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Generalized monotonicity and convexity of non-differentiable functions ☆

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Abstract

The relationships between (strict, strong) convexity of non-differentiable functions and (strict, strong) monotonicity of set-valued mappings, and (strict, strong, sharp) pseudo convexity of non-differentiable functions and (strict, strong) pseudo monotonicity of set-valued mappings, as well as quasi convexity of non-differentiable functions and quasi monotonicity of set-valued mappings are studied in this paper. In addition, the relations between generalized convexity of non-differentiable functions and generalized co-coerciveness of set-valued mappings are also analyzed.

Keywords: Generalized convexity; Generalized monotonicity; Generalized co-coerciveness; Clarke's generalized subdifferential mapping

1. Introduction

It is well known that the generalized monotonicity of set-valued mappings plays an important role in studying the existence and the sensitivity analysis of solutions for variational inequalities, variational inclusions, and complementarity problems. In the case of real-valued functions, this topic has been studied in the context of literature ([1–6] and references therein) and related to the generalized convexity of differentiable functions [3,7,8], which is a basis in investigating mathematical programming as well as economic theory [9]. Komlósi in [10] discussed the relationships between quasi (pseudo, strict pseudo) con-

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vexity of lower semi-continuous functions and quasi (pseudo, strict pseudo) monotonicity of their generalized derivatives by means of the generalized monotonicity of bifunctions which are still single-valued functions. However, few research has been done on the relationships of generalized convexity of non-differentiable functions and generalized monotonicity of set-valued mappings.

This paper is devoted to study the relationships between generalized convexity of non-differentiable functions and generalized monotonicity as well as generalized co-coerciveness of set-valued mappings. It is organized as follows. In Section 2, we recall some basic concepts and results used in the sequel. In Section 3, we establish the relationships between (strict, strong) convexity of non-differentiable functions and (strict, strong) monotonicity of set-valued mappings. In Section 4, we analyze the relationships between (strict, strong, sharp) pseudo convexity and (strict, strong, sharp) pseudo monotonicity, and in Section 5, we discuss the relation between quasi convexity and quasi monotonicity. In Section 6, we investigate the relationships between other types of generalized monotonicity and convexity, pseudo convexity as well as quasi convexity.

2. Preliminaries

Let X be a real Banach space endowed with a norm $\|\cdot\|$ and X^* its dual space with a norm $\|\cdot\|_*$. We denote by 2^{X^*}, $\langle\cdot,\cdot\rangle$, $[x,y]$, and (x,y) the family of all non-empty subsets of X^*, the dual pair between X and X^*, the line segment for $x,y\in X$, and the interior of $[x,y]$, respectively. Let K be a non-empty open convex subset of X, $T:X\to 2^{X^*}$ a set-valued mapping and $f:X\to R$ a non-differentiable real-valued function. The following concepts and results are taken from [11].

Definition 2.1. Let f be locally Lipschitz continuous at a given point $x\in X$ and v any other vector in X. The Clarke's generalized directional derivative of f at x in the direction v, denoted by $f^0(x;v)$, is defined by

$$f^0(x;v)=\limsup_{y\to x,\ t\downarrow 0}\frac{f(y+tv)-f(y)}{t}.$$

Definition 2.2. Let f be locally Lipschitz continuous at a given point $x\in X$ and v any other vector in X. The Clarke's generalized subdifferential of f at x, denoted by $\partial^c f(x)$, is defined as follows:

$$\partial^c f(x)=\left\{\xi\in X^*:\ f^0(x;v)\geqslant\langle\xi,v\rangle,\ \forall v\in X\right\}.$$

Lemma 2.1. *Let f be locally Lipschitz continuous with a constant L at $x\in X$. Then*:

(i) *$\partial^c f(x)$ is a non-empty convex weak*-compact subset of X^* and $\|\xi\|_*\leqslant L$ for each $\xi\in\partial^c f(x)$;*
(ii) *For every $v\in X$, $f^0(x;v)=\max\{\langle\xi,v\rangle:\ \xi\in\partial^c f(x)\}$.*

Lemma 2.2 (Mean value theorem). *Let x and y be points in X and suppose that f is Lipschitz continuous near each point of a non-empty closed convex set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that*

$$f(x) - f(y) \in \left\langle \partial^c f(u), x - y \right\rangle.$$

Lemma 2.3. *If f is convex on K and locally Lipschitz continuous at $x \in K$, then $\partial^c f(x)$ coincides with the subdifferential $\partial f(x)$ of f at x in the sense of convex analysis, and $f^0(x; v)$ coincides with the directional derivative $f'(x; v)$ for each $v \in X$, where*

$$\partial f(x) = \left\{\xi \in X^*\colon\ f(y) - f(x) \geqslant \langle \xi, y - x\rangle,\ \forall y \in K\right\},$$
$$f'(x; v) = \lim_{t\downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

3. Convexity and monotonicity

In this section, we establish the relations between (strict, strong) convexity of f and (strict, strong) monotonicity of its Clarke's generalized subdifferential mapping $\partial^c f$, which is a set-valued mapping.

