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Semilocal convergence of a continuation method with Hölder continuous second derivative in Banach spaces

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ABSTRACT

In this paper, the semilocal convergence of a continuation method combining the Chebyshev method and the convex acceleration of Newton's method used for solving nonlinear equations in Banach spaces is established by using recurrence relations under the assumption that the second Frěchet derivative satisfies the Hölder continuity condition. This condition is mild and works for problems in which the second Frěchet derivative fails to satisfy Lipschitz continuity condition. A new family of recurrence relations are defined based on two constants which depend on the operator. The existence and uniqueness regions along with a closed form of the error bounds in terms of a real parameter $\alpha \in [0, 1]$ for the solution x^* is given. Two numerical examples are worked out to demonstrate the efficacy of our approach. On comparing the existence and uniqueness regions for the solution obtained by our analysis with those obtained by using majorizing sequences under Hölder continuity condition on F'', it is found that our analysis gives improved results. Further, we have observed that for particular values of the α, our analysis reduces to those for the Chebyshev method ($\alpha = 0$) and the convex acceleration of Newton's method ($\alpha = 1$) respectively with improved results.

1. Introduction

One of the most important and challenging problems in scientific computing is that of finding efficiently the solutions of nonlinear equations by iterative methods in Banach spaces and establish their convergence analysis. There exists a large number of applications in chemical engineering, transportation, operation research etc. that give rise to thousands of such equations depending on one or more parameters. The boundary value problems appearing in kinetic theory of gases, elasticity and other applied areas are reduced to solving nonlinear equations. Dynamic systems are mathematically modeled by difference or differential equations and their solutions usually represent the equilibrium states of the system obtained by solving nonlinear equations. As a result, these problems are extensively studied and many methods of various orders along with their local, semilocal and global convergence analysis are developed in [1–4]. Basically, two different kinds of approaches are used for convergence analysis. The first one uses majorizing sequences obtained by the same iterative method applied to a scalar function to majorize the iterates. The other one uses recurrence relations which has some advantages over majorizing sequences because, one can reduce the initial problem in a Banach space to a simpler problem with real sequences and vectors. Besides, we find a symmetry between some special properties of the iteration method and the corresponding ones in the system of recurrence relations. Newton's method and its variants are the quadratically convergent iterative methods used to solve these equations. The well known Kantorovich theorem [5,6] gives sufficient conditions for the semilocal convergence of Newton's method as well as the error estimates and existence–uniqueness

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regions of solutions. Many researchers [7–12] have also studied the semilocal convergence of one point third-order iterative methods such as the Chebyshev method, the Halley method and the Super-Halley method used for solving nonlinear equations under Kantorovich-type assumptions [5] on the involved operator in Banach spaces. These methods are also used in many applications. For example, they can be used in stiff systems [13], where a quick convergence is required. Their main assumption for the convergence analysis was that the second Fréchet derivative satisfies Lipschitz continuity condition. However, it is not always true as the following example illustrates.

Example. Let us consider the integral equation of Fredholm type [14]:

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+p} dt,$$

with $s \in [0, 1]$, $x, f \in C[0, 1]$, $p \in (0, 1]$ and λ is a real number.

Under the sup-norm on the operator F, the second Fréchet derivative of F satisfies

$$\|F''(x) - F''(y)\| \le |\lambda|(1+p)(2+p)\log 2\|x-y\|^p, \quad x, y \in \Omega.$$

Hence, for $p \in (0, 1)$, F'' does not satisfy the Lipschitz continuity condition, but it satisfies the Hölder continuity condition. Hernández and Salanova [15] and Xintao Ye and Chong Li [16] studied the convergence of the Chebyshev method and the Euler–Halley method by using recurrence relations under the assumption that F'' satisfies the Hölder continuity condition. The convergence of the Chebyshev method and the convex acceleration of Newton's method using majorizing sequences under the Hölder continuity conditions are given in [17,18]. Further, A family of Newton-type iterative processes solving nonlinear equations in Banach spaces, that generalizes the usually iterative methods of R-order at least three is considered by Hernández and Romero in [19,20]. They weaken the usual semilocal convergence conditions, for this type of iterative processes, assuming that the second Fréchet derivative is ω-conditioned. This means that second Fréchet derivative satisfies $\|F''(x) - F''(y)\| \le \omega(\|x-y\|)$, where ω is a nondecreasing continuous real function.

