 2 that an algebra of semi-dihedral type with two simple modules is derived equivalent to $SD(2\mathcal{B})_1^{k,s}(c)$ with $k \geq 1$, $s \geq 2$ and $c \in \{0, 1\}$ or to $SD(2\mathcal{B})_2^{k,s}(c)$ with $k \geq 1$, $s \geq 2$, $k + s \geq 4$ and $c \in \{0, 1\}$.

Lemma 5.2. *Let A be the algebra $SD(2\mathcal{B})_1^{k,s}(c)$ or the algebra $SD(2B)_2^{k,s}(c)$.*
(1) *If K is of characteristic 2, then*

$$Z(A) \simeq K[u, v, w, t]/(u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

and $R(A) = K \cdot u^k \oplus K \cdot w$.
(2) *If K is of characteristic different from 2, then*

$$Z(A) \simeq K[u, v, t]/(u^{k+1}, v^{s+1}, t^2, uv, ut, vt)$$

and $R(A) = K \cdot u^k \oplus K \cdot v^s$.

Proof. By [6, IX 1.2 LEMMA], a basis of the centre of $SD(2\mathcal{B})_1^{k,s}(c)$ is given by

$$\{1; (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i; (\beta\gamma\alpha)^{k-1}\beta\gamma; (\alpha\beta\gamma)^k; \eta^j \mid 1 \leq i \leq k-1; 1 \leq j \leq s\}.$$

Now let

$$u = \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta, v = \eta, \qquad t = (\beta\gamma\alpha)^{k-1}\beta\gamma, \qquad w = (\alpha\beta\gamma)^k.$$

If $\mathrm{char}(K) = 2$, then $u^k = v^s$; otherwise, $u^k = v^s + 2w$. Hence, w may be eliminated from the relations by the equation $u^k = v^s + 2w$ in case $\mathrm{char}(K) \neq 2$. It is easy to verify all other relations. An argument of comparing dimensions gives the result.

As for $SD(2B)_2^{k,s}(c)$, by [6, IX 1.2 LEMMA], a basis of the centre of $SD(2B)_1^{k,s}(c)$ is given by

$$\{1; (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i; (\beta\gamma\alpha)^{k-1}\beta\gamma; (\alpha\beta\gamma)^k; \eta + (\alpha\beta\gamma)^{k-1}\alpha; \eta^j \mid 1 \leq i \leq k-1; 2 \leq j \leq s\}.$$

Now let

$$u = \alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta, \qquad v = \eta + (\alpha\beta\gamma)^{k-1}\alpha, \qquad t = (\beta\gamma\alpha)^{k-1}\beta\gamma, \qquad w = (\alpha\beta\gamma)^k.$$

A similar argument as above gives the result. □

It is important to know that in this presentation the element t is not in the socle of $SD(2\mathcal{B})_1^{k,s}(c)$ and can therefore not be in the projective centre (cf. Proposition 1.5).

The Cartan matrix of $SD(2\mathcal{B})_1^{k,s}(c)$ with $k \geq 1, s \geq 2$ and $c \in \{0, 1\}$ and of $SD(2B)_2^{k,s}(c)$ with $k \geq 1, s \geq 2, k + s \geq 4$ and $c \in \{0, 1\}$ is

$$\begin{pmatrix} 4k & 2k \\ 2k & s+k \end{pmatrix}.$$

The determinant of this matrix is $4ks$.

Recall that Holm proved in [10, Lemma 4.16] that the centre of $SD(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$ and $a \geq 2$ has a basis given by

$$\{1, (\beta\gamma + \gamma\beta)^{i_1}; (\kappa\lambda + \lambda\kappa)^{i_2}; (\delta\eta + \eta\delta)^{i_3}; (\beta\gamma)^a; (\lambda\kappa)^b; (\delta\eta)^c \mid 1 \leq i_1 < a, 1 \leq i_2 < b, 1 \leq i_3 < c\}$$

and so we get

Lemma 5.3.

$$\begin{aligned} Z(SD(3\mathcal{K})^{a,b,c}) \simeq K[A, B, C, S_1, S_2, S_3]/(&A^{a+1}, B^{b+1}, C^{c+1}, A^a - S_2 - S_3, B^b - S_3 - S_1, \\ &C^c - S_1 - S_2, AS_i, BS_i, CS_i, S_iS_j, AB, AC, BC; i, j \in \{1, 2, 3\}) \end{aligned}$$

and $R(SD(3\mathcal{K})^{a,b,c}) = K \cdot S_1 \oplus K \cdot S_2 \oplus K \cdot S_3$.