Definition 3.1. (i) f is said to be convex on K if for any $x, y \in K$ and any $t \in [0, 1]$,

$$f\big(tx + (1 - t)y\big) \leqslant tf(x) + (1 - t)f(y);$$

(ii) f is said to be strictly convex on K if for any $x, y \in K$ with $x \neq y$ and any $t \in (0, 1)$,

$$f\big(tx + (1 - t)y\big) < tf(x) + (1 - t)f(y);$$

(iii) [7] f is said to be strongly convex on K if there exists a constant $\alpha > 0$ such that for any $x, y \in K$ and any $t \in [0, 1]$,

$$f\big(tx + (1 - t)y\big) \leqslant tf(x) + (1 - t)f(y) - \alpha t(1 - t)\|x - y\|^2.$$

It is evident that strong convexity implies strict convexity and strict convexity implies convexity. But the converses are not true in general. For example, the function

$$f(x) = \begin{cases} 1 - x, & x \in [0, 1], \\ 0, & x \in [1, 2], \\ x - 2, & x \in [2, 3], \end{cases}$$

is convex but not strictly convex on the interval $[0, 3]$ and the function $f(x) = (1 + x^2)^{1/2}$ is strictly convex but not strongly convex on R, for details see [12].

Definition 3.2. (i) [7] T is said to be monotone on K if for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$ one has

$$\langle u - v, x - y\rangle \geqslant 0;$$

(ii) T is said to be strictly monotone on K if for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$ one has

$$\langle u - v, x - y\rangle > 0;$$

(iii) [7] T is said to be strongly monotone on K if there exists a constant $\alpha > 0$ such that for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$ one has

$$\langle u - v, x - y\rangle \geqslant \alpha\|x - y\|^2;$$

(iv) [13] T is said to be partially relaxed strongly monotone on K if there exists a constant $\beta > 0$ such that for any $x, y, z \in K$ and any $u \in T(x)$, $v \in T(y)$ one has

$$\langle u - v, z - y\rangle \geqslant -\beta\|x - z\|^2.$$

We can easily see from Definition 3.2 that the class of strictly monotone mappings includes the class of strongly monotone mappings and the class of monotone mappings includes the classes of strictly monotone and partially relaxed strongly monotone mappings. But the converses are not true in general, for the sake of convenience, we observe the following several real-valued functions.

It is clear that a constant function $T(x) = c$ is partially relaxed strongly monotone but neither strictly monotone nor strongly monotone on R.

The function $T(x) = x^2$ is strictly monotone but neither strongly monotone nor partially relaxed strongly monotone on the interval $[0, +\infty)$. Indeed, for any real number $\beta > 0$, let $x = 4\beta$, $y = 2\beta$, and $z = \beta$. Then

$$\bigl\langle T(x) - T(y), z - y\bigr\rangle + \beta\|x - z\|^2 = (x^2 - y^2)(z - y) + \beta(x - z)^2 = -3\beta^3 < 0,$$

which indicates that T is not partially relaxed strongly monotone on $[0, +\infty)$.

Similarly, we can prove that the function $T(x) = x^2$ is strongly monotone but not partially relaxed strongly monotone on the interval $[1, +\infty)$. In fact, it suffices to take $x = 4\beta + 1$, $y = 2\beta + 1$, and $z = \beta + 1$ for any real number $\beta > 0$.

Proposition 3.1. *Let f be locally Lipschitz continuous on K. Then*:

(i) *f is convex on K if and only if for any $x, y \in K$ and any $\eta \in \partial^c f(y)$ one has*

$$f(x) - f(y) \geqslant \langle \eta, x - y\rangle; \tag{3.1}$$

(ii) *f is strongly convex on K with a constant $\alpha > 0$ if and only if for any $x, y \in K$ and any $\eta \in \partial^c f(y)$ one has*

$$f(x) - f(y) \geqslant \langle \eta, x - y\rangle + \alpha\|x - y\|^2. \tag{3.2}$$

Proof. We only prove the assertion (ii). With the similar way, we can prove the if part of (i) and the only if part is trivial. Suppose first that f is strongly convex on K with a constant $\alpha > 0$. Then for any $x, y \in K$ and any $t \in (0, 1)$ one has

$$t\bigl[f\bigl(tx + (1-t)y\bigr) - f(x)\bigr] + (1-t)\bigl[f\bigl(tx + (1-t)y\bigr) - f(y)\bigr]$$
$$\leqslant -\alpha t(1-t)\|x - y\|^2.$$

Consequently, from Lemma 2.3 it follows that

$$f(y) - f(x) + f^0(y; x - y) \leqslant -\alpha\|x - y\|^2.$$

By Lemma 2.1 (ii), we can deduce that inequality (3.2) holds for any $\eta \in \partial^c f(y)$.