A continuation method is a parameter based method, giving a continuous connection between two functions f and g. Mathematically, a continuation method between two functions $f, g : X \to Y$, where X and Y are Banach spaces, is defined as a continuous map $h : [0, 1] \times X \to Y$ such that $h(\alpha, x) = \alpha f(x) + (1-\alpha)g(x), \alpha \in [0, 1]$ and $h(0, x) = g(x)$, $h(1, x) = f(x)$. The continuation method was known as early as 1930s. It was used by Kinemitician in the 1960s for solving mechanism synthesis problems. It also gives a set of certain answers and a simple iteration process to obtain solutions more exactly. For further literature survey on it, one can refer to the works of [21–23].

The aim of this paper is to study the semilocal convergence of a continuation method combining the Chebyshev method and the convex acceleration of Newton's method for solving nonlinear equations

$$F(x) = 0, \tag{1}$$

where, $F : \Omega \subseteq X \to Y$ be a nonlinear operator on an open convex domain Ω of a Banach space X with values in Banach space Y. This is done by using recurrence relations under the assumption that the second Frëchet derivative satisfies the Hölder continuity condition. The existence and uniqueness theorem is given. We have also derived a closed form of error bounds in terms of $\alpha \in [0, 1]$. Two numerical examples are worked out to demonstrate the efficacy of our approach. On comparing the existence and uniqueness regions and error bounds for the solution obtained by our analysis with those obtained by using majorizing sequences, it is found that our analysis gives improved results. Further, we have observed that for particular values of the α, our analysis reduces to those for the Chebyshev method ($\alpha = 0$) and the convex acceleration of Newton's method ($\alpha = 1$) respectively.

This paper is organized in six sections. Section 1 is the introduction. In Section 2, some preliminary results are given. Then, two real sequences are generated and their properties are studied. In Section 3, the recurrence relations are derived. In Section 4, a convergence theorem is established for the existence and uniqueness regions along with a priori error bounds for the solution. In Section 5, two numerical examples are worked out to demonstrate, the efficacy of our convergence analysis and a comparison of the existence and uniqueness regions for the solution obtained by our analysis with those obtained by using majorizing sequences under the Hólder continuity condition are done. Finally, conclusions form the Section 6.

2. Preliminary results

In this section, we shall derive a family of recurrence relations based on two constants in order to discuss the semilocal convergence of a continuation method combining two third order iterative methods namely, the Chebyshev method and the convex acceleration of Newton's method used for solving (1). Let us assume that $F'(x_0)^{-1} \in L(Y, X)$ exists at some point $x_0 \in \Omega$, where $L(Y, X)$ is the set of bounded linear operators from Y into X. The Chebyshev method and the convex Acceleration of Newton's method are defined as follows. For a suitable initial approximation $x_0 \in \Omega$, we define for $n = 0, 1, 2, \ldots,$ the iterations

$$x_{n+1} = J_0(x_n) = x_n - \left[I + \frac{1}{2}L_F(x_n)\right] F'(x_n)^{-1} F(x_n), \tag{2}$$

and

$$x_{n+1} = J_1(x_n) = x_n - \left[I + \frac{1}{2} L_F(x_n)(I - L_F(x_n))^{-1} \right] F'(x_n)^{-1} F(x_n) \tag{3}$$

where I is identity operator and $L_F(x)$ is the linear operator given by

$$L_F(x) = F'(x)^{-1} F''(x) F'(x)^{-1} F(x), \quad x \in X. \tag{4}$$

The continuation method between (2) and (3) can now be defined for a suitable initial approximation $x_{\alpha,0} \in \Omega$, $\alpha \in [0, 1]$ and for $n = 0, 1, 2, \ldots$, by the iteration

$$x_{\alpha,n+1} = \alpha J_1(x_{\alpha,n}) + (1 - \alpha) J_0(x_{\alpha,n}) \quad n \geq 0. \tag{5}$$

