Nest-Subalgebras of von Neumann Algebras*

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The structure of a certain class of separably acting reflexive operator algebras is investigated for which the nest algebras of J. Ringrose can be considered prototypes. To a fixed von Neumann algebra and a complete nest of projections contained therein one associates the algebra of all operators in the von Neumann algebra which leave every member of the nest invariant. A generalization of the Ringrose criterion for inclusion in the Jacobson radical of a nest algebra is given for this more general class of algebras. Further properties of the radical are studied.

INTRODUCTION

R. Kadison and I. M. Singer [9] introduced their theory of triangular operator algebras in 1959, and in 1963 J. Ringrose [14] introduced a theory of reflexive operator algebras with totally ordered invariant subspace lattices which he called nest algebras. A nest is a family of closed subspaces of a Hilbert space totally ordered by inclusion, and the associated nest algebra is the class of all bounded linear operators from the space into itself which leave invariant each member of the nest. These include the most tractable of the maximal triangular algebras, the hyper-intransitive ones, and can be regarded as prototypes of certain more general classes of reflexive operator algebras.

A natural generalization of Ringrose's original concept of nest algebra is that of a nest-subalgebra of a von Neumann algebra. In precise terms, to a fixed separably acting von Neumann algebra \( \mathcal{B} \) and a complete nest \( \mathcal{N} \) of projections contained in \( \mathcal{B} \) one associates the algebra \( \mathcal{A} \) of all bounded linear operators in \( \mathcal{B} \) which leave invariant every member of \( \mathcal{N} \). So \( \mathcal{A} = \mathcal{B} \cap \mathcal{A}_\mathcal{N} \), where \( \mathcal{A}_\mathcal{N} \) denotes the nest algebra of \( \mathcal{N} \) in \( L(H) \). \( \mathcal{A} \) is then a reflexive operator algebra with invariant subspace lattice equal to the reflexive lattice generated by \( \mathcal{N} \) together with the projections in the commutant of \( \mathcal{B} \) in \( L(H) \). This lattice is often highly noncommutative, but results show it is perhaps the most tractable of the noncommutative noncommplemented subspace lattices.

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A principal result of Ringrose [14] is an operator-theoretic characterization of the Jacobson radical of an arbitrary nest algebra (the Ringrose criterion) which has proven to be the key to much of the subsequent work accomplished in this area. Known results for nest algebras lead naturally to questions concerning this more general class. The study of such an algebra, and in particular its Jacobson radical, is at least in part motivated by the success enjoyed by Ringrose and others in their study of the corresponding properties of nest algebras. Also, applications of the theory of nest algebras are likely to have extensions to this more general setting, at least in certain cases. We remark that this study makes essential contact with certain aspects of the work of Loebl and Muhly [11], in which they showed that nest-subalgebras of von Neumann algebras are precisely the algebras of analytic operators with respect to certain one parameter groups of inner *automorphisms of the von Neumann algebras.

While nest-subalgebras of von Neumann algebras are natural generalizations of nest algebras they frequently bear little resemblance to their prototypes. This is most easily seen by observing that (countable) direct sums of nest algebras are members of this class. Thus while extensions of results for nest algebras to this class are sometimes possible, they are often nontrivial (e.g., the work on direct sums of nest algebras in [7]).

This paper is organized in the following fashion. Section 1 contains preliminaries and Section 2 concerns basic properties of nest-subalgebras of von Neumann algebras. In Section 3 we establish elements of a direct integral decomposition theory for separably acting commutative subspace lattices as needed in this paper. In Section 4 we examine certain properties of nest-subalgebras of von Neumann algebras via the decomposition of the algebra along the center of the von Neumann algebra. This section contains a simple criterion which specifies precisely when a given separably acting commutative subspace lattice is the invariant subspace lattice of some nest-subalgebra of a von Neumann algebra.

In Section 5 a version of the Ringrose criterion is established (5.7) valid for arbitrary nest-subalgebras of von Neumann algebras. The result is first established for nest-subalgebras of factors, and then "lifted" to the appropriate version for the general case. This result is then used in Section 6 where we investigate the relationship between rad \( \mathcal{A} \) and rad \( \mathcal{A}_f \). It is shown that given an arbitrary countable subset \( S \) of rad \( \mathcal{A} \) there always exists a nest \( \mathcal{K}_0 \) in lat \( \mathcal{A} \) with the property that rad \( \mathcal{A}_{f_0} \) contains \( S \). So every countably generated topologically nil ideal in \( \mathcal{A} \) is contained in the radical of some nest algebra relative to \( L(H) \). It is also shown that in all but the simplest cases the ideal rad \( \mathcal{A} \) is never entirely contained in the radical of any single nest algebra with nest in lat \( \mathcal{A} \).
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1. Preliminaries

Throughout this paper, all Hilbert spaces discussed will be separable, all operators will be bounded, all subspaces will be closed, and all projections will be self-adjoint. Let $H$ be a Hilbert space. We write $L(H)$ for the collection of all bounded operators on $H$, $C(H)$ for the unit ball of $L(H)$ (i.e., the set of contractions on $H$), and $P(H)$ for the lattice of projections in $L(H)$. The space $C(H)$ is to be regarded as equipped with the strong operator topology and the Borel structure subordinate to it; this makes $C(H)$ into a complete metric space.

Let $\mathcal{L}$ be a collection of subspaces containing $\{0\}$ and $H$ which form a lattice under the operations $\vee$ and $\wedge$, where $M \vee N$ is the subspace generated by $M$ and $N$ while $M \wedge N$ is the intersection $M \cap N$. $\mathcal{L}$ is commutative if the projections on the subspaces commute pairwise. $\mathcal{L}$ is a nest (usually denoted by $\mathcal{N}$) if the lattice is linearly ordered by inclusion; thus a nest is commutative.

For convenience we shall disregard the distinction between a subspace of $H$ and the orthogonal projection on it. Thus a lattice will consist of either subspaces or projections depending on the context in which it is used. As usual we write $\text{Lat} \ S$ for the lattice of all projections left invariant under every operator in a subset $S$ of $L(H)$, and dually $\text{Alg} \ Q$ denotes the algebra of all operators leaving each projection in a subset $Q$ of $P(H)$ invariant. The term subspace lattice will denote a strongly closed lattice of projections containing 0 and I. Unless otherwise stated, all lattices will be strongly closed and the term algebra will refer to a strongly closed operator algebra with identity. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = \text{Alg} \ \text{Lat} \mathcal{A}$, and dually a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = \text{Lat} \ \text{Alg} \mathcal{L}$. Subspace lattices need not be reflexive; however, commutative subspace lattice are reflexive [1]. Strongly closed lattices are complete as lattices, and the converse is true for commutative lattices; arbitrary complete lattices need not be strongly closed, however [18, 12].

Let $\mathcal{N}$ be a complete nest of projections. We use $\leq$ to denote inclusion, $\subset$ being reserved for proper inclusion. If $N \in \mathcal{N}$, $N \neq 0$, write $N_0 = \bigvee \{L \in \mathcal{N}: L < N\}$, and set $N_0 = 0$ for $N = 0$. If $N_0 \neq N$ we term $N_0$ the immediate predecessor of $N$ in $\mathcal{N}$. Also, if $N \neq I$ write $N_+ = \bigwedge \{M \in \mathcal{N}: M > N\}$, and set $N_+ = I$ for $N = I$. If $N_+ \neq N$ we term $N_+$ the immediate successor of $N$. If $\mathcal{A}$ is a nest we use the notation $\mathcal{A}_+$ to denote the nest algebra $\text{Alg} \mathcal{A}_+$. The core $\mathcal{C}_+$ is the von Neumann algebra generated by the projections in $\mathcal{A}_+$.

The Jacobson radical of an arbitrary algebra is defined to be the intersection of the kernels of all strictly transitive representations of the algebra [13]. A right or left ideal in a Banach algebra $\mathcal{A}$ is (nil, topologically nil) if each of its elements is (nilpotent, quasinilpotent). The following theorem is a
statement of well known properties of the radical of a Banach algebra (See, e.g., [13, p. 57].)

**Theorem 1.1.** The radical $R$ of a Banach algebra $\mathcal{A}$ is a closed two-sided topologically nil ideal; $R$ contains every topologically nil left or right ideal in $\mathcal{A}$.

If $\mathcal{A}$ is a Banach algebra with identity then it is apparent from the above theorem that

$$R = \{B \in \mathcal{A}: AB \text{ is quasinilpotent, } A \in \mathcal{A}\}$$

$$= \{B \in \mathcal{A}: BA \text{ is quasinilpotent, } A \in \mathcal{A}\}.$$  

Also, if $\sigma(A)$ denotes the spectrum of $A$ in $\mathcal{A}$ then $\sigma(A + B) = \sigma(A)$, $A \in \mathcal{A}$, $B \in R$.

We will usually denote the radical of an algebra $\mathcal{A}$ by $\text{rad } \mathcal{A}$.

If $\mathcal{N}$ is a nest of projections then a projection $E$ is an $\mathcal{N}$-interval if $E$ is of the form $E = M - N$, where $M, N \in \mathcal{N}$, $M > N$. The projections $M, N$ are called the upper and lower endpoints of $E$, respectively. The endpoints of a nonzero $\mathcal{N}$-interval are well defined, for suppose $E = M_i - N_i$, $i = 1, 2$, and $E \neq 0$. Then $M_1 > E$ and $N_2 \perp E$ implies $M_1 > N_2$; hence $M_1 > M_2$. Similarly $M_2 > M_1$. Thus $M_1 = M_2$, so also $N_1 = N_2$. A projection is simple if it is a finite sum of mutually orthogonal $\mathcal{N}$-intervals.

In this original paper on nest algebras [14] Ringrose presented an operator-theoretic characterization of the Jacobson radical of an arbitrary nest algebra (the Ringrose criterion) which has proven to be the key to much of the subsequent work accomplished in this area. Ringrose actually presented two such criteria, equivalent for nest algebras, each of which is useful to us in its proper context. For this reason we present these here. Let $R_{+}$ denote $\text{rad } \mathcal{A}_+$. 

**Theorem 1.2** (Ringrose). If $A \in \mathcal{A}_+$, then $A \in R_{+}$ if and only if both of the following are satisfied:

(i) Given $N \in \mathcal{N}$ with $N \neq 0$ and given $\varepsilon > 0$ there exists $L \in \mathcal{N}$ such that $L < N$ and $\| (N - L) A (N - L) \| < \varepsilon$.

(ii) Given $N \in \mathcal{N}$ with $N \neq 1$ and given $\varepsilon > 0$ there exists $M \in \mathcal{N}$ such that $N < M$ and $\| (M - N) A (M - N) \| < \varepsilon$.

**Theorem 1.3** (The Ringrose criterion). If $A \in \mathcal{A}_+$, then $A \in R_{+}$ if and only if for each $\varepsilon > 0$ there exists a finite set $\{E_i\}$ of mutually orthogonal $\mathcal{N}$-intervals with $\sum E_i = I$ such that $\|E_i A E_i\| < \varepsilon$, all $i$. 
2. Basic Properties

Let \( \mathcal{B} \) be a von Neumann algebra contained in \( L(H) \) and let \( \mathcal{A}' \) be a complete nest of projections contained in \( \mathcal{B} \). Let \( \mathcal{N} \) denote the nest algebra \( \text{alg} \mathcal{A}' \) and let \( \mathcal{A}' = \mathcal{N} \cap \mathcal{A}' \) denote the nest-subalgebra of \( \mathcal{B} \) relative to \( \mathcal{A}' \). In addition, let \( \mathcal{M} \) denote the lattice of projections in the commutant of \( \mathcal{B} \). Then \( \mathcal{A}' \) is reflexive, and if \( \mathcal{N}' \mathcal{M} \) denotes the smallest strongly closed lattice containing \( \mathcal{N} \mathcal{M} \) then \( \mathcal{A}' = \text{alg}(\mathcal{N}' \mathcal{M}) \). It is not known whether \( \mathcal{N}' \mathcal{M} \mathcal{N} \mathcal{M} \) is reflexive in general: i.e., is the join of the projection lattice of a von Neumann algebra with a complete nest in its commutant necessarily reflexive? We present certain basic properties of nest-subalgebras of von Neumann algebras relevant to the sequel. Our terminology is that established above.

**Lemma 2.1.** \( \mathcal{A}' \) contains a m.a.s.a. iff \( \mathcal{M} \) contains a m.a.s.a.

**Proof:** \( \mathcal{A}' \) contains a m.a.s.a. iff \( \text{lat} \mathcal{A}' \) is commutative iff \( \mathcal{M} \) is abelian iff \( \mathcal{A}' \) contains a m.a.s.a. \( \mathcal{M} \) contains a m.a.s.a.

So there is no ambiguity in speaking of a nest-subalgebra of a von Neumann algebra which contains a m.a.s.a. These algebras are studied in depth in Section 4.

A nest algebra is never semisimple unless its nest is the trivial lattice \( \{0, H\} \). The following observation will prove useful.

**Lemma 2.2.** Let \( \mathcal{N} \) be a nest in a Hilbert space \( H \), and let \( N \) be an arbitrary member of \( \mathcal{N} \). If \( A \) is an operator with support in \( N' \) and range in \( N \) then necessarily \( A \in \mathcal{A}' \). In fact \( A \) is in the radical of \( \mathcal{A}' \).

**Proof:** Fix \( M \in \mathcal{A}' \). If \( M \subseteq N \) then \( AM = (0) \subseteq M \). If \( M \supseteq N \), then \( AM \subseteq N \subseteq M \). So \( AM \subseteq M \) for every \( M \in \mathcal{N} \). So \( A \in \mathcal{A}' \). That \( A \in \text{rad}(\mathcal{A}') \) now follows from the fact that the projection \( P_N \) from \( H \) onto \( N \) is contained in \( \mathcal{A}' \), and its image under any transitive representation of \( \mathcal{A}' \), must be either 0 or the identity.

A nest subalgebra of a von Neumann algebra usually has nonzero radical. We state this formally.

**Lemma 2.3.** A nest-subalgebra of a von Neumann algebra is never semisimple except in the case in which it is a von Neumann algebra. (This case occurs iff \( \mathcal{N}' \) consists of central projections of \( \mathcal{B} \).)