Proof. Let

$$A = \beta\gamma + \gamma\beta, \qquad B = \kappa\lambda + \lambda\kappa, C = \delta\eta + \eta\delta, \qquad S_1 = \lambda\beta\delta, S_2 = \delta\lambda\beta, \qquad S_3 = \beta\delta\lambda.$$

Then it is a straight forward verification that A, B, C, S_1, S_2, S_3 satisfy the relations on the right-hand side. Now the isomorphism follows from a dimension argument. It is clear that the elements S_1, S_2 and S_3 are in the socle. The socle is three-dimensional, hence S_1, S_2 and S_3 span the Reynolds ideal. □

The Cartan matrix of $SD(3\mathcal{K})^{a,b,c}$ equals (cf. [6, tables pages 294ff])

$$\begin{pmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{pmatrix}$$

which has determinant $4abc$.

5.2. Quaternion type

As displayed in the table at the beginning of Section 2 an algebra of quaternion type with one simple module is Morita equivalent to $Q(1\mathcal{A})_1^\ell$ with $\ell \geq 2$ or to $Q(1\mathcal{B})_2^\ell(c, d)$ with $\ell \geq 2$ and $(c, d) \neq (0, 0)$. Again, by [6, IX.1.1 Proposition] the centre is of dimension $\ell + 3$ and the algebra is of dimension 4ℓ.

Let A be one of the above algebras. In the above presentation, the centre has a K-basis given by

$$\{1; (XY)^i + (YX)^i; (XY)^\ell; X(YX)^{\ell-1}; (YX)^{\ell-1}Y \mid 1 \leq i \leq \ell - 1\}.$$

Lemma 5.4. (1) *If K is of characteristic 2, then*

$$Z(A) \simeq K[U, T, V, W]/(U^k, T^2, V^2, W^2, UT, UV, UW, TV, TW, VW)$$

and $R(A) = Z(A) \cap soc(A) = K \cdot T$.

(2) *If K is of characteristic different from 2, then*

$$Z(A) \simeq K[U, V, W]/(U^{k+1}, V^2, W^2, UV, UW, VW)$$

and $R(A) = Z(A) \cap soc(A) = K \cdot U^k$.

Proof. The proof is a straight forward verification. □

As displayed in the table at the beginning of Section 2 an algebra of quaternion type with two simple modules is derived equivalent to $Q(2\mathcal{B})_1^{k,s}(a, c)$ with $k \geq 1, s \geq 3$ and $a \neq 0$. By [6, IX 1.2 LEMMA], the centre of $Q(2\mathcal{B})_1^{k,s}(a, c)$ has a basis

$$\{1; (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i, (\beta\gamma\alpha)^{k-1}\beta\gamma, (\alpha\beta\gamma)^k, \eta + (\alpha\beta\gamma)^{k-1}\alpha, \eta^j \mid 1 \leq i \leq k-1; 2 \leq j \leq s\}.$$

By a similar proof as that of Proposition 5.2, we have

Lemma 5.5. (1) *If* $char(K) = 2$*, then*

$$Z(Q(2\mathcal{B})_1^{k,s}(a,c)) \simeq K[u,v,w,t]/(u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

and $R(A) = Z(A) \cap soc(A) = K \cdot u^k \oplus K \cdot w$.

(2) *If* $char(K) \neq 2$*, then*

$$Z(Q(2\mathcal{B})_1^{k,s}(a,c)) \simeq K[u,v,t]/(u^{k+1}, v^{s+1}, t^2, uv, ut, vt)$$

and $R(A) = Z(A) \cap soc(A) = K \cdot u^k \oplus K \cdot v^s$.