Conversely, assume that inequality (3.2) holds for any $x, y \in K$ and any $\eta \in \partial^c f(y)$. Then for any $t \in [0, 1]$ and any $\gamma \in \partial^c f(tx + (1 - t)y)$, one has

$$tf(x) - tf\big(tx + (1 - t)y\big) \geqslant t(1 - t)\langle\gamma, x - y\rangle + \alpha t(1 - t)^2\|x - y\|^2,$$
$$(1 - t)f(y) - (1 - t)f\big(tx + (1 - t)y\big)$$
$$\geqslant -t(1 - t)\langle\gamma, x - y\rangle + \alpha t^2(1 - t)\|x - y\|^2.$$

Hence,

$$tf(x) + (1 - t)f(y) - f\big(tx + (1 - t)y\big) \geqslant \alpha t(1 - t)\|x - y\|^2,$$

which indicates that f is strongly convex on K with $\alpha > 0$. □

The following theorem was stated in [11, Proposition 2.2.9] without proof. For the sake of completeness, we provide a proof here.

Theorem 3.2. *Let f be locally Lipschitz continuous on K. Then f is convex on K if and only if its Clarke's generalized subdifferential mapping $\partial^c f$ is monotone on K.*

Proof. We first assume that f is convex on K. For any given $x, y \in K$, $\xi \in \partial^c f(x)$, and $\eta \in \partial^c f(y)$, according to inequality (3.1), we have

$$\langle\xi - \eta, x - y\rangle = -\langle\xi, y - x\rangle - \langle\eta, x - y\rangle \geqslant f(x) - f(y) + f(y) - f(x) = 0.$$

Hence, $\partial^c f$ is monotone on K.

Conversely, suppose that $\partial^c f$ is monotone on K. For any given $x, y \in K$ with $x \neq y$ and $t \in (0, 1)$, in view of Lemma 2.2, there exist $h, l \in (0, 1)$: $0 < h < t < l < 1$, $p \in \partial^c f(u)$, and $q \in \partial^c f(v)$ such that

$$f(x) - f\big(tx + (1 - t)y\big) = (1 - t)\langle p, x - y\rangle,$$
$$f(y) - f\big(tx + (1 - t)y\big) = -t\langle q, x - y\rangle,$$

where $u = y + l(x - y) \in (x, tx + (1 - t)y)$ and $v = y + h(x - y) \in (tx + (1 - t)y, y)$. In view of the monotonicity of $\partial^c f$, we can deduce that

$$\langle p - q, u - v\rangle = (l - h)\langle p - q, x - y\rangle \geqslant 0.$$

Therefore, for any $x, y \in K$ with $x \neq y$ and any $t \in (0, 1)$ one has

$$f\big(tx + (1 - t)y\big) \leqslant tf(x) + (1 - t)f(y). \tag{3.3}$$

It is clear that inequality (3.3) holds whenever $x = y$ or $t = 0$ and 1. The proof is complete. □

Theorem 3.3. *Let f be locally Lipschitz continuous on K. If $\partial^c f$ is strictly monotone on K, then f is strictly convex on K.*

The proof is similar to that of Theorem 3.2 and we omit it.

Theorem 3.4. *Let f be locally Lipschitz continuous on K. Then f is strongly convex on K with $\alpha > 0$ if and only if $\partial^c f$ is strongly monotone on K with 2α.*

Proof. The only if part is immediate consequence of Proposition 3.1(ii). It suffices to prove the if part. Let $\partial^c f$ be strongly monotone on K with $\beta > 0$. Assume to the contrary that f is not strongly convex on K, then, for any $\alpha > 0$, there exist $x_0, y_0 \in K$ with $x_0 \neq y_0$ and $\eta_0 \in \partial^c f(y_0)$ such that

$$f(x_0) - f(y_0) < \langle \eta_0, x_0 - y_0 \rangle + \alpha \|x_0 - y_0\|^2.$$

From Lemma 2.2 it follows that there exist $t_0 \in (0, 1)$ and $\gamma_0 \in \partial^c f(u_0)$ such that

$$\langle \gamma_0 - \eta_0, x_0 - y_0 \rangle < \alpha \|x_0 - y_0\|^2,$$

where $u_0 = t_0 x_0 + (1 - t_0) y_0$. Consequently, we have

$$\beta t_0^2 \|x_0 - y_0\|^2 = \beta \|u_0 - y_0\|^2 \leqslant \langle \gamma_0 - \eta_0, u_0 - y_0 \rangle < \alpha t_0 \|x_0 - y_0\|^2,$$

that is, $\alpha > t_0 \beta$, which contradicts the arbitrariness of α.