**Proof:** In the terminology of (2.1), clearly \( \mathcal{A}' = \mathcal{B} \) iff \( \mathcal{N}' \) is contained in the center of \( \mathcal{B} \). We remark that every self-adjoint algebra is semisimple [13]. On the other hand, if some projection \( N \in \mathcal{N} \), were not central then...
some nonzero $NBN^\perp$ is in $\mathcal{O}_\gamma$, by Lemma 2.2, and is also in $\mathcal{B}$, so $NBN^\perp \subseteq \mathcal{O}$. Also by (2.2) we have $NBN^\perp \subseteq \text{rad}(\mathcal{O}_\gamma)$, and since the radical is a two-sided ideal this implies $NBN^\perp \subseteq \text{rad}(\mathcal{O})$.

The above lemma shows that the radical plays a definite role in the structure of such an algebra. However, while $\mathcal{O} \subseteq \mathcal{O}_\gamma$ implies $\text{rad}(\mathcal{O}_\gamma) \subseteq \text{rad}(\mathcal{O})$, it does not in general follow that $\text{rad}(\mathcal{O}) \subseteq \text{rad}(\mathcal{O}_\gamma)$. Thus rad($\mathcal{O}_\gamma$) cannot be studied in general via the Ringrose criterion for rad($\mathcal{O}_\gamma$), and its structure is often considerably more complex than that of rad($\mathcal{O}_\gamma$). This is studied in detail in Sections 5 and 6. The pathology is illustrated by the following example.

**Example 2.4.** Let $\{\mathcal{N}_n\}$ be a countable family of nontrivial nests on Hilbert spaces $\{H_n\}$. Let $H = \bigoplus H_n$, and let $\mathcal{L}$ be the direct sum of the nests $\Sigma \oplus \mathcal{N}_n$. (That is, all projections of the form $P_1 \oplus P_2 \oplus \cdots \oplus P_i \in \mathcal{N}_i$.) Then $\mathcal{O} = \text{alg} \mathcal{L}$ is the direct sum of the nest algebras $\mathcal{O}_\gamma$. Let $E_n$ be the orthogonal projection onto $H_n$ for each $n$, and let $B$ be the commutant of the set $\{E_n : 1, 2, \ldots\}$. That is, $B = \Sigma \oplus \mathcal{L}(H_n)$. Let $\mathcal{N}$ be the ordinal sum of the $\mathcal{N}_n$ acting on $H$. That is, $\mathcal{N}$ is the nest on $H$ consisting of 0 and 1 together with all projections of the form $E_1 \oplus E_2 \oplus \cdots \oplus E_{n-1} \oplus P \oplus 0 \oplus 0 \cdots$ for $P \in \mathcal{N}_n$, for each $n = 1, 2, \ldots$. Then $\mathcal{O} = B \cap \mathcal{O}_\gamma$, so $\mathcal{O}$ is a nest-subalgebra of a von Neumann algebra. Now for each $n$ let $P_n$ be a nontrivial member of $\mathcal{N}_n$, and let $S_n$ be a partial isometry on $H_n$ with support in $(E_n - P_n)$ and range in $P_n$. Then $S_n \in \mathcal{O}_\gamma$, for each $n$ by (2.2) so that the direct sum $S = \Sigma \oplus S_n$ is in $\mathcal{O}$. It is easily seen that for each $A \in \mathcal{O}$ the product $SA$ is nilpotent of index 2, so $S \in \text{rad}(\mathcal{O})$. Now for each $n \geq 2$ let $V_n$ be a partial isometry in $L(H)$ with support in $\text{rang}(S_n)$ and range in $\text{sup}(S_{n-1})$. Then $V_n \in \mathcal{O}_\gamma$, by (2.2) and the sum $V = \sum V_n$ converges in the strong operator topology and is in $\mathcal{O}_\gamma$. However, $(VS)^n$ has norm one for all $n$ so $S$ is not in rad($\mathcal{O}_\gamma$). Thus rad($\mathcal{O}$) is not contained in rad($\mathcal{O}_\gamma$). We remark that the radical of a direct sum of nest algebras has been characterized in [7].

**Remark.** The above can be considered to be the discrete case in which $\mathcal{B}$ is the commutant of a purely atomic abelian von Neumann algebra. In other cases we must employ direct integral reduction techniques to study properties of nest-subalgebras of von Neumann algebras in terms of known properties of nest algebras.

**Remark 2.5.** A nest-subalgebra of a von Neumann algebra is always "large" in that it generates the von Neumann algebra it is defined in terms of. Indeed, the work of [11] shows that $\mathcal{O} + \mathcal{O}^*$ is ultraweakly dense in $\mathcal{B}$. This removes any ambiguity in the choice of $\mathcal{B}$ in the description of $\mathcal{O}$. (The
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nest \cdot f^* can often be chosen in different ways, however.) The following observation proves useful.

**Lemma 2.6.** Let \( \mathcal{N} \) be the nest-subalgebra of the von Neumann algebra \( \mathcal{B} \) with respect to a nest \( N \). Then the center of \( \mathcal{N} \) is equal to the center of \( \mathcal{B} \).

**Proof.** Clearly \( \text{cent}(\mathcal{N}) \supseteq \text{cent}(\mathcal{B}) \). For the converse, if \( A \in \text{cent}(\mathcal{N}) \) then \( A \) commutes with \( N \) since \( N \subseteq \mathcal{N} \) so \( A \in \mathcal{N} \cap \mathcal{N}^* \). In particular \( A^* \in \mathcal{N} \) also. Thus \( A \) commutes with its adjoint so it is normal. So by Fuglede's theorem \( A \) commutes with \( \mathcal{N}^* \) as well as \( \mathcal{N} \); hence it commutes with \( \mathcal{B} \) since \( \mathcal{N} + \mathcal{N}^* \) is ultraweakly dense in \( \mathcal{B} \).

## 3. Decomposition of Subspace Lattices

The purpose of this section is to establish the notion of a direct integral decomposition of a commutative subspace lattice which will be useful in investigating nest-subalgebras of von Neumann algebras. There is a natural topological criterion for a field of commutative subspace lattices to be attainable as the direct integral decomposition of a subspace lattice and in particular we show that a direct integral of nests is realizable as the decomposition of a nest lattice. The consideration of more general lattices is not undertaken here for two reasons. First, the applications we have in mind in this paper do not need a more general theory, and second, certain fundamentally different problems arise which we need not consider in the commutative case. Arbitrary nest-subalgebras of von Neumann algebras can be dealt with via the commutative theory, even though their invariant subspace lattice are generally noncommutative.

The ideas in this section parallel the notion of the direct integral of strongly closed algebras and for convenience and notation we briefly mention some germane results and definitions. For details of direct integral theory as regards von Neumann algebras we refer to [4, 16]. We begin by fixing, once and for all, a sequence of Hilbert spaces \( h_1 \subseteq h_2 \subseteq \cdots \subseteq h_\infty \) with \( h_n \) having dimension \( n \) and \( h_\infty \) spanned by the remaining \( h_n \)'s. Next, suppose we have a partitioned measure space \((A, \mu, \{e_\alpha\})\). This means \( A \) is a separable metric space, \( \mu \) is (the completion of) a regular Borel measure on \( A \) and \( \{e_\alpha\} \) is a Borel partition of \( A \); we also assume that \( \mu \) is \( \sigma \)-finite and \( A \) is almost \( \sigma \)-compact. Then we form the associated direct integral Hilbert space \( H = \int_A \oplus h(\lambda) \mu(d\lambda) \). This consists of all (equivalence classes of) measurable functions \( f \) from \( A \) into \( h_\infty \) such that \( f(\lambda) \in h(\lambda) = h_n \) for \( \lambda \in e_n \) and \( \int_A \| f(\lambda) \|^2 \mu(d\lambda) < \infty \). The element in \( H \) represented by the function \( \lambda \to f(\lambda) \) is denoted by \( \int_A \oplus f(\lambda) \mu(d\lambda) \).

An operator \( A \) on \( H \) is said to be decomposable if there exists a strongly \( \mu \)-
measurable operator-valued function $A(\cdot)$ defined on $\mathcal{A}$ such that $A(\lambda)$ is an operator on $h(\lambda)$ and for $f \in H$, $Af(\lambda) = A(\lambda)f(\lambda)$. We write $A = \int_A \oplus A(\lambda) \mu(d\lambda)$ for the equivalence class corresponding to $A(\lambda)$. If $A(\cdot)$ is a scalar multiple of the identity on $h(\lambda)$ for almost all $\lambda$, then $A$ is called digonal. The collection of all diagonal operators $\mathcal{Q}$ is called the diagonal algebra of $A$. It is an abelian von Neumann algebra and $\mathcal{Q}'$ is the algebra of decomposable operators. Moreover, each abelian von Neumann algebra on a Hilbert space $H$ corresponds to the algebra $\mathcal{Q}$ with respect to some direct integral decomposition of $H$.

Let $\mathcal{A}$ be a strongly closed subalgebra of $L(H)$ and let $\mathcal{D}$ be an abelian von Neumann algebra in the commutant of $\mathcal{A}$. As mentioned above we can regard $\mathcal{D}$ as the algebra of diagonal operators with respect to some decomposition $H = \int_A \oplus h(\lambda) \mu(d\lambda)$ of the Hilbert space. Each $A \in \mathcal{A}$ is decomposable and if $\{A_n\}$ is a strongly dense sequence in the unit ball of $\mathcal{A}$ choose a representation $\lambda \mapsto A_n(\lambda)$ for each $n$ and let $\mathcal{A}(\lambda)$ be he strongly closed algebra generated by $\{A_n(\lambda)\}$. This formal decomposition is denoted by $\mathcal{A} \sim \int_A \oplus \mathcal{A}(\lambda) \mu(d\lambda)$ and it was shown in [2] that this decomposition is well defined in the sense that if $\{A_n\}$ is another strongly dense sequence and $\mathcal{A}(\lambda)$ the corresponding algebras, there exists a $\mu$-null set $N$ such that $\mathcal{A}(\lambda) = \mathcal{A}(\lambda)$ off $N$. If $[\mathcal{A}, \mathcal{D}]$ denotes the strongly closed algebra generated by $\mathcal{A}$ and $\mathcal{D}$ then also $[\mathcal{A}, \mathcal{D}] \sim \int_A \mathcal{A}(\lambda) \mu(d\lambda)$, and furthermore $[\mathcal{A}, \mathcal{D}]$ is maximal with respect to this property. That is, if $A$ is decomposable, then $A \in [\mathcal{A}, \mathcal{D}]$ iff $A$ has a representative $\lambda \mapsto A(\lambda)$ with $A(\lambda) \in \mathcal{A}(\lambda)$ $\mu$-a.e. In case that $\mathcal{D} \subseteq \mathcal{A}$ we write $\mathcal{A} = \int_A \mathcal{A}(\lambda) \mu(d\lambda)$ and say that $\mathcal{A}$ is the direct integral of the algebras $\mathcal{A}(\lambda)$. For these and related results we will refer to [2].

One important result in the theory of decomposition of algebras is a criterion for when a field of algebras $\lambda \mapsto \mathcal{A}(\lambda)$ on a decomposed space $H$ is attainable as the decomposition of an algebra acting on $H$; that is, when there exists an algebra $\mathcal{A}$ on $H$ with $\mathcal{A} \subset \mathcal{D}'$ and $\mathcal{A} \sim \int_A \oplus \mathcal{A}(\lambda) \mu(d\lambda)$. The basic result utilizes the notion of a multifunction and for topological reasons one associates the algebra with its unit ball (this restriction is usually only implied in the notation). For von Neumann algebras it was shown by E. G. Effros [5] and for non-self-adjoint strongly closed algebras it was shown in [2] that a field $\lambda \mapsto \mathcal{A}(\lambda)$ of algebras on $H = \int_A \oplus h(\lambda) \mu(d\lambda)$ is attainable as an algebra $\mathcal{A}$ on $H$ if and only if $\lambda \mapsto \mathcal{A}(\lambda)$ is measurable as a multifunction from $A$ to the space $(C(h), \text{s.o.t.})$ of contraction operators on $h$ (here $h = h(\lambda)$ $\mu$-a.e.).

We may sometimes make the assumption in our proofs that for a direct integral decomposition $H = \int_A \oplus h(\lambda) \mu(d\lambda)$ the spaces $h(\lambda)$ are all equal to a single space $h$. When this is done a trivial modification adapts to the general situation. This enables us to use the terminology and theory of multifunctions in a simple fashion. Thus the values of the multifunctions we use will be closed sets in the space of contraction operators $C(h)$ on a
Hilbert space $h$ taken with the strong operator topology. Recall that we are assuming that our measure space $(A, \mu)$ is complete. Under these conditions the notions of weak and strong measurability for a multifunction from $\lambda \to C(h)$ coincide and we shall simply use the term measurable. The theorems of C. Castaing and others imply that in our case the measurability of a multifunction $F: \lambda \to C(h)$ is also equivalent to the measurability of its graph in the product measure space or the existence of a countable dense set of measurable selectors: that is, measurable functions $\phi_n: \lambda \to C(h)$ so that \{\phi_n(\lambda)\} is dense in $F(\lambda)$ $\mu$-a.e. Finally we may alter $F$ on a set of measure zero so that the graph of $F$ is a Borel set in the product Borel structure (cf. [2, 3, 6, 17]).

We begin our decomposition theory for commutative subspace lattices paralleling the theory for algebras above by letting $(A, \mu)$ be a partitioned measure space and $\mathcal{Q}$ the algebra of diagonal operators on $H = \int_A \oplus h(\lambda)\mu(d\lambda)$. We first define what is meant by the formal decomposition of a lattice relative to a countable generating subset. We will show (3.4) that this formal decomposition is in fact independent of the particular generating set chosen.

**Definition 3.1.** Let $\mathcal{L}$ be a strongly closed commutative lattice of projections in $\mathcal{L}'$. Let $\{P_n\}$ be a countable subset of $\mathcal{L}$ which generates $\mathcal{L}$ as a strongly closed lattice and fix a Borel representative $\lambda \to P_n(\lambda)$ for each $n$ with $P_n(\lambda)$ a projection for each $\lambda$. Define $\mathcal{L}(\lambda)$ to be the strongly closed lattice generated by $\{P_n(\lambda)\}$. Denote this formal decomposition by $\mathcal{L} \sim \int_A \oplus \mathcal{L}(\lambda)\mu(d\lambda)$.

**Remark 3.2.** It is elementary that every strongly closed lattice is complete, and for commutative lattices it is straightforward that completeness is equivalent to strong closure. A deep result of W. Arveson [1] shows that every commutative subspace lattice is reflexive, so that completeness, strong closure, and reflexivity are equivalent for separably acting commutative subspace lattice. Also, in the commutative case the lattice operations are strongly continuous, and it follows that the strong closure of a commuting lattice of projections is itself a lattice. These relationships do not hold in general. Our proofs depend upon theory established in [2] for algebras and upon properties of commutative lattices mentioned above. Thus while the formal decomposition (3.1) relative to a generating set "makes sense" for arbitrary subspace lattices the failure of the above relationships in the noncommutative case would make any noncommutative theory at least technically more difficult. We present the following simple example for sake of exposition.