The Cartan matrix of $Q(2\mathcal{B})_1^{k,s}(a,c)$ is

$$\begin{pmatrix} 4k & 2k \\ 2k & k+s \end{pmatrix}.$$

As displayed in the table at the beginning of Section 2 an algebra of quaternion type with three simple modules is derived equivalent to $Q(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$, $b \geq 2$ and $(a,b,c) \neq (2,2,1)$ or to $Q(3\mathcal{A})_1^{2,2}(d)$ with $d \notin \{0,1\}$.

The dimension of the centre of $Q(3\mathcal{K})^{a,b,c}$ is $a+b+c+1$ and the centre has a basis

$$\{1, (\beta\gamma+\gamma\beta)^{i_1}, (\kappa\lambda+\lambda\kappa)^{i_2}, (\delta\eta+\eta\delta)^{i_3}; (\lambda\kappa)^b; (\beta\gamma)^b; (\delta\eta)^c \mid 1 \leq i_1 < a; 1 \leq i_2 < b; 1 \leq i_3 \leq c\}.$$

The Cartan matrix of the algebra $Q(3\mathcal{K})^{a,b,c}$ is

$$\begin{pmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{pmatrix}.$$

The dimension of the centre of $Q(3\mathcal{A})_1^{2,2}(d)$ is 6 and the centre has a basis

$$\{1, \beta\gamma+\gamma\beta+d\eta\delta, \beta\gamma+\eta\delta+\delta\eta, (\beta\gamma)^2, (\gamma\beta)^2 = d(\delta\eta)^2, (\eta\delta)^2\}.$$

The Cartan matrix of $Q(3\mathcal{A})_1^{2,2}(d)$ is

$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

Indeed, the fact that the above elements are central is readily verified and the dimensions are as they should be. The statement on the Cartan matrix is taken from [6, Tables p.294 ff].

Lemma 5.6. *We have for* $a \geq b \geq c \geq 1$ *and* $b \geq 2$

$$\begin{aligned} Z(Q(3\mathcal{K})^{a,b,c}) &\simeq K[A,B,C,S_1,S_2,S_3]/(A^{a+1}, B^{b+1}, C^{c+1}, A^a - S_2 - S_3, B^b - S_3 - S_1, \\ &\qquad C^c - S_1 - S_2, AS_i, BS_i, CS_i, S_iS_j, AB, AC, BC; i,j \in \{1,2,3\}) \\ Z(Q(3\mathcal{A})_1^{2,2}) &\simeq K[A,B,C,S_1,S_2,S_3]/(A^3, B^3, C^2, A^2 - S_2 - S_3, B^2 - S_3 - S_1, \\ &\qquad C - S_1 - S_2, AS_i, BS_i, CS_i, S_iS_j, AB, AC, BC; i,j \in \{1,2,3\}) \end{aligned}$$

and $R(Q(3\mathcal{K})^{a,b,c}) = K \cdot S_1 \oplus K \cdot S_2 \oplus K \cdot S_3$.

Proof. The proof for $Q(3\mathcal{K})^{a,b,c}$ is identical to the one of Lemma 5.3. For $Q(3\mathcal{A})_1^{2,2}(d)$, let

$$\begin{aligned} A &:= \beta\gamma+\gamma\beta+d\eta\delta, \\ B &:= \beta\gamma+\eta\delta+\delta\eta, \\ C &:= (1-d)(\delta\eta)^2 + d^2(\eta\delta)^2, \\ S_1 &:= (1-d)(\delta\eta)^2, \\ S_2 &:= d^2(\eta\delta)^2, \\ S_3 &:= (\beta\gamma)^2 + d(\delta\eta)^2. \end{aligned}$$

The rest is a straight forward verification. □

Note that, in order to simplify the notation we may allow the parameters $(a,b,c) = (2,2,1)$ in $Q(3\mathcal{K})^{a,b,c}$ and obtain the centre and Cartan data of this algebra then becomes the centre and Cartan data of $Q(3\mathcal{A})_1^{2,2}(d)$.

6. Algebras with stable centres and Cartan data as for semi-dihedral and quaternion type

In what follows, we shall develop properties for algebras of semi-dihedral and quaternion type which will only depend on the Cartan data, the structure of their centre and the Reynolds ideal. We denote by $A_3^{a,b,c}$, $A_2^{k,s}$ and A_1^ℓ algebras having this particular centre, Reynolds ideal and these Cartan data. All results developed for these algebras apply then automatically to the corresponding algebras of semi-dihedral and quaternion type. In particular we shall not need to give a separate treatment for algebras of semi-dihedral and of quaternion type. Here are the details.

