ELSEVIER Journal of Pure and Applied Algebra 136 (1999) 231–265

JOURNAL OF PURE AND APPLIED ALGEBRA

Semigroup expansions using the derived category, kernel, and Malcev products

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Communicated by J. Rhodes; received 8 May 1995; received in revised form 5 May 1997

Abstract

Three new classes of expansions are defined in this paper. More precisely, three different expansions are associated to each semigroup variety V. It is shown that several previously defined expansions can be viewed as specific examples of these constructions, or slight variants there of. This method is then used to "smooth" an already existing expansion to one which is guaranteed to be functorial and is maximal in a sense that will be made precise. Perhaps more importantly, this method of construction provides a large resource of expansions to be used as needed in the future. © 1999 Elsevier Science B.V. All rights reserved.

AMS Classification: Primary 20M07; secondary 20M035, 68Q70

0. Introduction

The idea of a semigroup expansion was first formalized in [1]. However, the Rhodes expansion was used earlier in [14] to prove the Ideal Theorem, an important theorem in the understanding of the group complexity of a semigroup. A variation was also used in [3] to prove the Holonomy Theorem and more recently in [9] to give a second proof of the Holonomy Theorem and to get a general method for getting an interesting action of a semigroup on a tree.

Many other expansions appear as useful tools throughout the literature. In [7] an expansion is used to calculate initial objects in the category of X-generated E-unitary inverse monoids. A "non-functorial" form of an expansion was developed in [6] to show that every finite semigroup is the image of a semigroup in which the right stabilizers are idempotent. Another area where expansions have been useful is in answering the

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PII: S0022-4049(97)00172-2

following question. Let $\mathscr{V}$ be a variety of languages and $\boldsymbol{V}$ the corresponding variety of semigroups. Let $\mathscr{W}$ be the variety of languages generated by the languages of $\mathscr{V}$ and $\{L_1 \cdot L_2\colon L_1, L_2 \in A^*\mathscr{V}\}$, where $\cdot$ is some fixed binary relation on languages. How is the new corresponding semigroup variety $\boldsymbol{W}$ obtained from $\boldsymbol{V}$?

Although all these results have the use of expansions in common, the constructions of the various expansions have seemed somewhat ad hoc and unrelated. In this paper a unifying theme for these expansions is given. More precisely it is shown that these expansions, or slight variants on them which can usually be used in the same manner as the original expansions, arise from a common construction. This general construction can be used to create new expansions for future use.

1. Preliminaries and notation

A monoid is a semigroup with an identity element. If S is a semigroup, S^1 denotes the semigroup equal to S if S is a monoid and $S \cup \{1\}$ otherwise. In the latter case the multiplication on S is extended by setting $s1 = 1s = s$ for all $s \in S$. The identity of S will be denoted 1_S when it is necessary to distinguish the different identities of different monoids.

Given any semigroups S there is a natural right action of S on itself given by $s \cdot s' = ss'$ where ss' is just the product in S. The wreath product of S and T, written $T \circ S$, is the semidirect product $T^S * S$. Elements of $T^S * S$ are of the form (f, s) and

$$(h,s)(f,s') = (h +^s f, ss') \quad \text{where } [h + {}^s f](s_0) = h(s_0) + f(s_0 s).$$

Here T and T^S are written additively just for convenience of notation. They are not assumed to be commutative.

A variety of semigroups $\boldsymbol{V}$ is the class of all semigroups which satisfy some given set of equations. Birkhoff showed that an equivalent definition of a variety is a class of semigroups closed under taking products, homomorphic images, and subsemigroups. A variety of semigroups is called locally finite if every finitely generated member is finite.

1.1. Graphs and categories

The vertices or objects of a graph or category X will be denoted $Obj(X)$, An arrow or edge a from s to t will be denoted $[s, a, t]$. A path in X will mean a finite sequence of consecutive edges.

If S is a semigroup with generators A, the Cayley graph of S, $\Gamma(S, A)$, has as objects the elements of S, and there is an edge from s_1 to s_2 labelled by a if $s_1 a = s_2$. Note that the Cayley graph is dependent on the choice of generators.

If C is a category, and c and c' are in $Obj(C)$ then $C(c, c')$ denotes the set of all arrows in C from c to c'. If c equals c' we can multiply the elements of $C(c, c') = C(c, c)$.

The axioms for multiplication in a category ensure that $C(c,c)$ is a monoid. $C(c,c)$ is called a local monoid of C. An element of $C(c,c)$ will be called a loop.

If X is a graph, the free category over X, denoted X^*, is defined by

$$Obj(X^*) = Obj(X),$$

$$X^*(c,c') = \{p\colon c \to c'\colon\ p \text{ is a path in } X\}.$$

If C is a category the consolidated semigroup, denoted C_{cd}, is the semigroup whose objects are the edges of $C \cup \{0\}$ and with the multiplication $[r,a,s][t,b,u] = [r,ab,u]$ if $s = t$ and 0 otherwise.

Given a surjective homomorphism of semigroups $\phi : S \to T$, D_ϕ will denote the derived category:

- $Obj(D_\phi) = T$.
- There is an arrow $[t,s,t']$ from t to t' if $ts\phi = t'$. The arrows $[t,s,t']$ and $[t,s',t']$ are equal if $s_0 s = s_0 s'$ for all $s_0 \in t\phi^{-1}$.
- Multiplication of consecutive arrows is given by $[t,s,t'][t',s',t''] = [t,ss',t'']$.

For more discussion on the consolidated semigroup and derived category see [15].

1.2. Expansions

Given any finite set A, A^+ will denote the free semigroup over A, and A^* will denote the free monoid over A. We will denote by $\boldsymbol{S_A}$ the category of A generated semigroups. The objects of $\boldsymbol{S_A}$ are pairs (S,ϕ), where S is a semigroup and ϕ is a surjective homomorphism from A^+ to S. There is an arrow η from (S_1,ϕ_1) to (S_2,ϕ_2) if η is a homomorphism from S_1 to S_2 such that $\phi_1\eta = \phi_2$. $\boldsymbol{M_A}$ will denote the analogous category of A generated monoids.

A semigroup expansion will mean a functor F from $\boldsymbol{S_A}$ to itself, along with a natural transformation ε from F to the identity functor such that the arrows of ε are surjective morphisms. Similarly a monoid expansion will mean a functor F from $\boldsymbol{M_A}$ to itself and a natural transformation ε from F to the identity functor such that the arrows of ε are surjective morphisms. When there is no ambiguity $\varepsilon((M,\phi))$ will be denoted ε_M.

$$\begin{array}{ccc}
F((M_1,\phi_1)) & \xrightarrow{F(\eta)} & F((M_2,\phi_2)) \\
{\scriptstyle \varepsilon_{M_1}}\downarrow & & \downarrow{\scriptstyle \varepsilon_{M_2}} \\
(M_1,\phi_1) & \xrightarrow{\ \eta\ } & (M_2,\phi_2)
\end{array}$$

An important fact about working inside $\boldsymbol{M_A}$ or $\boldsymbol{S_A}$ is that if a homomorphism is well defined, then it is automatically surjective. So, in particular, when checking to see if a map is an isomorphism we need only check that it is well defined and injective. This fact will be used without comment throughout this paper.

2. The expansion S^V

2.1. The definition of (S^V, ϕ_V)

In this section we will define a class of monoid expansions and semigroup expansions, one of each for each semigroup variety $\mathbf{V}$. We will begin by working inside the category $\mathbf{M}_A$, i.e. all pairs of the form (S, ϕ) will be objects of $\mathbf{M}_A$. Later it will be shown how to extend the definitions to all semigroups and work inside the category $\mathbf{S}_A$. Let $\mathbf{V}$ be a fixed variety of semigroups. The image of (S, ϕ) under the expansion will be denoted (S^V, ϕ_V). The idea behind the expansion is to make S^V the largest monoid that has S as a homomorphic image and such that the local monoids of the derived category of the homomorphism are in $\mathbf{V}$. The following lemma describes the structure of the derived category of the homomorphism ϕ from A^* onto S.

Lemma 2.1. *Let $\Gamma(S,A)$ be the Cayley graph of S. Then $D_\phi \cong \Gamma(S,A)^*$.*

Proof. They have the same objects and it is clear that there is an arrow $[s,u,s']$ in D_ϕ if and only if there is a path in $\Gamma(S,A)$ from s to s' labelled by $u \in A^*$. We need to check that if $[s,u,s'] = [s,v,s']$ in D_ϕ then $u = v$. Suppose $w \in s\phi^{-1}$. As $[s,u,s'] = [s,v,s']$ we have $wu = wv$. As A^* is a cancellative semigroup, we have $u = v$ as desired. $\square$

The above lemma allows us to think of arrows in D_ϕ as arising from paths in $\Gamma(S,A)$. To say an arrow is labelled by $u \in A^*$ will mean that it arises from a path p in $\Gamma(S,A)$ such that the word obtained by reading the labels of the edges of p in consecutive order is u. We will be taking quotients of D_ϕ. Arrows in the quotients will be identified with equivalence classes of arrows in D_ϕ and may be labelled by more than one element of A^*.

Let τ_ϕ be the smallest congruence on D_ϕ such that the local monoids of $D = D_\phi/\tau_\phi$ are in $\mathbf{V}$. If $u \in A^*$ and $s \in S$ then there is an arrow in D_ϕ from s to $s(u)\phi$ labelled by u. Let $\overline{[s,u,s(u)\phi]}$ denote the τ_ϕ equivalence class of $[s,u,s(u)\phi]$.

We define $f_u : S \to D_{\mathrm{cd}}$ from S into the consolidation of D as the function taking s in S to the arrow leaving s and labelled by u, i.e. $f_u(s) = \overline{[s,u,s(u)\phi]}$.

Define $\phi_V : A^* \to D_{\mathrm{cd}} \circ S$ by

$$u\phi_V = (f_u, u\phi) \quad \text{for } u \in A^*,$$

so

$$u\phi_V w\phi_V = (f_u +^{u\phi} f_w, (uw)\phi).$$

As

$$f_u(s) +^{u\phi} f_w(s) = \overline{[s,u,s(u)\phi]} + \overline{[s(u)\phi, w, s(uw)\phi]} = f_{uw}(s)$$

ϕ_V is a homomorphism. Note that $f_u(s) \neq 0$ for all u in A^* and s in S. Let S^V denote the image of ϕ_V.

2.2. Properties of (S^V, ϕ_V)

Lemma 2.2. *There is a left action of S on D_ϕ given by*

$$s \cdot [s_1, u, s_1(u)\phi] = [ss_1, u, ss_1(u)\phi].$$

Proof. In Lemma 2.1 we saw that $[s,u,s'] = [s,v,s']$ if and only if $u = v$, and hence the action is well defined. □

Lemma 2.3. *The left action of S on D_ϕ induces a left action of S on D.*

Proof. We need to show that if s and s_1 are in S with $\overline{[s_1,u,s_1(u)\phi]} = \overline{[s_1,v,s_1(v)\phi]}$, then $\overline{[ss_1,u,ss_1(u)\phi]} = \overline{[ss_1,v,ss_1(v)\phi]}$. Let A^*/ρ be the relatively free A-generated monoid in V, or in other words ρ is the smallest congruence on A^* generated by the equations of V. As $\overline{[s_1,u,s_1(u)\phi]} = \overline{[s_1,v,s_1(v)\phi]}$ there is a sequence of steps which takes the path $[s_1,u,s_1(u)\phi]$ in $\Gamma(S,A) \cong D_\phi$ to the path $[s_1,v,s_1(v)\phi]$ by applying the equations of V to the local monoids of D_ϕ. In other words, there must be a sequence of n steps of the following form:

$$[s_1,x_i,s_1(x_i)\phi][s_1(x_i)\phi,u_i,s_1(x_iu_i)\phi][s_1(x_iu_i)\phi,y_i,s_1(x_iu_iy_i)\phi]$$

$$\equiv_{\tau_\phi} [s_1,x_i,s_1(x_i)\phi][s_1(x_i)\phi,v_i,s_1(x_iv_i)\phi][s_1(x_iv_i)\phi,y_i,s_1(x_iv_iy_i)\phi]$$

where $s_1(x_i)\phi = s_1(x_iu_i)\phi = s_1(x_iv_i)\phi$ (so the middle terms are elements of a local monoid), $u_i\rho v_i$, $u = x_1u_1y_1$, and $v = x_nv_ny_n$. Now $s_1(x_i)\phi = s_1(x_iu_i)\phi = s_1(x_iv_i)\phi$ implies $ss_1(x_i)\phi = ss_1(x_iu_i)\phi = ss_1(x_iv_i)\phi$. So the middle terms of the following are elements of a local monoid with $u_i\rho_iv_i$, and the sequence of steps

$$[ss_1,x_i,ss_1(x_i)\phi][ss_1(x_i)\phi,u_i,ss_1(x_iu_i)\phi][ss_1(x_iu_i)\phi,y_i,ss_1(x_iu_iy_i)\phi]$$

$$\equiv_{\tau_\phi} [ss_1,x_i,ss_1(x_i)\phi][ss_1(x_i)\phi,v_i,ss_1(x_iv_i)\phi][ss_1(x_iv_i)\phi,y_i,ss_1(x_iv_iy_i)\phi]$$

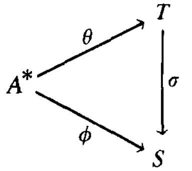
for $i = 1,\dots,n$ shows that $\overline{[ss_1,u,ss_1(u)\phi]} = \overline{[ss_1,v,ss_1(v)\phi]}$. □

This action implies that if we want to determine that the pair $\overline{[s,u,s']}$, $\overline{[s,v,s']}$ are equal it suffices to show that $\overline{[1,u,u\phi]} = \overline{[1,v,v\phi]}$. This leads us to the following corollary.

Corollary 2.4. $u\phi_V = v\phi_V$ *if and only if* $f_u(1) = f_v(1)$.

Proof. By definition $u\phi_V = v\phi_V$ implies $f_u = f_v$. Assume $f_u(1) = f_v(1)$ then $\overline{[1,u,u\phi]} = \overline{[1,v,v\phi]}$ and it follows that $u\phi = v\phi$. If s is in S then $f_u(s) = \overline{[s,u,s(u)\phi]}$, and it follows from Lemma 2.3 that this equals $\overline{[s,v,s(v)\phi]} = f_v(s)$. □

Lemma 2.5. *Let $\sigma:(T,\theta)\to(S,\phi)$ be a homomorphism. There is a congruence τ_σ on D_ϕ such that $D_\sigma \cong D_\phi/\tau_\sigma$. Moreover if $[1,u,s_1] \equiv_{\tau_\sigma} [1,v,s_1]$ then $[s,u,ss_1] \equiv_{\tau_\sigma} [s,v,ss_1]$ for all s in S.*



Proof. Define $[s_1,u,s_2] \equiv_{\tau_\sigma} [s_1,v,s_2]$ if $[s_1,u\theta,s_2]=[s_1,v\theta,s_2]$ in D_σ. We need to show that τ_σ is well defined and a congruence. It then follows from the definition of τ_σ that $D_\sigma \cong D_\phi/\tau_\sigma$.

