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Nonlinear Evolution Equations without Convexity

FRANCESCA PAPALINI

Department of Mathematics of Perugia University, Via Vanvitelli 1, Perugia, 06100, Italy

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1. INTRODUCTION

In the theory of the differential inclusions it is known (see [4]) that the initial value problem for evolution differential equations of the form

$$\begin{aligned} &x' \in -\partial V(x) + h \\ &x(0) = x_0, \qquad x_0 \in D(V), \end{aligned} \tag{I}$$

where ∂V is the subdifferential of a proper, convex, and lower semicontinuous function (l.s.c.) V, defined on a real separable Hilbert space H and with values in $\mathbb{R} \cup \{+\infty\}$, while $h \in L^2([0, b], H)$ is a single valued perturbation, has a (unique) solution. Moreover, some Authors (cf. [1, 10]) have proved the existence of solutions for the Cauchy problem (I), where the single-valued perturbation is replaced with a multivalued one.

In 1989, M. Tosques [11] has obtained an existence result for a quasi-autonomous evolution equation. In fact, he proved the existence and the uniqueness of the solutions to the Cauchy problem

$$\begin{aligned} &x' \in -\partial^- f(x) + h \\ &x(0) = x_0, \qquad x_0 \in D(f), \end{aligned} \tag{II}$$

where $\partial^- f$ is the Fréchet subdifferential of a function f defined on an open subset Ω of a real separable Hilbert space H, taking its values in $\mathbb{R} \cup \{+\infty\}$ and having a ψ-monotone subdifferential of order two, while h is a function belonging to $L^2([0, b], H)$. We recall that $\partial^- f$ is an extension of the notion of a subdifferential of a convex function and that it coincides with ∂f when f is convex (cf. [7, p. 224]).

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On the other hand, in 1989 Bressan, Cellina, and Colombo [3] proved an existence theorem for the problem

$$\begin{aligned} x' &\in F(x) \\ x(0) &= x_0, \end{aligned} \tag{III}$$

where $F: \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is an upper semicontinuous multifunction with compact and nonempty values not necessarily convex but contained in the subdifferential of a proper, convex, and l.s.c. function. In other words, F is cyclically monotone (cf. Remark 3).

In 1991 Cellina and Staicu [6] have improved the previous result by proving the existence of solutions to the initial value problem for evolution equations of the form

$$\begin{aligned} &x' \in -\partial V(x) + F(x) \\ &x(0) = x_0, \qquad x_0 \in D(\partial V), \end{aligned} \tag{IV}$$

by assuming that $V: \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex, and l.s.c. function and that $F: U(x_0) \to 2^{\mathbb{R}^n}$ is an upper semicontinuous and cyclically monotone multivalued operator defined on some neighborhood of x_0 and with compact and nonempty values.

In this article we consider the Cauchy problem of the form

$$\begin{aligned} &x' \in -\partial^- f(x) + F(x) \\ &x(0) = x_0, \qquad x_0 \in D(\partial^- f), \end{aligned} \tag{*}$$

to prove that (cf. Theorem 1) it has solutions by supposing that $f: \Omega \to \mathbb{R} \cup \{+\infty\}$ is a function with a ψ-monotone subdifferential of order 2, defined on an open subset $\Omega \subset \mathbb{R}^n$ and such that the function $x \mapsto \text{grad}^- f(x)$ is locally bounded in x_0, while $F: U(x_0) \to 2^{\mathbb{R}^n}$ is like in (IV) (cf. [6]).

To obtain our existence result we first establish a lemma which provides a sufficient condition for the function $\lambda: L^2([0, T], H) \to C([0, T], H)$, defined by $\lambda(h) = x_h$, where x_h is the unique solution of the problem (II), to have a closed graph in a bounded subset of $L^2([0, T], H)$, with respect to the weak topology in $L^2([0, T], H)$ and to the strong topology in $C([0, T], H)$. Since, as we said, the Fréchet subdifferential is an extension of the subdifferential of a proper, convex, and l.s.c. function and, on the other hand, every proper, convex, and l.s.c. function has a ψ-monotone subdifferential of order two, our problem (*) contains, in one sense, the problem (IV). In any case our proposition extends the mentioned theorem of Cellina and Staicu, in the sense that there exist functions which do not satisfy the assumptions of this proposition but which verify the conditions of our existence theorem (cf. Example 1). Moreover, if $x_0 \in \text{int}\, D(\partial^- f)$, our proposition strictly contains the existence result of Cellina and Staicu (cf. Remark 4).

