Two inverse eigenvalue problems for a special kind of matrices

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Abstract

In this paper, a special kind of matrices which are symmetric, all elements are equal to zero except for the first row, the first column and the diagonal elements, and the elements of the first row are positive except for the first one are considered. Two inverse problems are discussed. One is to construct one of this kind of matrices by the minimal and maximal eigenvalues of its all leading principal submatrices. The other is to construct one of this kind of matrix by one of its eigenpair and eigenvalues of its all leading principal submatrices. The necessary and sufficient conditions for the solvability of the two problems are derived. Furthermore, corresponding numerical algorithms and some examples are given.

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1. Introduction

Usually, there is some specific application background of inverse eigenvalue problems. For example, we often meet the following type of questions of non-linear regulation systems in modern control theory.
We consider the movement equation with some perturbations in regulation systems \([1,2]\)

\[
\dot{x}_i = \lambda_i x_i + f(\sigma), \quad f(\sigma) \in k(0, \infty) \quad (i = 1, 2, \ldots, n)
\]

\[
\dot{\sigma} = v^T X - \rho f(\sigma), \quad \rho > 0,
\]

where \(\lambda_i, \rho, v, X\) have been given and \(v = (v_1, v_2, \ldots, v_n)^T, X = (x_1, x_2, \ldots, x_n)^T\).

Now we consider the simplest Liapunov function

\[
V = \frac{1}{2} \sum_{i=1}^{n} x_i^2 + R \int_{0}^{\sigma} f(\sigma) \, d\sigma, \quad R > 0.
\]

The derivative of its solution along the system is

\[
\dot{V} = -\sum_{i=1}^{n} (-\lambda_i) x_i^2 - \rho R f^2(\sigma) + \sum_{i=1}^{n} (1 + Rv_i) x_i f(\sigma)
\]

\[
= -\left( \begin{array}{c} X \\ f(\sigma) \end{array} \right)^T W \left( \begin{array}{c} X \\ f(\sigma) \end{array} \right),
\]

where

\[
W = \begin{pmatrix} A & g \\ \rho R \\ g^T \end{pmatrix}, \quad A = \begin{pmatrix} -\lambda_1 & & \\ & \ddots & \\ & & -\lambda_n \end{pmatrix}, \quad g = \begin{pmatrix} -\frac{1}{2}(1 + Rv_1) \\ \vdots \\ -\frac{1}{2}(1 + Rv_n) \end{pmatrix}.
\]

So it's enough to study the properties of \(W\) instead of the system. This shows that it is practical to study the properties of matrix \(W\) and its inverse eigenvalue problem, which is the problem we will discuss.

For convenience, we discuss matrices with the following form

\[
A_n = \begin{pmatrix} a_1 & b_1 & b_2 & \cdots & b_{n-1} \\
 b_1 & a_2 & 0 & \cdots & 0 \\
 b_2 & 0 & a_3 & \cdots & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots \\
 b_{n-1} & 0 & 0 & \cdots & a_n \end{pmatrix},
\]

where \(a_i\) are distinct for all \(i = 2, 3, \ldots, n\) and all \(b_i\) are positive.

Without loss of generality, suppose that \(a_2 > a_3 > \cdots > a_n\). Clearly, discussing the inverse eigenvalue problem of \(W\) is equivalent to discussing the inverse eigenvalue problem of \(A_n\), because we can always get this kind of form of matrix \(A_n\) from \(W\) by permutation similarity transformations.

Throughout this paper, we use \(A_n\) to denote the special kind of matrices defined as in (1) and \(A_j\) to denote the \(j \times j\) leading principal submatrix of \(A_n\).

Notice that the matrix \(A_n\), the same with the \(n \times n\) Jacobi matrix, has \(2n - 1\) non-zero elements. Jacobi inverse eigenvalue problem has of great value for many applications, and a series of good conclusions have been made [3–7]. Similarly, some inverse eigenvalue problems for \(A_n\) have been
proposed, such as to construct a matrix $A_n$ by two sets of spectrum datas [8] and to construct a matrix $A_n$ by two given eigenpairs [9]. In this paper, we mainly consider the following problems:

**Problem I.** For $2n - 1$ given real number $\lambda^{(n)}_1 < \lambda^{(n-1)}_1 < \cdots < \lambda^{(2)}_1 < \lambda^{(1)}_1 < \lambda^{(2)}_1 < \cdots < \lambda^{(n)}_n$, finding an $n \times n$ matrix $A_n$, such that $\lambda^{(j)}_1$ and $\lambda^{(j)}_j$ are, respectively, the minimal and the maximal eigenvalues of $A_j$ for all $j = 1, 2, \ldots, n$.