If $[s_1,u,s_2]=[s_1,v,s_2]$ in D_ϕ then by Lemma 2.1 $u=v$, and $[s_1,u\theta,s_2]=[s_1,v\theta,s_2]$. Hence τ_σ is well defined.

Suppose $[s_1,u,s_2] \equiv_{\tau_\sigma} [s_1,v,s_2]$ and $[s_2,w\theta,s]$ is an arrow in D_σ leaving s_2. By the definition of τ_σ we have $[s_1,u\theta,s_2]=[s_1,v\theta,s_2]$, and so

$$[s_1,u\theta,s_2][s_2,w\theta,s]=[s_1,v\theta,s_2][s_2,w\theta,s].$$

This means $[s_1,u,s_2][s_2,w,s] \equiv_{\tau_\sigma} [s_1,v,s_2][s_2,w,s]$ in D_ϕ, and τ_σ is a right congruence. An analogous argument shows that τ_σ is a left congruence. By construction it is clear that $D_\phi/\tau_\sigma = D_\sigma$.

If $[1,u,s_1] \equiv_{\tau_\sigma} [1,v,s_1]$ then $t(u)\theta=t(v)\theta$ for all $t\in 1\sigma^{-1}$. As $1_T\in 1\sigma^{-1}$, $u\theta=v\theta$ and $[s,u,ss_1] \equiv_{\tau_\sigma} [s,v,ss_1]$ for all s in S. □

Proposition 2.6. *Let $\sigma:(T,\theta)\to(S,\phi)$ be a homomorphism. Let $\psi:A^*\to(D_\sigma)_{\mathrm{cd}}\circ S$ be the homomorphism sending u to $(g_u,u\phi)$ where $g_u(s)=[s,u\theta,s(u)\phi]$. The map $\eta:A^*\psi\to T$ which sends $u\psi$ to $u\theta$ is an isomorphism.*

Proof. If $u\psi=v\psi$ then $g_u(1)$ equals $g_v(1)$. As $1_T\in 1\sigma^{-1}$, $1_T(u)\theta=1_T(v)\theta$ and $u\theta=v\theta$. Hence η is well defined.

Reversely, if $u\theta=v\theta$ then $g_u(1)=g_v(1)$ and it follows from Lemma 2.5 that $g_u=g_v$. Hence η is injective. □

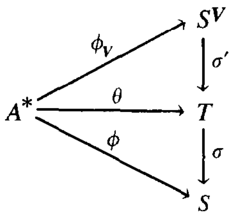
Proposition 2.7. *Let $\varepsilon_S:S^V\to S$ denote the restriction of the projection of $D_{\mathrm{cd}}\circ S$ onto S. The local monoids of D_{ε_S} are in V and $D_{\varepsilon_S}\cong D$.*

Proof. Using the notation in Lemma 2.5 $D_{\varepsilon_S}\cong D_\phi/\tau_{\varepsilon_S}$. By Lemmas 2.3 and 2.5 it suffices to show that $[1,u,s] \equiv_{\tau_\phi} [1,v,s]$ exactly when $[1,(f_u,s),s] \equiv_{\tau_{\varepsilon_S}} [1,(f_v,s),s]$.

As $f_u(1)=\overline{[1,u,s]}$ and $f_v(1)=\overline{[1,v,s]}$, $[1,u,s]\equiv_{\tau_\phi}[1,v,s]$ implies that $f_u(1)=f_v(1)$ and it follows from Corollary 2.4 that $f_u=f_v$ and $[1,(f_u,s),s] \equiv_{\tau_{\varepsilon_S}} [1,(f_v,s),s]$.

If $[1,(f_u,s),s] \equiv_{\tau_{\varepsilon_S}} [1,(f_v,s),s]$, then as $1_{S^V}\in 1\varepsilon_S^{-1}$ we have $(f_u,s)=(f_v,s)$. So $f_u(1)=\overline{[1,u,s]}=f_v(1)=\overline{[1,v,s]}$. □

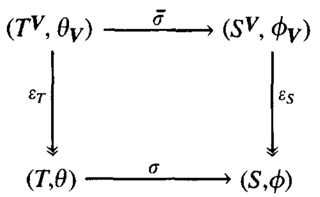
Theorem 2.8. *Let $\sigma : (T,\theta) \to (S,\phi)$ be as in Lemma 2.5. If the local monoids of D_σ are in $\mathbf{V}$ then there exists a surjective homomorphism $\sigma' : (S^V, \phi_V) \to (T,\theta)$.*



Proof. Recall that $D = D_\phi/\tau_\phi$ and $D_\sigma \cong D_\phi/\tau_\sigma$. As the local monoids of D_σ are in $\mathbf{V}$, $\tau_\phi \subseteq \tau_\sigma$. Therefore, the map $F : D \to D_\sigma$ which sends $\overline{[s,u,s(u)\phi]}$ to $[s,u\theta,s(u)\phi]$ is a well-defined homomorphism. We can then define $\sigma' : (f_u, s) \mapsto (g_u, s)\eta$ as in Proposition 2.6, with $g_u(t) = F(f_u(t))$. □

Theorem 2.9. *The functor taking (S,ϕ) to (S^V,ϕ_V), along with the natural transformation ε is an expansion.*

Proof. We need to define $\bar{\sigma}$ such that the following diagram commutes.



Let $C = D_\theta/\tau_\theta$ be the largest quotient of D_θ whose local monoids are in $\mathbf{V}$. The map $G : C_{\text{cd}} \to D_{\text{cd}}$ sending $\overline{[t,u,t(u)\theta]}$ to $\overline{[t\sigma,u,(t)\sigma(u)\phi]}$ and 0 to 0 is a surjective homomorphism. For u in A^* let $f'_u : T \to C_{\text{cd}}$ denote the function $f'_u(t) = \overline{[t,u,t(u)\theta]}$, and let $f_u : S \to D_{\text{cd}}$ denote the function $f_u(s) = \overline{[s,u,s(u)\phi]}$ as before. Then $G(f'_u(t)) = f_u(t\sigma)$. Since σ is onto, it follows that $f'_u = f'_v$ implies $f_u = f_v$. Define $\bar{\sigma}$ by $(f'_u, u\theta) \mapsto (f_u, u\phi)$. Then $(u\theta_V)\bar{\sigma} = ((f'_u, u\theta))\bar{\sigma} = (f_u, u\phi) = u\phi_V$ and the diagram commutes. □

2.3. Extending the definitions to $\mathbf{S}_A$

Let (S,θ) be an object in $\mathbf{S}_A$. The homomorphism θ can be extended to a homomorphism ϕ from A^* to S^1 by letting ϕ take the empty word to 1 and $\phi = \theta$ on A^+. By the previous section we know how to define $((S^1)^V, \phi_V)$. Let θ_V be the restriction of ϕ_V to A^+. Let S^V be the image of θ_V. The analogous theorems hold for (S^V, θ_V), and this defines an expansion on $\mathbf{S}_A$.

2.4. Finiteness properties of S^V

A category C is said to be locally finite if any subcategory generated by a finite set of arrows is finite. The following lemma is found in [6] and will not be proven here.

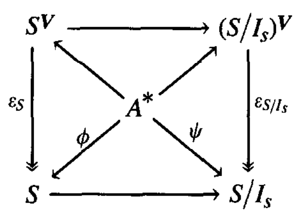
Lemma 2.10. *Let C be a category such that each of its base monoids $C(c,c)$ is locally finite. Then C is locally finite.*

Proposition 2.11. *Let S be finite and V be locally finite. Then S^V is finite.*

Proof. Each of the local monoids of D are in V, and so are locally finite. D is generated by the finite set of arrows $\overline{[s,a,s(a)\phi]}$, $a \in A$, $s \in S$, and so is finite by the above lemma. It follows that $S^V \subseteq D_{\text{cd}} \circ S$ is finite. □

A semigroup S is called finite $\mathcal{J}$ above if for each $s \in S$ $\{t \in S\colon t \geq_{\mathcal{J}} s\}$ is finite.

Corollary 2.12. *If S is finite $\mathcal{J}$ above and V is locally finite then S^V is finite $\mathcal{J}$ above.*



Proof. For s in S let $I_s = \{s' \varepsilon S\colon s' \not\geq_{\mathcal{J}} s\}$. I_s is an ideal of S. S/I_s is all elements of S which are $\mathcal{J}$ greater than or equal to s union zero. As S is finite $\mathcal{J}$ above, S/I_s is finite and it follows from Proposition 2.11 that $(S/I_s)^V$ is finite.

Let $\tilde{s} \in (s)\varepsilon_S^{-1}$ and $\tilde{I}_s = (I_s)\varepsilon_S^{-1}$. $\{\tilde{s}_1 \in S^V\colon \tilde{s}_1 \geq_{\mathcal{J}} \tilde{s}\}$ is a subset of $S^V/\tilde{I}_s$. We will show that $(S/I_s)^V \cong S^V/\tilde{I}_s$.

Let $u, v \in A^*$ with $u\phi, v\phi \notin I_s$. By Corollary 2.4 it suffices to show that if $[1,u,u\psi] \equiv_{\tau_\psi} [1,v,v\psi]$ then $[1,u,u\phi] \equiv_{\tau_\phi} [1,v,v\phi]$. By Lemma 2.1 we can consider $[1,u,u\phi]$ as a path in $\Gamma(S,A)$ and $[1,u,u\psi]$ as a path in $\Gamma(S/I_S,A)$. Since $u\phi \notin I_s$ the path $[1,u,u\psi]$ never passes through the vertex 0 in $\Gamma(S/I_S,A)$. This implies that if it passes twice through the same vertex in $\Gamma(S/I_S,A)$ then the path $[1,u,u\phi]$ passes twice through the same vertex in $\Gamma(S,A)$ and at the same places in u. It follows that any equation which can be applied to a loop in $[1,u,u\psi]$ corresponds to an equation which can be applied to a loop in $[1,u,u\phi]$ and we must have $[1,u,u\phi] \equiv_{\tau_\phi} [1,v,v\phi]$. □

There is a dual theory to everything above using the reverse derived category and the reverse wreath product. The reverse derived category of $\phi : S \to T$, denoted $D_{\phi,r}$, is defined similar to the derived category, only with arrows corresponding to multiplication on the left, instead of the right:

- $Obj(D_{\phi,r}) = T$.
- There is an arrow $[t', s, t]$ if $t' = (s)\phi t$. The arrows $[t', s, t]$ and $[t', s', t]$ are equal if $ss_0 = s's_0$ for all $s_0 \in t\phi^{-1}$.
- Multiplication of consecutive arrows is given by $[t'', s', t'][t', s, t] = [t'', s's, t]$.

The reverse wreath product, $S \circ^r T$ is the reverse semidirect product $S *^r T^S$ with multiplication

$$(s_1, f)(s_2, g) = (s_1 s_2, f^{s_2} + g)$$

where $[f^{s_2} + g](s) = f(s_2 s) + g(s)$.

Let $\tau_{\phi,r}$ be the smallest congruence such that the local monoids of $D_r = D_{\phi,r}/\tau_{\phi,r}$ are in $\boldsymbol{V}$. Let the homomorphism $\phi_{V,r} : A^* \to S \circ^r (D_r)_{\text{cd}}$ be the extension of the map

$$a\phi_{V,r} = (a\phi, f_a) \quad \text{for } a \in A \text{ where } f_a(s) = \overline{[(a)\phi s, a, s]}.$$

The image of $\phi_{V,r}$ will be denoted $S^{V,r}$.

Let $r(S)$ denote the reverse semigroup of S. The elements of $r(S)$ are $\{s^r : s \in S\}$, and multiplication is given by $s_1^r s_2^r = (s_2 s_1)^r$. For $u = a_1 a_2 \ldots a_{n-1} a_n$ an element of A^*, let $\tilde{u} = a_n a_{n-1} \ldots a_2 a_1$. If (S, ϕ) is an object of $\boldsymbol{M_A}$ (or $\boldsymbol{S_A}$) then $(r(S), \phi_r)$ with $u\phi_r = (\tilde{u}\phi)^r$ is also an object of $\boldsymbol{M_A}$ (or $\boldsymbol{S_A}$).

Consider the category $D_{\phi_r,r}$. There is an arrow $[s_1^r, a_1, s^r]$ if $(a_1)\phi_r s^r = ((a_1)\phi)^r s^r = (s(a_1)\phi)^r = s_1^r$, and $[s_2^r, a_2, s_1^r][s_1^r, a_1, s^r] = [s_2^r, a_2 a_1, s^r]$.

Lemma 2.13. *Let (S, ϕ) and $(r(S), \phi_r)$ be as above, then there is a reverse isomorphism from D_ϕ to $D_{\phi_r,r}$.*

Proof. The isomorphism is given by the map which takes $[s, u, s']$ in D_ϕ to $[(s')^r, \tilde{u}, s^r]$ in $D_{\phi_r,r}$. There is an arrow $[s, u, s']$ from s to s' in D_ϕ if and only if $s(u)\phi = s'$. Now $s(u)\phi = s'$ if and only if $(u\phi)^r s^r = (s')^r \cdot (u\phi)^r = (\tilde{u})\phi_r$, and $(\tilde{u})\phi_r s^r = (s')^r$ if and only there is an arrow $[(s')^r, \tilde{u}, s^r]$ in $D_{\phi_r,r}$. By Lemma 2.1, and the analogous result for $D_{\phi,r}$, if an arrow labelled by u is equal to an arrow labelled by v in either category then $u = v$. Noticing that

$$[s, u, s'][s', v, s''] = [s, uv, s''] \quad \text{in } D_\phi$$

and

$$[(s'')^r, \tilde{v}, (s')^r][(s')^r, \tilde{u}, s^r] = [(s'')^r, \tilde{v}\tilde{u}, s^r] \quad \text{in } D_{\phi_r,r},$$

and that $\tilde{v}\tilde{u} = \widetilde{uv}$ shows that the map is a reverse isomorphism. □

As the isomorphism takes loops to loops of the same form we have the following corollary.

Corollary 2.14. *If D is the largest image of D_ϕ such that the local monoids of D are in some variety $\mathbf{V}$, and D_r is the largest image of $D_{\phi_r,r}$ such that its local monoids are also in $\mathbf{V}$, then there is a reverse isomorphism from D to D_r.*

Theorem 2.15. *The semigroup you get by applying the reverse expansion to the reverse semigroup is the reverse of that you get by just applying the expansion to the original semigroup, i.e. $(r(S))^{\mathbf{V},r} \cong r(S^{\mathbf{V}})$.*

Proof. First note that the image of u in $(r(S))^{\mathbf{V},r}$ is $(u\phi_r, h_u) = ((\tilde{u}\phi)^r, h_u)$ where $h_u(s^r) = [(u)\phi_r s^r, u, s^r] = [(\tilde{u}\phi)^r s^r, u, s^r]$.