Replacing x_n by $x_{\alpha,n}$ in (2) and (3) and substituting the expressions for $J_0(x_{\alpha,n})$ from (2) and $J_1(x_{\alpha,n})$ from (3) in (5), we get

$$\left.\begin{aligned} y_{\alpha,n} &= x_{\alpha,n} - \Gamma_{\alpha,n} F(x_{\alpha,n}) \\ x_{\alpha,n+1} &= y_{\alpha,n} + \frac{1}{2} L_F(x_{\alpha,n}) G_\alpha(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n}) \end{aligned}\right\} \tag{6}$$

where,

$$G_\alpha(x_{\alpha,n}) = I + \alpha L_F(x_{\alpha,n}) H_\alpha(x_{\alpha,n})$$

for

$$H_\alpha(x_{\alpha,n}) = (I - L_F(x_{\alpha,n}))^{-1}$$

and

$$L_F(x_{\alpha,n}) = F'(x_{\alpha,n})^{-1} F''(x_{\alpha,n}) F'(x_{\alpha,n})^{-1} F(x_{\alpha,n}).$$

Let $\Gamma_{\alpha,0} = F'(x_{\alpha,0})^{-1} \in L(Y, X)$ exist at some $x_{\alpha,0}$ and, let the following assumptions hold.

$$\left.\begin{aligned} &1.\ \|\Gamma_{\alpha,0}\| = \|F'(x_{\alpha,0})^{-1}\| \leq \beta, \\ &2.\ \|F'(x_{\alpha,0})^{-1} F(x_{\alpha,0})\| \leq \eta, \\ &3.\ \|F''(x)\| \leq M, \forall x \in \Omega, \\ &4.\ \|F''(x) - F''(y)\| \leq N\|x - y\|^p, \forall x, y \in \Omega, p \in (0, 1] \end{aligned}\right\}. \tag{7}$$

Let

$$a_0 = M\beta\eta \quad \text{and} \quad b_0 = N\beta\eta^{1+p} \tag{8}$$

$$f(x) = \frac{2(1-x)}{2 - 4x + x^2 - (\alpha - 1)x^3}, \tag{9}$$

and

$$g(x, y) = \frac{\alpha x^2}{2(1-x)} + \frac{y(1 + (\alpha - 1)x)}{(1-x)(p+1)(p+2)} + \frac{x^2(1 + (\alpha - 1)x)}{2(1-x)} + \frac{x^3(1 + (\alpha - 1)x)^2}{8(1-x)^2}. \tag{10}$$

Now, define the real sequences $\{a_n\}$ and $\{b_n\}$, where,

$$a_{n+1} = a_n f(a_n)^2 g(a_n, b_n), \qquad b_{n+1} = b_n f(a_n)^{2+p} g(a_n, b_n)^{1+p}. \tag{11}$$

Let $r_0 = 0.380778$ be the smallest positive zero of the polynomial $p(x) = (2\alpha^2 - 4\alpha + 2)x^6 - (\alpha^2 + 2\alpha - 3)x^5 + (18\alpha - 16)x^4 - (8\alpha + 1)x^3 - (4\alpha - 36)x^2 - 32x + 8 = 0$ for $\alpha \in [0, 1]$. We now describe the properties of the sequences $\{a_n\}$ and $\{b_n\}$ through the following Lemmas.

Lemma 1. *Let f and g be the functions defined in (9) and (10), respectively. Then for $x \in (0, r_0)$*

(i) *f is a increasing function and $f(x) > 1$ in $(0, r_0]$ for $\alpha \in [0, 1]$.*
(ii) *g is increasing in both arguments for $y > 0$ and $\alpha \in [0, 1]$.*
(iii) *$f(\gamma x) < f(x)$ and $g(\gamma x, \gamma^{p+1} y) \leq \gamma^{p+1} g(x, y)$ for $\gamma \in (0, 1)$, $p \in (0, 1]$, and $\alpha \in [0, 1]$.*

Proof. The proof of Lemma 1 is trivial and hence omitted here. □

Lemma 2. *For a fixed $p \in (0, 1]$ and $\alpha \in [0, 1]$, define the function*

$$\Phi_p(x) = \frac{(p+1)(p+2)}{8} \left[\frac{(2\alpha^2 - 4\alpha + 2)x^6 - (\alpha^2 + 2\alpha - 3)x^5 + (18\alpha - 16)x^4}{8(1 + (\alpha - 1)x)(1 - x)} - \frac{(8\alpha + 1)x^3 - (4\alpha - 36)x^2 - 32x + 8}{8(1 + (\alpha - 1)x)(1 - x)} \right]. \tag{12}$$

If $0 < a_0 \le r_0$ *and* $0 \le b_0 \le \Phi_p(a_0)$*, then*

(i) $f(a_n)^2 g(a_n, b_n) \le 1$ *for all n.*
(ii) $\{a_n\}$ *and* $\{b_n\}$ *are decreasing and* $a_n \le r_0 < 1$, $\forall n$.