- For $\ell \geq 2$, let A_1^ℓ be a basic indecomposable symmetric algebra of dimension 4ℓ so that in case K is of characteristic 2,

  $$Z(A_1^\ell) \simeq K[U, T, V, W]/(U^\ell, T^2, V^2, W^2, UT, UV, UW, TV, TW, VW)$$

  and the Reynolds ideal $R(A_1^\ell) = K \cdot T$ and if K is of characteristic different from 2, then

  $$Z(A_1^\ell) \simeq K[U, V, W]/(U^{\ell+1}, V^2, W^2, UV, UW, VW)$$

  and $R(A_1^\ell) = K \cdot U^\ell$.
- For $k, s \geq 1$, let $A_2^{k,s}$ be a basic indecomposable symmetric algebra with Cartan matrix

  $$\begin{pmatrix} 4k & 2k \\ 2k & s+k \end{pmatrix}$$

  with determinant $4ks$. In case K is of characteristic 2, the centre is of the form

  $$Z(A_2^{k,s}) \simeq K[u, v, w, t]/(u^k - v^s, w^2, t^2, uv, uw, vw, tw, ut, vt)$$

  and the Reynolds ideal is $R(A_2^{k,s}) = Ku^k \oplus Kw$. If K is of characteristic different from 2, then

  $$Z(A_2^{k,s}) \simeq K[u, v, t]/(u^{k+1}, v^{s+1}, t^2, uv, ut, vt)$$

  and the Reynolds ideal is $R(A_2^{k,s}) = Ku^k \oplus Kv^s$.
- For $a, b, c \geq 1$, let $A_3^{a,b,c}$ be a basic indecomposable symmetric K-algebra with centre isomorphic to

  $$\begin{aligned} Z(A_3^{a,b,c}) \simeq\ & K[A, B, C, S_1, S_2, S_3]/(A^{a+1}, B^{b+1}, C^{c+1}, A^a - S_2 - S_3, B^b - S_3 - S_1, \\ & C^c - S_1 - S_2, AS_i, BS_i, CS_i, S_iS_j, AB, AC, BC; i, j \in \{1, 2, 3\}) \end{aligned}$$

  and Cartan matrix

  $$\begin{pmatrix} a+b & a & b \\ a & a+c & c \\ b & c & b+c \end{pmatrix}$$

  with determinant $4abc$ and the Reynolds ideal $R(A_3^{a,b,c}) = KS_1 \oplus KS_2 \oplus KS_3$.

  Observe that

  $$\dim(Z(A_1^\ell)) = \ell + 3, \quad \dim(Z(A_2^{k,s})) = k + s + 2 \ \text{ and } \ \dim(Z(A_3^{a,b,c})) = a + b + c + 1.$$

  The next result gives the precise structure of the centre modulo the Reynolds ideal for the above three classes of algebras.

Proposition 6.1. (1) *For A_1^ℓ we get*

$$Z(A_1^\ell)/R(A_1^\ell) \simeq K[U, V, W]/(U^\ell, V^2, W^2, UV, UW, VW).$$

(2) *For $A_2^{k,s}$ we get*

$$Z(A_2^{k,s})/R(A_2^{k,s}) \simeq K[u, v, t]/(u^k, v^s, t^2, uv, ut, vt).$$

(3) *For $A_3^{a,b,c}$ we get*

$$Z(A_3^{a,b,c})/R(A_3^{a,b,c}) \simeq K[A, B, C]/(A^a, B^b, C^c, AB, AC, BC).$$

Proof. Since the one-dimensional socle of A_1^ℓ is in the centre, the result follows for A_1^ℓ.

In case of the algebra $A_2^{k,s}$ and K is of characteristic 2, the Reynolds ideal is generated by u^k and w, and so the result follows in this case. If K is of characteristic different from 2, the result follows immediately as well.