Let $\beta : (r(S))^{\mathbf{V},r} \to r(S^{\mathbf{V}})$ be given by $(u\phi_r, h_u) \mapsto (f_{\tilde{u}}, \tilde{u}\phi)^r$. Then $u\phi_r = v\phi_r$ if and only if $(\tilde{u}\phi)^r = (\tilde{v}\phi)^r$ if and only if $\tilde{u}\phi = \tilde{v}\phi$. It follows from the proof of Lemma 2.13 and Corollary 2.14 that $h_u = h_v$ if and only if $f_{\tilde{u}} = f_{\tilde{v}}$.

It is left to check that the map respects multiplication.

$$(u\phi_r, h_u)(v\phi_r, h_v) = ((uv)\phi_r, h_{uv}) \quad \text{and}$$

$$(f_{\tilde{u}}, \tilde{u}\phi)^r (f_{\tilde{v}}, \tilde{v}\phi)^r = ((f_{\tilde{v}}, \tilde{v}\phi)(f_{\tilde{u}}, \tilde{u}\phi))^r = (f_{\tilde{v}\tilde{u}}, (\tilde{v}\tilde{u})\phi)^r = (f_{\widetilde{uv}}, (\widetilde{uv})\phi)^r$$

which completes the proof. □

3. Examples

As many of the proofs in the following sections require looking at the derived category of a homomorphism, we will once again assume that we are working in $\mathbf{M}_A$. The proofs go through if we restrict to $\mathbf{S}_A$, however they require first extending semigroup homomorphisms from S to T to monoid homomorphisms from S^1 to T^1 as described in Section 2.3, going through the proofs below, and then restricting back to elements of $\mathbf{S}_A$.

3.1. The expansion $S^{\mathbf{I},r}$

One of the first expansions was the Rhodes expansion [8, 14]. Let L denote the set of all strict $\mathscr{L}$-chains of S, i.e. $L = \{s_n < s_{n-1} < \cdots s_2 < s_1 \colon s_n, \ldots, s_1 \in S\}$, where $s < s'$ means s is strictly below s' in the $\mathscr{L}$ order. There is an associative multiplication on L given by

$$(s_n < \cdots < s_1)(t_m < \cdots < t_1) = Red(s_n t_m \le \cdots \le s_1 t_m \le t_m < \cdots < t_1)$$

where $Red(s_n t_m \le \cdots \le s_1 t_m \le t_m < \cdots < t_1)$ is the new chain you get by using the rule: whenever there is a string of $\mathscr{L}$ equivalent elements keep only the left most element of the string, i.e.

$$Red(s_n t_m \leq \cdots \leq s_1 t_m \leq t_m < \cdots < t_1) = s_n t_m < s_{i_j} t_m < \cdots s_{i_1} t_m < \cdots t_2 < t_1$$

where $s_{i_k} t_m \mathscr{L} s_{i_k - 1} t_m \mathscr{L} \cdots \mathscr{L} s_{i_{k-1}+1} t_m < s_{i_{k-1}} t_m$.

$(S^{\hat{}L}, \phi^{\hat{}L})$ is the Rhodes expansion cut to generators, where $\phi^{\hat{}L}$ is the homomorphism which sends a to $Red(a\phi \leq 1)$. (For (S, θ) in $\boldsymbol{S}_A$, $\theta^{\hat{}L}$ is defined by $a\theta^{\hat{}L} = a\theta$, considered as a string of length 1.) The map $\sigma : (S^{\hat{}L}, \phi^{\hat{}L}) \to (S, \phi)$ which sends $s_n < \cdots < s_1$ to s_n is a surjective homomorphism.

Lemma 3.1. *The local monoids of $D_{\sigma,r}$ are trivial.*

Proof. Let s be an object of $D_{\sigma,r}$ and let $\overline{[s,t,s]}$ be an element of $D_{\sigma,r}(s,s)$ with $t = s_n < \cdots < s_1$. It follows from the definition of $D_{\sigma,r}$ that $s = s_n s$. If $t' \in s\sigma^{-1}$ then t' is of the form $s < \cdots < s_1'$ and

$$tt' = (s_n < \cdots < s_1)(s < \cdots < s_1') = Red(s_n s \leq \cdots \leq s < \cdots < s_1') = s < \cdots < s_1' = t'.$$

In other words $\overline{[t,s,t]}$ acts like the identity of $D_{\sigma,r}$ and $\overline{[s,t,s]} = \overline{[s,1,s]}$. □

After realizing this, we can then ask how different $S^{\hat{}L}$ and $S^{\boldsymbol{I},r}$ are, where $\boldsymbol{I}$ is the variety whose only member is the trivial semigroup. $S^{\boldsymbol{I},r}$ turns out to be the Karnofsky–Rhodes expansion, an extension of the Rhodes expansion [11].

Let $A^0 = A \cup \{0\}$. Elements of the Karnofsky–Rhodes expansion are strings of elements of $A^0 \times S$ of the form $[(0, s_n)(a_{n-1}, s_{n-1}) \cdots (a_1, s_1)]$ where $s_{i+1} \mathscr{L} (a_i)\phi s_i <_{\mathscr{L}} s_i$. The product

$$[(0, s_n)(a_{n-1}, s_{n-1}) \cdots (a_1, s_1)] \cdot [(0, s_m')(a_{m-1}', s_{m-1}') \cdots (a_1', s_1')]$$

$$= [(0, s_n s_m')(a_{i_k}, s_{i_k} s_m') \cdots (a_{m-1}', s_{m-1}') \cdots (a_1', s_1')]$$

where

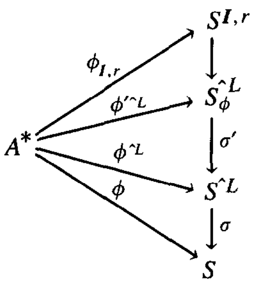
$$(s_n < \cdots < s_2 < s_1)(s_m' < \cdots < s_2' < s_1') = (s_n s_m' < s_{i_k} s_m' < \cdots < s_{m-1}' < s_2' < s_1')$$

in the Rhodes expansion $S^{\hat{}L}$. Let $S_\phi^{\hat{}L}$ be the monoid generated by the elements of the form $[(0, a\phi)(a, 1)]$ if $a\phi < 1$ and $[(0, a\phi)]$ if $a\phi \mathscr{L} 1$ for a in A, and let $\phi'^{\hat{}L}$ be the corresponding map from A^* to $S_\phi^{\hat{}L}$. The map $\sigma' : (S_\phi^{\hat{}L}, \phi'^{\hat{}L}) \to (S, \phi)$ sending $[(0, s_n) \ldots (a_1, s_1)]$ to s_n is a surjective homomorphism. The proof of the following lemma is essentially the same as in Lemma 3.1.

Lemma 3.2. *The local monoids of $D_{\sigma',r}$ are trivial.*

An edge $[c_1, u, c_2]$ of a graph is called a transition edge if there is no path from c_2 to c_1. Tilson proves the following in [15].

Lemma 3.3. *If X is a graph, $C = X^*$, and τ is the smallest congruence on C such that the local monoids of C/τ are trivial then two coterminal paths in C are τ equivalent if and only if they have the same transition edges.*



Theorem 3.4. *The Karnofsky–Rhodes expansion of S is the largest expansion in which the local monoids of the reverse derived category of the homomorphism down to S are in $\boldsymbol{I}$. That is $(S^{I,r}, \phi_{I,r}) \cong (S_\phi^{\hat{}L}, \phi'^{\hat{}L})$.*

Proof. It follows from Theorem 2.8 and Lemma 3.2 that $S_\phi^{\hat{}L}$ is a homomorphic image of $S^{I,r}$. Using the notation of Section 2.1 we have $D = D_{\phi,r}/\tau_{\phi,r}$. There is an arrow from s_1 to s_2 if and only if there is an $s \in S$ such that $ss_1 = s_2$. If X is the graph with vertices S and an edge $[s, a, s']$ from s' to s if $s = (a)\phi s'$ then by the analogous version of Lemma 2.1 for the reverse derived category we have $D_{\phi,r} \cong X^*$. So an arrow in $D_{\phi,r}$ can be thought of as a path in X which is labelled by a word $u \in A^*$. An arrow in D is an equivalence class of paths in X. The transition edges are exactly $\{\overline{[(a)\phi s, a, s]}: (a)\phi s < s\}$. Suppose $u\phi'^{\hat{}L} = v\phi'^{\hat{}L} = [(0, s_n)(a_{n-1}, s_{n-1}) \ldots (a_1, s_1)]$. The transition edges of $f_u(1)$ are

$$\{[(a_{n-1})\phi s_{n-1}, a_{n-1}, s_{n-1}], \ldots, [(a_1)\phi s_1, a_1, s_1]\},$$

which are also the transition edges of $f_v(1)$. By Lemma 3.3 we know $f_u(1) \equiv_{\tau_{\phi,r}} f_v(1)$, and it follows from Corollary 2.4 that $f_u = f_v$ and $u\phi_{I,r} = v\phi_{I,r}$. □

We see that $S^{I,r}$ and $S^{\hat{}L}$ are not isomorphic. However, they still have many of the same nice properties as illustrated by the following theorems. Proofs of these theorems can be found in [11].

Theorem 3.5. *Right stabilizers in $S^{I,r}$ and $S^{\hat{}L}$ are aperiodic.*

Theorem 3.6. *Let e be an idempotent of S. The inverse image of e in $S^{I,r}$ and in $S^{\hat{}L}$ each satisfy the identity $xy = y$.*

3.2. The expansion S^{SL}

Let $\boldsymbol{SL}$ denote the semigroup variety of semilattices, that is all semigroups which satisfy the equations $xx = x$ and $xy = yx$. In this section we will see that expanding by $\boldsymbol{SL}$ is the same as applying the Cayley expansion.

Once again (S,ϕ) will be an element of $\boldsymbol{M}_A$ and $\Gamma(S,A)$ its Cayley graph. There is a left action of S on $\Gamma(S,A)$ given by $s \cdot [s_1,a,s_2] = [ss_1,a,ss_2]$. Let Y be the set of all pairs (P,s) where $s \in S$ and P is a connected subgraph of $\Gamma(S,A)$ containing both 1 and s as vertices. There is an associative multiplication on Y given by

$$(P_1,s_1)(P_2,s_2) = (P_1 \cup s_1P_2, s_1s_2)$$

where s_1P_2 is the image of P_2 under the left action of s_1 described above.

Let ϕ_C be the homomorphism which takes a in A to $(P_a, a\phi)$, where P_a is the single edge $[1,a,a\phi]$. The image of ϕ_C, which will be denoted S^C, is the Cayley expansion of S.

Lemma 3.7. *Let $\sigma:(S^C,\phi_C) \to (S,\phi)$ be the homomorphism that takes (P,s) to s. The local monoids of D_σ are semilattices.*

Proof. Suppose $[s,(P_1,s_1),s]$ is an element of the local monoid $D_\sigma(s,s)$. Thus $ss_1 = s$. If $(P,s) \in s\sigma^{-1}$ then

$$(P,s)(P_1,s_1)(P_1,s_1) = (P \cup sP_1, s) = (P,s)(P_1,s_1).$$

Hence $([s,(P_1,s_1),s])^2 = [s,(P_1,s_1),s]$. If $[s,(P_2,s_2),s]$ is any other element of $D_\sigma(s,s)$, then

$$\begin{aligned}(P,s)(P_1,s_1)(P_2,s_2) &= (P \cup sP_1 \cup ss_1P_2, ss_1s_2)\\ &= (P \cup sP_1 \cup sP_2, s) = (P,s)(P_2,s_2)(P_1,s_1),\end{aligned}$$

and therefore $[s,(P_1,s_1),s][s,(P_2,s_2),s] = [s,(P_2,s_2),s][s,(P_1,s_1),s]$. □

If X is a graph and Y is a subgraph of X, the content of Y is the set of edges of Y. The following result is due to I. Simon. A proof of this lemma can be found in [2].

Lemma 3.8. *If X is a graph, $C = X^*$, and τ is the smallest congruence on C such that the local monoids of C/τ are semilattices, then two coterminal paths in C are τ equivalent if and only if they have the same content.*

Theorem 3.9. *The Cayley expansion of S is the largest expansion in which the local monoids of the derived category of the homomorphism down to S are in* **SL**. *That is $(S^{SL},\phi_{SL}) \cong (S^C,\phi_C)$.*

Proof. By Theorem 2.8 and Lemma 3.7 we know that S^C is a homomorphic image of S^{SL}. Once again using Lemma 2.1 we view arrows in D as paths in $\Gamma(S,A)$, and we see that the path $f_u(1)$ is the same as the picture P_u. If $(P_u,u\phi) = (P_v,v\phi)$ then $f_u(1) = f_v(1)$, and it follows from Corollary 2.4 that $f_u = f_v$ and $u\phi_{SL} = v\phi_{SL}$. □

4. The expansion $(S^{l,r})^{\mathbf{Z}_p}$

4.1. The expansion $S^{\mathbf{Z}_p}$

We will first describe the expansion $S^{\mathbf{Z}_p}$, where $\mathbf{Z}_p$ is the variety of finite p-modules. Throughout this section p will denote an integer strictly greater than 2, and S will be a finite monoid.

Let C and D be categories. C is said to divide D, written $C \prec D$, if there is a relation of categories $\eta : C \to D$ such that

- The object relation $\eta : Obj(C) \to Obj(D)$ is a function.
- For all $s \in C(c,c')$, $s\eta \neq \emptyset$. (The relation is fully defined.)
- If $s,s' \in C(c,c')$ with $c\eta \cap c'\eta \neq \emptyset$ then $s = s'$. (The relation is injective.)
- For every pair s,s' of consecutive arrows of C, $s\eta s'\eta \subseteq (ss')\eta$.
- For each object $c \in Obj(C)$, $1_{c\eta} \in 1_c\eta$.

We have seen that all the loops at a given object of a category can be regarded as monoid. Similarly, a monoid can be regarded as a category with one object, x, and a loop, $[x,m,x]$ for each element of m of the monoid. Multiplication is given by $[x,m_1,x][x,m_2,x] = [x,m_1m_2,x]$, where m_1m_2 is the product in the monoid. Note then when dealing with a division into a monoid M the function on the objects must take everything to the unique object of the category M, and every arrow will be taken to a loop at that object. A proof of the following result can be found in [13].

Lemma 4.1. *If C is a category whose local monoids are in $\mathbf{Z}_p$, $p > 2$, then $C \prec M$ for some $M \in \mathbf{Z}_p$* [13].

So if (S,ϕ) is in $\mathbf{M}_A$, and $D = D_\phi/\tau_\phi$ is the largest quotient of D_ϕ whose local monoids are in $\mathbf{Z}_p$, there is a division $H : D \prec M$, where M is in $\mathbf{Z}_p$.