Proof. (i) can easily be proved for all n, as from the definitions of f and g, we get

$$f(a_n)^2 g(a_n, b_n) \le 1$$

if and only if

$$\left[\frac{4(1-a_n)^2}{(2-4a_n+a_n^2-(\alpha-1)a_n^3)^2}\right] \times \left[\frac{\alpha a_n^2}{2(1-a_n)} + \frac{b_n(1+(\alpha-1)a_n)}{(1-a_n)(p+1)(p+2)} + \frac{a_n^2(1+(\alpha-1)a_n)}{2(1-a_n)} + \frac{a_n^3(1+(\alpha-1)a_n)^2}{8(1-a_n)^2}\right] \le 1$$

or,

$$b_n \le \frac{(p+1)(p+2)}{8}\left[\frac{(2\alpha^2-4\alpha+2)x^6-(\alpha^2+2\alpha-3)x^5+(18\alpha-16)x^4}{8(1+(\alpha-1)x)(1-x)} - \frac{(8\alpha+1)x^3-(4\alpha-36)x^2-32x+8}{8(1+(\alpha-1)x)(1-x)}\right]$$

or,

$$b_n \le \Phi_p(a_n).$$

To prove (ii), we shall use induction. From (i) for $0 < a_0 \le r_0$ and $0 \le b_0 \le \Phi_p(a_0)$, we get $f(a_0)^2 g(a_0, b_0) \le 1$. Using (11),

$$a_1 = a_0 f(a_0)^2 g(a_0, b_0) \le a_0 < 1$$

and as $f(x) > 1$ in $(0, r_0]$, we get

$$b_1 = b_0 f(a_0)^{2+p} g(a_0, b_0)^{1+p} < b_0 f(a_0)^2 g(a_0, b_0)[f(a_0)^2 g(a_0, b_0)]^p \le b_0.$$

Let us assume that (ii) holds for $n = k$. Then, proceeding similarly as above, one can easily show that $a_{k+1} \le a_k \le r_0 < 1$ and $b_{k+1} \le b_k$. Since, f and g are increasing functions, this gives

$$f(a_{k+1})^2 g(a_{k+1}, b_{k+1}) \le f(a_k)^2 g(a_k, b_k) \le 1.$$

Hence it is true for all n. This proves the Lemma 2. □

Lemma 3. *Let* $0 < a_0 < r_0$ *and* $0 < b_0 < \Phi_p(a_0)$*. Define* $\gamma = \frac{a_1}{a_0}$*, then for* $n \ge 1$ *we have*

(i) $a_n \le \gamma^{(2+p)^{n-1}} a_{n-1} \le \gamma^{((2+p)^n-1)/(1+p)} a_0$ *for* $n \ge 1$.
(ii) $b_n \le (\gamma^{(2+p)^{n-1}})^{(1+p)} b_{n-1} \le \gamma^{(2+p)^n-1} b_0$.
(iii) $f(a_n)g(a_n, b_n) \le \gamma^{(2+p)^n} \frac{f(a_0)g(a_0,b_0)}{\gamma} = \frac{\gamma^{(2+p)^n}}{f(a_0)}$, $n \ge 0$.

Proof. Induction will be used to prove (i) and (ii). Since $a_1 = \gamma a_0$ and $a_1 < a_0$ from Lemma 2(i), we get $\gamma < 1$. Also by Lemma 1(i), we get

$$b_1 = b_0 f(a_0)^{p+2} g(a_0, b_0)^{p+1} < (f(a_0)^2 g(a_0, b_0))^{1+p} b_0 = \left(\frac{a_1}{a_0}\right)^{1+p} b_0 = \gamma^{1+p} b_0.$$