**Example 3.3.** The closure in the strong operator topology (or even norm topology) of a complete lattice need not be a lattice. Let $H$ have
as an orthonormal basis and define $P_n$ to be the projection on \text{span}\{e_1 + (1/n) e_2, e_2 + (1/n) e_4\}$ and $Q$ the projection on \text{span}\{e_2, e_3\}. For each $n$, $Q \land P_n = 0$ while $Q \lor P_n = I$. Moreover, $P_n \land P_m = 0$ while $P_n \lor P_m = I$ if $n \neq m$. Then $\mathcal{L} = \{0, I, Q, P_1, P_2, \ldots\}$ is a complete lattice. However, $P$, the projection on span\{e_2\}, is in the strong closure of $\mathcal{L}$. If $\mathcal{F}$ were a lattice, then $R$, the projection on span\{e_2\}, would also be in $\mathcal{F}$. This is impossible since $\| (R - P_n) e_2 \| \geq 1/\sqrt{2}$ for all $n$ while $\| (R - Q) e_3 \| = 1$.

The following proposition is the basic result concerning the decomposition of commutative subspace lattices given in (3.1). We use the “join” notation $\mathcal{L} \lor \mathcal{M}$ to denote the smallest subspace lattice containing both $\mathcal{L}$ and $\mathcal{M}$.

**Proposition 3.4.** Let $\mathcal{L}$ be a commutative subspace lattice with formal decomposition $\mathcal{L} \sim \bigoplus \mathcal{L}(\lambda) \mu(d\lambda)$ with respect to a diagonal algebra $\mathcal{D}$. Let $\mathcal{M}$ be the lattice of projections in the diagonal algebra and let $\mathcal{L}_0 = \mathcal{L} \lor \mathcal{M}$. Then

1. $(\text{Alg } \mathcal{L}) \cap \mathcal{D}' = \text{Alg } \mathcal{L}_0 = \bigoplus \text{Alg } \mathcal{L}(\lambda) \mu(d\lambda)$ and
2. $\mathcal{L}_0 = \bigoplus \mathcal{L}(\lambda) \mu(d\lambda),$

where (2) means that $P \in \mathcal{L}_0$ if and only if $P(\lambda) \in \mathcal{L}(\lambda)$ $\mu$-a.e. Moreover two commutative subspace lattices $\mathcal{L}_1$ and $\mathcal{L}_2$ have the same decomposition $\mu$-a.e. if and only if

$$\mathcal{L}_1 \lor \mathcal{M} = \mathcal{L}_2 \lor \mathcal{M}.$$ 

**Proof.** Let $\{P_n\}$ be the generating set for $\mathcal{L}$ used to determine $\mathcal{L}(\lambda)$ and $\lambda \mapsto P_n(\lambda)$ fixed Borel representatives for $P_n$. Let $\mathcal{B} = \{(\lambda, A) : A \in C(h), AP_n(\lambda) = P_n(\lambda) AP_n(\lambda) \text{ for all } n\}$. Since composition is a continuous map in the s.o.t. this implies that $\mathcal{B}$ is a Borel set in $A \times C(h)$ and thus $\lambda \mapsto \text{Alg } \mathcal{L}(\lambda) \cap C(h)$ is a measurable multifunction from $A$ to $(C(h), \text{s.o.t.})$. Thus by Theorem 5.5 in [2], there exists a unique algebra $\mathcal{G}$ containing $\mathcal{B}$ such that

$$\mathcal{G} = \bigoplus A \mathcal{L}(\lambda) \mu(d\lambda).$$

If $A \in \text{Alg } \mathcal{L}_0$, then $A$ is decomposable and a.e. $A(\lambda) P_n(\lambda) = P_n(\lambda) A(\lambda) P_n(\lambda)$ for all $n$. Thus $A(\lambda) \in \text{Alg } \mathcal{L}(\lambda)$ a.e. and hence $A \in \mathcal{G}$ by Proposition 3.13 in [2]. Conversely if $A \in \mathcal{G}$, then $A(\lambda) \in \text{Alg } \mathcal{L}(\lambda)$ a.e. and $AP_n = P_n A P_n$ for all $n$ and hence $A \in \text{Alg } \mathcal{L}$. Moreover, since $A \in \mathcal{D}'$, then $AP = PA$ for all $P$ in $\mathcal{M}$ and hence $A \in \text{Alg } \mathcal{L}_0$. Finally, since $\mathcal{M}$ is complemented we have $\mathcal{D}' = \text{alg } \mathcal{M}$ so that $\text{Alg } \mathcal{L}_0 = \text{alg } \mathcal{L} \cap \mathcal{D}'$. Thus we have shown that (1) holds.
Since $\text{Alg } \mathcal{L}_0 = \int_A \oplus \text{Alg } \mathcal{L}(\lambda) \mu(d\lambda)$ let $\{A_n\}$ be a generating set in $\text{Alg } \mathcal{L}_0$ so that $\{A_n(\lambda)\}$ generate $\text{Alg } \mathcal{L}(\lambda)$ $\mu$-a.e. (Theorem 5.5 in [2].) If $P \in \mathcal{L}_0$, then $PA_nP = A_nP$ for all $n$ and hence $P(\lambda) A_n(\lambda) P(\lambda) = A_n(\lambda) P(\lambda)$ $\mu$-a.e. for all $n$. Thus $P(\lambda) \in \text{Lat}\{A_n(\lambda)\} = \text{Lat Alg } \mathcal{L}(\lambda) = \mathcal{L}(\lambda)$ because of the reflexivity of commutative lattices. The argument is reversible which completes part (2).

Let $\{P_n\}$ and $\{Q_n\}$ generate $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, and determine decompositions $\mathcal{L}_1 \sim \int_A \oplus \mathcal{L}_1(\lambda) \mu(d\lambda)$ and $\mathcal{L}_2 \sim \int_A \oplus \mathcal{L}_2(\lambda) \mu(d\lambda)$, respectively. If $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}_2 \cup \mathcal{L}_1$ and each $P_n$ has a representative $\lambda \to P_n(\lambda)$ so that $P_n(\lambda) \in \mathcal{L}_2(\lambda)$ $\mu$-a.e. by (2). Thus off a set of measure zero $\{P_n(\lambda)\} \subset \mathcal{L}_2(\lambda)$ and hence $\mathcal{L}_1(\lambda) \subset \mathcal{L}_2(\lambda)$ $\mu$-a.e. Similarly $\{Q_n(\lambda)\} \subset \mathcal{L}_1(\lambda)$ $\mu$-a.e. and $\mathcal{L}_2(\lambda) \subset \mathcal{L}_1(\lambda)$ $\mu$-a.e. Finally it follows directly from (2) that if $\mathcal{L}_1(\lambda) = \mathcal{L}_2(\lambda)$ $\mu$-a.e., then $\mathcal{L}_1 \cup \mathcal{L}_2 = \mathcal{L}_2 \cup \mathcal{L}_1$. [1]

Remark. If $\mathcal{L}$ has two sets of generators, then the above argument shows that they yield the same formal decomposition of $\mathcal{L}$ $\mu$-a.e. In addition we may clearly take the set $\{P_n\}$ to be a dense countable sublattice of $\mathcal{L}$. In this case we also have $\mu$-a.e. that $\{P_n(\lambda)\}$ is a lattice and the $\{P_n(\lambda)\} = \mathcal{L}(\lambda)$ $\mu$-a.e.

The converse of this decomposition is important and useful in several settings. As mentioned in the introduction to this section there are measurability conditions on a field of algebras to ascertain if they comprise the decomposition of an algebra on a direct integral of spaces. The following result shows the same is true for subspace lattices.

**Proposition 3.5.** Let $\mathcal{L} \sim \int_A \oplus \mathcal{L}(\lambda) \mu(d\lambda)$ be a decomposition of a commutative subspace lattice. Then $\lambda \to \mathcal{L}(\lambda)$ is measurable as a multifunction. Conversely if $\psi$ is a measurable multifunction taking as values commutative subspace lattices in $C(\mathcal{H})$, then there exists a commutative subspace lattice $\mathcal{L}$ on $\mathcal{H} = \int_A \oplus h(\lambda) \mu(d\lambda)$ so that $\mathcal{L}(\lambda) = \psi(\lambda)$ $\mu$-a.e.

**Proof.** As mentioned above, in this case measurability is equivalent to the existence of a dense set of measurable selectors. So the first statement follows from the remark above. Conversely let $\lambda \to P_n(\lambda)$ be a dense set of measurable selectors for $\lambda \to \psi(\lambda)$. The operators $P_n = \int \oplus P_n(\lambda) \mu(d\lambda)$ form a commutative family of projections on $\mathcal{H}$. If $\mathcal{L}$ is the subspace lattice they generate, then by (3.1) we have $\mathcal{L}(\lambda) = \psi(\lambda)$ $\mu$-a.e. [1]

For a measurable field $\lambda \to \mathcal{L}(\lambda)$ of lattices there may be many lattices which have $\lambda \to \mathcal{L}(\lambda)$ as their formal decompositions. If the lattices $\{\mathcal{L}(\lambda)\}$ have additional properties it is useful to know if these properties can be lifted to a lattice with the given decomposition. In case the lattices $\mathcal{L}(\lambda) = \mathcal{L}(\lambda)$ are nests $\mu$-a.e. the answer is yes; that is, there exists a nest $\mathcal{L}$ (in general nonunique) so that $\mathcal{L} \sim \int_A \oplus \mathcal{L}(\lambda) \mu(d\lambda)$.  

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PROPOSITION 3.6. Let \( \psi : A \to (C(h), \text{s.o.t.}) \) be measurable so that \( \psi(\lambda) = \mathcal{N}(\lambda) \) is a nest \( \mu \)-a.e. Then there exists a nest \( \mathcal{N} \subset \mathcal{D} \) so that \( \mathcal{N} \sim \int_A \mathcal{A} \), \( \mathcal{A}(\lambda) \mu(d\lambda) \).

Proof. Let \( \{P_n\} \) be a set of Borel functions on \( A \) so that \( \{P_n(\lambda)\} \) is dense in \( \psi(\lambda) \) \( \mu \)-a.e. We may assume that \( P_n(\lambda) \) is a projection \( \mu \)-a.e. for all \( n \). Assume that for \( 1 \leq i \leq k \) finite nests \( \mathcal{N}_1, \ldots, \mathcal{N}_k \) in \( \mathcal{D} \) are determined so that

\[
\{P_i, P_{i+1}\} = \bigcup_{A \in \mathcal{N}_i} \{P_i(A), P_{i+1}(A)\}, \quad \mathcal{A}(\lambda) = \{0, P_1(\lambda), \ldots, P_k(\lambda), I(\lambda)\}.
\]

Now suppose \( \mathcal{N}_k \) consists of \( Q_1, \ldots, Q_n_k \), where \( 0 = Q_1 \leq Q_2 \leq \cdots \leq Q_{n_k} = I \). Define \( \mathcal{N}_{k+1} \) to be the nest consisting of the projections in \( \mathcal{N}_k \) along with the projections \( R_{k+1,j} = (P_{k+1} \wedge Q_j) \vee Q_{j-1} \) for \( 2 \leq j \leq n_k \). Since \( \mathcal{N}(\lambda) \) is a nest almost everywhere, for almost all \( \lambda \) there is an \( i \) depending on \( \lambda \) so that \( P_i(\lambda) \leq P_{i+1}(\lambda) \leq P_i(\lambda) \) or else either \( P_{i+1}(\lambda) \geq P_i(\lambda) \) or \( P_i(\lambda) \leq P_{i+1}(\lambda) \) for \( 1 \leq i \leq k \). In any event it is clear that \( \{P_{i+1}(\lambda), P_i(\lambda)\} = \{P_i(\lambda), \ldots, P_{k+1}(\lambda)\} \). Since \( \mathcal{N}_k \) and \( \mathcal{N}_{k+1} \) are in \( \mathcal{D} \) it follows that \( \mathcal{N}_{k+1} \) is in \( \mathcal{D} \). Let \( \mathcal{N} \) be the completion of the nest \( \bigcup_k \mathcal{N}_k \). It is clear from our construction that \( \mathcal{N} \sim \int_A \mathcal{A}(\lambda) \mu(d\lambda) \).

In general we will start with a commutative lattice and decompose it and its algebra into direct integrals of lattices and algebras in such a way as to study properties of \( \mathcal{L} \) and \( \text{Alg} \mathcal{L} \).

LEMMA 3.1. Let \( \mathcal{L} \) be a commutative subspace lattice and let \( \mathcal{C} = \text{Alg} \mathcal{L} \). There exists a direct integral decomposition of \( \mathcal{C} \) and \( \mathcal{L} \), so that

\[
\mathcal{L} \sim \int_A \mathcal{C}(\lambda) \mu(d\lambda), \quad \mathcal{L} \sim \int_A \mathcal{L}(\lambda) \mu(d\lambda)
\]

such that \( \mathcal{C}(\lambda) = \text{Alg} \mathcal{L}(\lambda) \) is irreducible.

Proof. Let \( \mathcal{L} \) be the von Neumann algebra generated by the complemented projections \( \mathcal{M} \) in \( \mathcal{L} \). Since \( \mathcal{C} \subset \mathcal{C} \cap \mathcal{C} \), then \( \mathcal{C} \subset \mathcal{D} \), and

\[
\mathcal{L} = \{\mathcal{C}, \mathcal{D}\} = \int_A \mathcal{C}(\lambda) \mu(d\lambda) \] \quad \text{while} \quad \mathcal{L} = \mathcal{L} \cup \mathcal{M} = \int_A \mathcal{L}(\lambda) \mu(d\lambda).
\]

We have seen in the previous lemmas that \( \mathcal{C}(\lambda) = \text{Alg} \mathcal{L}(\lambda) \).

Let \( \{A_n\} \) be a generating family for \( \mathcal{C} \) and assume \( \lambda \to A_n(\lambda) \) is a Borel function for all \( n \). Let \( E = \{(\lambda, P) : A_n(\lambda) P = PA_n(\lambda), P \neq 0, I, \text{ for all } n\} \). Then \( E \) is a Borel set in \( A \times C(h) \) and by Measurable Selection there exists a measurable cross section \( P : A \to C(h) \) such that \( P(\lambda) = 0 \) if \( \lambda \) is not in \( \Pi_A(E) \) and \( (\lambda, P(\lambda)) \in E \) if \( \lambda \in \Pi_A(E) \). (See remark below.) However, \( P \) and \( I - P \) are in \( \text{Lat} \mathcal{C} = \mathcal{D} \) and hence in \( \mathcal{D} \). Hence, \( P(\lambda) = 0 \) or \( I \mu \)-a.e. and hence \( \Pi_A(E) = 0 \). Thus \( \mathcal{C}(\lambda) \) is irreducible \( \mu \)-a.e. and \( \mathcal{L}(\lambda) \) has no complemented members \( \mu \)-a.e.