Finally, we observe that, since if f is a constant function the problem (*) reduces itself to problem (III), our proposition strictly contains the mentioned theorem due to Bressan, Cellina, and Colombo [3].

2. Preliminaries

Let $[a, b]$ be an interval, μ the Lebesgue measure on it, and H a real separable Hilbert space, with norm $\|\cdot\|$ endowed by the scalar product $\langle \cdot, \cdot \rangle$. For $x \in H$ and $\varepsilon > 0$ we set $B(x, \varepsilon) = \{y \in H: \|y - x\| < \varepsilon\}$ and $\text{cl } B(x, \varepsilon) = \{y \in H: \|y - x\| \leq \varepsilon\}$ represents the closure of $B(x, \varepsilon)$; moreover, given a subset A of H, we put $B(A, \varepsilon) = \{x \in H: \rho(x, A) < \varepsilon\}$, where $\rho(x, A) = \inf\{\|y - x\|: y \in A\}$. For a closed and convex subset A of H, we denote by $m(A)$ the element of A such that

$$\|m(A)\| = \inf\{\|y\| : y \in A\rangle.$$

A function $V: H \to \mathbb{R} \cup \{+\infty\}$ is said to be *proper* if $D(V) \neq \emptyset$, where $D(V) = \{x \in H: V(x) < +\infty\}$. If V is proper, convex, and lower semicontinuous the multifunction $\partial V: H \to 2^H$, defined by

$$\partial V(x) = \{y \in H: V(\xi) - V(x) \geq \langle y, \xi - x\rangle, \forall \xi \in H\}, \qquad \forall x \in H,$$

is called the *subdifferential* of V. We denote by $D(\partial V)$ the set $D(\partial V) = \{x \in H: \partial V(x) \neq \emptyset\}$.

Given an open subset Ω of H and a function $f: \Omega \to \mathbb{R} \cup \{+\infty\}$, the multifunction $\partial^- f: \Omega \to 2^H$, defined as

$$\partial^- f(x) = \begin{cases} \emptyset, & \text{if } f(x) = +\infty, \\ \left\{\alpha \in H: \liminf\limits_{y \to x} \dfrac{f(y) - f(x) - \langle \alpha, y - x\rangle}{\|y - x\|} \geq 0\right\}, & \text{if } f(x) < +\infty, \end{cases}$$

is called the *Fréchet subdifferential* of f. We also put $D(f) = \{x \in \Omega: f(x) < +\infty\}$ and $D(\partial^- f) = \{x \in \Omega: \partial^- f(x) \neq \emptyset\}$.

Remark 1. The values of $\partial^- f$ are closed and convex (cf. [8, p. 1403]).

For every $x \in D(\partial^- f)$, we denote by $\text{grad}^- f(x)$ the element of the minimal norm of $\partial^- f(x)$.

If $f\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, we say that f has a *ψ-monotone subdifferential of order two* if there exists a continuous map $\psi\colon [D(f)]^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ such that

for every $x, y \in D(\partial^- f)$ and for every $\alpha \in \partial^- f(x)$ and $\beta \in \partial^- f(y)$, we have

$$\langle \alpha - \beta, x - y \rangle \geq -\psi(x, y, f(x), f(y))(1 + \|\alpha\|^2 + \|\beta\|^2)\|x - y\|^2. \tag{2.1}$$

Remark 2. If $f\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ is proper, convex, and lower semicontinuous, then $\partial^- f(x) = \partial f(x)$, $\forall x \in D(\partial^- f)$ (cf. [7, p. 224]).

A multifunction $F\colon \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is called *Hausdorff-lower* (*upper*) *semicontinuous* if $\forall x \in \mathbb{R}^n$ and $\forall \varepsilon > 0$ there exists $\delta > 0$ such that

$$F(x) \subset B(F(y), \varepsilon), (F(y) \subset B(F(x), \varepsilon)) \qquad \forall y \in B(x, \delta).$$

Moreover, F is said to have a "*closed graph*" if the set

$$\operatorname{Gr} F = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in F(x)\}$$

is closed in $\mathbb{R}^n \times \mathbb{R}^n$.

