Strange Conserved Quantities for the Wave Equation for Elastic Membranes

A. A. MINZONI AND C. A. VARGAS
FENOMEC-IIMAS, UNAM, México
Apdo. Postal 20-726, Admón. 20, Deleg. Alvaro Obregón
México, D.F.

(Received and accepted October 1998)

Abstract—In this letter, we examine conservation laws for the vibrating membrane which arise from Noether's Theorem. We show that for linear membranes Noether's Theorem gives conservation laws which appear to contradict the basic assumptions on the model. We then show that the appropriate interpretation of these laws must follow from the nonlinear theory. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Noether's Theorem, Conservation laws, Nonlinear membrane.

INTRODUCTION

In this note, we examine the problem of conserved quantities for both the linear and nonlinear equations for the vibrating membrane. Since the equations can be derived from a Lagrangian in both cases Noether's Theorem gives, using invariances of the Lagrangian, conservation laws which reflect these invariances.

For the linear membrane translation and rotation invariance, give two conserved quantities with no obvious interpretation [1]. In this note, we show using the nonlinear theory that, for the nonlinear membrane besides the conservation of the horizontal component of the linear momentum and the vertical component of the angular momentum, there are new conservation laws which stem from the parametrization invariance. One of them involves the tangential component of the linear momentum along the deformed membrane, the other a quantity related to the angular momentum due to the tangential velocity of the membrane. It is the combination of these two quantities which gives the conservation laws obtained in the linear theory. Since the linear theory for the membrane, to the leading term, parametrization invariance, translation, and rotational invariance are the same, it is not possible to use, without careful examination of the problem, the general idea of Gelfand and Fomin [2] which suggest to identify in general linear momentum as arising from translation invariance and components angular momentum as coming from rotational invariance. This general identification may be possible provided that the invariances are not degenerated as in the membrane problem that we now discuss.

We want to acknowledge S. Antman for careful reading of the manuscript and many helpful suggestions. The first author acknowledges R. Weinacht for helpful conversations in the early stages of the work.
FORMULATION

The Lagrangian for the linear homogeneous membrane whose equilibrium position is the region $\Omega$ in the $(\xi, \eta)$ plane, vertical displacement is $z(\xi, \eta, t)$ is given by

$$L = \int_0^1 \int_{\Omega} \left[ \frac{1}{2} (\dot{z}^2 - |\nabla z|^2) \right] \, d\xi \, d\eta \, dt. \tag{1}$$

To use Noether's Theorem [2] to determine the conservation equations, we recall that this result states that when the functional

$$J[u] = \int_R F(\xi, u, \nabla u) \, d\xi$$

is invariant under the family of transformations ($\epsilon \ll 1$)

$$\xi_i^* = \Phi_i(\xi, u, \nabla u; \epsilon) \sim \xi_i + \epsilon \varphi_i(\xi, u, \nabla u),$$
$$u^* = \Psi(\xi, u, \nabla u; \epsilon) \sim u + \epsilon \psi(\xi, u, \nabla u)$$

$(i = 1, \ldots, n)$ for an arbitrary region $R$, of $n$-dimensional space, then

$$\sum_{i=1}^n \frac{\partial}{\partial \xi_i} \left( F_{u_i} \psi + F \varphi_i \right) = 0$$

on each extremal surface of $J[u]$, where

$$\psi = \psi - \sum_{i=1}^n u_{\xi_i} \varphi_i.$$

In the special case of linear membrane taking $\Omega = \mathbb{R}^2$, $\xi_1 = \xi$, $\xi_2 = \eta$, we find that the Lagrangian is invariant under translations. Moreover if $z$ and its derivatives decay fast enough at infinity, we have using $\xi \rightarrow \xi + \epsilon$ as the transformation that Noether's Theorem gives

$$\frac{\partial}{\partial \xi} (z \dot{z} \dot{z}^2) = \frac{1}{2} \left( \frac{\partial}{\partial \xi} (z_\xi^2 + z_\eta^2 - z_\eta^2) \dot{z} + (z_\xi \dot{z}_\eta) \dot{\eta} \right). \tag{2}$$

This is to say that the quantity $z_\xi ^2 + z_\eta ^2$ is conserved. This has no obvious interpretation.

If $z_\xi z_\xi$ is to be identified with horizontal momentum with a straightforward application of the results [2], we arrive immediately to the presence of a horizontal motion in a linear membrane. This is not consistent with the physical picture that linear membranes have only a vertical motion.

In the same way due to the rotation invariance

$$\xi \rightarrow \xi \cos \epsilon + \eta \sin \epsilon, \quad \eta \rightarrow -\xi \sin \epsilon + \eta \cos \epsilon,$$

Noether's Theorem gives the conservation law

$$\frac{\partial}{\partial \xi} \left( \eta z_\eta^2 - \xi z_\xi z_\eta \right) = \left( \frac{1}{2} \eta \left( z_\xi^2 + z_\eta^2 - z_\eta^2 \right) \xi + \left( \frac{1}{2} \xi \left( z_\xi^2 - z_\xi^2 + z_\eta^2 \right) - \eta z_\xi z_\eta \right) \eta \right), \tag{3}$$

which again has the same problem as the previous quantity, since the vertical motion of the linear membrane cannot contribute to a $z$ component of the angular momentum as suggested by straightforward application of the interpretation in [2].