**Problem II.** For $n$ given real number $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$ and a real vector $X = (x_1, x_2, \ldots, x_n)^T$, finding an $n \times n$ matrix $A_n$, such that $\lambda^{(j)}$ is the eigenvalue of $A_j$ for all $j = 1, 2, \ldots, n$ and $(\lambda^{(n)}, X)$ is an eigenpair of $A_n$.

We first discussed some properties of $A_n$. Then the necessary and sufficient conditions for solvability of Problems I and II are derived respectively. Finally, two numerical algorithms and examples are given.

2. Properties of the matrix $A_n$

For convenience of discussion later on, define $b_0 = 1$, and let $\varphi_j(\lambda) = \det(\lambda I_j - A_j)$, $\varphi_0(\lambda) = 1$.

**Lemma 1.** For a given matrix $A_n$, the sequence $\{\varphi_j(\lambda)\}$ satisfies the recurrence relation

$$
\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b^2_{j-1} \prod_{i=2}^{j-1} (\lambda - a_i), \quad j = 2, 3, \ldots, n.
$$

**Proof.** It is easy to verify by expanding the determinant. \(\square\)

**Lemma 2.** The eigenvalues of matrix $A_n$ are real and distinct. Moreover, if the eigenvalues of $A_n$ are

$$
\lambda_1 > \lambda_2 > \cdots > \lambda_n,
$$

then

$$
\lambda_1 > a_2 > \lambda_2 > \cdots > \lambda_{n-1} > a_n > \lambda_n.
$$

**Proof.** Assume that $\lambda_0$ is an eigenvalue of $A_n$. Then, by Lemma 1, we get

$$
\varphi_n(\lambda_0) = \det(\lambda_0 I_n - A_n)
= (\lambda_0 - a_n)\varphi_{n-1}(\lambda_0) - b^2_{n-1} \prod_{i=2}^{n-1} (\lambda_0 - a_i)
= \prod_{i=1}^{n} (\lambda_0 - a_i) - \sum_{j=1}^{n-1} \left( b^2_{j} \prod_{i=2}^{n} (\lambda_0 - a_i) \right) = 0,
$$
so
\[
\prod_{i=1}^{n} (\lambda - a_i) = \sum_{j=1}^{n-1} \left( b_j^2 \prod_{i=2, i\neq j+1}^{n} (\lambda - a_i) \right). \tag{3}
\]

Assume that there is an integer \( k \) (\( 2 \leq k \leq n \)) such that \( a_k = \lambda_0 \), it is follows from (3) that
\[
b_1^2 (\lambda - a_3) (\lambda - a_4) \cdots (\lambda - a_n) = 0,
b_{k-1}^2 (\lambda - a_2) \cdots (\lambda - a_k-1) (\lambda - a_{k+1}) \cdots (\lambda - a_n) = 0, \quad k = 3, \ldots, n.
\]
Thus, there is an integer \( l \) (\( l \neq k \)) such that \( a_l = \lambda_0 \). We get \( a_k = \lambda_0 = a_l \), which yields a contradiction.

On the other hand, recall from Cauchy’s Interlacing Theorem that the eigenvalues of symmetric matrix \( A \) interlace those of its \((n-1)\times(n-1)\) principal submatrix, we have
\[
\lambda_1 \geq a_2 \geq \lambda_2 \geq a_3 \geq \cdots \geq a_n-1 \geq \lambda_{n-1} \geq a_n \geq \lambda_n.
\]
Since \( \lambda \neq a_i \) for all \( i = 2, \ldots, n \), we have
\[
\lambda_1 > a_2 > \lambda_2 > a_3 > \cdots > a_{n-1} > \lambda_{n-1} > a_n > \lambda_n. \tag*{□}
\]

**Lemma 3** [10]. Let \( A \) be an \( n \times n \) real symmetric matrix, \( x_1, x_2, \ldots, x_n \) be orthonormal eigenvectors of \( A \) associated with distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Then
\[
\text{adj}(\lambda_j I - A) = \varphi'_n(\lambda_j) x_j x_j^T, \quad j = 1, 2, \ldots, n, \tag{4}
\]
where \( \text{adj}(A) \) denotes the adjoint matrix of \( A \).