Suppose that f is strongly convex on K with $\alpha > 0$. We want to show that $\beta = 2\alpha$. For any given $x, y \in K$, $\xi \in \partial^c f(x)$, and $\eta \in \partial^c f(y)$, it follows from Proposition 3.1(ii) that

$$f(x) - f(y) \geqslant \langle \eta, x - y \rangle + \alpha \|x - y\|^2,$$
$$f(y) - f(x) \geqslant \langle \xi, y - x \rangle + \alpha \|x - y\|^2.$$

Proceeding to the next step, we have $\langle \xi - \eta, x - y \rangle \geqslant 2\alpha \|x - y\|^2$. Therefore, $\beta = 2\alpha$. The proof is complete. □

4. Pseudo convexity and pseudo monotonicity

This section discusses the relationships between strict (strong, sharp) pseudo convexity of the non-differentiable function f and strict (strong) pseudo monotonicity of its Clarke's generalized subdifferential mapping $\partial^c f$.

Definition 4.1. Let f be locally Lipschitz continuous on K. Then:

(i) f is said to be pseudo convex on K if for any $x, y \in K$ and any $\eta \in \partial^c f(y)$,

$$\langle \eta, x - y \rangle \geqslant 0 \quad \text{implies} \quad f(x) \geqslant f(y);$$

(ii) f is said to be strictly pseudo convex on K if for any $x, y \in K$ with $x \neq y$ and any $\eta \in \partial^c f(y)$,

$$\langle \eta, x - y \rangle \geqslant 0 \quad \text{implies} \quad f(x) > f(y);$$

(iii) f is said to be strongly pseudo convex on K if there exists a constant $\alpha > 0$ such that for any $x, y \in K$ and any $\eta \in \partial^c f(y)$,

$$\langle \eta, x - y \rangle \geqslant 0 \quad \text{implies} \quad f(x) \geqslant f(y) + \alpha \|x - y\|^2;$$

(iv) f is said to be sharply pseudo convex on K if there exists a constant $\alpha > 0$ such that for any $x, y \in K$, $t \in [0,1]$, and $\eta \in \partial^c f(y)$,

$$\langle \eta, x - y \rangle \geqslant 0 \quad \text{implies} \quad f(x) \geqslant f\big(x + t(x - y)\big) + \alpha t(1 - t)\|x - y\|^2.$$

Note that the concepts of pseudo convexity, strong pseudo convexity, and sharp pseudo convexity in Definition 4.1 are natural generalizations of the corresponding notions introduced in [3,7,14] for a differentiable function, respectively. It is obvious that the class of pseudo convex functions includes the class of strictly pseudo convex functions and the class of strictly pseudo convex functions includes the class of strongly pseudo convex functions. We will prove that the class of strongly pseudo convex functions includes the class of sharply pseudo convex functions.

Definition 4.2. (i) [5,7] T is said to be pseudo monotone on K if for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has that

$$\langle v, x - y \rangle \geqslant 0 \quad \text{implies} \quad \langle u, x - y \rangle \geqslant 0;$$

(ii) [5] T is said to be strictly pseudo monotone on K if for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$, one has that

$$\langle v, x - y \rangle \geqslant 0 \quad \text{implies} \quad \langle u, x - y \rangle > 0;$$

(iii) [5] T is said to be strongly pseudo monotone on K if there exists a constant $\alpha > 0$ such that for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has that

$$\langle v, x - y \rangle \geqslant 0 \quad \text{implies} \quad \langle u, x - y \rangle \geqslant \alpha \|x - y\|^2.$$

The pseudo monotonicity of a set-valued mapping was first introduced by Saigal [15] in a finite dimensional space setting and this concept is also a generalization of that first introduced by Karamardian [2]. From Definition 4.2, we can easily see that strict pseudo monotonicity implies pseudo monotonicity. But the converse is not true, as an example in [3] shows. Karamardian and Schaible established in [3] the relationship between pseudo convex and pseudo monotone functions. Namely, a differentiable function is pseudo convex if and only if its gradient is pseudo monotone. We now show that a similar result holds for strict pseudo convexity of a non-differentiable function.

Theorem 4.1. *Let f be locally Lipschitz continuous on K. Then f is strictly pseudo convex on K if and only if the set-valued mapping $\partial^c f$ is strictly pseudo monotone on K.*

Proof. We first prove the only if part. Suppose that f is strictly pseudo convex on K. For any given $x, y \in K$ with $x \neq y$, $\xi \in \partial^c f(x)$, and $\eta \in \partial^c f(y)$, let

$$\langle \eta, x - y \rangle \geqslant 0. \tag{4.1}$$

We want to show that $\langle \xi, x - y \rangle > 0$. Assume to the contrary that $\langle \xi, x - y \rangle \leqslant 0$, then the strict pseudo convexity of f indicates that

$$f(y) > f(x). \tag{4.2}$$

On the other hand, it follows from inequality (4.1) that $f(x) > f(y)$, which contradicts inequality (4.2). Hence, $\partial^c f$ is strictly pseudo monotone on K.