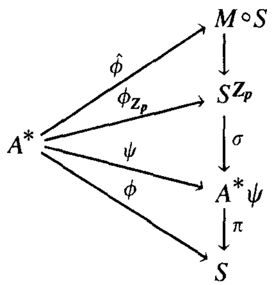
Lemma 4.2. *Let D, M and H be as described above. Then $S^{\mathbf{Z}_p} \prec M \circ S$.*

Proof. Let $\varepsilon : S^{\mathbf{Z}_p} \to S$ be the projection onto the first coordinate. Then $D_\varepsilon \prec D \prec M$. Thus, by Tilson's covering lemma [15], there is a division $h : S^{\mathbf{Z}_p} \to M \circ S$. □

Let $A = \{a_1,\ldots,a_n\}$ and $S = \{s_1,\ldots,s_m\}$. Let Z_p be the cyclic group of order p and let $G = Z_p^{nm}$. Let $g_{a_i} \in G^S$ be such that $g_{a_i}(s_j) = (0,\ldots,0,1,0,\ldots,0)$ where the one is in the $n(i-1)+j$ coordinate. Note that $g_{a_i}(s_j) = g_{a_{i'}}(s_{j'})$ implies $i = i'$ and $j = j'$. Let $\psi : A^* \to G \circ S$ be the homomorphism induced by the map $a \mapsto (g_a, a\psi)$. For $u = b_1b_2\ldots b_r$ in A^* denote by g_u the function $g_{b_1} + {}^{b_1\phi}g_{b_2} + \cdots + {}^{(b_1b_2\cdots b_{r-1})\phi}g_{b_r}$. It is a straight forward check that $u\psi = (g_u, u\phi)$.

Theorem 4.3. *$A^*\psi$ is the largest expansion of S such that the local monoids of the homomorphism down to S are in $\mathbf{Z}_p$. That is $(S^{\mathbf{Z}_p}, \phi_{\mathbf{Z}_p}) \cong (A^*\psi, \psi)$.*

Proof. Tilson shows in [15] that if $\pi : A^*\psi \to S$ is the restriction of the projection of $G \circ S$ onto the first coordinate then $D_\pi \prec G^S$. If $D_\pi \prec G^S$ then so do the local monoids of D_π, and hence the local monoids of D_π are in $\boldsymbol{Z_p}$. By Theorem 2.8 there exists a homomorphism $\sigma : (S^{\boldsymbol{Z_p}}, \phi_{\boldsymbol{Z_p}}) \to (A^*\psi, \psi)$.



For each $a \in A$ fix an $(F_a, a\phi)$ in $h(a\phi_{\boldsymbol{Z_p}})$, where h is the division shown to exist in Lemma 4.2, and define $\hat{\phi} : A^* \to M \circ S$ by $a\hat{\phi} = (F_a, a\phi)$. Let $u, v \in A^*$ with $u\psi = (g_u, s)$ and $v\psi = (g_v, s)$. If $u = b_1 \ldots b_r$ then

$$g_u = g_{b_1} + {}^{b_1\phi}g_{b_2} + \cdots + {}^{(b_1\cdots b_{r-1})\phi}g_{b_r}.$$

As G^S, i.e. functions from S to $G = Z_p^{nm}$, is a commutative semigroup and pg is the zero function for any $g \in G^S$, we can rearrange the terms and write g_u and g_v in the form

$$g_u = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j}{}^{s_j}g_{a_i} \quad \text{and} \quad g_v = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{i,j}{}^{s_j}g_{a_i} \quad \text{with } 0 \le \alpha_{i,j}, \beta_{i,j} \le p-1.$$

As $\{{}^{s_j}g_{a_i}(1)\colon\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ are linearly independent, the $\alpha_{i,j}$ and $\beta_{i,j}$ are uniquely determined. As $M \in \boldsymbol{Z_p}$, M^S is also a commutative semigroup with the same property and we can similarly rearrange the terms of

$$u\hat{\phi} = F_{b_1} + {}^{b_1\phi}F_{b_2} + \cdots + {}^{(b_1\cdots b_{r-1})\phi}F_{b_r}$$

and of F_v to get

$$u\hat{\phi} = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j}{}^{s_j}F_{a_i}, s \right) \quad \text{and} \quad v\hat{\phi} = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{i,j}{}^{s_j}F_{a_i}, s \right).$$

Now if $u\psi = v\psi$ then $g_u(1) = g_v(1)$. This in turn implies the $\alpha_{i,j} = \beta_{i,j}$. By the form above for F_u and F_v we see it follows that $u\hat{\phi} = v\hat{\phi}$, and therefore $u\phi_{\boldsymbol{Z_p}} = v\phi_{\boldsymbol{Z_p}}$. Thus σ is one to one and an isomorphism. □

If S is finite then for any fixed $s \in S$ there is a largest n such that there exists a string of the form $s = s_n <_{\mathscr{J}} s_{n-1} <_{\mathscr{J}} \cdots <_{\mathscr{J}} s_1$. This n is called the $\mathscr{J}$-depth of s. The $\mathscr{J}$-depth of S is the maximum of the $\mathscr{J}$-depths of its elements. From now on given any $\phi : A^* \to S$ the morphism $\phi_{\boldsymbol{Z_p}} : A^* \to S^{\boldsymbol{Z_p}}$ will denote the map ψ as defined above and p will denote a number greater than or equal to the $\mathscr{J}$-depth of S.

4.2. Right stabilizers in $(S^{I,r})^{Z_p}$

Given an element (S,ϕ) of M_A, a new monoid $({}^p\hat{S}_A,\tau)$ is defined in [6] as follows. Let $d:S\to N\backslash\{0\}$ be the $\mathscr{J}$-depth function of S. Let $Q'=Z_p^{d(S)}\times\hat{S}$. Elements of Q' are pairs of the form $(q,s_n<\cdots<s_1)$ where q is a map from $d(S)$ to Z_p.

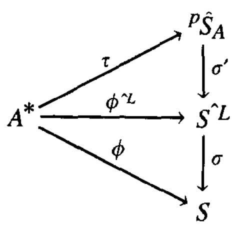
The alphabet A acts on Q' by

$$(q,s_n<\cdots<s_1)\cdot a=(g,Red(s_na\phi\leq\cdots\leq s_1a\phi))$$

where g is defined as follows. Set $s_0=1\in S$. For each i in $d(s)$, let j_i be maximal in $\{0,\dots,n\}$ such that $d(s_{j_i})\leq i$. Let

$$g(i)=\begin{cases} q(i) & \text{if } d(s_{j_i}(a)\phi)\leq i,\\ q(i)+1 & \text{if } d(s_{j_i}(a)\phi)>i.\end{cases}$$

Let $q_0=(\bar{0},1)$, where $\bar{0}$ is the constant zero function, and let $Q=\{q_0\cdot u\mid u\in A^*\}$. Let ${}^p\hat{S}_A$ be the transition monoid of $(Q,A,\cdot)$ and τ be the projection of A^* onto ${}^p\hat{S}_A$. Proofs of the following two results can be found in [6].



Theorem 4.4. *Right stabilizers in ${}^p\hat{S}_A$ satisfy the identities $x^2=x$ and $xyz=xy$.*

Theorem 4.5. *Let e be an idempotent of S and $\hat{e}$ an idempotent of $S^{\hat{}L}$.*
(a) *$\hat{e}\sigma'^{-1}$ satisfies the identities $xyz=xzy$ and $xy^p=x$ and is the direct product of an idempotent $\mathscr{L}$-class with Z_p^k.*
(b) *$(e\sigma^{-1})\sigma'^{-1}$ satisfies the identity $x^{p+1}=x$ and is a Rees matrix semigroup over Z_p^k.*

In [6] they claim the stronger result that $e\sigma^{-1}\sigma'^{-1}$ is the direct product of a rectangular band and of a subgroup of $Z_p^{d(e)-1}$. This is not true, and a correct computation of the second example in Section 5 of their paper is a counter example.

Lemma 4.6. *The local monoids of $D_{\sigma'}$ are in $\boldsymbol{Z_p}$.*

Proof. Let u,v in A^* be such that $(uv)\phi^{\hat{}L}=u\phi^{\hat{}L}$. Let $(q_w,w\phi^{\hat{}L})$ be any state with $w\phi^{\hat{}L}=u\phi^{\hat{}L}$. As $(wv^k)\phi^{\hat{}L}=w\phi^{\hat{}L}$ for all k, by checking the definition of $(q_w,w\phi^{\hat{}L})\cdot v,(q_{wv},(wv)\phi^{\hat{}L})\cdot v$, etc., we see that $q_{wv^k}(i)-q_{wv^{k-1}}(i)=q_{wv}(i)-q_w(i)$ for $k=1,2,3,\dots,$

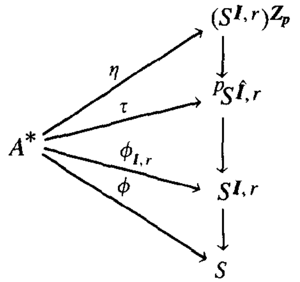
and so

$$\begin{aligned} q_{wv^p}(i) &= q_{wv^p}(i) - q_{wv^{p-1}}(i) + q_{wv^{p-1}}(i) - q_{wv^{p-2}}(i) + \cdots + q_{wv}(i) - q_w(i) + q_w(i) \\ &= p(q_{wv}(i) - q_w(i)) + q_w(i) = q_w(i) \end{aligned}$$

Hence $[u\phi^{\hat{L}}, v\tau, u\phi^{\hat{L}}]^p = [u\phi^{\hat{L}}, 1, u\phi^{\hat{L}}]$.

Suppose also that $(uv')\phi^{\hat{L}} = u\phi^{\hat{L}}$. Once again by definition of the action we see that $q_{wv'v}(i) - q_{wv'}(i) = q_{wv}(i) - q_w(i)$ and $q_{wvv'}(i) - q_{wv}(i) = q_{wv'}(i) - q_w(i)$. This implies $q_{wv'v}(i) = q_{wvv'}(i)$ and $[u\phi^{\hat{L}}, vv'\tau, u\phi^{\hat{L}}] = [u\phi^{\hat{L}}, v'v\tau, u\phi^{\hat{L}}]$. □

This lemma tells us that first doing the reverse expansion using $\boldsymbol{I}$ and then expanding by $\boldsymbol{Z_p}$ is bigger than the expansion defined in [6]. However, the derived categories of the maps down to the original semigroups are similar. That leaves us to ask if the larger expansion still has some of the same nice properties as the expansion defined in [6]. For clarity of notation the homomorphism from A^* to $(S^{\boldsymbol{I},r})^{\boldsymbol{Z_p}}$ which results from expanding $(S^{\boldsymbol{I},r}, \phi_{\boldsymbol{I},r})$ by $\boldsymbol{Z_p}$ will be denoted η.



Theorem 4.7. *Right stabilizers in $(S^{\boldsymbol{I},r})^{\boldsymbol{Z_p}}$ satisfy $x \leq_{\mathscr{L}} y \Rightarrow xy = x$.*

Proof. Let $u, v, w \in A^*$ with $v\eta$ and $w\eta$ both right stabilizers of $u\eta$, and $v\eta \leq_{\mathscr{L}} w\eta$. Let $w = b_1 \dots b_l$. As $(uw)\eta = u\eta$ we have

$$(g_u, u\phi_{\boldsymbol{I},r})(g_{b_1} + {}^{b_1\phi_{I,r}}g_{b_2} + \cdots + {}^{(b_1 \dots b_{l-1})\phi_{I,r}}g_{b_l}, w\phi_{\boldsymbol{I},r}) = (g_u, u\phi_{\boldsymbol{I},r})$$

and ${}^{u\phi_{I,r}}g_{b_1} + \cdots + {}^{(ub_1 \dots b_{l-1})\phi_{I,r}}g_{b_l} \equiv 0$. This means

$$ {}^{u\phi_{I,r}}g_{b_1} + \cdots + {}^{(ub_1 \dots b_{l-1})\phi_{I,r}}g_{b_l} = \sum_{i=1}^{n} \alpha_i{}^{t_i} g_{b_{k_i}} \tag{$*$}$$

where each α_i is congruent to 0 mod p. As we want $(vw)\eta = v\eta$, we need to show

$$ {}^{v\phi_{I,r}}g_{b_1} + \cdots + {}^{(vb_1 \dots b_{l-1})\phi_{I,r}}g_{b_l} \equiv 0.$$

To do this it suffices to show that $(vb_1 \dots b_i)\phi_{\boldsymbol{I},r} = (vb_1 \dots b_{i+j})\phi_{\boldsymbol{I},r}$ whenever $(ub_1 \dots b_i)\phi_{\boldsymbol{I},r} = (ub_1 \dots b_{i+j})\phi_{\boldsymbol{I},r}$, for then

$$ {}^{v\phi_{I,r}}g_{b_1} + \cdots + {}^{(vb_1 \dots b_{l-1})\phi_{I,r}}g_{b_l} = \sum_{i=1}^{n} \alpha_i{}^{t'_i} g_{b_{k_i}} \tag{$**$}$$

where if $t_i = ub_1 \dots b_j$ in $(*)$ then $t'_i = vb_1 \dots b_j$ in $(**)$.

Now $(uv)\tau = u\tau$ and it follows from Theorem 4.4 that $v\phi_{I,r}$ is idempotent. This means $u\phi_{I,r}$ and $v\phi_{I,r}$ are of the form

$$v\phi_{I,r} = [(0,e_k)(c_{k-1},x_{k-1})\dots(c_1,x_1)],$$

and

$$u\phi_{I,r} = [(0,x_m)\dots(c_k,x_k)(c_{k-1},x_{k-1})\dots(c_1,x_1)]$$

with $e_k \mathscr{L} x_k$ and e_k an idempotent of S. Now

$$(x_m b_1 \dots b_i)\phi = (ub_1 \dots b_i)\phi = (ub_1 \dots b_{i+j})\phi = (x_m b_1 \dots b_{i+j})\phi$$

implies

$$e_k(b_1 \dots b_i)\phi = (vb_1 \dots b_i)\phi = (vb_1 \dots b_{i+j})\phi = e_k(b_1 \dots b_{i+j})\phi.$$

Checking the definition of multiplication in $S^{I,r}$, i.e. in the Karnofsky–Rhodes expansion, we see this implies

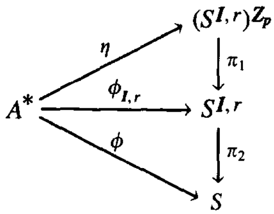
$$(vb_1 \dots b_i)\phi_{I,r} = (vb_1 \dots b_{i+j})\phi_{I,r}$$

and hence $(vw)\eta = v\eta$. □

Corollary 4.8. *Right stabilizers in $(S^{I,r})^{Z_p}$ are idempotent.*

Proof. This is Theorem 4.7 in the case $x = y$. □

Let π_1 and π_2 be as in the following diagram, and let $\pi = \pi_1\pi_2$.