Suppose (i) and (ii) hold for $n = k$, then

$$\begin{aligned} a_{k+1} &= a_k f(a_k)^2 g(a_k, b_k) \le \gamma^{(2+p)^{k-1}} a_{k-1} f(a_{k-1})^2 g(\gamma^{(2+p)^{k-1}} a_{k-1}, (\gamma^{(2+p)^{k-1}})^{1+p} b_{k-1}) \\ &\le \gamma^{(2+p)^{k-1}} a_{k-1} f(a_{k-1})^2 (\gamma^{(2+p)^{k-1}})^{1+p} g(a_{k-1}, b_{k-1}) = \gamma^{(2+p)^k} a_k. \end{aligned}$$

Hence,

$$\begin{aligned} a_{k+1} &\le \gamma^{(2+p)^k} a_k \le \gamma^{(2+p)^k} \gamma^{(2+p)^{k-1}} a_{k-1} \le \gamma^{(2+p)^k} \gamma^{(2+p)^{k-1}} \cdots \gamma^{(2+p)^0} a_0 \\ &\le \gamma^{((2+p)^{k+1}-1)/(1+p)} a_0. \end{aligned}$$

Again, from $f(x) > 1$ in $(0, r_0]$, we get

$$b_{k+1} = b_k f(a_k)^{2+p} g(a_k, b_k)^{1+p} \le b_k [f(a_k)^2 g(a_k, b_k)]^{1+p} \le b_k \left(\frac{a_{k+1}}{a_k}\right)^{1+p} \le (\gamma^{(2+p)^k})^{1+p} b_k.$$

Hence,

$$b_{k+1} = (\gamma^{(2+p)^k})^{1+p} b_k \le (\gamma^{(2+p)^k})^{1+p} (\gamma^{(2+p)^{k-1}})^{1+p} \cdots (\gamma^{(2+p)^0})^{1+p} b_0 \le \gamma^{(2+p)^{k+1}-1} b_0$$

(iii) follows from

$$\begin{aligned} f(a_n)g(a_n, b_n) &\le f(\gamma^{(2+p)^n - 1/(1+p)} a_0) g(\gamma^{(2+p)^n - 1/(1+p)} a_0, \gamma^{(2+p)^n - 1} b_0) \\ &\le \gamma^{(2+p)^n} \frac{f(a_0)g(a_0, b_0)}{\gamma} = \gamma^{(2+p)^n}/f(a_0) \end{aligned}$$

as $\gamma = a_1/a_0 = f(a_0)^2 g(a_0, b_0)$. Thus, the Lemma 3 is proved. □

3. Recurrence relations

In this section, the recurrence relations will be derived for the method (6) under the assumptions of the previous section. Now $y_{\alpha,0}$ exists as $\Gamma_{\alpha,0} = F'(x_{\alpha,0})^{-1}$ exists. Thus, we get

$$\|L_F(x_{\alpha,0})\| \le M\|\Gamma_{\alpha,0}\| \|\Gamma_{\alpha,0} F(x_{\alpha,0})\| \le a_0.$$

Using Banach Lemma, this gives

$$\|(I - L_F(x_{\alpha,0}))^{-1}\| \le \frac{1}{1 - a_0}$$

and

$$\|G_\alpha(x_{\alpha,0})\| \le 1 + \frac{\alpha a_0}{1 - a_0}.$$

Thus,

$$\|x_{\alpha,1} - y_{\alpha,0}\| \le \frac{a_0}{2}\left(1 + \frac{\alpha a_0}{1 - a_0}\right) \|y_{\alpha,0} - x_{\alpha,0}\|$$

and

$$\|x_{\alpha,1} - x_{\alpha,0}\| \le \|x_{\alpha,1} - y_{\alpha,0}\| + \|y_{\alpha,0} - x_{\alpha,0}\| \le \frac{(2 - a_0 + (\alpha - 1)a_0^2)}{2(1 - a_0)} \|y_{\alpha,0} - x_{\alpha,0}\|.$$

Also,

$$N\|\Gamma_{\alpha,0}\| \, \|y_{\alpha,0} - x_{\alpha,0}\|^{1+p} \le N\beta\eta^{1+p} = b_0. \tag{13}$$

We shall now prove the following inequalities for $n \ge 1, \alpha \in [0, 1]$.