For $A_3^{a,b,c}$ the Reynolds ideal is generated by S_1, S_2 and S_3. Hence $A^a = S_2 + S_3 \in R(A)$, $B^b = S_1 + S_3 \in R(A)$ and $C^c = S_1 + S_2 \in R(A)$. This implies

$$Z(A_3^{a,b,c})/R(A_3^{a,b,c}) \simeq K[A, B, C]/(A^a, B^b, C^c, AB, AC, BC). \quad \square$$

We shall study the centre of $A_3^{a,b,c}$ in a little more detail.

Let K be of characteristic different from 2. Concerning the relations of $Z(A_3^{a,b,c})$ we see that the elements $S_1+S_2, S_2+S_3, S_3+S_1$ generate the same space as S_1, S_2, S_3, since the base field is assumed to be of characteristic different from 2. Therefore we get

$$Z(A_3^{a,b,c}) \simeq K[A,B,C]/(A^{a+1}, B^{b+1}, C^{c+1}, AB, AC, BC)$$

in this case.

In case the characteristic of K is 2 the subspace of the socle of the algebra $A_3^{a,b,c}$ generated by $S_1+S_2, S_2+S_3, S_1+S_3$ is of codimension 1, namely given by the condition

$$(S_1+S_2)+(S_2+S_3)+(S_1+S_3)=0$$

and so $A^a+B^b+C^c=0$. Hence, in characteristic 2 we get

$$Z(A_3^{a,b,c}) \simeq K[A,B,C,S]/(A^{a+1}, B^{b+1}, C^{c+1}, S^2, A^a+B^b+C^c, AS, BS, CS, AB, AC, BC),$$

where we put $S := S_1$.

Theorem 6.2. *Let K be an algebraically closed field of characteristic $p \geq 0$. Then we get the following statements.*

(1) (a) *$A_3^{a,b,c}$ cannot be stably equivalent of Morita type to $A_2^{k,s}$.*
  (b) *$A_2^{k,s}$ cannot be stably equivalent of Morita type to A_1^{ℓ}.*
  (c) *$A_3^{a,b,c}$ cannot be stably equivalent of Morita type to A_1^{ℓ}.*

(2) (a) *Suppose A_1^{ℓ} is stably equivalent of Morita type to $A_1^{\ell'}$ for $\ell, \ell' \geq 2$. Then $\ell = \ell'$*
  (b) *Suppose $A_2^{k,s}$ is stably equivalent of Morita type to $A_2^{k',s'}$ for $k, s, k', s' \geq 1$. Then $(k,s)=(k',s')$ or $(k,s)=(s',k')$.*
  (c) *Suppose $A_3^{a,b,c}$ is stably equivalent of Morita type to $A_3^{a',b',c'}$ for $a \geq b \geq c \geq 1$ and $a' \geq b' \geq c' \geq 1$. Then $(a,b,c)=(a',b',c')$.*

Proof. We shall use Propositions 1.4, 1.8, Theorems 1.2 and 1.7. For the three classes of algebras which we discuss in the theorem, either the characteristic of the base field is positive or the Cartan matrices are always non-singular in case that the characteristic of the base field is zero. In each case, the hypothesis of Theorem 1.7 holds.

We shall now prove the statement (1a). Suppose that $A_2^{k,s}$ and $A_3^{a,b,c}$ are stably equivalent of Morita type.

The equality of the absolute values of Cartan matrices (cf. Proposition 1.8) gives $abc = ks$. Now Theorem 1.7 implies that $Z(A_2^{k,s})/R(A_2^{k,s}) \simeq Z(A_3^{a,b,c})/R(A_3^{a,b,c})$. By Proposition 6.1 we know that

$$Z(A_2^{k,s})/R(A_2^{k,s}) \simeq K[u,v,t]/(u^k, v^s, t^2, uv, ut, vt)$$

and

$$Z(A_3^{a,b,c})/R(A_3^{a,b,c}) \simeq K[A,B,C]/(A^a, B^b, C^c, AB, AC, BC).$$

Then, comparing the radical series, one gets $\{k,s,2\}=\{a,b,c\}$, taken with multiplicities. But this implies that $abc = 2ks$, which contradicts the equality of the absolute values of Cartan matrices. This shows statement (1a).