Theorem 4.9. *Let e be an idempotent of S and $\hat{e}$ an idempotent of $S^{I,r}$.*
(a) *$\hat{e}\pi_1^{-1}$ satisfies the identities $xyz = xzy$ and $xy^p = x$ and is the direct product of an idempotent $\mathscr{L}$-class with Z_p^k.*
(b) *$e\pi^{-1}$ satisfies the identity $x^{p+1} = x$ and is a Rees matrix semigroup over Z_p^k.*

Proof. (a) Let $(g_1,\hat{e})$, $(g_2,\hat{e})$, $(g_3,\hat{e})$ be in $\hat{e}\pi_1^{-1}$. Then

$$(g_1,\hat{e})(g_2,\hat{e})(g_3,\hat{e}) = (g_1 + {}^{\hat{e}}g_2 + {}^{\hat{e}}g_3, \hat{e}) = (g_1,\hat{e})(g_3,\hat{e})(g_2,\hat{e})$$

and $(g_1,\hat{e})(g_2,\hat{e})^p = (g_1 + p \cdot {}^{\hat{e}}g_2, \hat{e}) = (g_1,\hat{e})$. If $(g_1,\hat{e})$, and $(g_2,\hat{e})$ are idempotents then $(g_2,\hat{e})(g_2,\hat{e}) = (g_2 + {}^{\hat{e}}g_2, \hat{e}) = (g_2,\hat{e})$, so ${}^{\hat{e}}g_2$ is the zero function and $(g_1,\hat{e})(g_2,\hat{e}) = (g_1 + {}^{\hat{e}}g_2, \hat{e}) = (g_1,\hat{e})$.

(b) Every element of $e\pi_2^{-1}$ is idempotent, and so the identity follows from part a. For each $\hat{e} \in e\pi_2^{-1}$, $\hat{e}\pi_1^{-1}$ is an $\mathscr{L}$-class of $e\pi^{-1}$. Let $\bar{e} \in \hat{e}\pi_1^{-1}$ and $\bar{f} \in \hat{f}\pi_1^{-1}$, with $\hat{e}$, $\hat{f} \in e\pi_2^{-1}$. Then as $\hat{e}\hat{f} = \hat{f}$, $\bar{e}\bar{f}$ is in $\hat{f}\pi_1^{-1}$. So, $\bar{e}\bar{f}\mathscr{L}\bar{f}$ and $\bar{e} \geq_{\mathscr{J}} \bar{f}$. Dually we see $\bar{f} \geq_{\mathscr{J}} \bar{e}$ and $e\pi^{-1}$ is a single $\mathscr{J}$-class. □

5. The expansion $S^{K(V)}$

To get the expansion S^V we imposed restrictions on the local monoids of the derived category, consolidated the resulting category, took a wreath product of the consolidated category with the original monoid, and then restricted to an image of A^*. In the same way we can impose restrictions on the local monoids of the kernel, consolidate the resulting category, take a block product of the consolidated category with the original monoid, and restrict to an image of A^*.

The kernel of a monoid homomorphism can be thought of as a two side version of the derived category. Given a surjective morphism of monoids $\phi : S \to T$, the kernel, denoted K_ϕ, is the following category:

- $Obj(K_\phi) = T \times T$.
- There is an arrow $[(t_l,t_r),s,(t_l',t_r')]$ from (t_l,t_r) to (t_l',t_r') if $t_l(s)\phi = t_l'$ and $(s)\phi t_r' = t_r$. The arrows $[(t_l,t_r),s,(t_l',t_r')]$ and $[(t_l,t_r),s',(t_l',t_r')]$ are equal if $s_lss_r = s_ls's_r$ for all $s_l \in t_l\phi^{-1}$ and $s_r \in t_r'\phi^{-1}$.
- Multiplication of consecutive arrows is given by $[(t_l,t_r),s,(t_l',t_r')][(t_l',t_r'),s',(t_l'',t_r'')] = [(t_l,t_r),ss',(t_l'',t_r'')]$.

A double action of S on T is a function $S \times T \times S \to T$, $(s,t,s') \mapsto sts'$ satisfying

$$s(t_1+t_2)s' = st_1s' + st_2s', \quad s_1(s_2ts_2')s_2 = (s_1s_2)t(s_2's_1'),$$

$$1t1 = t, \quad s0s = 0$$

for all $s,s',s_1,s_2,s_1',s_2' \in S$ and all $t,t_1,t_2 \in T$.

Given a double action of S on T the double semidirect product $T**S$ is the set $T \times S$ with multiplication given by

$$(t,s)(t',s') = (1ts' + st'1, ss').$$

There is a natural double action of S on $T^{S\times S}$ given by

$$(t,f,t') \mapsto [t,f,t'] \text{ with } t_l[t,f,t']t_r = (t_lt)f(t't_r),$$

where $[t,f,t']$ denotes the result of the double action and the notation tft' denotes the result of evaluating f at (t,t'). Thus, $[t,f,t'] \in T^{S\times S}$ and $tft' \in T$.

The block product $T\square S$ is the double semidirect product $T^{S\times S}**S$ associated with this double action. Thus, $T\square S$ has as its underlying set $T^{S\times S} \times S$, and the multiplication is given by

$$(f,s)(f',s') = ([1,f,s'] + [s,f',1], ss').$$

For more on the kernel and block product see [10].

5.1. The definition of $(S^{K(V)}, \phi_{K(V)})$

Let (S,ϕ) and V be as in Section 2.1. The proofs in this section are analogous to those in Section 2.1 with the derived category replaced by the kernel and the wreath product replaced by the block product. The resulting expansion will be denoted $(S^{K(V)}, \phi_{K(V)})$.

Let τ_ϕ be the smallest congruence on K_ϕ, the kernel of ϕ, such that the local monoids of $K = K_\phi/\tau_\phi$ are in V. If $u \in A^*$ and $(s_l,s_r),(s'_l,s'_r) \in S \times S$ then there is an arrow in K_ϕ from (s_l,s_r) to (s'_l,s'_r) labelled by u if $s_l(u)\phi = s'_l$ and $(u)\phi s'_r = s_r$. Let $\overline{[(s_l,s_r),u,(s'_l,s'_r)]}$ denote the τ_ϕ equivalence class of $[(s_l,s_r),u,(s'_l,s'_r)]$.

For $u \in A^*$, $f_u : S \times S \to K^1_{\mathrm{cd}}$ will be the function taking the pair (s_1,s_2) to the arrow from $(s_1,(u)\phi s_2)$ to $(s_1(u)\phi, s_2)$ labelled by u, i.e. $s_1 f_u s_2 = \overline{[(s_1,(u)\phi s_2),u,(s_1(u)\phi,s_2)]}$. Once again, K_{cd} is written additively for convenience of notation, but is not in general commutative.

Define $\phi_{K(V)} : A^* \to K^1_{\mathrm{cd}} \square S$ by $u\phi_{K(V)} = (f_u, u\phi)$. Then

$$u\phi_{K(V)} w\phi_{K(V)} = ([1, f_u, w\phi_{K(V)}] + [u\phi_{K(V)}, f_w, 1], (uw)\phi),$$

and as

$$\begin{aligned} & s_1[1,f_u,w\phi]s_2 + s_1[u\phi,f_w,1]s_2 \\ &= \overline{[(s_1,(uw)\phi s_2),u,(s_1(u)\phi,(w)\phi s_2)]} + \overline{[(s_1(u)\phi,(w)\phi s_2),w,(s_1(uw)\phi,s_2)]} \\ &= s_1 f_{uw} s_2 \end{aligned}$$

$\phi_{K(V)}$ is a homomorphism. Note that $s_1 f_u s_2 \neq 0$ for all u in A^* and (s_1,s_2) in $S \times S$. Let $S^{K(V)}$ denote the image of $\phi_{K(V)}$.

5.2. Properties of $(S^{K(V)}, \phi_{K(V)})$

Let X be the graph with vertices $S \times S$ and an edge $[(s_l,s_r),a,(s'_l,s'_r)]$ from (s_l,s_r) to (s'_l,s'_r) labelled by $a \in A$ if $s_l(a)\phi = s'_l$ and $(a)\phi s'_r = s_r$.

Lemma 5.1. *The kernel of ϕ is isomorphic to the free category over X. That is $K_\phi \cong X^*$.*

Proof. They have the same objects and there is an arrow $[(s_l,s_r),u,(s'_l,s'_r)]$ in K_ϕ if an only if there is a path in X from (s_l,s_r) to (s'_l,s'_r) labelled by $u \in A^*$. We need to check that if $[(s_l,s_r),u,(s_ls,s'_r)] = [(s_l,s_r),v,(s_ls,s'_r)]$ in K_ϕ then $u=v$. Suppose $x \in s_l\phi^{-1}$ and $y \in s'_r\phi^{-1}$. Then if $[(s_l,s_r),u,(s_ls,s'_r)] = [(s_l,s_r),v,(s_ls,s'_r)]$ we have $xuy = xvw$. As A^* is a cancellative semigroup, this implies $u=v$. □

The above Lemma 5.1 is analogous to Lemma 2.1 and allows us to think of arrows in K and K_ϕ as arising from paths in X. Notice that in K, as was in D, it is possible for a single arrow to be labelled by two different words in A^*.

Lemma 5.2. *There is a left action of S on K_ϕ given by*

$$s \cdot [(s_l, s_r), u, (s'_l, s'_r)] = [(ss_l, s_r), u, (ss'_l, s'_r)]$$

and a right action given by

$$[(s_l, s_r), u, (s'_l, s'_r)] \cdot s = [(s_l, s_r s), u, (s'_l, s'_r s)].$$

Proof. If $s_l u\phi = s'_l$ then $ss_l u\phi = ss'_l u\phi$ and similarly for the second action. The action is well defined as in the proof of the above lemma we saw that $[(s_l, s_r), u, (s'_l, s'_r)] = [(s_l, s_r), v, (s'_l, s'_r)]$ if and only if $u = v$. □

Lemma 5.3. *The above actions induce left and right actions of S on K.*

Proof. We need to show that if

$$\overline{[(s_l, s_r), u, (s'_l, s'_r)]} = \overline{[(s_l, s_r), v, (s'_l, s'_r)]},$$

then

$$\overline{[(ss_l, s_r s'), u, (ss'_l, s'_r s')]} = \overline{[(ss_l, s_r s'), v, (ss'_l, s'_r s')]}.$$

Let A^*/ρ be the relatively free A-generated monoid in $\boldsymbol{V}$, i.e. ρ is the smallest congruence on A^* generated by the equations of $\boldsymbol{V}$. As $\overline{[(s_l, s_r), u, (s'_l, s'_r)]} = \overline{[(s_l, s_r), v, (s'_l, s'_r)]}$ there is a sequence of steps which take $[(s_l, s_r), u, (s'_l, s'_r)]$ to $[(s_l, s_r), v, (s'_l, s'_r)]$ by applying the equations of $\boldsymbol{V}$ to the local monoids of K_ϕ. Thinking of an arrow in K_ϕ as a path in X, this means there is a sequence of n steps of the following form

$$\begin{aligned}&[(s_l, s_r), x_i, (s_1, s_2)][(s_1, s_2), u_i, (s_1, s_2)][(s_1, s_2), y_i, (s'_l, s'_r)]\\ &\quad \equiv_{\tau_\phi} [(s_l, s_r), x_i, (s_1, s_2)][(s_1, s_2), v_i, (s_1, s_2)][(s_1, s_2), y_i, (s'_l, s'_r)],\end{aligned}$$

where the middle terms are elements of a local monoid, $u_i \rho v_i$, $u = x_1 u_1 y_1$, and $v = x_n v_n y_n$. If $s_1(x_i)\phi = s_1$ and $s_2 = (x_i)\phi s_2$ then $ss_1(x_i)\phi = ss_1$ and $s_2 s' = (x_i)\phi s_2 s'$. Hence, the action will take the loop $[(s_1, s_2), x_i, (s_1, s_2)]$ to another loop, $[(ss_1, s_2 s'), x_i, (ss_1, s_2 s')]$. We can then apply the same equation to this loop and get the sequence of steps

$$\begin{aligned}&[(ss_l, s_r s'), x_i, (ss_1, s_2 s')][(ss_1, s_2 s'), u_i, (ss_1, s_2 s')][(ss_1, s_2 s'), y_i, (ss'_l, s'_r s')]\\ &\quad \equiv_{\tau_\phi} [(ss_l, s_r s'), x_i, (ss_1, s_2 s')][(ss_1, s_2 s'), v_i, (ss_1, s_2 s')][(ss_1, s_2 s'), y_i, (ss'_l, s'_r s')]\end{aligned}$$

for $i = 1, \ldots, n$ which shows that

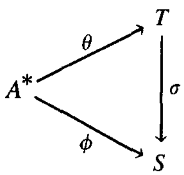
$$\overline{[(ss_l, s_r s'), u, (ss'_l, s'_r s')]} = \overline{[(ss_l, s_r s'), v, (ss'_l, s'_r s')]}. \qquad \square$$

This action implies that if we want to determine if a pair of coterminal paths labelled by u and v are equal, it is sufficient to know that the arrow from $(1,u\phi)$ to $(u\phi,1)$ labelled by u is equal to the arrow from $(1,v\phi)$ to $(v\phi,1)$ labelled by v.

Corollary 5.4. *$u\phi_{K(V)} = v\phi_{K(V)}$ if and only if $1f_u1 = 1f_v1$.*

Proof. By definition $u\phi_{K(V)} = v\phi_{K(V)}$ implies $f_u = f_v$. Assume $1f_u1 = 1f_v1$ then $\overline{[(1,u\phi),u,(u\phi,1)]} = \overline{[(1,v\phi),v,(v\phi,1)]}$ and it follows that $u\phi = v\phi$. If (s_l,s_r) is in $S\times S$ then $s_lf_us_r = \overline{[(s_l,(u)\phi s_r),u,(s_l(u)\phi,s_r)]}$, and it follows from Lemma 5.3 that this equals $\overline{[(s_l,(v)\phi s_r),v,(s_l(v)\phi,s_r)]} = s_lf_vs_r$. □

Lemma 5.5. *Let $\sigma:(T,\theta)\to(S,\phi)$ be a homomorphism. There is a congruence τ_σ on K_ϕ such that $K_\sigma \cong K_\phi/\tau_\sigma$. Moreover if $[(1,s_1),u,(s_1,1)] \equiv_{\tau_\sigma} [(1,s_1),v,(s_1,1)]$ then $[(s,s_1s'),u,(ss_1,s')] \equiv_{\tau_\sigma} [(s,s_1s'),v,(ss_1,s')]$ for all (s,s') in $S\times S$.*



Proof. Define τ_σ by $[(s_l,s_r),u,(s'_l,s'_r)] \equiv_{\tau_\sigma} [(s_l,s_r),v,(s'_l,s'_r)]$ if the arrows $[(s_l,s_r),u\theta,(s'_l,s'_r)]$ and $[(s_l,s_r),v\theta,(s'_l,s'_r)]$ are equal in K_σ. If we show that τ_σ is a well defined congruence it will then follow from the definition that $K_\sigma \cong K_\phi/\tau_\sigma$.

If $[(s_l,s_r),u,(s'_l,s'_r)] = [(s_l,s_r),v,(s'_l,s'_r)]$ in K_ϕ then $xuy = xvy$ for all $x,y\in A^*$ with $x\phi = s_l$ and $y\phi = s'_r$. A^* is a cancellative semigroup and thus $u = v$ and $[(s_l,s_r),u\theta,(s'_l,s'_r)] = [(s_l,s_r),v\theta,(s'_l,s'_r)]$.