In order to give the proper interpretation of these quantities we now consider a nonlinear model, which linearizes to (1), where the conservation laws have a consistent physical meaning. We then study these laws in the linear limit to obtain the appropriate understanding and interpretation of equations (2) and (3).
CONSERVATION LAWS FOR THE NONLINEAR MEMBRANE

We now consider a special case for the nonlinear membrane [3] which for small amplitude (to be made precise later) linearize to equation (1). In this case the Lagrangian is

\[ L = \int_{t_0}^{t} \int_{\Omega} \left( \frac{1}{2} \left[ \frac{dx}{dt} \right]^2 - F \left( |x_\xi \wedge x_\eta| - 1 \right) \right) d\xi d\eta dt. \]

In this formulation, a typical material point of the membrane is defined with its coordinates \((\xi, \eta)\) in the planar equilibrium configuration by \(x(\xi, \eta)\). The position of this material point at the time \(t\) is given by \(x(\xi, \eta, t)\).

The first term corresponds to the Euclidean norm of the velocity and is identified with the kinetic energy. The elastic energy \(F\) is to be specified.

Linear Momentum for the Nonlinear Membrane

The Lagrangian \(L\) is invariant under displacement of the membrane position. Thus, using Noether's Theorem with the transformation \(x^* = x + \epsilon, y^* = y, z^* = z\) and no changes \(\xi, \eta, \) and \(t\) we obtain, denoting \(|x_\xi \wedge x_\eta| - 1\) by \(u\), the following conservation law:

\[ \text{(4)} \]

which clearly is the conservation of linear momentum in the \(x\) direction, as for a system of particles. A new conservation law arises since the Lagrangian is invariant under displacement of the parametrization, \(\xi^* = \xi + \epsilon, \eta^* = \eta^*, t^* = t^*\) and no changes in \(x, y, \) and \(z\). This invariance gives the conservation law

\[ \text{(5)} \]

Note that the left-hand side of (4) is the time variation of the horizontal component in the \(x\) direction of the linear momentum. On the other hand, the left-hand side of (5) is the time variation of one tangential component (with respect to the membrane) of the linear momentum.

A new conservation law is obtained by subtraction of (4) from (5). Using the linear approximation, i.e., when \(x = \xi + \bar{x}(\xi, \eta, t), y = \eta + \bar{y}(\xi, \eta, t), z = z(\xi, \eta, t)\) which neglects \(\bar{x}, \bar{y}\) as second order as compared to the small vertical displacement \(z\). In the difference between equations (4) and (5), we obtain

\[ \text{(6)} \]

For the case \(F(u) = u^2/2 + u\), which is equivalent to \((z_x^2 + z_y^2)/2\), we obtain

\[ \text{(7)} \]

which coincides with (2).

This conservation law can now be interpreted as follows: the momentum can be decomposed as the vertical \((0, 0, z_t)\) and horizontal components, this is,

\[ x_t = (0, 0, z_t) + (\bar{x}_t, \bar{y}_t, 0). \]
Thus, the left-hand side of the subtraction of equations (4) and (5) can be written in the form
\[ (0, 0, z_t) \cdot (0, 0, z_t) = (0, 0, z_t) \cdot ((1, 0, z_\xi) - (1, 0, 0)) + (0, 0, z_t) \cdot (\ddot{x}_t, \ddot{y}_t, 0) \cdot (\ddot{x}_t, \ddot{y}_t, z_\xi). \]

We observe that to leading order \( z_t z_\xi \) is nothing but
\[ (0, 0, z_t) \cdot (1, 0, z_\xi), \]
which is the projection of the vertical momentum along the tangent vector (in the direction of the invariance) of the linearized membrane position. This gives an interpretation consistent with the linear theory.

This peculiar conservation law arises from the fact that, in the linear approximation, parametrization, and translation invariance are indistinguishable to leading order. Due to the fact in the linear approximation, the contributions of the horizontal momentum cancel from (4) and (5) leaving only the appropriate projection of the vertical component.

Angular Momentum for the Membrane

We now consider the situation for the angular momentum. We give the same arguments as before. The Lagrangian is invariant under rotation of the membrane position. Thus, using Noether’s Theorem with the transformation
\[ x^* = x \cos \epsilon + y \sin \epsilon, \quad y^* = -x \sin \epsilon + y \cos \epsilon, \]
we obtain the usual conservation of the vertical component of the angular momentum
\[ ((x \wedge x_t)_3) = \left[ \frac{F'(u)}{|x_\xi \wedge x_\eta|} \left( |x_\xi|^2 |x_\eta|^2 - (x_\xi \cdot x_\eta)^2 \right) \right]_\xi + \left[ \frac{F'(u)}{|x_\xi \wedge x_\eta|} \left( |x_\xi|^2 |x_\eta|^2 - (x_\xi \cdot x_\eta)^2 \right) \right]_\eta, \tag{8} \]
where \( U_3 \) indicates the vertical component of the vector \( U \).