We shall show the statement (1b). Suppose $A_2^{k,s}$ with $k,s \geq 1$ is stably equivalent of Morita type to A_1^{ℓ} with $\ell \geq 2$.

Recall from Proposition 6.1 that

$$Z(A_2^{k,s})/R(A_2^{k,s}) \simeq K[u,v,t]/(u^k, v^s, t^2, uv, ut, vt)$$

and

$$Z(A_1^{\ell})/R(A_1^{\ell}) \simeq K[U,V,W]/(U^{\ell}, V^2, W^2, UV, UW, VW).$$

Hence, comparing the radical structure of these rings, taking into account that they need to be isomorphic by Theorem 1.7, we get that ($k=2$ and $s=\ell$) or ($s=2$ and $k=\ell$). Suppose without loss of generality that $k=\ell$ and $s=2$. The Cartan determinant of A_1^{ℓ} is 4ℓ whereas the Cartan determinant of $A_2^{\ell,2}$ is $4ks=8\ell$. The absolute values of the Cartan determinants have to be equal by Proposition 1.8. This contradiction shows the statement (1b).

We shall prove (1c) now. Suppose $A_3^{a,b,c}$ is stably equivalent of Morita type to A_1^{ℓ} with $\ell \geq 2$, and suppose without loss of generality $a \geq b \geq c \geq 1$. We apply Proposition 6.1 and Theorem 1.7 which determines the quotient of the centres modulo the Reynolds ideal and which shows that these quotients have to be isomorphic. We compare the radical series of the centres modulo the Reynolds ideals and obtain $a=\ell$ and $b=c=2$. The absolute values of the Cartan determinants of the two algebras have to coincide by Proposition 1.8. The Cartan determinant of $A_3^{a,b,c}$ is $4abc=8\ell$. The Cartan determinant of the local algebra A_1^{ℓ} coincides with its dimension 4ℓ. This is a contradiction and proves (1c).

Finally we shall prove (2). As in (1b) and (1c), we use the two invariants: the algebra structure of the centre modulo the Reynolds ideal and the absolute value of the Cartan determinant. Therefore, (2) is actually an immediate consequence of Proposition 6.1 applied to the various situations. □

Although we cannot classify completely algebras of semi-dihedral and quaternion type up to stable equivalences of Morita type, we can nevertheless prove the following

Corollary 6.3. *Let A be an indecomposable algebra which is stably equivalent of Morita type to an algebra B of semi-dihedral type or of quaternion type. Then A has the same number of simple modules as B.*

Proof. This is an immediate consequence of the above Theorem 6.2. □

7. The main theorem and concluding remarks

We summarise the results of this paper in a single theorem. We use the notations introduced above, which coincides with the notations in [6] or [10].

Theorem 7.1. *Let K be an algebraically closed field.*
Suppose A and B are indecomposable algebras which are stably equivalent of Morita type.

- *If A is an algebra of dihedral type, then B is of dihedral type. If A is of semi-dihedral type, then B is of semi-dihedral type. If A is of quaternion type then B is of quaternion type.*
- *If A and B are of dihedral, semi-dihedral or quaternion type, then A and B have the same number of simple modules.*
- *Let A be an algebra of dihedral type.*

  (1) *If A is local, then A is stably equivalent of Morita type to an algebra in exactly one of the families* (a), (b), (c), (d) *or* (e) *in the following list:*
  (a) $A_1(m,n)$ *with* $m \geq n \geq 2$ *and* $m+n>4$;
  (b) $D(1\mathcal{A})_1^1$;
  (c) $D(1\mathcal{A})_1^k$ *with* $k \geq 2$;
  (d) $D(1\mathcal{A})_2^k(d)$ *with* $k \geq 2$ *and* $d \in \{0,1\}$, *in case the characteristic of K is* 2.
  (e) B_1, *in case the characteristic of K is* 2.
  If $A(m,n)$ is stably equivalent of Morita type to $A(m',n')$ for $m \geq n \geq 2$ and $m' \geq n' \geq 2$ and $(m,n) \neq (2,2) \neq (m',n')$, then $(m,n)=(m',n')$.
  If $D(1\mathcal{A})_1^k$ is stably equivalent of Morita type to $D(1\mathcal{A})_1^{k'}$ for $k,k' \geq 2$, then $k=k'$.
  If $p=2$ and $D(1\mathcal{A})_2^k(d)$ is stably equivalent of Morita type to $D(1\mathcal{A})_2^{k'}(d')$ for $k,k' \geq 2$ and $d,d' \in \{0,1\}$, then $k=k'$.
  (2) *If A has two simple modules, then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(2\mathcal{B})^{k,s}(0)$ with $k \geq s \geq 1$ or if $p=2$, $D(2\mathcal{B})^{k,s}(1)$ with $k \geq s \geq 1$.*
  (3) *If A has three simple modules then A is stably equivalent of Morita type to one and exactly one of the following algebras: $D(3\mathcal{K})^{a,b,c}$ with $a \geq b \geq c \geq 1$ or $D(3\mathcal{R})^{k,s,t,u}$ with $s \geq t \geq u \geq k \geq 1$ and $t \geq 2$.*