To show τ_σ is a right congruence suppose that

$$[(s_l,s_r),u,(s'_l,s'_r)] \equiv_{\tau_\sigma} [(s_l,s_r),v,(s'_l,s'_r)]$$

and that $[(s'_l,s'_r),w\phi,(s''_l,s''_r)]$ is any arrow in D_ϕ leaving (s'_l,s'_r). By the definition of τ_σ we have

$$[(s_l,s_r),u\theta,s(s'_l,s'_r)] = [(s_l,s_r),v\theta,(s'_l,s'_r)]$$

and thus

$$[(s_l,s_r),u\theta,(s'_l,s'_r)][(s'_l,s'_r),w\theta,(s''_l,s''_r)] = [(s_l,s_r),v\theta,(s'_l,s'_r)][(s'_l,s'_r),w\theta,(s''_l,s''_r)].$$

This implies

$$[(s_l,s_r),u,(s'_l,s'_r)][(s'_l,s'_r),w,(s''_l,s''_r)] \equiv_{\tau_\sigma} [(s_l,s_r),v,(s'_l,s'_r)][(s'_l,s'_r),w,(s''_l,s''_r)].$$

An analogous argument shows that τ_σ is a left congruence. By construction it is clear that $K_\phi/\tau_\sigma = K_\sigma$.

To show the action is well defined, suppose

$$[(1,s),u,(s,1)] \equiv_{\tau_\sigma} [(1,s),v,(s,1)].$$

By the definition of τ_σ,

$$[(1,u\phi),u\theta,(u\phi,1)] = [(1,v\phi),v\theta,(v\phi,1)]$$

and $t(u)\theta t' = t(v)\theta t'$ for all $t,t' \in 1\sigma^{-1}$. As $1_T \in 1\sigma^{-1}$, $u\theta = v\theta$ and

$$[(s,s_1s'),u\theta,(ss_1,s')] \equiv_{\tau_\sigma} [(s,s_1s'),v\theta,(ss_1,s')],$$

and hence

$$[(s,s_1s'),u,(ss_1,s')] \equiv_{\tau_\sigma} [(s,s_1s'),v,(ss_1,s')]. \qquad \square$$

Proposition 5.6. *Let $\sigma:(T,\theta)\to(S,\phi)$ be a homomorphism. Let $\psi:A^*\to(K_\sigma)_{\mathrm{cd}}\square S$ be the homomorphism sending u to $(g_u,u\phi)$ where*

$$sg_us' = [(s,(u)\phi s'),u\theta,(s(u)\phi,s')].$$

The map $\eta:A^*\psi\to T$ which sends $u\psi$ to $u\theta$ is an isomorphism.

Proof. If $u\psi = v\psi$ then $1g_u1$ equals $1g_v1$, i.e. $[(1,u\phi),u\theta,(u\phi,1)] = [(1,v\phi),v\theta,(v\phi,1)]$. As $1_T \in 1\sigma^{-1}$, $u\theta = v\theta$ and η is well defined.

Reversely, if $u\theta = v\theta$ then $1g_u1 = 1g_v1$ and it follows from Lemma 5.5 that $g_u = g_v$ and η is injective. $\square$

Proposition 5.7. *Let $\varepsilon_S : S^{K(V)} \to S$ be the restriction of the projection of $K^1_{\mathrm{cd}}\square S$ onto S. The local monoids of K_{ε_S} are in V and $K_{\varepsilon_S} \cong K$.*

Proof. Using the notation in Lemma 5.5 $K_{\varepsilon_S} \cong K_\phi/\tau_{\varepsilon_S}$. Suppose $u\phi = v\phi = s$. By Lemmas 5.3 and 5.5 it suffices to show that

$$[(1,s),u,(s,1)] \equiv_{\tau_\phi} [(1,s),v,(s,1)]$$

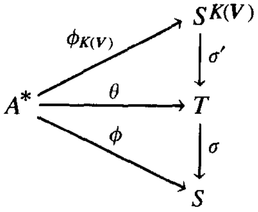
exactly when

$$[(1,s),(f_u,s),(s,1)] \equiv_{\tau_{\varepsilon_S}} [(1,s),(f_v,s),(s,1)].$$

$1f_u1 = \overline{[(1,s),u,(s,1)]}$ and $1f_v1 = \overline{[(1,s),v,(s,1)]}$. So $1f_u1 = 1f_v1$ whenever $[(1,s), u,(s,1)] \equiv_{\tau_\phi} [(1,s),v,(s,1)]$. It follows from Corollary 5.4 that $f_u = f_v$, which implies $[(1,s),(f_u,s),(s,1)] \equiv_{\tau\varepsilon_S} [(1,s),(f_v,s),(s,1)]$.

If $[(1,s),(f_u,s),(s,1)] \equiv_{\tau_{\varepsilon_S}} [(1,s)(f_v,s),(s,1)]$, then as the identity of $S^{K(V)}$ is in the inverse image under ε_S of the identity of S, we have $(f_u,s) = (f_v,s)$. Hence $1f_u1 = \overline{[(1,s),u,(s,1)]} = 1f_v1 = \overline{[(1,s),v,(s,1)]}$. $\square$

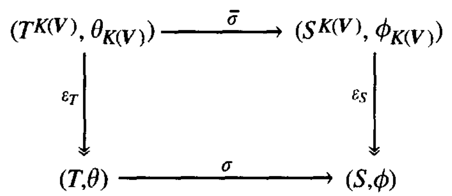
Theorem 5.8. *If the local monoids of K_σ are in V then there exists a surjective homomorphism σ': $(S^{K(V)}, \phi_{K(V)}) \to (T, \theta)$.*



Proof. $K = K_\phi/\tau_\phi$ and $K_\sigma \cong K_\phi/\tau_\sigma$. As the local monoids of K_σ are in V, $\tau_\phi \subseteq \tau_\sigma$. Therefore the map $F : K_{\text{cd}} \to (K_\sigma)_{\text{cd}}$ which sends $\overline{[(s_l, s_r), u, (s'_l, s'_r)]}$ to $[(s_l, s_r), u\theta, (s'_l, s'_r)]$) is a well defined homomorphism. We can then define $\sigma' : (f_u, s) \mapsto (g_u, s)\eta$ as in Proposition 5.6, with $tg_ut' = F(tf_ut')$. □

Theorem 5.9. *The functor taking (S, ϕ) to $(S^{K(V)}, \phi_{K(V)})$, along with the natural transformation ε is an expansion.*

Proof. We need to define $\bar{\sigma}$ such that the following diagram commutes.



Let $C = K_\theta/\tau_\theta$ be the largest quotient of K_θ whose local monoids are in V. The map $G : C_{\text{cd}} \to K_{\text{cd}}$ sending $\overline{[(t_l, t_r), u, (t'_l, t'_r)]}$ in C to $\overline{[(s_l, s_r), u, (s'_l, s'_r)]}$ in K where $t_l\sigma = s_l$ and $t'_r\sigma = s'_r$, and sending 0 to 0 is a surjective homomorphism. For u in A^* let $f'_u : T \times T \to C_{\text{cd}}$ denote the function $tf'_ut' = \overline{[(t, (u)\theta t'), u, (t(u)\theta, t')]}$, and let $f_u : S \times S \to K_{\text{cd}}$ denote the function $sf_us' = \overline{[(s, (u)\phi s'), u, (s(u)\phi, s')]}$ as before. Then $G(tf'_ut') = t\sigma f_u t'\sigma$. Since σ is onto, it follows that $f'_u = f'_v$ implies $f_u = f_v$. Define $\bar{\sigma}$ by $(f'_u, u\theta) \mapsto (f_u, u\phi)$. Then $u\theta_V\bar{\sigma} = ((f'_u, u\theta))\bar{\sigma} = (f_u, u\phi) = u\phi_{K(V)}$ and the diagram commutes. □

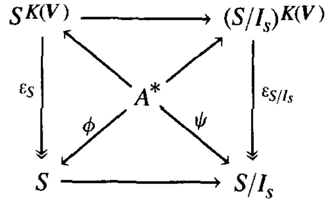
The expansion can be modified the same way as in Section 2.3 to get a semigroup expansion based on the kernel.

5.3. Finiteness properties of $S^{K(V)}$

Proposition 5.10. *Let S be finite and V be locally finite. Then $S^{K(V)}$ is finite.*

Proof. Each of the local monoids of K are in V and so are locally finite. K is generated by the finite set of arrows $\overline{[(s_l, s_r), a, (s'_l, s'_r)]}$, $a \in A$ and $s_l, s_r, s'_l, s'_r \in S$ with $s_lu\phi = s'_l$ and $u\phi s'_r = s_r$. It follows from Lemma 2.10 that K is finite. As the block product of finite monoids is finite, $K^1_{\text{cd}} \Box S$ is finite. As $S^{K(V)} \subseteq K_{\text{cd}} \Box S$ it must also be finite. □

Corollary 5.11. *If S is finite $\mathcal{J}$ above and V is locally finite then $S^{K(V)}$ is finite $\mathcal{J}$ above.*



Proof. As in Corollary 2.12, let $I_s = \{s'$ in S: $s' \not\geq_{\mathcal{J}} s\}$. As S is finite $\mathcal{J}$ above S/I_s is finite, and it follows from Proposition 5.10 that $(S/I_s)^{K(V)}$ is finite. Let ψ be the natural map from A^* to S/I_S. Let $\tilde{s} \in s\varepsilon_S^{-1}$ and $I_{\tilde{s}} = \{\tilde{t} \in S^{K(V)}\colon\ \tilde{t} \not\geq_{\mathcal{J}} \tilde{s}\}$. If $\tilde{I}_s = I_s\varepsilon_S^{-1}$, then $\tilde{I}_s \subseteq I_{\tilde{s}}$ and thus $S^{K(V)}/I_{\tilde{s}} \subseteq S^{K(V)}/\tilde{I}_s$. We will show that $(S/I_s)^{K(V)} \cong S^{K(V)}/\tilde{I}_s$. It follows that $S^{K(V)}/I_{\tilde{s}}$ is finite and $S^{K(V)}$ is finite $\mathcal{J}$ above.

Let $u, v \in A^*$ with $u\phi, v\phi \notin I_s$. By Corollary 5.4 it suffices to show that if

$$[(1, u\psi), u, (u\psi, 1)] \equiv_{\tau_\psi} [(1, v\psi), v, (v\psi, 1)]$$

then

$$[(1, u\phi), u, (u\phi, 1)] \equiv_{\tau_\phi} [(1, v\phi), v, (v\phi, 1)].$$

As $u\phi, v\phi \notin I_s$, $\psi(u) = \phi(u) = \phi(v) = \psi(v)$. By Lemma 5.1 the arrow $[(1, u\psi), u, (u\psi, 1)]$ in D_ψ can be thought of as a path in a graph Y. Likewise the arrow $[(1, u\phi), u, (u\phi, 1)]$ is a path in the graph X as in Lemma 5.1. As $u\phi \notin I_s$, the path $[(1, u\psi), u, (u\psi, 1)]$ never passes through the vertex 0. Moreover, any path which does pass through the vertex 0 stays there and never goes back through any other vertex. Hence, any equation of V which can be applied to a loop in the path $[(1, u\psi), u, (u\psi, 1)]$ corresponds to a place where that same equation can be applied to a loop in the path $[(1, u\phi), u, (u\phi, 1)]$. So by using the same sequence of equations which show $[(1, u\psi), u, (u\psi, 1)] \equiv_{\tau_\psi} [(1, v\psi), v, (v\psi, 1)]$, and applying them to the corresponding local monoids of D_ϕ, we get $[(1, u\phi), u, (u\phi, 1)] \equiv_{\tau_\phi} [(1, v\phi), v, (v\phi, 1)]$. □

5.4. *The example $S^{K(SL)}$*

Once again for simplicity of notation we will assume that (S, ϕ) is an object in $\boldsymbol{M}_A$. The expansion $\hat{S}^{(2)}$ is due to Karsten Henckell [1]. As with the other expansions in this paper, we will describe $\hat{S}^{(2)}$ cut to generators. Let $\mathcal{P}_{\rm f}(S \times S)$ be the set of all finite subsets of $S \times S$. Define $\theta : A^* \to \mathcal{P}_{\rm f}(S \times S)$ by

$$a_1 a_2 \ldots a_k \mapsto \left\{ \prod_{1=1}^{m} a_i\phi,\ \prod_{i=m+1}^{k} a_i\phi\colon\ 0 \le m \le k \right\}.$$

Let $v = a_{k+1} \dots a_{k+h}$. There is an associative multiplication on $A^*\theta$ given by

$$\left\{\prod_{1=1}^{m} a_i\phi, \prod_{i=m+1}^{k} a_i\phi\colon 0 \le m \le k\right\} \cdot \left\{\prod_{i=k+1}^{r} a_i\phi, \prod_{i=r+1}^{k+h} a_i\phi\colon k \le r \le k+h\right\}$$
$$= \left\{\prod_{1=1}^{n} a_i\phi, \prod_{i=n+1}^{k+h} a_i\phi\colon 0 \le n \le k+h\right\}.$$

It follows from the definition of multiplication that $u\theta v\theta = (uv)\theta$ and θ is a homomorphism. The image of θ is $\hat{S}^{(2)}$. Let $\sigma : \hat{S}^{(2)} \to S$ be such that $\theta\sigma = \phi$, i.e. if $(s_1, s_2) \in u\theta$ then $u\theta\sigma = s_1 s_2 = u\phi$. It is easy to check that σ is a well defined homomorphism and this construction defines a monoid expansion.