Let $\mathcal{A}$ be the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}^n$; the multifunction F is called "*measurable*" if for any closed subset $C \subset \mathbb{R}^n$, we have

$$\{x \in \mathbb{R}^n : F(x) \cap C \neq \emptyset\} \in \mathcal{A}.$$

The multivalued operator F is said to be "*cyclically monotone*" if for every cyclical sequence

$$x_0, x_1, \ldots, x_N = x_0$$

and for every sequence $y_1, \ldots, y_N$ such that $y_i \in F(x_i)$, $i = 1, \ldots, N$, we have

$$\sum_{i=1}^{N} \langle y_i, x_i - x_{i-1} \rangle \geq 0.$$

Remark 3. We recall that (cf. [4, Theorem 2.5]) F is cyclically monotone iff there exists a proper, convex, lower semicontinuous function $W\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$F(x) \subset \partial W(x) \qquad \forall x \in \mathbb{R}^n.$$

We state here for reference a version of the Theorem 3.6 of Tosques (cf. [11, p. 82]).

PROPOSITION 1. *Let Ω be an open subset of H and let $f: \Omega \to \mathbb{R} \cup \{+\infty\}$ be a function with ψ-monotone subdifferential of order 2. Then $\forall x_0 \in D(f)$, $\forall M \geq 0$ such that $f(x_0) \leq M$, $\exists T^* > 0$ with the property: $\forall T \in]0, T^*]$ and $\forall h \in L^2([0, T], H))$ with $\|h\|_2 \leq M$, there exists a unique function x_h: $[0, T] \to H$ that is the solution of the Cauchy problem*:

$$\begin{aligned} &x' \in -\partial^- f(x) + h \\ &x(0) = x_0, \end{aligned} \tag{II}$$

and that satisfies the properties:

(I) *x_h is continuous on $[0, T]$ and absolutely continuous on compact subsets of $]0, T[$*;

(II) *$x_h(t) \in D(\partial^- f)$ a.e. in $[0, T]$ and $x_h'(t) \in -\partial^- f(x_h(t)) + h(t)$ a.e. in $[0, T]$*;

(III) *$x_h(0) = x_0$*;

(IV) *$x_h' \in L^2([0, T], H)$*;

(V) *$\int_0^t \|x_h'(s)\|^2\, ds \leq 2(f(x_0) - f(x_h(t))) + \int_0^t \|h(s)\|^2\, ds$, $\forall t \in [0, T]$*;

(VI) *$f0x_h$ is absolutely continuous on $[0, T]$*;

(VII) *$(f0x_h)'(t) = \langle h(t) - x_h'(t), x_h'(t)\rangle$ a.e. in $[0, T]$.*

Now let Ω be an open subset of H, $f: \Omega \to \mathbb{R} \cup \{+\infty\}$ is a function with ψ-monotone subdifferential of order 2, and $x_0 \in D(\partial^- f)$ such that there exist $k, r > 0$ with the property $\|\text{grad}^- f(x)\| \leq k$, $\forall x \in \text{cl}\, B(x_0, r) \cap D(\partial^- f)$. Moreover, for a fixed $M \geq 0$ such that $f(x_0) \leq M$, choose T^* according to Proposition 1.

In these conditions, being ψ: $[D(f)]^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ the continuous map satisfying (2.1), it is possible to find two positive numbers R, L with the properties:

$$f(x) \geq f(x_0) - 1 \qquad \forall x \in \text{cl}\, B(x_0, R), \tag{2.2}$$

$$L := \sup\Big\{\psi(x_1, x_2, y_1, y_2): x_1, x_2 \in \text{cl}\, B(x_0, R) \cap D(f),$$

$$y_1, y_2 \in \Big[f(x_0) - 1, f(x_0) + \frac{1}{2}M^2\Big]\Big\}, \tag{2.3}$$

$$\|\text{grad}^- f(x)\| \leq k \qquad \forall x \in \text{cl}\, B(x_0, R) \cap D(\partial^- f). \tag{2.4}$$

Set $T' = \min\{T^*, R^2/(2 + M^2)\}$, $T \in]0, T']$, and $\mathcal{K} = \{h \in L^2([0, T], H): \|h\|_2 \leq M\}$; we consider, now, the map $\lambda: \mathcal{K} \to C([0, T], H)$ defined by $\lambda(h) = x_h$, where x_h is the unique solution of the problem (II) (cf. Proposition 1); thus, the following holds.