$$\left.\begin{array}{l} \text{(I) } \|\Gamma_{\alpha,n}\| = \|F'(x_{\alpha,n})^{-1}\| \le f(a_{n-1})\|\Gamma_{\alpha,n-1}\|, \\ \text{(II) } \|\Gamma_{\alpha,n} F(x_{\alpha,n})\| \le f(a_{n-1}) g(a_{n-1}, b_{n-1}) \|\Gamma_{\alpha,n-1} F(x_{\alpha,n-1})\|, \\ \text{(III) } \|L_F(x_{\alpha,n})\| \le M\|\Gamma_{\alpha,n}\| \, \|\Gamma_{\alpha,n} F(x_{\alpha,n})\| \le a_n, \\ \text{(IV) } N\|\Gamma_{\alpha,n}\| \, \|\Gamma_{\alpha,n} F(x_{\alpha,n})\|^{1+p} \le b_n, \\ \text{(V) } \|x_{\alpha,n+1} - x_{\alpha,n}\| \le \left(\dfrac{(2 - a_n + (\alpha - 1)a_n^2)}{2(1 - a_n)}\right) \|\Gamma_{\alpha,n} F(x_{\alpha,n})\|, \\ \text{(VI) } y_{\alpha,n}, x_{\alpha,n+1} \in B(x_{\alpha,0}, R\eta), \text{ for } R = \dfrac{2 - a_0 + (\alpha - 1)a_0^2}{2(1 - a_0)(1 - \gamma\Delta)} \text{ and } \Delta = \dfrac{1}{f(a_0)} \end{array}\right\}. \tag{14}$$

This will require the following Lemma.

Lemma 4. *Let the sequence $\{x_{\alpha,n}\}$ and $\{y_{\alpha,n}\}$ be generated by (6). Then, for all $n \in \mathbb{Z}_+$, using Taylor's theorem, we get*

$$\begin{aligned} F(x_{\alpha,n+1}) = &\int_0^1 F''(x_{\alpha,n} + t(y_{\alpha,n} - x_{\alpha,n}))I - G_\alpha(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n})^2(1 - t)dt \\ &+ \int_0^1 [F''(x_{\alpha,n} + t(y_{\alpha,n} - x_{\alpha,n})) - F''(x_{\alpha,n})] G_\alpha(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n})^2(1 - t)dt \\ &+ \int_0^1 F''(x_{\alpha,n} + t(y_{\alpha,n} - x_{\alpha,n}))(y_{\alpha,n} - x_{\alpha,n})(x_{\alpha,n+1} - y_{\alpha,n})dt \\ &+ \int_0^1 F''(x_{\alpha,n} + t(x_{\alpha,n+1} - y_{\alpha,n}))(1 - t)dt(x_{\alpha,n+1} - y_{\alpha,n})^2. \end{aligned}$$

Now, conditions (I)–(VI) can be proved by using induction. For $x_{\alpha,1} \in \Omega$, we get

$$\begin{aligned}\|I - \Gamma_{\alpha,0}F'(x_{\alpha,1})\| &\le M\|\Gamma_{\alpha,0}\|\,\|x_{\alpha,1} - x_{\alpha,0}\| \\ &\le \frac{(2 - a_0 + (\alpha-1)a_0^2)}{2(1-a_0)}M\|\Gamma_{\alpha,0}\|\,\|y_{\alpha,0} - x_{\alpha,0}\| \\ &\le \frac{a_0(2 - a_0 + (\alpha-1)a_0^2)}{2(1-a_0)} < 1.\end{aligned}$$

Hence, $\Gamma_{\alpha,1} = F'(x_{\alpha,1})^{-1}$ exists. By using Banach Lemma, we get

$$\begin{aligned}\|\Gamma_{\alpha,1}\| &\le \frac{\|\Gamma_{\alpha,0}\|}{1 - M\|\Gamma_{\alpha,0}\|\,\|x_{\alpha,0} - x_{\alpha,1}\|} \\ &\le \frac{2(1-a_0)}{2 - 4a_0 + a_0^2 - (\alpha-1)a_0^3}\|\Gamma_{\alpha,0}\| = f(a_0)\|\Gamma_{\alpha,0}\|.\end{aligned} \tag{15}$$