Remark 3.8. Throughout the paper we shall invoke the principle of Measurable Selection. We shall refer to it as Measurable Selection or von Neumann's Selection Theorem. For a proof of a version sufficient for our cases von Neumann's original version or that of J. Schwartz [16, p. 26] will
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suffice. For a detailed exposition of the result and the related results we use concerning multifunctions we refer the reader to Wagner's work [17].

4. Decomposition of Nest-Subalgebras and a Criterion for Commutative Subspace Lattices

In this section it is shown that the decomposition of a nest-subalgebra of a von Neumann algebra \( \mathcal{A} \) along the center of the von Neumann algebra leads to algebras with simpler structure. This is used here and in the sequel to obtain results for the algebras which are not directly related to the decomposition. We first consider the case in which \( \mathcal{A} \) contains a m.a.s.a. (or equivalently, \( \text{lat}(\mathcal{A}) \) is commutative). Recall (Lemma 2.1) that this is also equivalent to \( W^*(\mathcal{A}) \) containing a m.a.s.a.

**Theorem 4.1.** Let \( \mathcal{A} \) be a reflexive operator algebra which contains a m.a.s.a. Then \( \mathcal{A} \) is a nest-subalgebra of a von Neumann algebra if and only if \( \mathcal{A} \) is (equivalent to) a direct integral of nest algebras. This is true iff \( \text{lat}(\mathcal{A}) \) is (equivalent to) a direct integral of nests.

**Proof.** Let \( \mathcal{A} \) be the direct integral \( \mathcal{A} = \int \lambda \oplus \mathcal{A}(\lambda) \mu(d\lambda) \) of nest algebras with diagonal operators \( \mathcal{E} \). Then \( \lambda \rightarrow \mathcal{A}(\lambda) \) is measurable from \( \lambda \) to \( (C(h), \text{s.o.t.}) \) and consequently \( \lambda \rightarrow \text{Lat}(\mathcal{A}(\lambda)) = \mathcal{F}(\lambda) \) is measurable [2]. Now by Proposition 3.6, there exists a nest \( \mathcal{E} \) on \( H \), so that \( \mathcal{F} \sim \int \lambda \oplus \mathcal{F}(\lambda) \mu(d\lambda) \). Proposition 3.4 states that if \( \mathcal{L} = \bigvee \mathcal{F}(\lambda) \mu(d\lambda) \), then \( \text{Alg}(\mathcal{L}) = \mathcal{F} \) and \( \text{Alg}(\mathcal{L}) = \mathcal{L} \cap \text{Alg}(\mathcal{F}) \).

Conversely let \( \mathcal{A} \) be the nest-subalgebra with respect to the nest \( \mathcal{E} \) and the von Neumann algebra \( \mathcal{B} \) which contains a m.a.s.a. Decompose the nest and \( \mathcal{B} \) with respect to the abelian algebra \( \mathcal{B}' \). Letting \( \mathcal{B}' \) be \( \mathcal{E} \) in Proposition 3.4, we see that \( \mathcal{E} = \bigvee \mathcal{F}(\lambda) \mu(d\lambda) \) and

\[
\text{Alg}(\mathcal{E}) \cap \mathcal{B} = \int \lambda \oplus \text{Alg}(\mathcal{F}(\lambda) \mu(d\lambda)).
\]

In concrete situations it is often a priori unknown whether a given reflexive operator algebra is a nest-subalgebra of a von Neumann algebra, and this question is not always easily decidable even when one has a reasonable description of its lattice. Arveson has shown that every separably acting commutative subspace lattice admits a representation as the lattice of increasing subsets of some standard partially ordered measure space, and a concrete commutative subspace lattice is very often described in terms of such a representation [1, Theorem 1.3.1]. Lattices \( \mathcal{L} \) which are nests have the following trivial characterization: given \( P, Q \in \mathcal{L} \) either \( PQ^\perp = 0 \) or
Theorem 4.2 gives an analogous simple criterion for commutative subspace lattices which have representations as direct integrals of nests.

THEOREM 4.2. A reflexive operator algebra with commutative subspace lattice \( \mathcal{L} \) is a nest-subalgebra of a von Neumann algebra iff its lattice \( \mathcal{L} \) satisfies the following criterion: given \( P, Q \in \mathcal{L} \) the projections \( PQ^\perp \) and \( P^\perp Q \) are contained in complemented members of \( \mathcal{L} \) (i.e., there exists an \( R \in \mathcal{L} \) with \( R^\perp \) also in \( \mathcal{L} \) such that \( PQ^\perp \leq R \) and \( P^\perp Q \leq R^\perp \)).

Proof. Let \( \mathcal{A} \) be a nest-subalgebra of a von Neumann algebra \( \mathcal{B} \) which contains a m.a.s.a. Then \( \text{Lat} \mathcal{A} = N \vee \mathcal{M} \), where \( \mathcal{M} \) are the projections in \( \mathcal{B} \) such that \( \mathcal{M}^\perp \) is the nest in \( \mathcal{B} \) and by Proposition 3.4 we have \( \mathcal{L} = \int_A \oplus \mathcal{L}(\lambda) \mu(d\lambda) \). Now if \( P, Q \in \mathcal{L} \), then let \( A_1 = \{ \lambda : P(\lambda) > Q(\lambda) \} \) and define \( R(\lambda) = x_A(\lambda) I(\lambda) \). Again by Proposition 3.4, \( R \) and \( R^\perp \) are in \( \mathcal{L} \) and \( PQ^\perp \leq R \) while \( P^\perp Q \leq R^\perp \).

Now assume the lattice \( \mathcal{L} \) is commutative and satisfies the lattice property in the theorem. By Lemma 3.7, there exists a diagonal algebra \( \mathcal{D} \) such that with respect to \( \mathcal{D} \), \( \mathcal{L} = \int_A \oplus \mathcal{L}(\lambda) \mu(d\lambda) \), \( \text{Alg} \mathcal{L} = \int_A \oplus \text{Alg} \mathcal{L}(\lambda) I(\lambda) \mu(d\lambda) \) with \( \mathcal{L}(\lambda) \) having no nontrivial complemented members. Let \( \{ P_n \} \) be strongly dense in \( \mathcal{L} \). For each pair \( m, n \) there is an \( R \) with \( R, R^\perp \in \mathcal{L} \) such that \( \mu \)-a.e. we have \( P_n(\lambda) \leq R(\lambda) \) while \( P_n(\lambda)^\perp P_m(\lambda) \leq R(\lambda)^\perp \), where we note that for each \( \lambda \) either \( R(\lambda) = 0 \) or \( R(\lambda) = I \) since \( \mathcal{L}(\lambda) \) has no nontrivial complemented projections. This implies that \( \mu \)-a.e. the set \( \{ P_n(\lambda) \} \) is a nest and hence so must be its completion, and this is equal to \( \mathcal{L}(\lambda) \) \( \mu \)-a.e.

EXAMPLE 4.3. Let \( S \) be the unit square \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) and let \( \mathcal{L}_0 \) be the class of all Borel subsets \( E \) of \( S \) with the property that whenever \((x_0, y_0) \in E \) then \((x, y_0) \in E \) for all \( x \geq x_0 \). If \( \mu \) is a finite Borel measure on \( S \) the multiplication operators corresponding to characteristic functions of sets in \( \mathcal{L}_0 \) constitute a complete lattice of projections \( \mathcal{L} \) acting on the Hilbert space \( L^2(S; \mu) \). An application of the above criterion shows that \( \text{Alg} \mathcal{L} \) is a nest-subalgebra of a von Neumann algebra and that \( \mathcal{L} \) is a direct integral of nests.

We have indicated above the simplest class of nest-subalgebras of von Neumann algebras. More generally reduction of an arbitrary such algebra via the center of the von Neumann algebra leads to a representation as a direct integral of nest-subalgebras of factor von Neumann algebras. These latter algebras prove amenable to computational techniques (as we show in later sections) and certain results for nest algebras extend to this setting.

THEOREM 4.4. Every separably acting nest-subalgebra of a von Neumann algebra is (equivalent to) a direct integral of nest-subalgebras of
factor von Neumann algebras. Conversely, if an operator algebra \( \mathcal{A} \) admits such a representation then it is the nest-subalgebra of \( W^*(\mathcal{A}) \) with respect to some nest \( \mathcal{V} \) contained in \( W^*(\mathcal{A}) \).

**Proof.** Let \( \mathcal{B} \) be the center of \( \mathcal{A} \). Since \( \mathcal{B} \) and \( \mathcal{V} \) are the same, they can be decomposed as \( \mathcal{B} = \bigoplus \mathcal{B}(\lambda) \mu(d\lambda) \) and \( \mathcal{V} = \bigoplus \mathcal{V}(\lambda) \mu(d\lambda) \), respectively. By Proposition 3.4, \( \mathcal{V} \cap \mathcal{V}' = \bigoplus \mathcal{V}(\lambda) \mu(d\lambda) \). The graphs of \( \lambda \rightarrow \mathcal{A} \cap \mathcal{V}(\lambda) \) and \( \lambda \rightarrow \mathcal{B}(\lambda) \) are measurable in \( \mathcal{A} \times C(h) \) with respect to the \( \sigma \)-algebra generated by the \( \mu \)-measurable sets on \( \mathcal{A} \) and the Borel sets on \( (C(h), s.o.t.) \). Hence their intersection is a measurable set in \( \mathcal{A} \times C(h) \). Thus \( \lambda \rightarrow \mathcal{A} \cap \mathcal{V}(\lambda) \cap \mathcal{B}(\lambda) \) is measurable and by Proposition 5.2 in [2] there is an algebra \( \mathcal{E} \) containing \( \mathcal{B} \) such that \( \mathcal{E} = \bigoplus \mathcal{A} \cap \mathcal{V}(\lambda) \cap \mathcal{B}(\lambda) \mu(d\lambda) \). It follows from Corollary 3.4 in [2] that \( \mathcal{E} = ((\mathcal{V} \cap \mathcal{V}') \cap \mathcal{B}) \). However, \( \mathcal{V} \cap \mathcal{V}' \cap \mathcal{B} = \mathcal{A} \cap \mathcal{B} \) and \( \mathcal{V} \cap \mathcal{B}(\lambda) \) is a nest-subalgebra of a factor. 

The following example illustrates this decomposition. In case \( \mathcal{A} \) contained a m.a.s.a. the decomposition in Theorem 4.1 also illustrates this result.

**Example 4.5.** In case \( \mathcal{V} \) is a nest in a type I von Neumann algebra \( \mathcal{B} \) this decomposition yields a decomposition into nest-subalgebras of type I factors. Let \( \mathcal{B} \) be a type I_m factor: i.e., \( \mathcal{B} = M_n \otimes I_m \), where \( I_m \) is the identity on a Hilbert space \( H_m \) of dimension \( m \) and \( M_n \) is the algebra of \( n \times n \) matrices on \( H_n \). If we assume \( \mathcal{V} \subseteq \mathcal{B} \), then each \( P \) in \( \mathcal{V} \) is of the form \( P = \mathcal{P} \otimes I_m \), where \( \mathcal{P} \) is a projection in \( M_n \). If we denote by \( \mathcal{V} \) the nest \( \{ \mathcal{P} \otimes I_m \subseteq \mathcal{B} \} \), then \( \mathcal{A} \cap \mathcal{V} \cap \mathcal{B} = (\mathcal{A} \cap \mathcal{V}) \otimes I_m \). Thus a nest-subalgebra of a type one von Neumann algebra is decomposed into the direct integral of nest algebras tensor identities. We remark that the lattice of \( \mathcal{A} \cap \mathcal{V} \otimes I_m \) is noncommutative in case \( m > 1 \).

**5. THE JACOBSON RADICAL**

If \( \mathcal{A} \subseteq \mathcal{B} \) are Banach algebras with \( \mathcal{A} \subseteq \mathcal{B} \) it is not in general true that \( \text{rad}(\mathcal{A}) \subseteq \text{rad}(\mathcal{B}) \), or equivalently, \( \text{rad}(\mathcal{A}) = \mathcal{A} \cap \text{rad}(\mathcal{B}) \). \( \mathcal{B} \) could be semisimple and \( \mathcal{A} \) radical, for instance. In fact for nest algebras \( \mathcal{A} \subseteq \mathcal{B} \) it follows from the Ringrose criterion that the reverse inclusion \( \text{rad}(\mathcal{B}) \subseteq \text{rad}(\mathcal{A}) \) holds. If \( \mathcal{A} \) is a nest algebra and \( \mathcal{A} \) an arbitrary subalgebra, either inclusion (or neither) is possible. It becomes of interest as to when such inclusion holds. In particular, if \( \text{rad}(\mathcal{A}) \subseteq \text{rad}(\mathcal{B}) \) then the Ringrose criterion for \( \mathcal{A} \) applies to \( \mathcal{B} \) as well. That is, elements in the radical of \( \mathcal{A} \) can be studied via the Ringrose criterion for \( \mathcal{B} \). We will show that if \( \mathcal{B} \) is a factor von Neumann algebra, if \( \mathcal{V} \) is a complete nest of projections contained in \( \mathcal{B} \), and if \( \mathcal{A} \) is the nest-subalgebra of \( \mathcal{B} \) associated with \( \mathcal{V} \), then the radical
of \( \mathcal{O} \) is contained in the radical of the nest algebra \( \mathcal{O}_{\mathcal{A}} \). Example 2.4 shows that this need not hold if \( \mathcal{B} \) is not a factor.

Next we show that the radical of an arbitrary nest-subalgebra of a von Neumann algebra is contained in the direct integral of the radicals of the summands for a decomposition of the algebra. This containment will in general be proper just as in the analogous case for direct sums or integrals of quasinilpotent operators. Then using these results we will be able to "lift" the Ringrose criterion for the radical of a nest algebra to the appropriate version of such a criterion for the radical of a nest-subalgebra of a von Neumann algebra.

The following is a convenient alternative version of the Ringrose criterion for nest algebras which proves useful.