We now prove the if part. Suppose that $\partial^c f$ is strictly pseudo monotone on K. For any given $x, y \in K$ with $x \neq y$ and $\eta \in \partial^c f(y)$, let inequality (4.1) hold. We want to show that $f(x) > f(y)$. Assume to the contrary that $f(x) \leqslant f(y)$. By Lemma 2.2, there exist $t \in (0,1)$ and $\lambda \in \partial^c f(tx + (1-t)y)$ such that $f(x) - f(y) = \langle \lambda, x - y\rangle \leqslant 0$. Consequently,

$$\bigl\langle \lambda, y - \bigl(tx + (1-t)y\bigr)\bigr\rangle = -t\langle \lambda, x - y\rangle \geqslant 0.$$

The strict pseudo monotonicity of $\partial^c f$ implies that

$$\bigl\langle \eta, y - \bigl(tx + (1-t)y\bigr)\bigr\rangle = -t\langle \eta, x - y\rangle > 0,$$

i.e., $\langle \eta, x - y\rangle < 0$. This contradicts inequality (4.1). The proof is complete. □

It notices that the relationship between pseudo convexity of f and pseudo monotonicity of $\partial^c f$ is not clear.

Theorem 4.2. *Let f be locally Lipschitz continuous on K. Then it is sharply pseudo convex on K with $\beta > 0$ if and only if the set-valued mapping $\partial^c f$ is strongly pseudo monotone on K with β.*

Proof. Suppose first that f is sharply pseudo convex on K with $\beta > 0$. For any given $x, y \in K$, $\xi \in \partial^c f(x)$, $\eta \in \partial^c f(y)$, and $t \in [0,1]$, let inequality (4.1) hold. From Definition 4.1(iv), we can deduce that

$$\limsup_{t\downarrow 0} \frac{f(x + t(y-x)) - f(x)}{t} + \beta\|x - y\|^2 \leqslant 0.$$

Since f is locally Lipschitz continuous and K is a non-empty open convex set, for any given $\varepsilon > 0$ and $t \in (0,1)$ small enough, there exists a constant $\delta > 0$ such that for any x' with $\|x' - x\| < \delta$, one has $x', x' + t(y-x) \in K$ and

$$\frac{f(x' + t(y-x)) - f(x')}{t} \leqslant \frac{f(x + t(y-x)) - f(x)}{t} + \varepsilon.$$

Hence,

$$\begin{aligned} &f^0(x; y-x) + \beta\|x - y\|^2 \\ &\quad = \limsup_{x'\to x,\ t\downarrow 0} \frac{f(x' + t(y-x)) - f(x')}{t} + \beta\|x - y\|^2 \\ &\quad \leqslant \limsup_{t\downarrow 0} \frac{f(x + t(y-x)) - f(x)}{t} + \varepsilon + \beta\|x - y\|^2 \leqslant \varepsilon. \end{aligned}$$

The arbitrariness of ε indicates that $f^0(x; y-x) + \beta\|x - y\|^2 \leqslant 0$. According to Lemma 2.1(ii), we have

$$\langle \xi, x - y\rangle \geqslant \beta\|x - y\|^2.$$

This shows that $\partial^c f$ is strongly pseudo monotone on K with β.

Now we prove the inverse implication. For any given $x, y \in K$, $\eta \in \partial^c f(y)$, and $t \in (0,1)$, let $x_t := x + t(y-x)$ and inequality (4.1) hold. By Lemma 2.2, there exist $h \in (0,t)$ and $\gamma \in \partial^c f(x_h)$ such that $f(x) - f(x_t) = t\langle \gamma, x-y\rangle$, where $x_h = x + h(y-x)$. Since $\langle \eta, x-y\rangle \geqslant 0$ implies $\langle \eta, x_h - y\rangle \geqslant 0$, by Definition 4.2(iii), we have

$$(1-h)\langle \gamma, x-y\rangle = \langle \gamma, x_h - y\rangle \geqslant \beta\|x_h - y\|^2 = \beta(1-h)^2\|x-y\|^2.$$

Therefore,

$$f(x) - f\big(x + t(y-x)\big) \geqslant \beta t(1-h)\|x-y\|^2 \geqslant \beta t(1-t)\|x-y\|^2.$$

This shows that f is sharply pseudo convex on K with $\beta > 0$. The assertion is proven completely. □

Theorem 4.3. *Let f be locally Lipschitz continuous on K. If the set-valued mapping $\partial^c f$ is strongly pseudo monotone on K with $\beta > 0$, then f is strongly pseudo convex on K with $\beta/4$.*