Thus, $y_{\alpha,1}$ also exists. Using Lemma 4, we get

$$\begin{aligned}\|F(x_{\alpha,1})\| \le\ & \frac{M}{2}\|I - G_\alpha(x_{\alpha,0})\|\,\|y_{\alpha,0} - x_{\alpha,0}\|^2 + N\int_0^1 t^p(1-t)dt\|y_{\alpha,0} - x_{\alpha,0}\|^{2+p}\|G_\alpha(x_{\alpha,0})\| \\ & + M\|y_{\alpha,0} - x_{\alpha,0}\|\,\|x_{\alpha,1} - y_{\alpha,0}\| + \frac{M}{2}\|x_{\alpha,1} - y_{\alpha,0}\|^2.\end{aligned}$$

From

$$\|I - G_\alpha(x_{\alpha,0})\| \le \frac{\alpha a_0}{1-a_0},$$

we get

$$\begin{aligned}\|F(x_{\alpha,1})\| \le\ & \frac{M}{2}\frac{\alpha a_0}{1-a_0}\|y_{\alpha,0} - x_{\alpha,0}\|^2 + \frac{N}{(p+1)(p+2)}\|y_{\alpha,0} - x_{\alpha,0}\|^{p+2}\left(\frac{(1+(\alpha-1)a_0)}{(1-a_0)}\right) \\ & + M\|y_{\alpha,0} - x_{\alpha,0}\|^2\frac{a_0(1+(\alpha-1)a_0)}{2(1-a_0)} + \frac{M}{8}\|y_{\alpha,0} - x_{\alpha,0}\|^2\left(\frac{a_0^2(1+(\alpha-1)a_0)^2}{(1-a_0)^2}\right) \\ \le\ & \frac{M}{2}\frac{\alpha a_0}{1-a_0}\eta^2 + \frac{N}{(p+1)(p+2)}\left(\frac{(1+(\alpha-1)a_0)}{(1-a_0)}\right)\eta^{p+2} \\ & + M\frac{a_0(1+(\alpha-1)a_0)}{2(1-a_0)}\eta^2 + \frac{M}{8}\eta^2\left(\frac{a_0^2(1+(\alpha-1)a_0)^2}{(1-a_0)^2}\right).\end{aligned}$$

This gives

$$\begin{aligned}\|\Gamma_{\alpha,1}F(x_{\alpha,1})\| &\le \|\Gamma_{\alpha,1}\|\,\|F(x_{\alpha,1})\| \\ &\le f(a_0)\left[\frac{\alpha a_0^2}{2(1-a_0)} + \frac{b_0(1+(\alpha-1)a_0)}{(p+1)(p+2)(1-a_0)} + \frac{a_0^2(1+(\alpha-1)a_0)}{2(1-a_0)} + \frac{a_0^3(1+(\alpha-1)a_0)^2}{8(1-a_0)^2}\right]\eta \\ &\le f(a_0)g(a_0,b_0)\eta.\end{aligned} \tag{16}$$

Also, we have

$$\begin{aligned}\|L_F(x_{\alpha,1})\| \le M\|\Gamma_{\alpha,1}\|\,\|\Gamma_{\alpha,1}F(x_{\alpha,1})\| &\le M\|\Gamma_{\alpha,0}\|f(a_0)^2g(a_0,b_0)\|y_{\alpha,0} - x_{\alpha,0}\| \\ &\le a_0f(a_0)^2g(a_0,b_0) = a_1\end{aligned} \tag{17}$$

and

$$\begin{aligned}N\|\Gamma_{\alpha,1}\|\,\|\Gamma_{\alpha,1}F(x_{\alpha,1})\|^{1+p} &\le N\|\Gamma_{\alpha,0}\|f(a_0)f(a_0)^{1+p}g(a_0,b_0)^{1+p}\|y_{\alpha,0} - x_{\alpha,0}\| \\ &\le b_0f(a_0)^{2+p}g(a_0,b_0)^{1+p} = b_1.\end{aligned} \tag{18}$$

By using Banach Lemma, this gives

$$\|(I - L_F(x_{\alpha,1}))\| \le \frac{1}{1-a_1}$$

as $\|L_F(x_{\alpha,1})\| \le 1$. Thus,

$$\|G_\alpha(x_{\alpha,1})\| \le 1 + \frac{\alpha a_1}{1-a_1}.$$

Hence,

$$\|x_{\alpha,2} - y_{\alpha,1}\| \le \frac{a_1(1 + (\alpha - 1)a_1)}{2(1 - a_1)} \|y_{\alpha,1} - x_{\alpha,1}\|$$

and

$$\begin{aligned} \|x_{\alpha,2} - x_{\alpha,1}\| &\le \|x_{\alpha,2} - y_{\alpha,1}\| + \|y_{\alpha,1} - x_{\alpha,1}\| \\ &\le \frac{(2 - a_1 + (\alpha - 1)a_1^2)}{2(1 - a_1)} \|y_{\alpha,1} - x_{\alpha,1}\|. \end{aligned} \tag{19}$$