- *Let A be an algebra of semi-dihedral type.*

  (1) *If A has one simple module then A is stably equivalent of Morita type to one of the following algebras: $SD(1\mathcal{A})_1^k$ for $k \geq 2$ or $SD(1\mathcal{A})_2^k(c,d)$ for $k \geq 2$ and $(c,d) \neq (0,0)$ if the characteristic of K is* 2. *Different parameters k yield algebras in different stable equivalence classes of Morita type.*
  (2) *If A has two simple modules then A is stably equivalent of Morita type to $SD(2\mathcal{B})_1^{k,s}(c)$ for $k \geq 1$; $s \geq 2$; $c \in \{0,1\}$ or to $SD(2\mathcal{B})_2^{k,s}(c)$ for $k \geq 1$; $s \geq 2$; $c \in \{0,1\}$; $k+s \geq 4$.*
  If $SD(2\mathcal{B})_1^{k,s}(c)$ is stably equivalent of Morita type to $SD(2\mathcal{B})_1^{k',s'}(c')$ for $k,k' \geq 1$, for $s,s' \geq 2$, and for $c,c' \in \{0,1\}$, then $(k,s)=(k',s')$ or $(k,s)=(s',k')$.
  (3) *If A has three simple modules, then A is stably equivalent of Morita type to one and only one algebra of the type $SD(3\mathcal{K})^{a,b,c}$ for $a \geq b \geq c \geq 1$.*

- *Let A be an algebra of quaternion type.*

  (1) *If A has one simple modules, then A is stably equivalent of Morita type to one of the algebras $Q(1\mathcal{A})_1^k$ for $k \geq 2$ or $Q(1\mathcal{A})_2^k(c,d)$ for $k \geq 2$, $(c,d) \neq (0,0)$ if the characteristic of K is* 2. *Different parameters k yield algebras in different stable equivalence classes of Morita type.*
  (2) *If A has two simple modules then A is stably equivalent of Morita type to one of the algebras $Q(2\mathcal{B})_1^{k,s}(a,c)$ for $k \geq 1$; $s \geq 3$; $a \neq 0$.*
  If $Q(2\mathcal{B})_1^{k,s}(a,c)$ is stably equivalent of Morita type to $Q(2\mathcal{B})_1^{k',s'}(a',c')$ for $k,k' \geq 1$, for $s,s' \geq 3$ and for $a,a' \neq 0$, then $(k,s)=(k',s')$ or $(k,s)=(s',k')$.
  (3) *If A has three simple modules, then A is stably equivalent of Morita type to one of the algebras $Q(3\mathcal{K})^{a,b,c}$ for $a \geq b \geq c \geq 1$; $b \geq 2$; $(a,b,c) \neq (2,2,1)$ or $Q(3\mathcal{A})_1^{2,2}(d)$ for $d \in K \setminus \{0,1\}$. Different parameters a,b,c yield algebras in different stable equivalence classes of Morita type.*

Proof. The first point is Proposition 2.1 and the second point is Corollaries 4.6 and 6.3. The third point is Propositions 4.1, 4.4 and 4.5. The fourth point is Theorem 6.2 together with Section 5.1 and the fifth point is Theorem 6.2 together with Section 5.2. □

Remark 7.2. For algebras of dihedral type, we proved in Section 4 that the classification up to stable equivalences of Morita type coincide with derived equivalence classification, up to a scalar problem in $D(1\mathcal{A})_2^k(d)$. The only piece that is missing for a complete classification is the question if $D(1\mathcal{A})_2^k(0)$ is stably equivalent of Morita type to $D(1\mathcal{A})_2^k(1)$.