Proof. For any given $x, y \in K$ and $\eta \in \partial^c f(y)$, let $z = (1/2)x + (1/2)y$ and inequality (4.1) hold. Then by Lemma 2.2, there exist $h, l \in (0,1)$ with $0 < h < 1/2 < l < 1$, $\gamma \in \partial^c f(u)$, and $\sigma \in \partial^c f(v)$ such that

$$f(x) - f(z) = (1/2)\langle \gamma, x-y\rangle,$$
$$f(z) - f(y) = (1/2)\langle \sigma, x-y\rangle,$$

where $u = x + h(y-x)$ and $v = x + l(y-x)$. It is obvious that inequality (4.1) implies that

$$\langle \eta, u-y\rangle \geqslant 0 \quad \text{and} \quad \langle \eta, v-y\rangle \geqslant 0.$$

From Definition 4.2(iii), it follows that

$$\begin{aligned} f(x) - f(z) &= \big(1/2(1-h)\big)\langle \gamma, u-y\rangle \geqslant \big(1/2(1-h)\big)\beta\|u-y\|^2 \\ &= (1/2)\beta(1-h)\|x-y\|^2. \end{aligned}$$

Similarly, we can also deduce that $f(z) - f(y) \geqslant (1/2)\beta(1-l)\|x-y\|^2$. Hence,

$$\begin{aligned} f(x) - f(y) &\geqslant (1/2)\beta\big((1-h) + (1-l)\big)\|x-y\|^2 \\ &\geqslant (1/2)\beta(1-h)\|x-y\|^2 > (\beta/4)\|x-y\|^2. \end{aligned}$$

This indicates that the assertion is true. □

From Theorems 4.2 and 4.3, we can obtain the following corollary.

Corollary 4.4. *Let f be locally Lipschitz continuous on K. If f is sharply pseudo convex on K with $\beta > 0$, then it is strongly pseudo convex on K with $\beta/4$.*

5. Quasi convexity and quasi monotonicity

In this section, we establish the relationship between quasi convexity of the non-differentiable function f and quasi monotonicity of the set-valued mapping $\partial^c f$.

Definition 5.1 [3,10]. f is said to be quasi convex on K if for any $x, y \in K$ and any $t \in [0,1]$, one has that

$$f(x) \leqslant f(y) \quad \text{implies} \quad f\big(tx + (1-t)y\big) \leqslant f(y),$$

or

$$f\big(tx + (1-t)y\big) \leqslant \max\big\{f(x), f(y)\big\}.$$

Definition 5.2. (i) T is called quasi monotone on K if for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has that

$$\langle v, x - y\rangle > 0 \quad \text{implies} \quad \langle u, x - y\rangle \geqslant 0;$$

(ii) T is called partially relaxed strongly quasi monotone on K if there exists a constant $\beta > 0$ such that for any $x, y, z \in K$ and any $u \in T(x)$, $v \in T(y)$, one has that

$$\langle v, z - y\rangle > 0 \quad \text{implies} \quad \langle u, z - y\rangle \geqslant -\beta\|x - z\|^2.$$

Remark 5.1. (i) The concept of the quasi monotonicity here is an extension of that introduced in [3].

(ii) We can see from Definitions 4.2 and 5.2 that the pseudo monotonicity implies the quasi monotonicity. But the converse is not true in general, as an example in [3] shows.

For a differentiable function, the following hybrid characterization presented by Arrow and Enthoven in [16] is well known.

If f is differentiable, then it is quasi convex on K if and only if for any $x, y \in K$, one has that

$$f(x) \leqslant f(y) \quad \text{implies} \quad \big\langle \nabla f(y), x - y\big\rangle \leqslant 0.$$

This result was extended by Diewert to the following radially lower semi-continuous[1] function, for details see [10,17].

Lemma 5.1. *Let f be radially lower semi-continuous on K. Then it is quasi convex on K if and only if for any $x, y \in K$, one has that*

$$f(x) \leqslant f(y) \quad \textit{implies} \quad f^D(y; x - y) \leqslant 0,$$

where $f^D(a; d)$ denotes the directional upper derivative of f at a in the direction d and is defined by

$$f^D(a; d) := \limsup_{t \downarrow 0} \frac{f(a + td) - f(a)}{t}.$$

[1] f is called radially lower semi-continuous on K if the function $S(t) := f(a + t(b - a))$ is lower semi-continuous on $[0,1]$ for each $a, b \in K$.

From Definition 2.1 and Lemma 2.1(ii), we can extend the above result to the form below:

Proposition 5.1. *Let f be locally Lipschitz continuous on K. Then it is quasi convex on K if and only if for any $x, y \in K$ and any $\eta \in \partial^c f(y)$, one has that*

$$f(x) \leqslant f(y) \quad \textit{implies} \quad \langle \eta, x-y \rangle \leqslant 0. \tag{5.1}$$

Proof. For any given $x, y \in K$, it is assumed that $f(x) \leqslant F(y)$.