LEMMA 1. *Let $(h_m)_m \subset \mathcal{K}$ be a sequence which converges weakly to $\hat{h}$ in $L^2([0, T], H)$. If there exists $\gamma \in L^2([0, T], \mathbb{R}^+)$ such that $\|h_m(t)\| \leq \gamma(t)$, $\forall m \in \mathbb{N}$, a.e. in $[0, T]$, and if the sequence $(x_m)_m \subset C([0, T], H)$, $x_m = \lambda(hm)$, converges to x in $C([0, T], H)$, then $x = \lambda(\hat{h})$.*

We start by observing that, from our assumptions, for every $h \in \mathcal{K}$, we have that

$$\|x_h(t) - x_0\| \leq R \qquad \forall t \in [0, T]. \tag{2.5}$$

Indeed, from (V), $\forall t \in [0, T]$ it follows that

$$\begin{aligned} \|x_h(t) - x_0\| &\leq \int_0^t \|x_h'(s)\|\, ds \leq \sqrt{t}\left(\int_0^t \|x_h'(s)\|^2\, ds\right)^{1/2} \\ &\leq \sqrt{t}\,[2(f(x_0) - f(x_h(t))) + M^2]^{1/2}. \end{aligned} \tag{2.6}$$

Let $\bar{T} = \sup\{t \in [0, T]: \|x_h(s) - x_0\| \leq R, \forall s \in [0, t]\}$; by the continuity of x_h we have that $0 < \bar{T} \leq T$. To obtain (2.5) it is sufficient to prove that $\bar{T} = T$. If $\bar{T} < T$ by (2.6), (2.2), and the choice of T' it follows that $\|x_h(\bar{T}) - x_0\| < \sqrt{T'}\ [2 + M^2]^{1/2} \leq R$. Since x_h is continuous on $[0, T]$, this contradicts the definition of $\bar{T}$. Therefore (2.5) is proved.

Now, from the last statement of the proof of Lemma 3.21 of [11] and from the property (VII) of the Proposition 1, $\forall h \in \mathcal{K}$ we have that

$$\|x_h'(t)\| \leq \|\operatorname{grad}^- f(x_h(t))\| + \|h(t)\| \qquad \text{a.e. in } [0, T],$$

therefore, by (2.5), (2.4), and our assumptions, it follows that

$$\|x_h'(t)\| \leq k + \gamma(t) \qquad \text{a.e. in } [0, T]. \tag{2.7}$$

Moreover, by (2.5), (2.2), and property (V) of Proposition 1, we obtain

$$f(x_h(t)) \in [f(x_0) - 1, f(x_0) + \tfrac{1}{2}M^2] \qquad \forall h \in \mathcal{K}, \forall t \in [0, T]. \tag{2.8}$$

Denoted by $\hat{x}$ the function $\lambda(\hat{h})$, having f a ψ-monotone subdifferential or order 2, by (2.5), (2.8), (2.3), and (2.7), for a.e. $t \in [0, T]$, we have

$$\begin{aligned}\langle x_m'(t) - \hat{x}'(t), x_m(t) - \hat{x}(t)\rangle &\le \langle h_m(t) - \hat{h}(t), x_m(t) - \hat{x}(t)\rangle \\ &\quad + L\,(1 + \|h_m(t) - x_m'(t)\|^2 + \|\hat{h}(t) - \hat{x}'(t)\|^2)\|x_m(t) - \hat{x}(t)\|^2 \\ &\le \langle h_m(t) - \hat{h}(t), x_m(t) - \hat{x}(t)\rangle + \delta(t)\|x_m(t) - \hat{x}(t)\|^2 \qquad \forall m \in \mathbb{N},\end{aligned}$$

where $\delta(t) = L[1 + 8(\gamma(t))^2 + 8k\gamma(t) + 4k^2]$, $\delta \in L^2([0, T], \mathbb{R}^+)$. By integrating we obtain

$$\frac{1}{2}\|x_m(t) - \hat{x}(t)\|^2 \le \int_0^t \langle h_m(s) - \hat{h}(s), x_m(s) - \hat{x}(s)\rangle\, ds + \int_0^t \delta(s)\|x_m(s) - \hat{x}(s)\|^2\, ds \qquad \forall t \in [0, T], \forall m \in \mathbb{N}.$$