**Lemma 5.1.** Let \( \mathcal{V} \) be a complete nest and let \( \mathcal{R}_\mathcal{V} \) be the Jacobson radical of \( \mathcal{O}_{\mathcal{V}} \). If \( A \in \mathcal{O}_{\mathcal{V}} \), then \( A \in \mathcal{R}_\mathcal{V} \) iff both of the following conditions are satisfied:

1. (i) \( EAE = 0 \) for every minimal \( \mathcal{V} \)-interval \( E \).
2. (ii) \( \|E_nAE_n\| \to 0 \) for every sequence \( E_n \) of mutually orthogonal \( \mathcal{V} \)-intervals.

**Proof.** First suppose \( A \in \mathcal{R}_\mathcal{V} \). If \( E \) is a minimal \( \mathcal{V} \)-interval then \( EAE \in \mathcal{O}_{\mathcal{V}} \), for every \( B \in \mathcal{L}(H) \), so in particular \( EA^*E \in \mathcal{O}_{\mathcal{V}} \). Since \( A \in \mathcal{R}_\mathcal{V} \), so must \( (EA^*E)(EAE) \); but a positive quasinilpotent must be zero, so necessarily \( EAE = 0 \).

Next, suppose \( A \in \mathcal{O}_{\mathcal{V}} \) and suppose \( \|E_nAE_n\| \geq \varepsilon > 0 \) for an infinite sequence \( E_n \) of mutually perpendicular \( \mathcal{V} \)-intervals. If \( F_i \) is any finite set of mutually perpendicular \( \mathcal{V} \)-intervals, it follows from the linear order of \( \mathcal{V} \) that at least one of the \( F_i \) must contain some \( E_n \). For this \( F_i \), we have \( \|F_iAF_i\| \geq \varepsilon \). It follows from the Ringrose criterion (1.3) that \( A \in \mathcal{R}_\mathcal{V} \). Thus every operator in \( \mathcal{R}_\mathcal{V} \) satisfies (ii).

Now suppose \( A \in \mathcal{O}_{\mathcal{V}} \) and both (i) and (ii) are satisfied. Fix \( N \in \mathcal{V} \), \( N \neq 0 \), and let \( \alpha = \inf\{\|(N - L)A(N - L)\| : L \in \mathcal{V}, L < N, L \neq N\} \). We claim \( \alpha = 0 \). If \( N \) has an immediate predecessor (i.e., if \( N_\prec \neq N \) then \( E = N - N_\prec \) is a minimal \( \mathcal{V} \)-interval and so \( \alpha = 0 \) by (i). If \( N_\prec = N \), let \( \mathcal{S} \) be the set of members of \( \mathcal{V} \) strictly contained in \( N \) directed upward by inclusion. Suppose \( \alpha > 0 \). Then for fixed \( L \in \mathcal{S} \) the set \( (L' - L)A(L' - L) \), \( L' \in \mathcal{S} \), converges strongly to \( (N - L)A(N - L) \), so by lower semicontinuity of the norm in the strong operator topology there is an \( L' \in \mathcal{S} \), \( L' > L \), such that \( \|(L' - L)A(L' - L)\| > \frac{1}{2}\alpha \). That is, for each \( L \in \mathcal{S} \) there exists \( L' \in \mathcal{S} \), \( L' > L \) such that \( \|(L' - L)A(L' - L)\| > \frac{1}{2}\alpha \). Inductively obtain a sequence (setting \( L_{n+1} = L_n \)) such that if \( F_n = L_{n+1} - L_n \) we have \( \|F_nAF_n\| > \frac{1}{2}\alpha \) for every \( n \), thus contradicting (ii). Thus necessarily \( \alpha = 0 \).
We have shown that for each \( N \in \mathcal{N} \) it is true that \( \inf \{ \| (N - L) A(N - L) \| \} : L \in \mathcal{I}, L < N, L \neq N \} = 0 \). A similar argument shows that \( \inf \{ \| (L - N) A(L - N) \| \} : L \in \mathcal{I}, L > N, L \neq N \} = 0 \). A version of the Ringrose criterion (1.2) then implies that \( A \in r_n \).

An immediate consequence of the above version of the Ringrose criterion is the well known result that every compact operator in \( \mathcal{L} \) with zero diagonal is in \( \mathcal{R}_n \). A direct proof of this from the Ringrose criterion would follow the later part of the above proof. This lemma is useful in part because it points out the precise manner in which operators in the radical of a nest algebra "behave" like compact operators, and is interesting because the theory of nest algebras has at least some of its roots in the study of superdiagonal forms for compact operators.

If \( \mathcal{I} \) is a nest, we will say the \( \mathcal{I} \)-intervals \( E \) and \( F \) are strictly ordered, and write \( E \triangleleft F \), if the upper endpoint of \( E \) is contained in the lower endpoint of \( F \). Thus \( E \triangleleft F \) iff \( E \perp F \) and \( EAF \in \mathcal{L} \), for every \( A \in L(H) \). It is clear that \( \triangleleft \) is a transitive relation.

**Theorem 5.2.** Let \( \mathcal{I} \) be a complete nest of projections contained in a factor von Neumann algebra \( \mathcal{B} \), and let \( \mathcal{L} = \mathcal{B} \cap \mathcal{L} \), be the corresponding nest-subalgebra of \( \mathcal{B} \). Then the Jacobson radical of \( \mathcal{L} \) is contained in the Jacobson radical of \( \mathcal{L} \). We have \( \text{rad}(\mathcal{L}) = \mathcal{B} \cap \text{rad}(\mathcal{L}) \).

**Proof.** Suppose \( T \in \text{rad}(\mathcal{L}) \). We will show that \( T \) must satisfy conditions (i) and (ii) of Lemma 5.1 and is thus contained in \( \text{rad}(\mathcal{L}) \).

If \( E \) is a minimal \( \mathcal{I} \)-interval then (as in (5.1)) the operator \( ET*E \) is in \( \mathcal{L} \), and is also in \( \mathcal{B} \) so is in \( \mathcal{L} \), and hence \( ET*ETE \) is a positive operator in \( \text{rad}(\mathcal{L}) \) and so must be zero. So \( ETE = 0 \).

Now suppose that \( \| E, TE, \| > \alpha > 0 \) for some infinite sequence \( E, \) of mutually orthogonal \( \mathcal{I} \)-intervals. We will construct a partial isometry \( S \in \mathcal{L} \) such that \( ST \) is not quasinilpotent, thus contradicting our hypothesis that \( T \in \text{rad}(\mathcal{L}) \). We can assume \( \alpha = 1 \) by normalizing \( T \) if necessary. Since every pair of mutually orthogonal intervals for a nest is strictly ordered, by reordering and dropping to a subsequence if necessary we can assume the \( E, \) either have ordering \( E, \perp E, \perp E, \perp \ldots \) or ordering \( E, \perp E, \perp E, \perp \ldots \). We assume the former; the proof for the latter is analogous.

For each \( i \) let \( T_i = E, TE, \) and let \( T_i = U, T, \) be its polar decomposition. Let \( P_i \) be a spectral projection for \( T_i \) so that \( \| T_i P_i, x \| > \| P_i, x \|, x \in H \). Note that \( T_i, U_i, | T, \) \( P_i \) are in \( \mathcal{B} \). Let \( Q, \) be the range projection of \( T_i P_i, \) so \( Q, \in \mathcal{B} \), and (since \( \mathcal{B} \) is a factor) let \( S, \) be a nonzero partial isometry in \( \mathcal{B} \) with initial projection a subprojection of \( Q, \) and final projection a subprojection of \( P, \) [4, III.1]. Now let \( Q_2 \) be the range projection of \( T, S, T, P, \). Continue inductively, for each \( i \geq 2 \) obtaining a nonzero partial isometry \( S, \) in \( \mathcal{B} \) from a subprojection of the range projection \( Q, \) of the product
Let \( T_i S_{i-1} T_{i-1} \cdots S_j T_j P_j \) to a subprojection of \( P_{i+1} \). By our construction for each \( i \) the product \( S_i T_i S_{i-1} T_{i-1} \cdots S_j T_j P_j \) has norm \( \geq 1 \). Moreover, each \( S_j \) is in \( \mathcal{G} \) by Lemma 2.2. Since the supports of the \( S_j \) are mutually \( \perp \) and similarly for the ranges, the sum \( \sum S_i \) converges strongly to a partial isometry \( S \) in \( \mathcal{G} \). We claim \( \| (ST)^n \| \geq 1 \) for \( n \geq 1 \). Indeed, the facts that \( E_{n+1} S = S_n = S E_n \) and that for each \( A \in \mathcal{G} \) we have \( S_i A S_j = 0 \) whenever \( i \leq j \) imply that for each \( n \geq 1 \) we have

\[
E_{n+1} (ST)^n E_1 = S_n T(S_{n-1} + \cdots S_1) T \cdots (S_2 + S_1) TS_1 TE_1
= S_n TS_{n-1} TS_{n-2} T \cdots S_1 TE_1
= S_n TS_{n-1} T_{n-1} \cdots S_1 T_1
\]

which has norm \( \geq 1 \), thus justifying our claim. So \( ST \) is not quasinilpotent, thus obtaining the desired contradiction.

**Remark.** The proof shows that if \( E_1, \ldots, E_k \) are \( \perp \) intervals so that \( \| E_i TE_i \| \geq \varepsilon \) then there exists a partial isometry \( S \) in \( \mathcal{G} \) so that \( \| (ST)^n \| \geq \varepsilon \) for \( m = 1, \ldots, k-1 \).

Now let \( \mathcal{B} \) be an arbitrary von Neumann algebra, \( \mathcal{G} \) a complete nest in \( L(H) \), and \( \mathcal{G} \) the associated nest-subalgebra of \( \mathcal{B} \). Let \( \mathcal{G} = \bigoplus_{\lambda} \mathcal{G}(\lambda) \mu(d\lambda) \) and \( \mathcal{B} = \bigoplus_{\lambda} \mathcal{B}(\lambda) \mu(d\lambda) \) be decompositions with respect to some algebra of diagonal operators \( \mathcal{D} \) in the center of \( \mathcal{B} \). \( \mathcal{D} \) will often, but not always, taken to be cent(\( \mathcal{B} \)). By (3.7) and (4.4) it follows that \( \mathcal{G}(\lambda) \) is a nest-subalgebra of \( \mathcal{B}(\lambda) \) \( \mu \)-a.e. Of course if \( \mathcal{D} = \text{cent}(\mathcal{B}) \), then each \( \mathcal{B}(\lambda) \) is a factor. The following lemma gives a relationship between the radical of \( \mathcal{G} \) and the radicals of the \( \mathcal{G}(\lambda) \). As mentioned in the introduction to this section, the containment will usually be proper.

**Lemma 5.3.** In the above terminology, if \( T \in \text{rad}(\mathcal{G}) \), then \( T(\lambda) \in \text{rad}(\mathcal{G}(\lambda)) \) \( \mu \)-a.e. That is, the radical of \( \mathcal{G} \) is contained in the direct integral of the radicals of the summands.

**Proof.** Suppose \( T \in \text{rad}(\mathcal{G}) \) and \( \lambda \to T(\lambda) \) is a Borel representation of \( T \). Let \( \text{Gr}(\mathcal{D}) \) denote the graph of the measurable multifunction \( \lambda \to \mathcal{G}(\lambda) \) from \( A \) to \( (C(h), \text{s.o.t.}) \); i.e., \( \{ (\lambda, A) : A \in L(h), A \in \mathcal{G}(\lambda) \} \). For \( \varepsilon > 0 \) consider the set

\[
E_\varepsilon = \{ (\lambda, B) : B \in L(h) \text{ and } \| (T(\lambda) B)^n \|^{1/n} > \varepsilon \text{ for all } n \} \cap \text{Gr}(\mathcal{D})
\]

The set \( E_\varepsilon \) is a Borel set in \( A \times C(h) \) because the maps in the sequence

\[
(\lambda, B) \to (T(\lambda), B) \to (T(\lambda) B)^n \to \| (T(\lambda) B)^n \|^{1/n}
\]

are each Borel functions and by altering \( \mathcal{G}(\lambda) \) on a \( \mu \)-null set we may assume
Gr(\mathcal{C}) is a Borel set in \( A \times C(h) \) (cf. (4.5)). Moreover, it follows that \( T(\lambda) \) is in \( \text{rad}(\mathcal{C}(\lambda)) \) precisely if \( \lambda \) is not in \( \Pi_A(E_{1/k}) \) for every \( k \) (\( \Pi_A \) denotes the projection map of \( A \times C(h) \) onto \( A \)). If however, the analytic set \( \Pi_A(E_{1/k}) \) has nonzero measure for some \( k \), then using Measurable Selection (3.8), there is a \( \mu \)-measurable map \( \lambda \rightarrow B(\lambda) \) from \( \Pi_A(E_{1/k}) \) to \( C(h) \) so that \( (\lambda, B(\lambda)) \) is in \( E_{1/k} \) \( \mu \text{-a.e.} \). Now extend this map to \( A \) by setting it to zero on the complement of \( \Pi_A(E_{1/k}) \) and denote by \( B \) the operator in \( L(H) \) which this function represents. From the definition of \( E_{1/k} \) it follows that \( B \in \mathcal{C} \) and from the functional calculus of direct integral operators it follows that \( \| (TB)^n \|^{1/n} \geq 1/k \) for all \( n \). This contradicts the fact that \( T \) is in \( \text{rad}(\mathcal{C}) \).

**Remark 5.4.** Recall that if \( \mathcal{L} \) is a lattice then an \( \mathcal{L} \)-interval is a projection of the form \( Q - P \), where \( Q, P \in \mathcal{L} \), \( P \leq Q \). In the following discussion let \( \mathcal{E} \) denote the set of all \( \mathcal{L} \)-intervals. If \( \mathcal{L} = \text{lat}(\mathcal{C}) \) for some algebra \( \mathcal{C} \), then \( \mathcal{E} \) consists precisely of all the semi-invariant projections for \( \mathcal{C} \) by [15]. Whenever \( \mathcal{L} \) is reflexive then \( \mathcal{E} \) is necessarily closed in the strong operator topology. Indeed, we need only check that if \( E \) is a strong limit point of \( \mathcal{E} \) then \( E \) is semi-invariant for \( \text{alg} \mathcal{L} \); i.e., that \( EAE^+BE = 0 \) for all \( A, B \in \text{alg} \mathcal{L} \). But this follows from strong continuity of multiplication in the unit ball of \( L(H) \) together with the fact that every projection in \( \mathcal{E} \) has this property. The following lemma relates the intervals of a reflexive lattice \( \mathcal{L} \) to the intervals of its integrand lattices.