We first prove the only if part. Definition 5.1 indicates that $f(y+t(x-y)) \leqslant f(y)$ for any $t \in (0,1)$. Since f is locally Lipschitz continuous on K, for any $\varepsilon > 0$ small enough, one has

$$\begin{aligned} f^0(y; x-y) &= \limsup_{y' \to y,\ t \downarrow 0} \frac{f(y'+t(x-y)) - f(y')}{t} \\ &\leqslant \limsup_{t \downarrow 0} \frac{f(y+t(x-y)) - f(y)}{t} + \varepsilon \leqslant \varepsilon. \end{aligned}$$

The arbitrariness of ε shows that $f^0(y; x-y) \leqslant 0$. From Lemma 2.1(ii), it follows that implication (5.1) is true.

Now we prove the if part. In view of implication (5.1) and Lemma 2.1(ii), we can obtain that $f^0(y; x-y) \leqslant 0$. Consequently,

$$\begin{aligned} f^D(y; x-y) &= \limsup_{t \downarrow 0} \frac{f(y+t(x-y)) - f(y)}{t} \\ &\leqslant \limsup_{y' \to y,\ t \downarrow 0} \frac{f(y'+t(x-y)) - f(y')}{t} = f^0(y; x-y) \leqslant 0. \end{aligned}$$

The locally Lipschitz continuity of f implies for any given $a, b \in K$ and $t_0 \in [0,1]$, one has

$$\liminf_{t \to t_0} S(t) = \liminf_{t \to t_0} f\big(a + t(b-a)\big) = S(t_0).$$

This shows that f is radially lower semi-continuous on K. According to Lemma 5.1, f is quasi convex on K. The proof is complete. □

Using the above proposition, we can prove the main result of this section.

Theorem 5.2. *Let f be locally Lipschitz continuous on K. Then it is quasi convex on K if and only if the set-valued mapping $\partial^c f$ is quasi monotone on K.*

Proof. Assume first that f is quasi convex on K. For any given $x, y \in K$, $\xi \in \partial^c f(x)$, and $\eta \in \partial^c f(y)$, let

$$\langle \eta, x-y \rangle > 0. \tag{5.2}$$

We want to show that $\langle \xi, x-y \rangle \geqslant 0$. Suppose to the contrary that $\langle \xi, x-y \rangle < 0$, then from Proposition 5.1 it follows that $f(y) > f(x)$, which implies that $\langle \eta, x-y \rangle \leqslant 0$. This contradicts inequality (5.2). Hence, the only if part of the assertion is true.

We now prove the if part. Suppose that f is not quasi convex on K. By Definition 5.1, there exist $x, y \in K$ with $x \neq y$ and $t \in (0,1)$ such that $f(x) \leqslant f(y)$ and $f(z) > f(y)$, where $z = x + t(y-x)$. By Lemma 2.2, there exist $h, l \in (0,1)$ with $0 < h < t < l < 1$, $\alpha \in \partial^c f(u)$, and $\beta \in \partial^c f(v)$ such that

$$f(z) - f(x) = \langle \alpha, z - x \rangle > 0,$$
$$f(z) - f(y) = \langle \beta, z - y \rangle > 0,$$

where $u = x + h(y-x) \in (x, z)$ and $v = x + l(y-x) \in (z, y)$. Proceeding to the next step, we have

$$\langle \alpha, v - u \rangle = (l-h)\langle \alpha, y - x \rangle = \frac{l-h}{t}\langle \alpha, z - x \rangle > 0,$$
$$\langle \beta, v - u \rangle = -(l-h)\langle \beta, x - y \rangle = -\frac{l-h}{1-t}\langle \beta, z - y \rangle < 0.$$

This contradicts the quasi monotonicity of $\partial^c f$. The proof is complete. □

6. Generalized convexity and generalized co-coerciveness

This section gives some relationships between the generalized convexity of the non-differentiable function f and the generalized co-coerciveness of the set-valued mapping $\partial^c f$.