Applying Gronwall's inequality (cf. [9, p. 36]), we get

$$\|x_m(t) - \hat{x}(t)\|^2 \le \alpha_m(t) + \int_0^t 2\delta(s)|\alpha_m(s)|\exp\left(\int_0^x 2\delta(u)\,du\right) ds \qquad \forall t \in [0, T], \forall m \in \mathbb{N},$$

where $\alpha_m(t) = 2\int_0^t \langle h_m(s) - \hat{h}(s), x_m(s) - \hat{x}(s)\rangle\, ds$.

Since (cf. [5, Proposition III.5])

$$\lim_{m \to +\infty} \alpha_m(t) = 0 \qquad \forall t \in [0, T],$$

and

$$|\alpha_m(t)| \le 8(\|x_0\| + R)\|\gamma\|_1 \qquad \forall t \in [0, T], \forall m \in \mathbb{N},$$

we have $\lim_{m \to +\infty}\|x_m(t) - \hat{x}(t)\|^2 = 0$, $\forall t \in [0, T]$; therefore $\hat{x} = x$, which was to be proved.

3. Existence Result

We consider the Cauchy problem

$$\begin{aligned} &x' \in -\partial^- f(x) + F(x) \\ &x(0) = x_0, \qquad x_0 \in D(\partial^- f), \end{aligned} \qquad (+)$$

where $f\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ (Ω is an open subset of $\mathbb{R}^n$) and $F\colon U(x_0) \to 2^{\mathbb{R}^n}$ ($U(x_0)$ is a neighbourhood of x_0) verify respectively the properties:

(α) f has a ψ-monotone subdifferential of order 2;
($\alpha\alpha$) $\exists\, k, r > 0 : \|\text{grad}^- f(x)\| \leq k, \forall x \in \text{cl}\, B(x_0, r) \cap D(\partial^- f)$;
(β) $F(x)$ is non empty and compact, $\forall x \in U(x_0)$;
($\beta\beta$) F is upper semicontinuous;
($\beta\beta\beta$) F is cyclically monotone;

A function x: $[0, T] \to \mathbb{R}^n$ *is called a* solution *of the Cauchy problem* (*) *if there exists a selection* $u \in L^2([0, T], \mathbb{R}^n)$ *of* $F(x(.))$ (i.e., $u(t) \in F(x(t))$ a.e. in $[0, T]$):

(a) *x is continuous on* $[0, T]$ *and absolutely continuous on the compact subsets of* $]0, T[$;
(b) $x(t) \in D(\partial^- f)$ a.e. *in* $[0, T]$ *and* $x'(t) \in -\partial^- f(x(t)) + u(t)$ a.e. *in* $[0, T]$;
(c) $x(0) = x_0$.

Our existence result is the following.

THEOREM 1. *Let f and F satisfy the conditions* (α), ($\alpha\alpha$), (β), ($\beta\beta$), ($\beta\beta\beta$). *Then there exist* $T > 0$ *and a solution* x: $[0, T] \to \mathbb{R}^n$ *of the Cauchy problem* (*).

We start by observing that from (α), ($\alpha\alpha$), (β), and ($\beta\beta$) it is possible to find two positive numbers R and M with the properties:

$$f(x) \geq f(x_0) - 1 \qquad \forall x \in \text{cl}\, B(x_0, R), \qquad f(x_0) \leq M, \tag{3.1}$$

$$\|y\| < M \qquad \forall y \in F(x), \forall x \in \text{cl}\, B(x_0, R), \tag{3.2}$$

$$\|\text{grad}^- f(x)\| \leq k \qquad \forall x \in \text{cl}\, B(x_0, R) \cap D(\partial^- f). \tag{3.3}$$

Let T' be such that $0 < T' < R^2/(2 + M^2)$ and let T^* be the positive number that, according to Proposition 1, there exists a correspondence of f, x_0, and M. Put $T = \min\{T', T^*, 1\}$.