**Lemma 5.5.** Let \( \lambda \rightarrow \mathcal{L}(\lambda) \) be a measurable field of reflexive lattice and for each \( \lambda \) let \( \mathcal{E}(\lambda) \) denote the set of \( \mathcal{L}(\lambda) \)-intervals. Let \( \mathcal{L} = \bigcup \lambda \mathcal{L}(\lambda) \mu(\mathcal{d}\lambda) \) and let \( \mathcal{E} \) denote the set of \( \mathcal{L} \)-intervals. Then \( \lambda \rightarrow \mathcal{E}(\lambda) \) is measurable and \( \mathcal{E} = \bigcup \lambda \mathcal{E}(\lambda) \mu(\mathcal{d}\lambda) \).

**Proof.** Let \( E \in \mathcal{E} \). Then \( E = P - Q \), where \( Q \leq P \) are in \( \mathcal{L} \) and moreover \( E \) decomposes into projections \( P(\lambda) - Q(\lambda) \) with \( P(\lambda), Q(\lambda) \) in \( \mathcal{L}(\lambda) \) \( \mu \text{-a.e.} \). By the functional calculus of decomposable operators it follows that \( Q(\lambda) \leq P(\lambda) \) \( \mu \text{-a.e.} \) so that \( E(\lambda) = P(\lambda) - Q(\lambda) \in \mathcal{E}(\lambda) \) \( \mu \text{-a.e.} \).

Conversely let \( E = \int \lambda \mu(\mathcal{d}\lambda) \), where \( \mathcal{E}(\lambda) \in \mathcal{E}(\lambda) \). Let \( \text{Gr}^2(\mathcal{L}) = \{(\lambda, P, Q), P, Q \in L(h) \text{ and } P, Q \in \mathcal{L}(\lambda)\} \). Consider the set \( S = \{(\lambda, P, Q) \in A \times C(h) \times C(h); P \geq Q \text{ and } P - Q = E(\lambda)\} \cap \text{Gr}^2(\mathcal{L}) \). This set is a Borel set in \( A \times C(h) \times C(h) \) since by altering \( \mathcal{L}(\lambda) \) on a \( \mu \)-null set we may assume that \( \text{Gr}^2(\mathcal{L}) \) is a Borel set (4.5). Hence by Measurable Selection (3.8), we can obtain a \( \mu \)-measurable map \( F: \lambda \rightarrow \text{Gr}(h) \times C(h) \) so that \( (\lambda, F(\lambda)) \in S \) for \( \lambda \in \Pi_A(S) = A \). That is, \( F(\lambda) = (P(\lambda), Q(\lambda)) \), where \( \lambda \rightarrow P(\lambda) \) and \( \lambda \rightarrow Q(\lambda) \) are measurable, \( P(\lambda), Q(\lambda) \) are in \( \mathcal{L}(\lambda) \), \( P(\lambda) \geq Q(\lambda) \), and \( E(\lambda) = P(\lambda) - Q(\lambda) \) \( \mu \text{-a.e.} \). Clearly if we denote by \( P \) (resp. \( Q, F) \) the map \( \lambda \rightarrow P(\lambda) \) (resp. \( Q(\lambda), P(\lambda) - Q(\lambda) \)) it follows that \( F = \int \lambda \mu(\mathcal{d}\lambda) \) and \( E = F \in \mathcal{E} \).
Now let $\mathcal{L}$ be a subspace lattice, $\mathcal{A} = \text{alg} \mathcal{L}$, and suppose $\mathcal{L}_0$ is a commutative sublattice of $\mathcal{L}$. If $\varepsilon > 0$ we say that an operator $A \in \mathcal{A}$ has an $\mathcal{L}_0$ $\varepsilon$-paving if there exists a finite set $E_1, \ldots, E_n$ of mutually orthogonal $\mathcal{L}_0$-intervals with $\sum E_i = I$ such that $\|E_iAE_i\| \leq \varepsilon$ for each $i$. Such intervals are said to $\varepsilon$-pave $A$. For each $A$ with an $\mathcal{L}_0$ $\varepsilon$-paving we define the $\mathcal{L}_0$ $\varepsilon$-paving number $P_\varepsilon(A)$ to be the minimum cardinality among sets of $\mathcal{L}_0$-intervals which $\varepsilon$-pave $A$. (Note that we do not require the projections in $\mathcal{L}_0$ to be contained in $\mathcal{P}$.)

**Lemma 5.6.** Let $\mathcal{L}$ be a subspace lattice, and suppose $\mathcal{L}_0$ is a commutative sublattice of $\mathcal{L}$. If an operator $A \in \text{alg}(\mathcal{L})$ admits an $\mathcal{L}_0$ $\varepsilon$-paving for every $\varepsilon > 0$ then necessarily $A$ is in the Jacobson radical of $\text{alg} \mathcal{L}$.

**Proof.** Let $\mathcal{A} = \text{alg} \mathcal{L}$, $\mathcal{A}_0 = \text{alg} \mathcal{L}_0$, so $\mathcal{A}_0 \supseteq \mathcal{A}$ and $\mathcal{A}_0 \supseteq \mathcal{L}_0$. We will show that $A \in \text{rad}(\mathcal{A}_0)$ and hence $A \in \text{rad}(\mathcal{A})$. For this, let $\Pi$ be any strictly transitive representation of $\mathcal{A}_0$, and note that, since $\mathcal{L}_0 \subseteq \mathcal{A}_0$, for each $P \in \mathcal{L}_0$ the image $\Pi(P)$ must be either 0 or the identity. It follows that the image of every $\mathcal{L}_0$-interval is 0 or the identity. So if $E_1, \ldots, E_n$ are mutually orthogonal $\mathcal{L}_0$-intervals with $\sum E_i = I$ then $\Pi(A) = \Pi(\sum E_iAE_i)$, and hence $A - \sum E_iAE_i \in \text{kernel}(\Pi)$. But $\text{kernel}(\Pi)$ is necessarily closed [13], and the fact that $A$ can be $\varepsilon$-paved for every $\varepsilon$ implies that $A$ is the norm-limit of operators of the form $A - \sum E_iAE_i$. Hence $A \in \text{kernel}(\Pi)$. Since $\Pi$ was arbitrary, $A \in \text{rad}(\mathcal{A}_0)$. So $A \in \text{rad}(\mathcal{A})$ since $\mathcal{A} \cap \text{rad}(\mathcal{A}_0) \subseteq \text{rad}(\mathcal{A})$. 

In essence, the Ringrose criterion (1.3) provides the converse to Lemma 5.6 in the special case in which $\mathcal{L}$ is a nest an $\mathcal{L}_0 = \mathcal{L}$. The converse, with $\mathcal{L}_0 = \mathcal{L}$, has been shown to hold for certain more general commutative subspace lattices and progress has been made on the general commutative case in [7, 8]. This is the "radical problem" emphasized in [8].

The following theorem is the main result of this section and gives the appropriate generalization of Ringrose's characterization of the radical of a nest algebra to the general case of a nest-subalgebra of a von Neumann algebra.

**Theorem 5.7.** Let $\mathcal{A}$ be a nest-subalgebra of a von Neumann algebra $\mathcal{B}$ with nest $\mathcal{N}$ and let $\mathcal{L}_0$ be the complete lattice generated by $\mathcal{N}$ and the central projections in $\mathcal{B}$. If $T \in \mathcal{A}$ then $T \in \text{rad}(\mathcal{A})$ if and only if given $\varepsilon > 0$ there exists a finite set $\{E_i\}$ of mutually orthogonal $\mathcal{L}_0$-intervals with $\sum E_i = I$ for which $\|E_iTE_i\| \leq \varepsilon$ for all $i$.

**Proof.** Since $\mathcal{A}$ is reflexive the "if" part follows from Lemma 5.6. For the converse, let $\mathcal{A} = \int_A \oplus \mathcal{A}(\lambda) \mu(d\lambda)$ and $\mathcal{B} = \int_A \oplus \mathcal{B}(\lambda) \mu(d\lambda)$ be decompositions along the center of $\mathcal{B}$. By Theorem 4.4 each $\mathcal{A}(\lambda)$ is a nest-subalgebra of the factor $\mathcal{B}(\lambda)$ and in fact $\mathcal{A}(\lambda) = \mathcal{B}(\lambda) \cap \mathcal{A}(r_A \lambda)$. Suppose
$T \in \text{rad}(\mathcal{A})$ and let $\lambda \to T(\lambda)$ be a Borel representative for $T$. Then by Lemma 5.3 there is a $\mu$-null set $N$ such that $T(\lambda) \in \text{rad}(\mathcal{A}(\lambda))$ off $N$, so by Theorem 5.2 $T(\lambda) \in \text{rad}(\mathcal{A}(\mathcal{F}(\lambda)))$ off $N$. We may modify $T$ so that $T(\lambda) = 0$ on $N$. The Ringrose criterion for nests (1.3) implies that for each $\lambda$, $T(\lambda)$ can be $\mathcal{F}(\lambda)$- $\varepsilon$-paved for each $\varepsilon > 0$. Let $P_\varepsilon(\lambda)$ denote the $\mathcal{F}(\lambda)$- $\varepsilon$-paving number of $T(\lambda)$. We will show that the $P_\varepsilon(\lambda)$ are "essentially" bounded: that is, we will show that there exists a $\mu$-null set $\tilde{N}$ such that for each $\varepsilon > 0$ the set of positive integers $\{P_\varepsilon(\lambda): \lambda \in A \sim \tilde{N}\}$ is uniformly bounded. This will lead to a proof that $T$ can be $\mathcal{L}_0$- $\varepsilon$-paved for each $\varepsilon > 0$.

Let $\mathcal{F}$ denote the set of $\mathcal{L}_0$-intervals and denote by $G^*_{n}(\mathcal{F})$ the graph of the multifunction $\lambda \to \mathcal{F}(\lambda)^n$; that is, $((\lambda, E_1, \ldots, E_n); E_i \in L(h) \text{ and } E_i \in \mathcal{F}(\lambda))$. By Lemma 5.5 and (4.5) we may assume that $G^*_{n}(\mathcal{F})$ is a Borel set in $A \times C(h)^n$. Let $I(\lambda)$ denote the identity on $h_\lambda = h$ and set

$$E^\varepsilon_n = \{(\lambda, E_1, \ldots, E_n); \|E, T(\lambda), E_i\| \leq \varepsilon, \sum E_i = I(\lambda), E_i E_j = \delta_{ij} E_i \} \cap G^*_{n}(\mathcal{F}).$$

As in earlier proofs we see that $E^\varepsilon_n$ is a Borel set in $A \times C(h)^n$ for each $n$ and hence $\Pi_{n}(E^\varepsilon_n) = \{\lambda: P_\varepsilon(\lambda) \leq n\}$ is analytic in $A$. To show that $P_\varepsilon(\lambda)$ is essentially bounded we must show for some integer $N$ that $\mu(A - \Pi_{n}(E^\varepsilon_n)) = 0$. Note that for fixed $\varepsilon > 0$ the sets $E^\varepsilon_n$ form an increasing sequence. For convenience let $G^*_{n}$ denote the sets $A \setminus \Pi_{n}(E^\varepsilon_n)$, and note that for each $\varepsilon > 0$ the intersection $\cap G^*_{n}$ is empty.

To show that for each $\varepsilon > 0$ we have $\mu(G^*_{n}) = 0$ for some integer $N$ we argue by contradiction. Suppose that for some $\varepsilon > 0$ we have $\mu(G^*_{n}) > 0$ for all $n$. Then since $\cap G^*_{n} = \emptyset$ there exist mutually disjoint measurable sets $G_k \subset G^*_{n}$ with $\mu(G_k) > 0$ for $k = 1, 2, \ldots$. Now fix $k \geq 3$, fix $\lambda \in G_k$ and choose $n$ so that $2n < k$. We have $P_\varepsilon(\lambda) \geq k > 2n$. Let $k_0 = P_\varepsilon(\lambda)$ and let $Q_1, \ldots, Q_{k_0}$ be an $\mathcal{F}(\lambda)$- $\varepsilon$-paving for $T(\lambda)$. Since $k_0$ is the $\varepsilon$-paving number of $T(\lambda)$ the sum of any two adjacent intervals from $\{Q_1, \ldots, Q_{k_0}\}$ is an interval for which the compression of $T(\lambda)$ to it has norm $>\varepsilon$. Combining adjacent intervals in this fashion we obtain at least $n$ mutually orthogonal $\mathcal{F}(\lambda)$-intervals for which the compression of $T(\lambda)$ to each has norm $>\varepsilon$. Now by the remark following Theorem 5.2, there is a $B$ in $C(h) \cap \mathcal{F}(\lambda)$ so that $\|(BT(\lambda))^m\|^1/m > \varepsilon, m = 1, 2, \ldots, n - 1$.

If we denote $\text{Gr}(\mathcal{A}) = \{(\lambda, A) \in A \times C(h); A \in \mathcal{A}(\lambda)\}$ and

$$F^\varepsilon_n = \bigcup_{m = 1}^{n} \{(\lambda, B) \subset A \times C(h); \|(BT(\lambda))^m\| \geq \varepsilon^m\} \cap \text{Gr}(\mathcal{A}).$$

then as in previous arguments we may take $F^\varepsilon_n$ to be a Borel set in $A \times C(h)$. 
The preceding paragraph now implies that \( G_k \subset \Pi_A(F^c_{n-1}) \) for \( 2n \leq k \). In particular \( G_{3k} \subset \Pi_A(F^c_k) \), \( k \geq 2 \). Now Measurable Selection (3.8) implies the existence of measurable cross sections \( \lambda \to B_{3k}(\lambda) \) for which \( B_{3k}(\lambda) = 0 \) for \( \lambda \not\in G_{3k} \) and \( (\lambda, B_{3k}(\lambda)) \in F_k^c \) for \( \lambda \in G_{3k} \), that is, \( \|(B_{3k}(\lambda) T(\lambda))_k^k\| \geq \varepsilon^k \) for \( \lambda \in G_{3k} \). Now let \( B_{3k} = \int_A \oplus B_{3k}(\lambda) \mu(d\lambda) \) and note that the \( B_{3k} \) have mutually orthogonal supports. Set \( B = \sum B_{3k} \); then \( \|B\| \leq 1 \) and \( B \in \mathcal{A} \). Furthermore

\[
\| (BT)^k \| = \text{ess sup}_{\lambda \in A} \| (B(\lambda) T(\lambda))^k \| \\
\geq \text{ess sup}_{\lambda \in G_{3k}} \| (B(\lambda) T(\lambda))^k \| \geq \varepsilon^k
\]

for \( k = 2, 3, \ldots \). So \( BT \) is not quasinilpotent, thus contradicting the hypothesis that \( T \in \text{rad} \mathcal{A} \). We have thus proved that \( \lambda \to P_{c}(\lambda) \) is essentially bounded for each \( \varepsilon > 0 \).