Definition 6.1. (i) [13] The set-valued mapping T is said to be co-coercive on K if there exists a constant $\beta > 0$ such that for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle u - v, x - y \rangle \geqslant \beta \|u - v\|_*^2;$$

(ii) The set-valued mapping T is said to be strictly co-coercive on K if there exists a constant $\beta > 0$ such that for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$, one has

$$\langle u - v, x - y \rangle > \beta \|u - v\|_*^2;$$

(iii) The set-valued mapping T is said to be strictly pseudo co-coercive on K if there exists a constant $\beta > 0$ such that for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$, one has that

$$\langle v, x - y \rangle \geqslant 0 \quad \text{implies} \quad \langle u, x - y \rangle > \beta \|u - v\|_*^2;$$

(iv) The set-valued mapping T is said to be quasi co-coercive on K if there exists a constant $\beta > 0$ such that for any $x, y \in K$ with $x \neq y$ and any $u \in T(x)$, $v \in T(y)$, one has that

$$\langle v, x - y \rangle > 0 \quad \text{implies} \quad \langle u, x - y \rangle \geqslant \beta \|u - v\|_*^2.$$

Remark 6.1. (i) The co-coerciveness and pseudo co-coerciveness here are natural generalizations of the corresponding notions, which are also called Dunn property and pseudo Dunn property, introduced in [8,13] for a real-valued function, respectively.

(ii) It appears that the concept of quasi co-coerciveness is first introduced here.

Definition 6.2. Let $T : K \to BC(X^*)$, where $BC(X^*)$ denotes the family of all non-empty bounded closed subsets of X^*. Then:

(i) T is called Lipschitz continuous on K if there exists a constant $\alpha > 0$ such that for all $x, y \in K$, one has

$$M\big(T(x), T(y)\big) \leqslant \alpha \|x - y\|,$$

where $M(\cdot\,,\cdot)$ denotes the Hausdorff metric on $BC(X^*)$;

(ii) T is called strongly Lipschitz continuous on K if there exists a constant $\alpha > 0$ such that for all $x, y \in K$, $u \in T(x)$, and $v \in T(y)$, one has

$$\|u - v\|_* \leqslant \alpha \|x - y\|.$$

Proposition 6.1. *If the set-valued mapping T is co-coercive on K with a constant $\beta > 0$, then it is partially relaxed strongly monotone on K with $1/(4\beta)$.*

Proof. The assertion can be proven by using the inequality

$$\|u\|^2 + \langle u, v\rangle \geqslant -(1/4)\|v\|^2, \quad \forall u, v \in X. \qquad \square$$

Theorem 6.2. *Let f be locally Lipschitz continuous on K. Then*:

(i) *If the set-valued mapping $\partial^c f$ is partially relaxed strongly monotone on K with a constant $\beta > 0$, then f is convex on K*;

(ii) *If the set-valued mapping $\partial^c f$ is co-coercive on K with a constant $\beta > 0$, then f is convex and $\partial^c f$ is strongly Lipschitz continuous with $1/\beta$ on K.*

Proof. From Definition 3.2 (i) and (iv), we can see that $\partial^c f$ is monotone if it is partially relaxed strongly monotone on K. Hence, assertion (i) is an immediate consequence of Theorem 3.2.

We now prove assertion (ii). In view of Proposition 6.1, it suffices to prove that $\partial^c f$ is strongly Lipschitz continuous on K. Lemma 2.1(i) indicates that $\partial^c f(x) \in BC(X^*)$ for any $x \in K$. On the other hand, Definition 6.1(i) implies that for any $x, y \in K$ and any $\xi \in \partial^c f(x)$, $\eta \in \partial^c f(y)$, one has

$$\|\xi - \eta\|_* \cdot \|x - y\| \geqslant \langle \xi - \eta, x - \rangle \geqslant \beta \|\xi - \eta\|_*^2,$$

that is, $\|\xi - \eta\|_* \leqslant (1/\beta)\|x - y\|$. This shows that assertion (ii) holds. $\square$

Theorem 6.3. *Let f be locally Lipschitz continuous on K. If the set-valued mapping $\partial^c f$ is strictly co-coercive on K, then f is strictly convex on K.*

Proof. We can easily see from Definitions 3.2(ii) and 6.1(ii) that $\partial^c f$ is strictly monotone if it is strictly co-coercive on K. Therefore, the assertion is an immediate consequence of Theorem 3.3. $\square$

Theorem 6.4. *Let f be locally Lipschitz continuous on K. If the set-valued mapping $\partial^c f$ is strictly pseudo co-coercive on K, then f is strictly pseudo convex on K.*

Proof. From Definitions 4.2(ii) and 6.1(iii), it follows that $\partial^c f$ is strictly pseudo monotone if it is strictly pseudo co-coercive on K. Therefore, the assertion is an immediate consequence of Theorem 4.1. □

Theorem 6.5. *Let f be locally Lipschitz continuous on K. Then*:

(i) *If the set-valued mapping $\partial^c f$ is quasi co-coercive on K, then f is quasi convex on K*;
(ii) *If the set-valued mapping $\partial^c f$ is partially relaxed strongly quasi monotone on K, then f is quasi convex on K.*

Proof. In view of Definitions 5.2 and 6.1(iv), we can easily see that $\partial^c f$ is quasi monotone if it is quasi co-coercive or partially relaxed strongly monotone on K. Therefore, the assertions are immediate consequences of Theorem 5.2. □

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