Lemma 5.12. *The local monoids of K_σ are semilattices.*

Proof. First we need to show that if $\overline{[(s_l, s_r), u\theta, (s_l, s_r)]}$ is an element of a local monoid of K_σ, then it is equal to $\overline{[(s_l, s_r), (u^2)\theta, (s_l, s_r)]}$. If $w, v \in A^*$ are such that $w\theta\sigma = s_l$ and $v\theta\sigma = s_r$, we need to show $(wuv)\theta = (wu^2v)\theta$. Let $w = a_1 \dots a_k$, $u = a_{k+1} \dots a_m$, and $v = a_{m+1} \dots a_h$.

$$(wu^2v)\theta = \left\{\prod_{i=1}^{n} a_i\phi, \left(\prod_{i=n+1}^{m} a_i\phi\right)(uv)\phi\colon 0 \le n \le m\right\}$$
$$\cup \left\{(wu)\phi\left(\prod_{i=k+1}^{n} a_i\phi\right), \prod_{i=n+1}^{h} a_i\phi\colon m \le n \le h\right\}.$$

As $s_r = (uv)\phi = v\phi = \prod_{i=m+1}^{h} a_i\phi$ and $s_l = (wu)\phi = w\phi = \prod_{i=1}^{k} a_i\phi$, we can simplify this expression to

$$\left\{\prod_{i=1}^{n} a_i\phi, \left(\prod_{i=n+1}^{m} a_i\phi\right) v\phi\colon 0 \le n \le m\right\}$$
$$\cup \left\{w\phi\left(\prod_{i=k+1}^{n} a_i\phi\right), \prod_{i=n+1}^{h} a_i\phi\colon m \le n \le h\right\}$$
$$= \left\{\prod_{i=1}^{n} a_i\phi, \prod_{n+1}^{h} a_i\phi\colon 0 \le n \le h\right\} = wuv\theta.$$

Let $\overline{[(s_l, s_r), x\theta, (s_l, s_r)]}$ be another loop at the same object with $x = b_1 \dots b_p$.

$$(wuxv)\theta = \left\{\prod_{i=1}^{n} a_i\phi, \left(\prod_{i=n+1}^{k} a_i\phi\right)(uxv)\phi\colon 0 \le n \le k\right\}$$
$$\cup \left\{w\phi\left(\prod_{i=k+1}^{n} a_i\phi\right), \left(\prod_{i=n+1}^{m} a_i\phi\right)(xv)\phi\colon k+1 \le n \le m\right\}$$

$$\cup \left\{ (wu)\phi \left(\prod_{i=11}^{n} b_i\phi \right), \left(\prod_{i=n+1}^{p} b_i\phi \right) v\phi\colon\ 0 \le n \le p \right\}$$
$$\cup \left\{ (wux)\phi \left(\prod_{i=m+1}^{n} a_i\phi \right), \prod_{i=n+1}^{h} a_i\phi\colon\ m+1 \le n \le h \right\}.$$

As $(wu)\phi = w\phi = (wx)\phi$ and $(uv)\phi = v\phi = (xv)\phi$ we can simplify this expression to

$$(wuxv)\phi = \left\{ \prod_{i=1}^{n} a_i\phi, \left(\prod_{i=n+1}^{k} a_i\phi \right) v\phi\colon\ 0 \le n \le k \right\}$$
$$\cup \left\{ w\phi \left(\prod_{i=k+1}^{n} a_i\phi \right), \left(\prod_{i=n+1}^{m} a_i\phi \right) v\phi\colon\ k+1 \le n \le m \right\}$$
$$\cup \left\{ w\phi \left(\prod_{i=11}^{n} b_i\phi \right), \left(\prod_{i=n+1}^{p} b_i\phi \right) v\phi\colon\ 0 \le n \le p \right\}$$
$$\cup \left\{ w\phi \left(\prod_{i=m+1}^{n} a_i\phi \right), \prod_{i=n+1}^{h} a_i\phi\colon\ m+1 \le n \le h \right\}.$$

Similarly we have

$$(wxuv)\theta = \left\{ \prod_{i=1}^{n} a_i\phi, \left(\prod_{i=n+1}^{k} a_i\phi \right) (xuv)\phi\colon\ 0 \le n \le k \right\}$$
$$\cup \left\{ w\phi \left(\prod_{i=11}^{n} b_i\phi \right), \left(\prod_{i=n+1}^{p} b_i\phi \right) (uv)\phi\colon\ 0 \le n \le p \right\}$$
$$\cup \left\{ (wx)\phi \left(\prod_{i=k+1}^{n} a_i\phi \right), \left(\prod_{i=n+1}^{m} a_i\phi \right) v\phi\colon\ k+1 \le n \le m \right\}$$
$$\cup \left\{ (wxu)\phi \left(\prod_{i=m+1}^{n} a_i\phi \right), \prod_{i=n+1}^{h} a_i\phi : m+1 \le n \le h \right\},$$

and simplifying in the same manner as above we have

$$(wxuv)\phi = \left\{ \prod_{i=1}^{n} a_i\phi, \left(\prod_{i=n+1}^{k} a_i\phi \right) v\phi\colon 0 \le n \le k \right\}$$
$$\cup \left\{ w\phi \left(\prod_{i=11}^{n} b_i\phi \right), \left(\prod_{i=n+1}^{p} b_i\phi \right) v\phi\colon 0 \le n \le p \right\}$$

$$\cup \left\{ w\phi \left(\prod_{i=k+1}^{n} a_i\phi \right), \left(\prod_{i=n+1}^{m} a_i\phi \right) v\phi \colon k+1 \le n \le m \right\}$$

$$\cup \left\{ w\phi \left(\prod_{i=m+1}^{n} a_i\phi \right), \prod_{i=n+1}^{h} a_i\phi \colon m+1 \le n \le h \right\}.$$

So $(wuxv)\theta = (wxuv)\theta$ and

$$\overline{[(s_l,s_r),u\theta,(s_l,s_r)]}\ \overline{[(s_l,s_r),x\theta,(s_l,s_r)]}$$
$$= \overline{[(s_l,s_r),x\theta,(s_l,s_r)][(s_l,s_r),u\theta,(s_l,s_r)]}. \qquad \square$$

As before we can now ask how different is $S^{K(SL)}$ from $\hat{S}^{(2)}$. We know by Theorem 5.8 that it will be larger. It turns out that the relationship between $\hat{S}^{(2)}$ and $S^{K(SL)}$ is very similar to the relationship between the Rhodes expansion and $S^{I,r}$, which turned out to be the Karnofsky–Rhodes expansion.

Let X be the graph defined in Section 5.2. It follows from Corollary 5.4 and Lemma 3.8 that the elements of $\hat{S}^{(2)}$ are determined by looking at the coterminal paths in X from objects of the form $(1,s)$ to $(s,1)$ and identifying those which have the same content.

If $u = a_1 \dots a_k$ is a word in A^* the content of the path starting at $(1,u\phi)$ and labelled by u is the set of edges

$$\left\{ \left[\left(\prod_{i=1}^{n-1} a_i\phi, \prod_{i=n}^{k} a_i\phi \right), a_n, \left(\prod_{i=1}^{n} a_i\phi, \prod_{i=n+1}^{k} a_i\phi \right) \right] : 0 \le n \le k \right\}.$$

Comparing this to $u\theta$, we see that $u\theta$ is the set of all the vertices that this path passes through. However, it does not remember the labels on the edges which are transversed. This is the exact same difference as that between the Rhodes expansion and $S^{I,r}$.

Define ψ as the map which takes a word to the set of transition edges associated to that word as above. There is an associative multiplication on $A^*\psi$ given by

$$\left\{ \left[\left(\prod_{i=1}^{n-1} a_i\phi, \prod_{i=n}^{k} a_i\phi \right), a_n, \left(\prod_{i=1}^{n} a_i\phi, \prod_{i=n+1}^{k} a_i\phi \right) \right] : 0 \le n \le k \right\}$$
$$\cdot \left\{ \left[\left(\prod_{i=1}^{n-1} b_i\phi, \prod_{i=n}^{h} b_i\phi \right), b_n, \left(\prod_{i=1}^{n} b_i\phi, \prod_{i=n+1}^{h} b_i\phi \right) \right] : 0 \le n \le b \right\}$$
$$= \left\{ \left[\left(\prod_{i=1}^{n-1} a_i\phi, \left(\prod_{i=n}^{k} a_i\phi \right) (b_1 \dots b_h)\phi \right), a_n, \right.\right.$$
$$\left.\left. \left(\prod_{i=1}^{n} a_i\phi, \left(\prod_{i=n+1}^{k} a_i\phi \right) (b_1 \dots b_h)\phi \right) \right] : 0 \le n \le k \right\}$$

$$\cup \left\{ \left[\left((a_1 \ldots a_k)\phi \left(\prod_{i=1}^{n-1} b_i\phi \right), \prod_{i=n}^{h} b_i\phi \right), \right. \right.$$
$$\left. \left. b_n, \left((a_1 \ldots a_k)\phi \left(\prod_{i=1}^{n} b_i\phi \right), \prod_{i=n+1}^{h} b_i\phi \right) \right] : 0 \le n \le b \right\}.$$

Moreover this multiplication is such that $u\psi \cdot v\psi = uv\psi$, thus ψ is a homomorphism. The discussion above essentially proves the following.

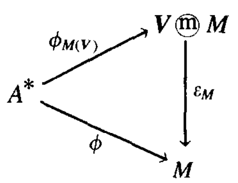
Theorem 5.13. $(S^{K(SL)}, \phi_{K(SL)}) \cong (A^*\psi, \psi)$.

6. The expansion $V ⓜ M$

6.1. *The definition of* $V ⓜ M$

If $\eta : T \to S$ is a semigroup homomorphism such that for all idempotents $e \in S$, $e\phi'^{-1} \in V$ then T is said to be a Malcev product of S over V. In this section an expansion which takes a monoid M (semigroup S) to the largest Malcev product of M (S) over a given variety V which is in $\mathbf{M}_A$ ($\mathbf{S}_A$) is defined. The definitions for monoids and semigroups are similar and will be given concurrently, with the relevant changes for semigroups given in parenthesis. V will be a locally finite variety, (M, ϕ) will denote an element of $\mathbf{M}_A$, and (S, ϕ') will denote an element of $\mathbf{S}_A$.

Let θ_ϕ ($\theta_{\phi'}$) be the congruence on A^* (A^+) generated by imposing the identities of V on each $e\phi^{-1}$ ($e\phi'^{-1}$) for e an idempotent of M (S). Define $u\phi_{M(V)}$ ($u\phi'_{M(V)}$) to be the θ_ϕ ($\theta_{\phi'}$) congruence class of u and $V ⓜ M = A^*/\theta_\phi$ ($V ⓜ S = A^+/\theta_{\phi'}$). Let ε_M (ε_S) be such that $\phi = \phi_{M(V)}\varepsilon_M$ ($\phi' = \phi'_{M(V)}\varepsilon_S$).



The following is a well known theorem.

Theorem 6.1 (Brown). *If T is a locally finite semigroup and α is a homomorphism from S to T such that the inverse image of idempotents is locally finite, then $T\alpha^{-1}$ is locally finite.*

The next theorem is a direct consequence of Brown's theorem.

Theorem 6.2. *Let M (S) be finite and V be locally finite. Then $V ⓜ M$ ($V ⓜ S$) is finite.*

By construction it is clear that if $\eta : (N,\alpha) \to (M,\phi)$ is any other homomorphism with $e\eta^{-1} \in V$ for each idempotent e of M, then there is a surjective homomorphism $\pi_2 : (V ⓜ M, \phi_{M(V)}) \to (N,\eta)$. That is, $V ⓜ M$ is the largest way to expand M by a Malcev product over V.

Theorem 6.3. *Given a homomorphism $\sigma : (N,\alpha) \to (M,\phi)$ there is a homomorphism $\bar{\sigma}$ from $(V ⓜ N, \alpha_{M(V)})$ to $(V ⓜ M, \phi_{M(V)})$ such that the following diagram commutes, and the functor taking (M,ϕ) to $(V ⓜ M, \phi_{M(V)})$ along with the natural transformation ε is a monoid expansion. Similarly the functor taking (S,ϕ') to $(V ⓜ S, \phi'_{M(V)})$ gives rise to a semigroup expansion.*

$$\begin{array}{ccc} (V ⓜ N, \alpha_{M(V)}) & \xrightarrow{\bar{\sigma}} & (V ⓜ M, \phi_{M(V)}) \\ \downarrow \varepsilon_N & & \downarrow \varepsilon_M \\ (N,\alpha) & \xrightarrow{\sigma} & (M,\phi) \end{array}$$

Proof. Suppose $u, v \in e\alpha^{-1}$, with $e = e^2$ an idempotent of N and $(u,v) \in \theta_\alpha$. If $f = e\sigma$, then f is an idempotent of M, and $u, v \in f\phi^{-1}$. If w is any element of $e\alpha^{-1}$, then $w \in f\phi^{-1}$, so imposing the equations of V on $f\phi^{-1}$ yields $(u,v) \in \theta_\phi$. Hence $\theta_\alpha \subseteq \theta_\phi$, and thus we can define $\bar{\sigma}$ by $\alpha_{M(V)}\bar{\sigma} = \phi_{M(V)}$. Replacing N and M by semigroups does not change the proof. □

It is important to know which category you are working in as the following example illustrates. Let $V = A^\ell$ be the semigroup variety defined by the equation $xy = x$. Let S be the cyclic monoid $\langle z \mid z^4 = z \rangle$. The members of S are z, z^2, and $z^3 = 1$. Let $A = \{a\}$. Define $\phi' : A^+ \to S$ as the extension of the map $a\phi' = z$. Define $\phi : A^* \to S$ as the extension of the map $a\phi = z$ and $\varepsilon\phi = z^3 = 1$, where ε is the empty word.

$1\phi'^{-1} = \{a^3, a^6, a^9, \ldots\}$. Imposing the identity $xy = x$ on this set we get $a^3 \equiv_{\theta_{\phi'}} a^6 \equiv_{\theta_{\phi'}} a^9 \cdots$. It follows that when working in the category S_A, $V ⓜ S$ is the 5 element cyclic semigroup generated by $a\phi'_{M(V)}$ with $a^3\phi'_{M(V)} = a^6\phi'_{M(V)}$. Note that this semigroup is not a monoid.

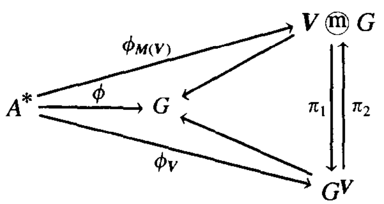
$1\phi^{-1} = \{\varepsilon, a^3, a^6, a^9, \ldots\}$, and imposing the identity $xy = y$ on this set implies $\varepsilon \equiv_{\theta_\phi} a^3 \equiv_{\theta_\phi} a^6 \cdots$. It follows that when working inside the category M_A, $V ⓜ S$ is the 3 element monoid $\{\varepsilon\phi_{M(V)}, a\phi_{M(V)}, (a^2)\phi_{M(V)}\}$, with $(a^3)\phi_{M(V)} = \varepsilon\phi_{M(V)} = 1$. In this case $V ⓜ S \cong S$.

6.2. The equivalence on groups

Theorem 6.4. *If (G,ϕ) is an object of M_A, G a group, then $(G^V, \phi_V) \cong (V ⓜ G, \phi_{M(V)})$.*

Proof. Let $\varepsilon_G : G^V \to G$ be as in Section 2.2. The only idempotent in G is 1. Therefore to show that G^V is a Malcev product of G over V, we only need show that $1\varepsilon_G^{-1}$ is in V. Let $u, v \in A^*$ be such that $u\phi = v\phi = 1$, i.e. using the notation of Section 2.1 $u\phi_V = (f_u, 1)$ and $v\phi_V = (f_v, 1)$. By Corollary 2.4 $(f_u, 1) = (f_v, 1)$ if and only if $f_u(1) = f_v(1)$. Now $(f_u, 1)(f_v, 1) = (f_u{}^1 f_v, 1) = (f_{uv}, 1)$. So $1\varepsilon_G^{-1}$ is isomorphic to the local monoid of $D = D_\phi/\pi_\phi$ at 1. The local monoids of D are in V and thus so is $1\varepsilon_G^{-1}$. It follows that there is a surjective homomorphism $\pi_1 : (V ⓜ G, \phi_{M(V)}) \to (G^V, \phi_V)$.