**Lemma 4.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be eigenvalues of \( A_n \), and \( x_1, x_2, \ldots, x_n \) be corresponding normal orthogonal eigenvectors. Then for all \( j \),

1. If \( 1 < \mu < \gamma \leq n \)
   \[
   \varphi'_n(\lambda_j) x_{\mu j} x_{\gamma j} = b_{\mu-1} b_{\gamma-1} \prod_{i=2, i\neq \mu, \gamma}^{n} (\lambda_j - a_i). \tag{5}
   \]
2. If \( 1 = \mu < \gamma \leq n \)
   \[
   \varphi'_n(\lambda_j) x_{1 j} x_{\gamma j} = b_{\gamma-1} \prod_{i=2, i\neq \gamma}^{n} (\lambda_j - a_i). \tag{6}
   \]
3. If \( 1 < \mu = \gamma \leq n \)
   \[
   \varphi'_n(\lambda_j) x_{\mu j}^2 = \prod_{i=1, i\neq \mu}^{n} (\lambda_j - a_i) - \sum_{k=1}^{n-1} \left( b_k^2 \prod_{i=2, i\neq k+1}^{n} (\lambda_j - a_i) \right). \tag{7}
   \]
If \( 1 = \mu = \gamma \)

\[
\varphi_n'(\lambda_j) x_{1j}^2 = \prod_{i=2}^{n} (\lambda_j - a_i),
\]  

(8)

where \( x_{\mu j} \) denotes the \( \mu \)th component of the vector \( x_j \).

**Proof.** (1) If \( 1 < \mu < \gamma \leq n \), By Lemma 3, we have

\[
\text{adj}(\lambda_j I - A_n) = \varphi_n'(\lambda_j) x_j x_j^T.
\]  

(9)

Compare the \((\mu, \gamma)\)-element on each side of (9). The \((\mu, \gamma)\)-element of the right side is \( \varphi_n'(\lambda_j) x_{\mu j} x_{\gamma j} \). Since \( \lambda_j I - A_n \) is an symmetric matrix, the \((\mu, \gamma)\)-element of \( \lambda_j I - A_n \) is equal to the \((\gamma, \mu)\)-element of \( \lambda_j I - A_n \). By direct computation, it is easy to get that the \((\mu, \gamma)\)-element of the right side of (9) is \( b_{\mu-1} b_{\gamma-1} \prod_{i=2}^{\gamma} (\lambda_j - a_i) \). Hence (5) holds. Similarly, (6)–(8) hold. □

**Lemma 5.** Let \( \lambda_1, \lambda_2, \ldots, \lambda_n \) be eigenvalues of \( A_n \), and \( x_1, x_2, \ldots, x_n \) be corresponding orthonormal eigenvectors. Then \( x_{\mu j} \neq 0 \) for all \( j, \mu = 1, 2, \ldots, n \), where \( x_{\mu j} \) is the \( \mu \)th component of the vector \( x_j \).

**Proof.** Since \( \lambda_i \) are distinct and \( \lambda_i \neq a_k \) for all \( i = 1, 2, \ldots, n, k = 2, 3, \ldots, n \), it is easy to derive that \( x_{1j} \neq 0 \) and \( \varphi_n'(\lambda_j) \neq 0 \) from (8). From (6) and \( b_i > 0 \) we can also get \( x_{\gamma j} \neq 0 \) for all \( \gamma = 2, 3, \ldots, n \). Consequently, the result holds. □

**Lemma 6.** The characteristic polynomial sequence \( \{\varphi_i(\lambda)\} \) form a Sturm sequence, satisfying

1. \( \varphi_0(\lambda) \) has everywhere the same sign;
2. Two successive \( \varphi_i(\lambda) \) cannot be simultaneously zero;
3. When \( \varphi_i(\lambda) \) vanishes, \( \varphi_{i+1}(\lambda) \) and \( \varphi_{i-1}(\lambda) \) are non-zero and have opposite signs.

**Proof**

(1) It is obvious that property (1) holds.