Remark 7.3. Derived equivalent local algebras are Morita equivalent as is shown by Roggenkamp and the second author (cf. [25]). Observe that tame local symmetric algebras are classified in [6, Chapter III]. Actually, the classification coincides with the algebras with one simple module we already dealt with in the text. So, a complete classification of the algebras of dihedral, semi-dihedral or quaternion type with one simple module up to stable equivalence of Morita type would give a classification up to stable equivalence of Morita type of tame local symmetric algebras.

Corollary 7.4. *The Auslander–Reiten conjecture holds for tame local symmetric algebras, i.e. if A is a tame local symmetric algebra and if B is an algebra without simple direct factor which is stably equivalent of Morita type to A, then B is local tame symmetric as well.*

Proof. By Liu [18] B is indecomposable since A is indecomposable. Erdmann classified tame local symmetric algebras [6, III.1 Theorem]. The classification coincides with the list of local algebras of dihedral, semi-dihedral or quaternion type. □

We cannot give any answer to the classification of algebras of dihedral, semi-dihedral or quaternion type up to derived equivalence beyond the information that is already known. Nevertheless, one more statement for algebras of semi-dihedral type was obtained by Holm and the second author.

Theorem 7.5 (*Holm and Zimmermann [12]*). *Let K be an algebraically closed field of characteristic 2.*

(1) *For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_1^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Put $B_c^{k,s} := SD(2\mathcal{B})_1^{k,s}(c)$. Suppose that if $k = 2$ then $s \geq 3$ is odd, and if $s = 2$ then $k \geq 3$ is odd. Then the factor rings $Z(B_0^{k,s})/T_1(B_0^{k,s})^\perp$ and $Z(B_1^{k,s})/T_1(B_1^{k,s})^\perp$ are not isomorphic.*

   In particular, the algebras $SD(2\mathcal{B})_1^{k,s}(0)$ and $SD(2\mathcal{B})_1^{k,s}(1)$ are not derived equivalent.

(2) *For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_2^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Put $C_c^{k,s} := SD(2\mathcal{B})_2^{k,s}(c)$. If the parameters k and s are both odd, then the factor rings $Z(C_0^{k,s})/T_1(C_0^{k,s})^\perp$ and $Z(C_1^{k,s})/T_1(C_1^{k,s})^\perp$ are not isomorphic.*

   In particular, for k and s odd, the algebras $SD(2\mathcal{B})_2^{k,s}(0)$ and $SD(2\mathcal{B})_2^{k,s}(1)$ are not derived equivalent.

We get the following positive result.

Corollary 7.6. *Let K be an algebraically closed field of characteristic 2.*

(1) *For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_1^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Suppose that if $k = 2$ then $s \geq 3$ is odd, and if $s = 2$ then $k \geq 3$ is odd. Then the algebras $SD(2\mathcal{B})_1^{k,s}(0)$ and $SD(2\mathcal{B})_1^{k,s}(1)$ are not stably equivalent of Morita type.*
(2) *For any given integers $k, s \geq 1$, consider the algebras of semi-dihedral type $SD(2\mathcal{B})_2^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. If the parameters k and s are both odd, then the algebras $SD(2\mathcal{B})_2^{k,s}(0)$ and $SD(2\mathcal{B})_2^{k,s}(1)$ are not stably equivalent of Morita type.*

Proof. Since the quotients $Z^{st}(A) := Z(A)/Z^{pr}(A)$ and $T_n^\perp(A)^{st} := T_n(A)^\perp/Z^{pr}(A)$ are invariants under stable equivalences of Morita type (cf. Theorem 1.2 and Proposition 1.6), so are the quotients $Z^{st}(A)/T_n^\perp(A)^{st} = Z(A)/T_n^\perp(A)$.

Hence the parameters in the theorem yield not only algebras in different derived equivalence classes, but also algebras in different equivalence classes up to stable equivalences of Morita type. □

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