However, just as in the case of the linear momentum, we have parametrization invariance under the transformation
\[ \xi^* = \xi \cos \epsilon + \eta \sin \epsilon, \quad \eta^* = -\xi \sin \epsilon + \eta \cos \epsilon. \]

Thus, Noether’s Theorem gives the conservation law
\[ (x_t \cdot (\xi x_\eta - \eta x_\xi))_t = \xi \left[ \frac{F'(u)}{|x_\xi \wedge x_\eta|} \left( |x_\xi|^2 |x_\eta|^2 - (x_\xi \cdot x_\eta)^2 \right) \right]_\eta - \eta \left[ \frac{F'(u)}{|x_\xi \wedge x_\eta|} \left( |x_\xi|^2 |x_\eta|^2 - (x_\xi \cdot x_\eta)^2 \right) \right]_\xi. \tag{9} \]

To compare these two conservation laws with the corresponding one of the linear membrane given in equation (3), we need to approximate the vector \( x \) around \((\xi, \eta, 0)\) up to second-order terms. Let \( x = (\xi \ddot{x}) (\xi + \ddot{x}(\xi, \eta, t), \eta + \ddot{y}(\xi, \eta, t), z), \) with \( \ddot{x}, \ddot{y} \) being terms of second order. Thus, up to second-order terms the conservation law (8) becomes
\[ (\ddot{y}_t - \eta \ddot{x}_t)_t = -\eta \left[ \frac{1}{2} F''(0) \left( z_\eta^2 - z_\xi^2 \right) + F'(0) \left( 2z_\eta^2 \right) \right]_\xi - \xi \left[ F'(0) z_\xi z_\eta \right]_\xi \tag{10} \]

\[ + \xi \left[ \frac{1}{2} F''(0) \left( z_\eta^2 - z_\xi^2 \right) + F'(0) \left( 2z_\eta^2 \right) \right]_\eta + \eta \left[ F''(0) z_\xi z_\eta \right]_\eta. \]
and (9) takes the form:

\[
(\xi (z_t z_\eta + \bar{y}_t) - \eta (z_t z_\xi + \bar{x}_t))_t = \xi \left[ F''(0) \left( \bar{x}_\xi + \bar{y}_\eta + \frac{1}{2} (z_\xi^2 + z_\eta^2) \right) + \frac{1}{2} z^2_\xi \right]_\eta \\
- \eta \left[ F''(0) \left( \bar{z}_\xi + \bar{y}_\eta + \frac{1}{2} (z_\xi^2 + z_\eta^2) \right) + \frac{1}{2} z^2_\xi \right]_\xi.
\] (11)

As in (8), the left-hand side of (10) is the time variation of the vertical component of the angular momentum, up to second-order approximation. Meanwhile, the left-hand side of (11) is the vertical component of the angular momentum due to the tangential component of the velocity. To see this, consider the following two independent tangential vectors to the membrane \( T_1 = x_\xi / |x_\xi| \) and \( T_2 = x_\eta / |x_\eta| \), then the tangential component of the velocity is

\[
v_T = (\mathbf{v} \cdot T_1) T_1 + (\mathbf{v} \cdot T_2) T_2.
\]

Considering only second-order terms, we have

\[
v_T = (\bar{x}_t + z_t z_\xi, \bar{y}_t + z_t z_\eta, 0).
\]

If we subtract (11) from (10), one gets the time variation of the vertical component of the angular momentum due to the velocity normal to the membrane and it is given by

\[
[\xi z_t z_\eta - \eta z_t z_\xi]_t = \eta \left[ \frac{1}{2} F''(0) (z_\eta^2 - z_\xi^2) - \frac{1}{2} z^2_\xi \right]_\xi + \xi [F'(0) z_\xi z_\eta]_\xi \\
- \xi \left[ \frac{1}{2} F''(0) (z_\xi^2 - z_\eta^2) - \frac{1}{2} z^2_\eta \right]_\eta - \eta [F'(0) z_\xi z_\eta]_\eta,
\]

which is equal to (3) when \( F'(0) = 1 \). As before, the conserved quantity in the left-hand side of equation (3) is the angular momentum, relative to the \( z \) axis produced by the projection of the vertical velocity of the membrane \( (0, 0, z_t) \) along the normal to the membrane as given by the linear theory. In this, the radius is taken to be \( (\xi, \eta) \) which is the underformed one. Again this gives a consistent interpretation in the linear approximation.

CONCLUSIONS

We have shown how strange conservation laws for the linear membrane arise as limits of simple conservation laws for the full nonlinear membrane. It was shown how a consistent interpretation is possible in terms of the linear theory. However this interpretation does not follow from the proposal of Gelfand and Fomin since in the linear membrane two different invariances become indistinguishable. In conclusion, care must be exercised since straightforward interpretation of approximations may lead to apparent inconsistencies.

REFERENCES