Let $g \in G$ and $t \in V ⓜ G$. If $g(t)\phi_{M(V)} = g$, then $(t)\phi_{M(V)} = 1$ and the local monoid of $D_{\phi_{M(V)}}$ at g is a homomorphic image of $1\phi_{M(V)}^{-1}$. As $1\phi_{M(V)}^{-1} \in V$, and varieties are closed under homomorphic images, we see that the local monoids of $D_{\phi_{M(V)}}$ are in V. By Theorem 2.8 there is a surjective homomorphism $\pi_2 : (G^V, \phi_V) \to (V ⓜ G, \phi_{M(V)})$, and we have the following commutative diagram.



It follows that π_1 and π_2 are inverses. □

6.3. Expanding groups by bands

In Section 3.1 the Rhodes expansion $S^{\hat{}L}$ was defined. There is an analogous Rhodes expansion, $S^{\hat{}R}$, using the $\mathscr{R}$ order instead of the $\mathscr{L}$ order of S. That is, if we let R be the set of all strict $\mathscr{R}$-chains of S, i.e. $R = \{s_1 > s_2 \cdots s_{n-1} > s_n : s_1, \dots, s_n \in S\}$, where $s > s'$ means s is strictly above s' in the $\mathscr{R}$ order, then there is an associative multiplication on R given by

$$(s_1 > \cdots > s_n)(t_1 > \cdots > t_m) = Red(s_1 > \cdots > s_n \geq s_n t_1 \geq \cdots \geq s_n t_m)$$

where $Red(s_1 > \cdots > s_n \geq s_n t_1 \geq \cdots \geq s_n t_m)$ is the new chain you get by using the rule: whenever there is a string of $\mathscr{R}$ equivalent elements keep only the right most element of the string.

The Birget–Rhodes expansion, cut to generators and considered as a monoid expansion, is the projective limit of S, $S^{\hat{}L}$, $S^{\hat{}L\hat{}R}$, $S^{\hat{}L\hat{}R\hat{}L}, \cdots$ in the category $\boldsymbol{M_A}$.

$$\begin{aligned}(S^{\mathrm{BR}}, \phi_{\mathrm{BR}}) &= \lim(\cdots \to (S^{\hat{}L\hat{}R}, \phi^{\hat{}L\hat{}R}) \to (S^{\hat{}L}, \phi^{\hat{}L}) \to (S, \phi)) \\ &= \lim(\cdots \to (S^{\hat{}R\hat{}L}, \phi^{\hat{}R\hat{}L}) \to (S^{\hat{}R}, \phi^{\hat{}R}) \to (S, \phi)).\end{aligned}$$

Let (G, ϕ) be an element of $\boldsymbol{M_A}$ with G a group. Let $\Gamma(G, A)$ be the Cayley graph of G, and let E be the set of edges of $\Gamma(G, A)$. Let $\boldsymbol{B}$ denote the semigroup variety of bands (semigroups satisfying the identity $x^2 = x$) and let $FB(E)$ be the free band on E.

A variety V of semigroups is called local if for any category C whose local monoids are in V there is some $S\in V$ such that $C\prec S$.

Theorem 6.5. *The variety of bands is local.*

A proof of this theorem can be found in [5, 12]. Let C be a category whose local monoids are bands. If X is a subgraph of C that generates C, and F is the free band on the edges of X, then a path in X corresponds to an element of F and also gives rise to an arrow in C. Define R from the arrows of C to subsets of F by $R(\alpha)$ equals the set of all elements of F which correspond to some path in X that gives rise to the arrow α.

Corollary 6.6. *R as defined above is a division.*

Let $G^{\mathrm{C,BR}}$ denote the image of first applying the Cayley expansion to G and then following with the Birget–Rhodes expansion. The corresponding homomorphism will be denoted $\phi_{\mathrm{C,BR}}$.

Lemma 6.7. *Let $\theta:(G^{\mathrm{C,BR}},\phi_{\mathrm{C,BR}})\to(G,\phi)$ be the natural homomorphism in M_A. The local monoids of D_θ are bands.*

Proof. The only right stabilizer in G is 1. The inverse image of 1 under the Cayley expansion is a band. An inverse image of an idempotent under the Birget–Rhodes expansion is idempotent. So $1\theta^{-1}$ is a band. If $[g_1,u\phi_{\mathrm{C,BR}},g_1]$ is an element of the local monoid $D_\theta(g_1,g_1)$ then $u\phi=1$. Thus $u\phi_{\mathrm{C,BR}}$ is in $1\theta^{-1}$ and therefore is idempotent. □

Theorem 6.8. *The largest Malcev product of a group over the variety of bands is the same as expanding the group maximally such that the derived category is locally in the variety of bands, and both are the same as first applying the Cayley expansion and then the Birget–Rhodes expansion. That is $(\boldsymbol{B}\,ⓜ\,G,\phi_{M(\mathrm{B})})\cong(G^{\mathrm{B}},\phi_{\mathrm{B}})\cong(G^{\mathrm{C,BR}},\phi_{\mathrm{C,BR}})$.*

Before giving the proof of this theorem we will establish a couple of useful lemmas. Let $c(u)$, called the content of u, be the set of edges appearing in P_u, where $u\phi_C=(P_u,u\phi)$. Let θ_R be the natural map from A^* onto $(G^C)^{\hat{}R}$ and let θ_L be the natural map from A^* onto $(G^C)^{\hat{}L}$. It is easy to check that $u\phi_C>(ua)\phi_C$ in the $\mathscr{R}$-order if and only if $c(u)$ is strictly smaller than $c(ua)$. Likewise, $(au)\phi_C<u\phi_C$ in the $\mathscr{L}$-order if and only if the size of $c(u)$ is less than the size of $c(au)$.

Lemma 6.9. *Let $u=a_1\dots a_m$ and $v=b_1\dots b_p$ be in A^* such that*

$$u\theta_R=(P_1,g_1)>\cdots>(a_1\dots a_j)\phi_C>u\phi_C$$

and

$$v\theta_R=(P_2,g_2)>\cdots>(b_1\dots b_{j'})\phi_C>v\phi_C.$$

If $u\phi_{\mathrm{C,BR}} = v\phi_{\mathrm{C,BR}}$ *then* $(a_1 \dots a_j)\phi_{\mathrm{C,BR}} = (b_1 \dots b_{j'})\phi_{\mathrm{C,BR}}$. *Dually, if*

$$u\theta_L = (a_1 \dots a_m)\phi_C < (a_k \dots a_m)\phi_C < \cdots < (P_1', g_1')$$

and

$$v\theta_L = (b_1 \dots b_p)\phi_C < (b_{k'} \dots b_p)\phi_C < \cdots < (P_2', g_2')$$

then $u\phi_{\mathrm{C,BR}} = v\phi_{\mathrm{C,BR}}$ *implies* $(a_k \dots a_m)\phi_{\mathrm{C,BR}} = (b_{k'} \dots b_p)\phi_{\mathrm{C,BR}}$.

Proof. Let $\theta_1 = \theta_R$, θ_2 be the natural map from A^* onto $(G^C)^{\hat{}R\hat{}L\hat{}R}$, and in general let θ_n be the natural map from A^* onto $(G^C)^{\hat{}R\hat{}L\hat{}R\cdots\hat{}L\hat{}R}$, where $\hat{}R$ has been applied n times. Let ψ_1 be the natural map from A^* onto $(G^C)^{\hat{}R\hat{}L}$, and in general let ψ_n be the natural map from A^* onto $(G^C)^{\hat{}R\hat{}L\cdots\hat{}R\hat{}L}$, where $\hat{}L$ has been applied n times.

If $(a_1 \dots a_j)\phi_{\mathrm{C,BR}} \neq (b_1 \dots b_{j'})\phi_{\mathrm{C,BR}}$ there must be some n such that $(a_1 \dots a_j)\psi_n \neq (b_1 \dots b_{j'})\psi_n$. As $(a_1 \dots a_j)\phi_C > (a_1 \dots a_j a_{j+1})\phi_C$ and there is a homomorphism from $(G^C)^{\hat{}R\hat{}L\dots\hat{}R\hat{}L}$ onto G^C, we have $(a_1 \dots a_j)\psi_n > (a_1 \dots a_j a_{j+1})\psi_n$, and

$$u\theta_{n+1} = (a_1 \dots a_{j_1})\psi_n > \cdots > (a_1 \dots a_j)\psi_n > \cdots > u\psi_n.$$

Similarly

$$v\theta_{n+1} = (b_1 \dots b_{j_1'})\psi_n > \cdots > (b_1 \dots b_{j'})\psi_n > \cdots > v\psi_n.$$

As $u\phi_{\mathrm{C,BR}} = v\phi_{\mathrm{C,BR}}$, $u\theta_{n+1} = v\theta_{n+1}$. So $(a_1 \dots a_j)\psi_n = (b_1 \dots b_i)\psi_n$ for some i. If $i > j'$ then $c(b_1 \dots b_i) = c(u) = c(v)$, which is strictly greater than $c(a_1 \dots a_j)$. This is a contradiction as $(a_1 \dots a_j)\psi_n = (b_1 \dots b_i)\psi_n$ implies $(a_1 \dots a_j)\phi_C = (b_1 \dots b_i)\phi_C$. If $j' > i$ then $(b_1 \dots b_{j'})\phi_n = (a_1 \dots a_{i'})\phi_n$ for some $i' > j$, and we get a similar contradiction. The proof of the second part is gotten by interchanging the roles of $\mathscr{R}$ and $\mathscr{L}$ above. $\square$

For $u = a_1 a_2 \dots a_n \in A^*$ define $\tilde{u} \in FB(E)$ in the following way. Let $(P_0, 1) = \varepsilon\phi_C$ and $(P_i, (a_1 \dots a_i)\phi) = (a_1 \dots a_i)\phi_C$. Denote by e_i "the last edge drawn" when drawing P_i, i.e. $e_i = [(a_1 \dots a_{i-1})\phi, a_i, (a_1 \dots a_i)\phi]$. Note that if $P_i \neq P_{i-1}$ then e_i is exactly the edge that they differ by. Let $\tilde{u} = e_1 e_2 \dots e_n$.

Lemma 6.10. *Let* $u, v \in A^*$. *Then* $u\phi_{\mathrm{C,BR}} = v\phi_{\mathrm{C,BR}}$ *implies* $\tilde{u} = \tilde{v}$.

Proof. If $u\phi_{\mathrm{C,BR}} = v\phi_{\mathrm{C,BR}}$ then $u\phi_C = v\phi_C$ and $c(u) = c(v)$. We will induct on the size of $c(u)$. Note that $|c(u)| = 0$ if and only if u is the empty word.

$|c(u)| = 1$: If $|c(u)| = 1$ then $u = a^r$ and $v = a^s$ for some $a \in A$. If either r or s is greater than 1 then $[1, a, 1]$ is a loop, and $\tilde{u} = \tilde{v} = \tilde{a}$.

$|c(u)| = n$: Let $u = a_1 \dots a_m$, $\tilde{u} = e_1 \dots e_m$, $v = b_1 \dots b_p$, and $\tilde{v} = f_1 \dots f_p$. Let $u_1 = a_1 \dots a_j$ be the largest initial segment of u whose content is strictly smaller than that of u. That is, $\{e_1, \dots, e_j\} \subset c(u)$, but is not equal to $c(u)$. Let $x_u = e_{j+1}$ be the single member of $c(u) \backslash c(u_1)$. Let $u_2 = a_i \dots a_m$ be the largest terminal segment of u such that $\{e_i, \dots, e_m\}$ is strictly smaller than $c(u)$. Let $y_u = e_{i-1}$. Define similarly $v_1 = b_1 \dots b_{j'}$,

$x_v = f_{j'+1}$, $v_2 = b_{i'} \dots b_p$, and $y_v = f_{i'-1}$. By a well known property of bands, see [4], $\tilde{u} = e_1 \dots e_j x_u y_u e_i \dots e_m$, and $\tilde{v} = f_1 \dots f_{j'} x_v y_v f_{i'} \dots f_p$.

As you drop in the $\mathscr{R}$-order in G^C if and only if you see a new edge,

$$u\theta_R = s_1 > \cdots > (a_1 \dots a_j)\phi_C > u\phi_C$$

and

$$v\theta_R = s_1 > \cdots > (b_1 \dots b_{j'})\phi_C > v\phi_C.$$

By Lemma 6.9 and the inductive hypothesis, we have $e_1 \dots e_j = f_1 \dots f_{j'}$, and as $P_u = P_v$ we must also have $x_u = x_v$.

Recall that there is a left action of G on $\Gamma(G,A)$, as defined in Section 3.2. As G is a group $(a_1 \dots a_{i-1})\phi$ acts as a graph isomorphism. If $(P,g) = u_2\phi_C$ and $(P',g') = (a_{i-1}u_2)\phi_C$, then $\{e_i, \dots, e_m\}$ are exactly the edges appearing in $(a_1 \dots a_{i-1})\phi \cdot P$ and $P_u = (a_1 \dots a_{i-2})\phi \cdot P'$. So similarly we have

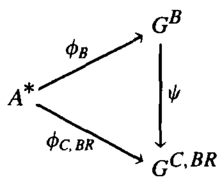
$$u\theta_L = u\phi < (a_i \dots a_m)\phi < \cdots < t_1$$

and

$$v\theta_L = v\phi < (b_{i'} \dots b_p)\phi < \cdots < t_1.$$

Again by Lemma 6.9 and the inductive hypothesis, $\tilde{u_2} = \tilde{v_2}$ and $y_u = y_v$. As G is a group and $(a_i \dots a_m)\phi = (b_{i'} \dots b_p)\phi$, $(a_1 \dots a_{i-1})\phi = (b_1 \dots b_{i'-1})\phi$. Let $\tilde{u_2} = e'_i \dots e'_m$ and $\tilde{v_2} = f'_{i'} \dots f'_p$. As G acts as isomorphisms, $e'_k = f'_l$, $k \geq i$ and $l \geq i'$, implies $e_k = f_l$, and if $e'_k = e'_l$, for $k, l \geq i$, then $e_k = e_l$, and similarly for the f_k. Hence $e_i \dots e_m = f_{i'} \dots f_p$ and $\tilde{u} = \tilde{v}$ as desired. □

Proof of Theorem 6.8. The first isomorphism is Theorem 6.4 with $\boldsymbol{V} = \boldsymbol{B}$. It follows from Lemma 6.7 that there exists a homomorphism ψ making the following diagram commute.



By Corollary 2.4 we know that $u\phi_{\mathrm{B}} = (f_u, g_1) = v\phi_{\mathrm{B}} = (f_v, g_2)$ if and only if $f_u(1) = f_v(1)$. Now $f_u(1)$ and $f_v(1)$ are arrows in $D = D_\phi / \pi_\phi$, which we consider as equivalence classes of paths in $\Gamma(G,A)$. Lemma 6.10 showed that $u\phi_{\mathrm{C,BR}} = v\phi_{\mathrm{C,BR}}$ implies $\tilde{u} = \tilde{v}$. It follows from Corollary 6.6 that $u\phi_{\mathrm{B}} = v\phi_{\mathrm{B}}$ whenever $\tilde{u} = \tilde{v}$. □

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