(2) By direct computation it is easy to see that there is no common solution between \( \varphi_0(\lambda), \varphi_1(\lambda) \) and \( \varphi_2(\lambda) \). For \( j = 3, 4, \ldots, n \), assume that there is a real number \( C \) such that \( \varphi_j(C) = 0 = \varphi_{j-1}(C) \). By Lemma 1 and \( b_{j-1} > 0 \) it follows immediately that \( \prod_{i=2}^{j-1} (C - a_i) = 0 \), i.e., there exist a positive integer \( k \) (\( 2 \leq k \leq j - 1 \)) such that \( a_k = C \). Substituting \( \lambda = C \) into (2) to compute \( \varphi_{j+1}(C) \), we also obtain \( \varphi_{j+1}(C) = 0 \). By (2), we finally have \( \varphi_n(C) = 0 \), that is to say \( C \) is an eigenvalue of \( A_n \), we obtain a contradiction. So property (2) holds.

(3) If \( \varphi_1(\lambda_0) = 0 \), then \( \varphi_2(\lambda_0) = (\lambda_0 - a_2)\varphi_1(\lambda_0) - b_1^2 = -b_1^2 \). It is clear that \( \varphi_0(\lambda_0) \) and \( \varphi_2(\lambda_0) \) have opposite signs.

When \( 2 \leq j \leq n - 1 \), from (2) we have

\[
\varphi_j(\lambda) = (\lambda - a_j)\varphi_{j-1}(\lambda) - b_{j-1}^2 \prod_{i=2}^{j-1} (\lambda - a_i),
\]
\[
\varphi_{j+1}(\lambda) = (\lambda - a_{j+1})\varphi_j(\lambda) - b_j^2 \prod_{i=2}^{j}(\lambda - a_i).
\]

If \( \varphi_j(\lambda_0) = 0 \), substituting \( \lambda = \lambda_0 \) into the equations above, we get

\[
(\lambda_0 - a_j)\varphi_{j-1}(\lambda_0) = b_{j-1}^2 \prod_{i=2}^{j-1}(\lambda_0 - a_i),
\]

\[
\varphi_{j+1}(\lambda_0) = -b_j^2 \prod_{i=2}^{j}(\lambda_0 - a_i).
\]

Hence

\[
\frac{\varphi_{j+1}(\lambda_0)}{\varphi_{j-1}(\lambda_0)} = -\frac{b_j^2}{b_{j-1}^2} (\lambda_0 - a_j)^2 < 0,
\]

that is, \( \varphi_{j+1}(\lambda_0) \) and \( \varphi_{j-1}(\lambda_0) \) have opposite sign. Property (3) holds. \( \square \)

**Lemma 7.** The zeros of \( \varphi_j(\lambda) \) and \( \varphi_{j-1}(\lambda) \) are respectively

\[
\lambda_1^{(j)}, \lambda_2^{(j)}, \ldots, \lambda_j^{(j)},
\]

and

\[
\lambda_1^{(j-1)}, \lambda_2^{(j-1)}, \ldots, \lambda_{j-1}^{(j-1)}.
\]

Then

\[
\lambda_1^{(j)} < \lambda_1^{(j-1)} < \lambda_2^{(j)} < \cdots < \lambda_{j-1}^{(j-1)} < \lambda_j^{(j)},
\]

that is, the zeros of \( \varphi_{j-1}(\lambda) \) strictly separate those of \( \varphi_j(\lambda) \).

**Proof.** The zeros of \( \varphi_{j-1}(\lambda) \) are the eigenvalues of \( A_{j-1} \). Then from Cauchy’s Interlacing theorem, they separate the eigenvalues of \( A_j \), which are the zeros of \( \varphi_j(\lambda) \). That is, \( \lambda_1^{(j)} \leq \lambda_1^{(j-1)} \leq \lambda_2^{(j)} \leq \cdots \leq \lambda_{j-1}^{(j-1)} \leq \lambda_j^{(j)} \). From Lemma 6 (2), we have \( \lambda_i^{(j)} \neq \lambda_i^{(j-1)} \). Hence, the result holds. \( \square \)

As an immediate consequence of the proof of Lemma 7, we have

**Corollary 1.** Denote \( \lambda_1^{(j)} \) and \( \lambda_j^{(j)} \), respectively, the minimal zero and the maximal zero of \( \varphi_j(\lambda) \). Then

(1) If \( \mu < \lambda_1^{(j)} \), \( (-1)^j \varphi_j(\mu) > 0 \);

(2) If \( \mu > \lambda_j^{(j)} \), \( \varphi_j(\mu) > 0 \), for all \( j = 1, 2, \ldots, n \).