Now we shall consider a sequence of functions defined in $[0, T]$ and prove that a subsequence converges to a solution of the Cauchy problem (*).

For every $m \in \mathbb{N}$ we set $I_{m,i} = [0, i(T/m)]$ $\forall i \in \{1, ..., m\}$, and we are construct two functions h_m, x_m: $[0, T] \to \mathbb{R}^n$. Choose $y_{m,0} \in F(x_0)$ and define h_m on $I_{m,1}$ by $h_m(t) = y_{m,0}$ $\quad \forall t \in I_{m,1}$. Since $h_m \in L^2(I_{m,1}, \mathbb{R}^n)$ and (cf. (3.2)) $\|h_m\|_2 \leq M$ by Proposition 1, there exists a unique function $x_{m,1}$: $I_{m,1} \to \mathbb{R}^n$ satisfying the conditions (I) $\cdots$ (VII). Moreover, as we saw in Lemma 1, we have that

$$\|x_m(t) - x_0\| \leq R \qquad \forall t \in I_{m,1}. \tag{3.4}$$

Now, assuming that h_m and x_m have been defined on the initial interval $I_{m,1}$, we shall extend these functions to the interval $I_{m,i+1}$, $\forall i \in \{1, \ldots, m-1\}$.

Taking $y_{m,i} \in F(x_m(iT/m))$, we define h_m on $]iT/m, (i+1)T/m]$ by

$$h_m(t) = y_{m,i} \qquad \forall t \in]iT/m, (i+1)T/m].$$

We have that $h_m \in L^2(I_{m,i+1}, \mathbb{R}^n)$ and (cf. (3.2) and (3.4)) $\|h_m\|_2 \leq M$, so by the Proposition 1 there exists a unique function x_m: $I_{m,i+1} \to \mathbb{R}^n$ with the properties (I) $\cdots$ (VII) and, moreover,

$$\|x_m(t) - x_0\| \leq R \qquad \forall t \in I_{m,i+1}.$$

So we have obtained two sequences of functions, $(h_m)_m$ and $(x_m)_m$, defined on $[0, T]$ and with values in $\mathbb{R}^n$.

Now we set δ_m: $[0, T] \to [0, T]$ defined by $\delta_m(t) = \sum_{i=1}^m (i-1)(T/m) \chi_{I_{m,i}}(t)$, $\forall t \in [0, T]$, and we observe that $h_m(t) = \sum_{i=1}^m y_{m,i-1}\chi_{I_{m,i}}(t)$, $\forall t \in (0, T]$, where $\chi_{I_{m,i}}$ is the characteristic function of the set $I_{m,i}$.

Moreover, by construction, we have

$$\delta_m(t) \to t \qquad \text{uniformly in } [0, T], \tag{3.5}$$

$$h_m(t) \in F(x_m(\delta_m(t))) \qquad \forall t \in [0, T], \forall m \in \mathbb{N}, \tag{3.6}$$

$$\|h_m(t)\| \leq M \qquad \forall t \in [0, T], \forall m \in \mathbb{N}. \tag{3.7}$$

$$\|x_m(t) - x_0\| \leq R \qquad \forall t \in [0, T], \forall m \in \mathbb{N}. \tag{3.8}$$

Since x_m is the function constructed by Proposition 1 in connection with $h = h_m$, it satisfies the corresponding conditions (I) $\cdots$ (VII).

By using the property (V) of the Proposition 1, from (3.8), (3.1), and (3.7) we have that $(x_m)_m$ is bounded in $L^2([0, T], \mathbb{R}^n)$; hence, by taking the Arzelà–Ascoli theorem and Theorem III.27 of [5] into account, it follows that there exist a subsequence of $(x_m)_m$, still denoted by $(x_m)_m$, and an absolutely continuous function x: $[0, T] \to \mathbb{R}^n$ such that

$$(x_m)_m \text{ converges uniformly to } x \tag{3.9}$$

and

$$(x'_m)_m \text{ converges weakly in } L^2([0, T], \mathbb{R}^n) \text{ to } x'. \tag{3.10}$$

Moreover, by (3.7) and Theorem III.27 of [5], we can assume that

$$(h_m)_m \text{ converges weakly in } L^2([0, T], \mathbb{R}^n) \text{ to } h. \tag{3.11}$$