Finally we can show that \( T \) has an \( \mathcal{L}_0 \) \( \varepsilon \)-paving for all \( \varepsilon > 0 \). Given \( \varepsilon > 0 \), let \( P_{c}(\lambda) \) be \( \mu \)-essentially bounded by \( n \) so that \( \lambda \in \Pi A E_n^c \mu \)-a.e. Thus using Measurable Selection (3.8), and recalling the definition of \( E_n^c \), there exists \( n \) maps \( \lambda \to E_i(\lambda) \) satisfying the conditions of the set \( E_n^c \). Now by (5.5) there exists \( E_1, \ldots, E_n \) in \( \mathfrak{B} \) so that \( E_i = \int_A \oplus E_i(\lambda) \mu(d\lambda) \). Clearly the set \( \{E_i\} \varepsilon \)-paves \( T \). 

The following is an immediate corollary of the above theorem and generalizes the result of Theorem 5.2 concerning factors. This result should be compared with the results in the next section where corollaries of (5.6) in a different direction are given.

**Corollary 5.8.** Let \( \mathcal{A} \) be a nest-subalgebra of \( \mathcal{B} \) with respect to the nest \( \mathcal{N} \). If \( \mathfrak{B} \) is the center of \( \mathcal{B} \), and \( \overline{\mathcal{A}} \) is the nest-subalgebra of \( \mathfrak{B} \) with respect to \( \mathcal{N} \), then \( \text{rad} \mathfrak{B} = \text{rad} \mathcal{A} \cap \mathfrak{B} \).

**Remark.** The nest-subalgebra \( \overline{\mathcal{A}} \) has commutative lattice \( \mathcal{L} = \mathcal{M} \vee \mathcal{N} \), where \( \mathcal{M} \) are the projections in \( \mathfrak{B} \). Another way to think of \( \overline{\mathcal{A}} \) is as the direct integral of the nest algebras which are involved in the decomposition of \( \mathcal{A} \) into nest-subalgebras of factors. That is, if \( \mathfrak{B} = \int \oplus \mathcal{A}_{x,} \cap \mathcal{B}_{x} \mu(d\lambda) \), where \( \mathcal{B}_{x} \) is a factor a.e., then \( \overline{\mathcal{A}} = \int \oplus \mathcal{A}_{x,} \cap \mathcal{B}_{x} \mu(d\lambda) \).

6. \( \text{rad} \mathcal{A} \) verses \( \text{rad} \mathcal{A}_{x} \).

In the main theorem in Section 5 we gave a characterization of the radical of a nest-subalgebra \( \mathcal{A} \) of a von Neumann algebra \( \mathcal{B} \) with respect to a nest \( \mathcal{N} \subset \mathcal{B} \). This characterization along with the result (5.2) that \( \text{rad} \mathcal{A}_{x} \supset \text{rad} \mathcal{A} \) whenever \( \mathcal{B} \) is a factor leads to questions concerning the relationship...
between \( \text{rad } \mathcal{A} \) and \( \text{rad } \mathcal{A} \). While it is usually true that \( \text{rad } \mathcal{A} \subseteq \text{rad } \mathcal{A} \) (2.4), we are led by Theorem 5.7 to consider this relationship for other nests in \( \mathcal{B} \). We show in this section that in fact \( \text{rad } \mathcal{A} \subseteq \bigcup \{ \text{rad } \mathcal{A} \} \), where \( \mathcal{L} = \mathcal{V} \vee \mathcal{M} \) is, as in Section 5, the join of \( \mathcal{V} \) and the projections in \( \text{cent}( \mathcal{B} ) \). More striking is the fact that a countable subset of \( \text{rad } \mathcal{A} \) can be included in a single \( \text{rad } \mathcal{A} \). However, in general the entire radical of \( \mathcal{A} \) cannot be included in the radical of a single nest algebra. Hence radicals of nest-subalgebras of von Neumann algebras are not, in general, contained in radicals of nest algebras. In addition to the factor case it is shown that \( \text{rad } \mathcal{A} \subseteq \text{rad } \mathcal{A} \) whenever \( \mathcal{V} \) is a finite nest. We show in Example 6.8 that these are essentially the only cases.

We begin by giving a technical lemma concerning finite subnests of a commutative subspace lattice.

**Lemma 6.1.** Let \( \mathcal{L} \) be a commutative lattice, and let \( \mathcal{A}_1, \mathcal{A}_2 \) be finite nests in \( \mathcal{L} \). Then there exists a finite nest \( \mathcal{A}_3 \) in \( \mathcal{L} \) such that

(i) \( \mathcal{A}_1 \subseteq \mathcal{A}_3 \),

(ii) every interval of \( \mathcal{A}_2 \) is the finite join of intervals of \( \mathcal{A}_3 \),

(iii) every minimal interval of \( \mathcal{A}_3 \) is a subprojection of a minimal interval of \( \mathcal{A}_1 \) and of \( \mathcal{A}_2 \).

**Proof.** Let \( \mathcal{A}_1 = \{ P_i \}_{i=0}^k \) and \( \mathcal{A}_2 = \{ Q_j \}_{j=0}^l \), where \( 0 = P_0 \leq P_1 \leq \cdots \leq P_k = I \) and similarly for \( \{ Q_j \} \). Define \( R_{i,j} = (P_i \vee Q_j) \wedge P_{i+1}, \ 0 \leq i \leq k-1 \) and \( 0 \leq j \leq l \). Notice that \( R_{i,j} - P_i \) is just the part of \( Q_j \) lying between \( P_i \) and \( P_{i+1} \). The family \( \mathcal{A}_3 = \{ R_{ij} \} \) is a nest ordered lexicographically; i.e.,

\[
0 = R_{00} \leq \cdots \leq R_{0l} = P_1 \leq R_{10} \leq \cdots \leq R_{1l} = P_2 \leq \cdots \leq R_{k-1,l} = P_k = I,
\]

where the \( R_{ij} \) are generally not distinct. Every minimal interval of \( \mathcal{A}_3 \), \( R_{ij} - R_{i,j-1} \) or \( R_{i,0} - R_{i-1,i} \), is a subinterval of \( Q_j - Q_{j-1} \) and \( P_{i+1} - P_i \) or respectively \( I - Q_{j-1} \) and \( P_i - P_{i-1} \). On the other hand \( Q_j - Q_{j-1} = \sum_{i=0}^{k-1} R_{ij} - R_{i,j-1} \) which are distinct intervals in \( \{ R_{ij} \} = \mathcal{A}_3 \).

In the following theorem let \( \mathcal{L} = \mathcal{V} \vee \mathcal{M} \) be the lattice generated by the nest \( \mathcal{V} \) together with the projections \( \mathcal{M} \) in \( \text{cent}( \mathcal{B} ) \).

**Theorem 6.2.** Let \( \mathcal{A} \) be a nest-subalgebra of a von Neumann algebra \( \mathcal{B} \) with respect to a nest \( \mathcal{V} \). If \( \{ T_n \} \) is any countable subset of \( \text{rad}( \mathcal{A} ) \) there is a nest \( \mathcal{A}_0 \subseteq \mathcal{L} \) such that \( \{ T_n \} \subseteq \text{rad}( \mathcal{A}_0 ) \).

**Proof.** Decompose the nest-subalgebra \( \mathcal{A} \) via its center. Then \( \mathcal{A} = \int \Lambda \oplus \mathcal{A}_\Lambda \mu(\mathcal{A}_\Lambda) \), where \( \mathcal{A}_\Lambda \) is a nest-subalgebra of a factor. Fix a \( T \) in \( \text{rad}( \mathcal{A} ) \) and
let $\varepsilon > 0$ be given. By Theorem 5.7 there exists intervals $E_1, \ldots, E_n$ from $\mathcal{L} = \mathcal{N} \setminus \mathcal{M}$ so that $||E_i T E_i|| < \varepsilon$ for all $i$ and $\sum E_i = I$. Now $E_i = \int_\Lambda \oplus E_\lambda(\mu)\,d\lambda$, and $\{E_1(\lambda), \ldots, E_n(\lambda)\}$ are intervals from the nest $\mathcal{N}_\lambda$ $\mu$-a.e. with $I(\lambda) = \sum E_\lambda(\lambda)$. Considering endpoints of the intervals $E_\lambda(\lambda)$, by Measurable Selection (3.8) obtain a finite nest $\mathcal{N}$ in $\mathcal{L}$ with minimal intervals $E_1, \ldots, E_n$ perhaps different from $E_1, \ldots, E_n$ but such that the sets $\{E_1(\lambda), \ldots, E_n(\lambda)\}$ and $\{E_1(\lambda), \ldots, E_n(\lambda)\}$ agree $\mu$-a.e. except perhaps for order. Now $||E_i T E_i|| = \text{ess sup} \{E_i(\lambda) T(\lambda) E_i(\lambda)|| < \varepsilon$ for all $i$ since $\mu$-a.e. $||E_j(\lambda) T(\lambda) E_j(\lambda)|| < \varepsilon$ for all $j$.

We have shown that for each fixed $T \in \text{rad}(\mathcal{O})$ and fixed $\varepsilon > 0$ there exists a finite nest $\mathcal{N}$ in $\mathcal{L}$-intervals from which $\varepsilon$-pave $T$. So if $(T_n)$ is a countable subset of $\text{rad}(\mathcal{O})$ there exist finite nests $\mathcal{N}_n$ in $\mathcal{L}$ such that $T_n$ is $1/j$-paved by intervals from $\mathcal{N}_n$. Let $\kappa(n)$ be an enumeration of all pairs of natural numbers and using Lemma 6.1 inductively generate a nested sequence of nests $\mathcal{N}_{\kappa(n)}$. Since the $\mathcal{N}_{\kappa(n)}$ are nested their union is a nest in $\mathcal{L}$. Let $\mathcal{N}_0$ be its completion. Then each $T_n$ can be $\varepsilon$-paved by intervals from $\mathcal{N}_0$ for every $\varepsilon > 0$. Since $\{T_n\} \subset \mathcal{O}$ and $\mathcal{N}_0 \subset \text{lat}(\mathcal{O})$ we have $\{T_n\} \subset \mathcal{O}_0$, so the Ringrose criterion for nest algebras (1.3) now shows that $\{T_n\} \subset \text{rad}(\mathcal{O}_{0})$.

If $T \in \text{rad}(\mathcal{O})$ then there exists a nest $\mathcal{N}_T \subset \mathcal{L}$ such that $T \in \text{rad}(\mathcal{O}_{0})$. That is, every $T$ in $\text{rad}(\mathcal{O})$ can be $\varepsilon$-paved by intervals from a nest $\mathcal{N}_T$ in $\text{lat}(\mathcal{O})$ which depends on $T$ but not on $\varepsilon$. The following corollary is actually equivalent to Theorem 5.7.

**Corollary 6.3.** $\text{rad}(\mathcal{O}) = \bigcup \{\text{rad}(\mathcal{O}_{0}) \cap \mathcal{B} : \mathcal{N}_0 \text{ nest in } \mathcal{L} = \mathcal{N} \setminus \mathcal{M}\}$.

**Corollary 6.4.** Every countably generated topologically nil ideal in $\mathcal{O}$ is contained in the radical of some nest algebra.

**Remark.** Corollary 6.3 raises the question of whether containment of an operator in the radical of an arbitrary reflexive operator algebra implies containment in the radical of some nest algebra. Note that if $T$ is contained in the radical of any reflexive operator algebra then $T$ is contained in the radical of $\text{Alg Lat}(T)$. Which quasinilpotents have this property? It is a result of Ringrose that every compact quasinilpotent is contained in the radical of some nest algebra and thus has this property.

The following results show that except for the simplest cases one cannot obtain a single nest $\mathcal{N}_0^\prime$ in $\mathcal{L}$ which will pave every $T$ in $\text{rad}(\mathcal{O})$. This is striking since we have shown above that a single nest exists, intervals of which $\varepsilon$-pave for every $\varepsilon$ a set whose strong closure contains $\text{rad}(\mathcal{O})$. If $\text{rad}(\mathcal{O})$ were norm separable or even countably generated as a norm closed ideal it would follow from (6.2) that $\text{rad}(\mathcal{O})$ would be entirely contained in the radical of some nest algebra. That this property fails points out in particular the failure of $\text{rad}(\mathcal{O})$ to be countably generated.
We begin with a lemma concerning elements of the radical of a nest algebra $\mathcal{N}$ and subnests $\mathcal{N}_0$ of $\mathcal{N}$ which are necessary to $\varepsilon$-pave them.

**Lemma 6.5.** Let $\mathcal{N}$ be a nest-subalgebra of a factor and $P_0$ an element of the nest $\mathcal{N}$ so that either $P_{0-} = P_0 = P_{0+}$ or $P_{0-} \neq P_0 \neq P_{0+}$. Then there exists an element $T_0$ in $\text{rad} \mathcal{N}$ so that any $\varepsilon$-paving from $\mathcal{N}$ must contain an interval with $P_0$ as an endpoint.

**Proof.** In case $P_{0-} = P_0 = P_{0+}$, let $E_n$ and $F_n$ be sequences of disjoint intervals from $\mathcal{N}$ such that the endpoints of $E_n$ are increasing, are less than $P_0$ and are strongly approaching $P_0$ and respectively for $F_n$, decreasing, greater than $P_0$ and strongly approaching $P_0$. By the Comparison Theorem there exist partial isometries $S_n$ in $\mathcal{B}$ with initial space in $F_n$ and final space in $E_n$. These partial isometries are in $\text{rad} \mathcal{N}$, by (2.2) and so in $\text{rad} \mathcal{N}$ and any $\varepsilon$-paving for $S_n$ must contain as an endpoint some element of $\mathcal{N}$ which lies between the lower endpoint of $E_n$ and the upper endpoint of $F_n$. For example, the intervals $\{P_0, I - P_0\}$ $\varepsilon$-pave $S_n$ for all $\varepsilon$ and $n$. The sum $S = \sum S_n$ also lies in $\text{rad} \mathcal{N}$ by (2.2). Notice that any paving of $S$ from $\mathcal{N}$ which has $P_0$ as an interior point for some interval contains some intervals $E_n$ and $F_n$ for large enough $n$. Hence since $\|S_n\| = 1$ such a paving cannot be an $\varepsilon$-paving for $\varepsilon < 1$. For case $P_{0-} \neq P_0 \neq P_{0+}$ only one partial isometry is needed and the argument follows as before. $\blacksquare$

**Remark 6.6.** This lemma may fail if $P_{0-} \neq P_0 = P_{0+}$ or vice-versa. The following is a concrete example of this lemma and illustrates the fact that for paving the radical the nest cannot be reduced by even a single element. It is important to note that one cannot in general restrict pavings to intervals from denumerable (even dense) subsets of $\mathcal{N}$.