3. The solutions of Problems I and II

We first consider the solvability conditions of Problem I.
Theorem 1. There is a unique solution of Problem I if and only if
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\[(10)\]
Since $\lambda_1^{(j)} < \lambda_1^{(j-1)} < \lambda_j^{(j-1)} < \lambda_j^{(j)}$, and $\lambda_j^{(j)}$, $\lambda_j^{(j)}$ are, respectively, the minimal and maximal eigenvalues of $A_j$, from Corollary 1 we have $(-1)^{j-1}\varphi_{j-1}(\lambda_1^{(j)}) > 0$, $\varphi_{j-1}(\lambda_j^{(j)}) > 0$. This, together with (10) gives $b_{j-1}^2 > 0$.

Necessity: Suppose that Problem I has a unique solution, that is, (14) have solutions $a_j$, $b_{j-1}$ satisfying $b_{j-1}^2 > 0$. Similar to the proof of the sufficiency, we also obtain (11)–(13). Applying Corollary 1 again, we have $(-1)^{j-1}\varphi_{j-1}(\lambda_1^{(j)}) > 0$, $\varphi_{j-1}(\lambda_j^{(j)}) > 0$. Thus, (10) holds. □

Now we discuss the solutions of Problem II.

**Theorem 2.** There are solutions of Problem II if and only if

1. $x_i \neq 0$, $i = 1, 2, \ldots, n$.
2. There is a positive solution $\alpha$ of the equation

$$\alpha^2 \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) - \alpha x_1 \varphi_{j-1}(\lambda^{(j)})/x_j + (\lambda^{(n)} - \lambda^{(j)}) \varphi_{j-1}(\lambda^{(j)}) = 0.$$  \hspace{1cm} (19)

Moreover, the solutions can be obtained from the following

$$a_1 = \lambda^{(1)},$$  \hspace{1cm} (20)

$$b_{j-1}^2 \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) - b_{j-1} x_1 \varphi_{j-1}(\lambda^{(j)})/x_j + (\lambda^{(n)} - \lambda^{(j)}) \varphi_{j-1}(\lambda^{(j)}) = 0,$$  \hspace{1cm} (21)

$$a_j = \lambda^{(n)} - b_{j-1} x_1/x_j.$$  \hspace{1cm} (22)

**Proof.** Sufficiency: To solve Problem II is equivalent to solve the equations

$$\varphi_j(\lambda^{(j)}) = 0,$$  \hspace{1cm} (23)

$$A_n X = \lambda^{(n)} X.$$  \hspace{1cm} (24)

Moreover, $b_{j-1}$ is positive for all $j$.

From (2) and (23), we have

$$(\lambda^{(j)} - a_j) \varphi_{j-1}(\lambda^{(j)}) - b_{j-1}^2 \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) = 0.$$  \hspace{1cm} (25)

Clearly, we can obtain $a_1 = \lambda^{(1)}$.

Eq. (24) can be transformed into the following form

$$a_1 x_1 + b_1 x_2 + \ldots + b_{n-1} x_n = \lambda^{(n)} x_1,$$  \hspace{1cm} (26)

$$b_{j-1} x_1 + a_j x_j = \lambda^{(n)} x_j, \quad j = 2, 3, \ldots, n.$$  \hspace{1cm} (27)

Since condition (1) holds, by (27) we get

$$a_j = \lambda^{(n)} - b_{j-1} x_1/x_j.$$  \hspace{1cm} (28)
Substituting (28) into (25), we have
\[ b_j^2 \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) - b_{j-1} x_1 \varphi_{j-1} (\lambda^{(j)})/x_j + (\lambda^{(n)} - \lambda^{(j)}) \varphi_{j-1} (\lambda^{(j)}) = 0. \] (29)

Condition (2) holds, which implies \( b_{j-1} > 0 \). It follows that Problem II has a solution.

Necessity: Assume that Problem II has a solution. Condition (1) can be obtained by Lemma 5. By the same way in the proof of the sufficiency we can also get (20)–(22). Problem II has a solution, that is, \( b_{j-1} > 0 \) for all \( j \). So condition (2) holds. \( \square \)

4. Numerical algorithms and examples

From the discussion of Section 3, it is natural that we should propose the following algorithms for solving Problems I and II.