On the other hand, from (3.6), (3.5), and (3.9) we obtain

$$\lim_{m\to+\infty} \rho((x_m(t), h_m(t)), GrF) \leq \lim_{m\to+\infty} \|x_m(t) - x_m(\delta_m(t))\| = 0 \qquad \forall t \in [0, T]. \tag{3.12}$$

From $(\beta\beta)$, (3.9), (3.11), (3.12) and from the convergence theorem 1.4.1. of [2], there exists (cf. $(\beta\beta\beta)$ and Remark 3) a proper, convex, and lower semicontinuous function W: $\mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that

$$h(t) \in \partial W(x(t)) \qquad \text{a.e. in } [0, T];$$

hence, by Lemma 3.3 of [4], it follows that

$$W(x(T)) - W(x_0) = \int_0^T < h(s),\ x'(s) > ds. \tag{3.13}$$

On the other hand, by (3.6) and by the definition of ∂W, we have

$$W(x_m(i\,(T/m))) - W(x_m((i-1)(T/m))) \geq \left\langle y_{m,i-1}, \int_{(i-1)T/m}^{iT/m} x_m'(s)\, ds \right\rangle$$
$$= \int_{(i-1)T/m}^{iT/m} \langle h_m(s), x_m'(s)\rangle\, ds \quad \forall i \in \{1, \ldots, m\}, \forall m \in \mathbb{N},$$

and, by adding $i = 1, \ldots, m$, we obtain

$$W(x_m(T)) - W(x_0) \geq \int_0^T \langle h_m(s), x_m'(s)\rangle\, ds \qquad \forall m \in \mathbb{N}.$$

Hence, by taking (3.9), Proposition 2.12 of [4], and (3.13) into account, we have

$$\limsup_{m\to+\infty} \int_0^T \langle h_m(s), x_m'(s)\rangle\, ds \leq \int_0^T \langle h(s), x'(s)\rangle\, ds. \tag{3.14}$$

Now, by the property (VII) of Proposition 1, it follows that

$$\int_0^T \|x_m'(s)\|^2\, ds \tag{3.15}$$
$$= f(x_0) - f(x_m(T)) + \int_0^T \langle h_m(s), x_m'(s)\rangle\, ds \qquad \forall m \in \mathbb{N}.$$

Analogously, by taking Lemma 1 into account, we obtain

$$\int_0^T \|x'(s)\|^2\,ds = f(x_0) - f(x(T)) + \int_0^T \langle h(s), x'(s)\rangle\,ds. \tag{3.16}$$

Therefore, by (3.15), (3.14), (3.16), and the lower semicontinuity of f, we obtain

$$\limsup_{m\to+\infty} \|x_m'\|_2 \le \|x'\|_2$$

Therefore (cf. (3.10) and Proposition III.30 of [5]),

$$(x_m')_m \textit{ converges strongly} \text{ in } L^2([0, T], \mathbb{R}^n) \text{ to } x';$$

hence (cf. [5, Theorem IV.9]), there exists a subsequence of $(x_m')_m$, still denoted $(x_m')_m$, which converges pointwise a.e. in $[0, T]$ to x'.

Now, set $G\colon [0, T] \to 2^{\mathbb{R}^n}$, η_m, $\eta\colon [0, T] \to \mathbb{R}^n$, defined as

$$G(t) = F(x(t)) - x'(t), \qquad \eta_m(t) = h_m(t) - x_m'(t),$$
$$\eta(t) = h(t) - x'(t) \qquad \text{a.e. in } [0, T]$$

By construction, $\eta_m(t) \in F(x_m(\delta_m(t))) - x_m'(t)$, a.e. in $[0, T]$ (cf. (3.6)) and, since from the last statement of the proof of Lemma 3.21 of [11] and from the property (VII) of Proposition 1, we have that

$$\|x_m'(t)\| \le \|\text{grad}^- f(x_m(t))\| + \|h_m(t)\|, \qquad \text{a.e. in } [0, T],$$

then it follows that (cf. (3.3) and (3.7)) $\|\eta_m(t)\| \le 2M + k$, a.e. in $[0, T]$. Moreover

$$\rho(\eta_m(t), G(t)) \le \|x_m'(t) - x'(t)\| + \sup\{\rho(z, F(x(t)))\colon z \in F(x_m(\delta_m(t)))\}, \qquad \text{a.e. in } [0, T], \forall m \in \mathbb{N}.$$