Let $H = L_2(0, 1)$ with Lebesgue measure. $\mathcal{N} = \{N_t : N_t = L_2(t, 1), t \in [0, 1]\}$ and $\mathcal{B} = L(H)$. Fix a $t_0$ in $(0, 1)$, let $\delta < \min(t_0, 1 - t_0)$ and define the map $S$ on $L_2(0, 1)$ by $S(f) = g$, where $g(t) = \chi_{(t_0 - \delta, t_0)}(t)f(2t_0 - t)$. This partial isometry reflects the part of $f$ between $t_0$ and $t_0 + \delta$ to $t_0 - \delta$ to $t_0$ about $t_0$. It is clear as in the proof of the lemma that $S$ is in $\text{rad} \mathcal{N}$ and any $\varepsilon$-paving of $S$ from $\mathcal{N}$ must contain an interval with an endpoint $N_{t_0}$.

The following is a useful corollary of this lemma.

**Corollary 6.7.** Let $P$ and $Q$ be members of a nest $\mathcal{N}$ which determines a nonminimal interval $E$ of $\mathcal{N}$ and let $\mathcal{B}$ be a factor von Neumann algebra containing $\mathcal{N}$.

There exists a partial isometry supported on $E$ which is in $\text{rad} \mathcal{N} \cap \mathcal{B}$ and for which every $\varepsilon$-paving ($\varepsilon < 1$) from $\mathcal{N}$ must contain an interval with an endpoint which lies between $P$ and $Q$.

**Proof.** Assume $P < R < Q$, where $R$ is in $\mathcal{N}$ and the ordering is strict.
Let $\mathcal{I}_0$ be the nest $\{0, P, R, Q, I\}$. Using (6.6) there exists a partial isometry $S$ in $\mathcal{B}$ and in rad $\mathcal{I}, \mathcal{I}_0$ so that $R$ must be an endpoint for some interval in every $\mathcal{N}_0$ $\varepsilon$-paving ($\varepsilon < 1$) for $S$. Thus every $\mathcal{N}$ $\varepsilon$-paving must use as an endpoint a member of $\mathcal{N}$ between $P$ and $Q$.

Before proving the next theorem we must consider how the size of the nest $\mathcal{I}$ is related to the von Neumann algebra $\mathcal{B}$. We shall call a nest $\mathcal{I}$ $\mathcal{B}$-finite if there exists a finite nest $\mathcal{I}_0$ in $\mathcal{B}$ so that the nest-subalgebras of $\mathcal{B}$ generated by $\mathcal{I}$ and $\mathcal{I}_0$ are the same. We shall call a nest $\mathcal{I}$ $\mathcal{B}$-infinite otherwise. The following is an example of an infinite nest $\mathcal{I}$ which is $\mathcal{B}$-finite with respect to a von Neumann algebra $\mathcal{B}$ containing it.

**Example 6.8.** Let $H = \sum_{n=0}^{\infty} H_n$, and $\mathcal{B} = \sum_{n=0}^{\infty} L(H_n)$. Furthermore let $\mathcal{N}_1 = \{0, P_1, I_1\}$, $0 \neq P_1 \neq I_1$, be a nest on $H_1$. Let $\mathcal{I}$ be the ordinal sum of these nests. Then $\mathcal{I}$ is $L(H)$-infinite (i.e., infinite) yet it is $\mathcal{B}$-finite.

Moreover we may exclude from consideration a part of a nest-subalgebra $\mathcal{A}$ which is self-adjoint. That is, a nest-subalgebra will be called completely non-self-adjoint if there is no central decomposition of the space whereon one part $\mathcal{A}$ is a self-adjoint algebra. Given a nest-subalgebra $\mathcal{A}$ of a von Neumann algebra $\mathcal{B}$, there is a unique central decomposition of the space so that correspondingly $\mathcal{A}$ is decomposed into completely non-self-adjoint parts and a self-adjoint part. Notice that the self-adjoint part is equal to the corresponding part of the von Neumann algebra $\mathcal{B}$.

In the theorem below we assume as usual that $\mathcal{A}$ is a nest-subalgebra of $\mathcal{B}$ with respect to the nest $\mathcal{I}$ and denote by $\mathcal{M}$ the commutative lattice generated by $\mathcal{I}$ and the projections $\mathcal{M}$ in the center of $\mathcal{B}$.

**Theorem 6.9.** Let $\mathcal{A}$ be a completely non-self-adjoint nest-subalgebra of a von Neumann algebra $\mathcal{B}$ with nest $\mathcal{I}$. Then there exists a nest $\mathcal{I}_0$ in $\mathcal{M}$ so that rad $\mathcal{A}$ = rad $\mathcal{A} \cap \mathcal{B}$ if and only if there exists a central projection $P_0$ so that

\begin{enumerate}
  \item $\mathcal{I} \mid P_0 H$ is $\mathcal{B} \mid P_0 H$-finite and
  \item $\mathcal{B} \mid P_0 H$ is the finite sum of factors.
\end{enumerate}

**Proof.** That such a central decomposition of the space for which (i) and (ii) are satisfied is sufficient follows from earlier results. On $P_0 H$, the result is true by taking the ordinal sum of the nests obtained by restricting $\mathcal{A}$ to the minimal central projections of $\mathcal{B} \mid P_0 H$ and applying Theorem 5.7. On $P_0 H$ we take a finite nest $\mathcal{N}_1$ so that $\mathcal{N}_1 \mid P_0 H$ generates $\mathcal{A} \mid P_0 H$. Letting $\mathcal{I}_0$ be the ordinal sum of these two nests, one on $P_0 H$ and one on $P_0 H$, the result follows.

To show the converse we let $\mathcal{I}_0$ be a nest in $\mathcal{M}$ and assume that such a
central decomposition of the algebras as given by (i) and (ii) does not exist. Let $H = \int_{\Lambda} \oplus H_{\lambda} \mu(\lambda) \mu(\lambda)$ be the decomposition of $H$ via the center of $\mathcal{A}$ and $\mathcal{H}$ be the projections in the center of $\mathcal{B}$. Thus we have $\mathcal{B} = \int_{\Lambda} \oplus \mathcal{B}(\lambda) \mu(\lambda) \mu(\lambda)$. By (3.4) it follows that $\mathcal{B}(\lambda)$ is central a.e. If $\mathcal{B}(\lambda) < N$ for $\lambda \in \Lambda_n$, and if $P_0$ is the central projection with support $\Lambda_n$, it follows that $\mathcal{A} P_0 H$ is $\mathcal{A} P_0 H$-finite. Thus if (i) and (ii) do not hold, there are measurable sets $A_n$ and corresponding nontrivial central projections $P_n$ such that $\mathcal{B}(\lambda) > n \mu$-a.e. on $\Lambda_n$, $n = 1, 2, \ldots$.

Consider first the case when $\mathcal{H}$ is a finite nest $0 = R_1 < R_2 < \cdots < R_n = 1$. Fix some $k, k > n$. Using a pigeon hole argument there exists an $i$ and a $Q_k$ in $\mathcal{H}$ supported on $A_k$ so that $R_i < Q_k < R_i+1$ and $R_i(\lambda) < Q_k(\lambda) < R_i+1(\lambda)$ on a set $A'_k$ of positive measure in $\Lambda_k$. (In this proof $< \mu$ will also mean $\mu$.) Using (6.7) for $\lambda$ in $A'_k$ there exists a partial isometry in $\mathcal{N}$ for which every $\varepsilon$-paving by $\mathcal{A}$ requires an interval with an endpoint strictly between $R_i(\lambda)$ and $R_i+1(\lambda)$. By an argument using Measurable Selection (3.8) there then exists a nonzero partial isometry $S$ supported on $A_k$ and in $\mathcal{H}$ so that every $\varepsilon$-paving of $S$ requires an interval of $\mathcal{A}$ which lies strictly between $R_i$ and $R_i+1$. Thus by (1.3) $S$ cannot be in $\mathcal{H}$. The case when $\mathcal{H}$ is infinite is similar however more technical. Begin by fixing a dense set $\{Q_n\}$ in $\mathcal{H}$ and fixing Borel representatives $Q_n(\lambda)$ of them. Recall that $\{Q_n(\lambda)\}$ is dense (s.o.t.) in $\mathcal{H}(\lambda)$ $\mu$-a.e. If for some $n$ card$\{\mathcal{H}(\lambda)\} < n$ on a set of positive measure in $\Lambda_n$, then by the argument in the preceding paragraph we obtain an operator $S$ in $\mathcal{H}$ which is not in rad $\mathcal{H}$. Thus we may assume that card$\{\mathcal{H}(\lambda)\} \geq n$ $\mu$-a.e. on $\Lambda_n$. In fact when card$\{\mathcal{H}(\lambda)\}$ is finite then card$\{\mathcal{H}(\lambda)\} = card\{Q_n(\lambda)\}$. Thus for each $k$, there exists a set of positive measure $A'_k$ and $k$ distinct members $Q_{k1}, \ldots, Q_{kk}$ of $\{Q_n\}$ so that $0 = Q_{k1}(\lambda) < Q_{k2}(\lambda) < \cdots < Q_{kk}(\lambda) = 1(\lambda)$ for $\lambda$ in $A'_k$. Next by dropping to a subsequence of $A_n$ and taking every other $Q_{ni}$ if necessary, we may assume that each interval of the set $\{Q_{nk}\}$ is nonminimal with respect to $\mathcal{H}$. In fact we need and can have that on $\Lambda'_n$ each interval of $\{Q_{nk}(\lambda)\}$ will be nonminimal with respect to $\mathcal{H}(\lambda)$.

We shall obtain, using a Bolzano–Weierstrass type argument, a sequence $n_i$ and corresponding disjoint intervals $\{E_{ni}\}$ of $\mathcal{H}$ so that $E_{ni} P_n H$ is nonminimal with respect to $\mathcal{H} P_n H$ and in fact for $\lambda$ in $A'_n$, $E_{ni}(\lambda)$ will be nonminimal with respect to $\mathcal{H}(\lambda)$. From this we will show that rad $\mathcal{H} \subseteq$ rad $\mathcal{H}$. Let $n_1 = 3$ and consider $0 = Q_{31} < Q_{32} < Q_{33} = 1$. Since $\mathcal{H}$ is a nest, for each $n > 3$, we have $Q_{ni} \leq Q_{n+1} \leq Q_{n+1} (\lambda)$ for some $i$. Thus $n/2$ elements of $\{Q_{ni}\}$ lie above or below $Q_{33}$. Hence there exists a sequence $m_i$ so that $m_i/2$ of the elements of $\{Q_{mi}\}$ lie above $Q_2$ (or if not a sequence so that they all lie below). In the former case set $E_{ni} = Q_{32}$ and $F_{ni} = I - Q_{32}$ and in the latter case reverse them. Now assume that $n_1, \ldots, n_k$ are chosen and corresponding mutually disjoint intervals $E_{n1}, \ldots, E_{nk}, F_{n1}, F_{nk}$, where $E_{ni}$ are from
\{Q_{n_{j},i}\} and \{F_{n_{k}}\} from \mathcal{I}_{0} so that there exists a sequence \(m_{i}\) depending on \(k\) for which \(m_{i}/2^{k}\) elements of \{Q_{m_{i},j}\} lie between the endpoints of \(F_{n_{k}}\). Choose \(n_{k+1}\) to be \(m_{i_{0}}\), where \(m_{i_{0}}/2^{n_{k}} > 3\), and fix three elements \(P < Q < R\) in \{Q_{n_{k+1},j}\} which lie between the endpoints of \(F_{n_{k}} = S_{n_{k}} - T_{n_{k}}\). As in the argument above, for each \(m_{i} \geq m_{i_{0}} = n_{k+1}\) half the members of \{Q_{m_{i},j}\} which lie between the endpoints of \(F_{n_{k}}\) (or \(\geq m_{i}/2^{k+1}\)) must be either greater or less than \(Q\). Thus there exists a subsequence \(m'_{i}\) of \(m_{i}\) which depends on \(k + 1\) so that those aforementioned members of \(Q_{m',j}\) lie above \(Q\) (or if not a subsequence so they all lie below). If the former let \(E_{k+1} = Q - P\) and \(F_{k+1} = S_{n_{k}} - Q\) and if the latter let \(E_{k+1} = R - Q\) and \(F_{n_{k+1}} = Q - T_{n_{k}}\). Thus by induction we obtain the disjoint intervals \(E_{n_{k}}\) of \(\mathcal{I}_{0}\).

The balance of the proof follows similarly to the case where \(\mathcal{I}_{0}\) was finite. For each \(n_{j}\) we use (6.7) and Measurable Selection (3.8) to construct a partial isometry \(S_{n_{j}}\) in \(\text{rad } \mathcal{I}\) supported on \(A_{n_{j}}'\) and on \(E_{n_{j}}\) so that \(S_{n_{j}}\) has an \(\mathcal{I}\) \(\varepsilon\)-paving number of 2 for all \(\varepsilon\) and every \(\varepsilon\)-paving (\(\varepsilon < 1\)) must contain an interval with an endpoint strictly between the endpoints of \(E_{n_{j}}\). Since \(A_{n_{j}}\) are disjoint, \(S = \sum S_{n_{j}}\) is a partial isometry. By the construction it can be \(\varepsilon\)-paved by \(\mathcal{I}\)' and so it is in \(\text{rad } \mathcal{I}\) by (5.7). However, any \(\mathcal{I}_{0}\) interval with an endpoint in \(E_{n_{j}}\) must have its endpoint in \(E_{n_{j}}\). Thus in order to \(\mathcal{I}_{0}\) \(\varepsilon\)-pave \(S\) we would need intervals with endpoints in each \(E_{n_{j}}\). Since these are a denumerable collection of disjoint intervals this would be impossible.

**Remark.** In the proof we could make the construction so that \(S\) can be \(\varepsilon\)-paved with intervals from \(\mathcal{I}_{0} \setminus \mathcal{I}\). In fact \(S\) is \(\varepsilon\)-paved with intervals from \(\mathcal{I} = \mathcal{I}_{0} \setminus \mathcal{I}\) but cannot be \(\varepsilon\)-paved (\(\varepsilon < 1\)) by intervals from \(\mathcal{I}_{0}\).


**References**