Algorithm 1. (To solve Problem I)
Step 1. \( j = 2, a_1 = \lambda^{(1)} \).
Step 2. If \( j = n + 1 \), ending the algorithm.
Step 3. Substituting \( a_1, a_2, \ldots, a_{j-1} \) and \( b_1, b_2, \ldots, b_{j-2} \) into (10) to compute
\[ (-1)^{j-1} \left( \varphi_{j-1} (\lambda^{(j)}) \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) - \varphi_{j-1} (\lambda^{(j)}) \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) \right). \]
If \((-1)^{j-1} \left( \varphi_{j-1} (\lambda^{(j)}) \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) - \varphi_{j-1} (\lambda^{(j)}) \prod_{i=2}^{j-1} (\lambda^{(j)} - a_i) \right) > 0\), computing \( a_j \) and positive \( b_{j-1} \) by (12) and (13) successively, \( j := j + 1 \), go to Step 2; otherwise, ending the algorithm.

Algorithm 2. (To solve Problem II)
Step 1. \( j = 2, a_1 = \lambda^{(1)} \).
Step 2. If \( j = n + 1 \), ending the algorithm.
Step 3. Substituting \( a_1, a_2, \ldots, a_{j-1} \) and \( b_1, b_2, \ldots, b_{j-2} \) into (22) to finding the two solutions \( \alpha_1, \alpha_2 \).
Step 4. If \( \alpha_1 < 0 \) and \( \alpha_2 < 0 \), ending the algorithm.
   If \( \alpha_1 > 0 \), then \( b_{j-1} = \alpha_1 \), computing \( a_j \) by (22);
   If \( \alpha_2 > 0 \), then \( b_{j-1}^{'} = \alpha_1 \), computing \( a_j^{'} \) by (22).
Step 5. \( j := j + 1 \), go to Step 2.

Example 1. For given 17 real number \( \lambda^{(1)}_1 = 2.33, \lambda^{(2)}_1 = 1.8, \lambda^{(2)}_2 = 6.59, \lambda^{(5)}_1 = -6.74, \lambda^{(5)}_5 = 12.3, \lambda^{(6)}_1 = -7.88, \lambda^{(6)}_6 = 13.88, \lambda^{(7)}_1 = -8.4, \lambda^{(7)}_7 = 14, \lambda^{(8)}_1 = -9.1, \lambda^{(8)}_8 = 15.3 \). Finding a \( 8 \times 8 \) matrix \( A_8 \) such that \( \lambda^{(j)}_1 \) and \( \lambda^{(j)}_i \) are the minimal and the maximal eigenvalue of its \( j \times j \) leading principal submatrix.
By applying Algorithm 1, we get the unique solution

\[
A_8 = \begin{pmatrix}
1.5026 & 6.0600 & 0 & 0 & 0 & 0 & 0 & 0 \\
4.2092 & 0 & -2.5136 & 0 & 0 & 0 & 0 & 0 \\
3.5497 & 0 & 0 & 6.4474 & 0 & 0 & 0 & 0 \\
6.6798 & 0 & 0 & 0 & 5.1713 & 0 & 0 & 0 \\
5.5520 & 0 & 0 & 0 & 0 & 4.7106 & 0 & 0 \\
2.2061 & 0 & 0 & 0 & 0 & 0 & -3.8785 & 0 \\
4.8522 & 0 & 0 & 0 & 0 & 0 & 0 & 6.7005
\end{pmatrix}
\]

From the above 8 × 8 matrix \( A_8 \) obtained by algorithm 1, we re-computing the spectrum of \( A_j \), which is denoted by \( \sigma(A_j) \), and get

\[
\sigma(A_1) = (2.3300),
\]
\[
\sigma(A_2) = (1.8000, 6.5900),
\]
\[
\sigma(A_3) = (-5.0000, 3.9764, 6.9000),
\]
\[
\sigma(A_4) = (-5.3000, 2.3271, 6.1167, 9.1800),
\]
\[
\sigma(A_5) = (-6.7400, -0.2599, 5.9766, 6.2184, 12.3000),
\]
\[
\sigma(A_6) = (-7.8800, -0.9639, 4.8920, 6.0113, 6.2663, 13.8800),
\]
\[
\sigma(A_7) = (-8.4000, -3.6359, -0.8083, 4.8923, 6.0117, 6.2673, 14.0000),
\]
\[
\]

These obtained data show that Algorithm 1 is quite efficient.