Then, by taking (3.9), (3.5), and $(\beta\beta)$ into account, we have

$$\lim_{m\to+\infty} \rho(\eta_m(t), G(t)) = 0, \qquad \text{a.e. in } [0, T].$$

Therefore, by Lemma 3.2 of [6], it follows that the multifunction $\phi\colon [0, T] \to 2^{\mathbb{R}^n}$, defined by

$$\phi(t) = \bigcap_{m\in\mathbb{N}} \text{cl}\Big(\bigcup_{i\ge m} \{\eta_i(t)\}\Big)$$

is such that $\phi(t)$ is nonempty and compact, a.e. in $[0, T]$, ϕ is measurable in $[0, T]$, and

$$\phi(t) \subset F(x(t)) - x'(t), \qquad \text{a.e. in } [0, T]. \tag{3.17}$$

Consider now, the multifunction G^* defined by $G^*(t) = \partial^- f(x(t)) \cap \text{cl } B(0, 2M + k)$. Since $\eta_m(t) \in \partial^- f(x_m(t)) \cap \text{cl } B(0, 2M + k)$, a.e. in $[0, T]$, and the multifunction $x \mapsto \partial^- f(x) \cap \text{cl } B(0, 2M + k)$ is upper semicontinuous in the set $S = \text{cl } B(x_0, R) \cap \{x \in D(\partial^- f): f(x) \leq f(x_0) + \frac{1}{2} M^2, \|\text{grad}^- f(x)\| \leq k\}$ (cf. [8, Theorem 1.18; 2, Corollary 1.1.1]), taking into account that $(x_m(t))_m$ is included in S (cf. (3.8), (3.3), and the property (V) of Proposition 1), we have

$$\lim_{m \to +\infty} \rho(\eta_m(t), G^*(t)) = 0, \text{a.e. in } [0, T];$$

hence, by using Lemma 3.2 of [6],

$$\phi(t) \subset \partial^- f(x(t)) \cap \text{cl } B(0, 2M + k), \qquad \text{a.e. in } [0, T]. \tag{3.18}$$

Let v: $[0, T] \to \mathbb{R}^n$ be a measurable selection of ϕ, and set u: $[0, T] \to \mathbb{R}^n$, $u(t) = v(t) + x'(t)$. By (3.17) and (3.18), we have that $u \in L^2([0, T], \mathbb{R}^n)$, $u(t) \in F(x(t))$, and $x'(t) \in -\partial^- f(x(t)) + u(t)$, a.e. in $[0, T]$; since $x_m(0) = x_0$, $\forall m \in \mathbb{N}$, it follows that x is a solution of the Cauchy problem (*).

The following example shows that, as we already said in the Introduction, our proposition extends the existence theorem of Cellina and Staicu [6].

EXAMPLE 1: Let f: $\mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$f(x) = |x| - x^2, \qquad x \in \mathbb{R}.$$

it is obvious that f is not convex but that it has a ψ-monotone subdifferential of order 2 and, since

$$\text{grad}^- f(x) = \begin{cases} |1 - 2x|, & x > 0, \\ 0, & x = 0, \\ |1 + 2x|, & x < 0, \end{cases}$$

it satisfies condition $(\alpha\alpha)$ in every point of $\mathbb{R}$.

Remark IV. In order to prove that, as we already said in the Introduction, if $x_0 \in \text{int } D(\partial^- f)$, our proposition strictly contains the mentioned theorem of Cellina and Staicu, observe that every proper, convex, and lower semicontinuous function $f\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ has a ψ-monotone subdifferential of order 2 and it satisfies condition $(\alpha\alpha)$ because (cf. [2, Theorem 0.7.2]) the multifunction $x \mapsto \partial f(x)$ is upper semicontinuous in x_0 and $\partial f(x_0)$ is a compact subset of $\mathbb{R}^n$. On the other hand, the function of the previous example does not satisfy, as we said, the conditions of the theorem of [6], but it verifies the assumptions of our proposition and $D(\partial^- f) = \mathbb{R}$.

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