**Example 2.** For given 8 real number \( \lambda^{(1)} = 4.26, \lambda^{(2)} = -3.66, \lambda^{(3)} = 7.67, \lambda^{(4)} = -7.3, \lambda^{(5)} = -2.9, \lambda^{(6)} = 2.38, \lambda^{(7)} = 5.4, \lambda^{(8)} = 13.32 \) and a real vector

\[
X = (0.76, 0.10, 0.26, 0.47, 0.18, 0.14, 0.22, 0.15)^T,
\]
\[
X/0.76 = (1.0000, 0.1316, 0.3421, 0.6184, 0.2368, 0.1842, 0.2895, 0.1974)^T.
\]

Finding a 8 × 8 matrix \( A_8 \) such that \( \lambda^{(j)} \) is its eigenvalue of \( j \times j \) leading principal submatrix, moreover \( \lambda^{(8)}, X \) is its eigenpair.

By applying Algorithm 1, we get two solutions

\[
A_8 = \begin{pmatrix}
2.1569 & -3.0726 & 0 & 0 & 0 & 0 & 0 & 0 \\
2.8981 & 0 & 4.8487 & 0 & 0 & 0 & 0 & 0 \\
8.3438 & 0 & 0 & -0.1721 & 0 & 0 & 0 & 0 \\
3.4634 & 0 & 0 & 0 & -1.3032 & 0 & 0 & 0 \\
1.9909 & 0 & 0 & 0 & 0 & 2.5123 & 0 & 0 \\
2.2445 & 0 & 0 & 0 & 0 & 0 & 5.5663 & 0 \\
3.9926 & 0 & 0 & 0 & 0 & 0 & 0 & -6.9092
\end{pmatrix}
\]
From the above $8 \times 8$ matrices $A_8$ and $A_8^*$ obtained by Algorithm 2, we re-computing the spectrum of $A_j$ and $A_j^*$ which are denoted by $\sigma(A_j)$ and $\sigma(A_j^*)$, and get

$$
\sigma(A_1) = (4.2600),
\sigma(A_2) = (-3.6600, 4.8474),
\sigma(A_3) = (-3.7356, 2.1016, 7.6700),
\sigma(A_4) = (-7.2996, -2.8409, 4.3251, 11.6793),
\sigma(A_5) = (-8.0086, -2.9000, -1.1237, 4.3797, 12.2133),
\sigma(A_6) = (-8.1510, -2.9042, -1.1261, 2.3796, 4.4239, 12.4508),
\sigma(A_7) = (-8.2917, -2.9074, -1.1277, 2.3727, 4.3315, 5.4000, 12.8618),
\sigma(A_8) = (-10.3517, -5.3837, -2.8833, 1.1216, 2.3801, 4.3644, 5.4059, 13.3200),
$$

the eigenvectors $Y$ corresponding to the eigenvalue 13.3200 is

$$
Y = (-7.6040, -1.001, -2.6010, -4.7020, -1.8010, -1.4010, -2.2010, -1.5010)^T,
Y/(-7.6040) = (1.0000, 0.1316, 0.3421, 0.6184, 0.2368, 0.1842, 0.2895, 0.1974)^T.
$$

$$
\sigma(A_1^*) = (4.2600),
\sigma(A_2^*) = (-3.6600, 4.8474),
\sigma(A_3^*) = (-6.9914, -3.1358, 7.6700),
\sigma(A_4^*) = (-7.3000, -3.1372, 3.6624, 10.9125),
\sigma(A_5^*) = (-8.0942, -3.1740, -2.8999, 4.1710, 11.4210),
\sigma(A_6^*) = (-8.1914, -3.1749, -2.9009, 2.3798, 4.5761, 11.6372),
\sigma(A_7^*) = (-8.3388, -3.1761, -2.9023, 2.1859, 3.6877, 5.4001, 12.0847),
\sigma(A_8^*) = (-46.6999, -7.4186, -3.1715, -2.8974, 2.3756, 3.8364, 5.5052, 13.3200),
$$

the eigenvectors $Y^*$ corresponding to the eigenvalue 13.3200 is

$$
Y^* = (7.6040, 1.001, 2.6010, 4.7020, 1.8010, 1.4010, 2.2010, 1.5010)^T,
Y^*/7.6040 = (1.0000, 0.1316, 0.3421, 0.6184, 0.2368, 0.1842, 0.2895, 0.1974)^T.
$$

These obtained data show that Algorithm 2 is quite efficient.
References