Now, for $\Delta = \frac{1}{f(a_0)}$, we have

$$\|y_{\alpha,1} - x_{\alpha,0}\| \le \frac{(2 - a_0 + (\alpha - 1)a_0^2)}{2(1 - a_0)(1 - \gamma\Delta)} \eta = R\eta \tag{20}$$

and

$$\|x_{\alpha,2} - x_{\alpha,0}\| \le \frac{2 - a_0 + (\alpha - 1)a_0^2}{2(1 - a_0)(1 - \gamma\Delta)} \eta = R\eta. \tag{21}$$

Using (15)–(21), the conditions (I)–(VI) hold for $n = 1$. Now let us assume that the conditions (I)–(VI) hold for $n = k$ and $x_{\alpha,k} \in \Omega$ for all $\alpha \in [0, 1]$. Proceeding similarly as above, we can prove that these conditions also hold for $n = k + 1$. Hence, by induction they hold for all n.

4. Convergence theorem

In this section we shall establish the convergence theorem and a closed form of the error bounds based on $\alpha \in [0, 1]$ for the method (6). Let us denote $\gamma = a_1/a_0$, $\Delta = 1/f(a_0)$ and $R = \frac{2-a_0+(\alpha-1)a_0^2}{2(1-a_0)(1-\gamma\Delta)}$ for all $\alpha \in [0, 1]$. Let $\mathcal{B}(x_{\alpha,0}, R\eta) = \{x \in \mathbb{X} : \|x - x_{\alpha,0}\| < R\eta\}$ and $\overline{\mathcal{B}}(x_{\alpha,0}, R\eta) = \{x \in \mathbb{X} : \|x - x_{\alpha,0}\| \le R\eta\}$ represent the open and closed balls around $x_{\alpha,0}$. Also, let us assume that $0 < a_0 \le r_0$ and $0 \le b_0 \le \Phi_p(a_0)$ hold, where r_0 be the smallest positive zero of the polynomial $(2\alpha^2 - 4\alpha + 2)x^6 - (\alpha^2 + 2\alpha - 3)x^5 + (18\alpha - 16)x^4 - (8\alpha + 1)x^3 - (4\alpha - 36)x^2 - 32x + 8 = 0$ for all $\alpha \in [0, 1]$.

Theorem 1. *Under the assumptions given in (7) and $\overline{\mathcal{B}}(x_{\alpha,0}, R\eta) \subseteq \Omega$, the method (6) starting from $x_{\alpha,0}$ generates a sequence of iterates $\{x_{\alpha,n}\}$ converging to the root x^* of $F(x) = 0$ with R-order at least $(2 + p)$. In this case $x_{\alpha,n}$, $y_{\alpha,n}$ and x^* lie in $\overline{\mathcal{B}}(x_{\alpha,0}, R\eta)$ and x^* is unique in $\mathcal{B}(x_{\alpha,0}, \frac{2}{M\beta} - R\eta) \cap \Omega$. Further the error bounds on x^* is given by*

$$\|x^* - x_{\alpha,n}\| \le \frac{(2 - \gamma^{((2+p)^n-1)/(1+p)}a_0 + (\alpha - 1)\gamma^{((2+p)^n-1)/(1+p)}a_0^2)}{2(1 - \gamma^{((2+p)^n-1)/(1+p)}a_0)} \frac{\gamma^{((2+p)^n-1)/(1+p)}\Delta^n\eta}{1 - \gamma^{(2+p)^n}\Delta}. \tag{22}$$

Proof. It is sufficient to show that $\{x_{\alpha,n}\}$ is a Cauchy sequence in order to establish the convergence of $\{x_{\alpha,n}\}$. Using (14), we can give

$$\begin{aligned} \|y_{\alpha,n} - x_{\alpha,n}\| &\le f(a_{n-1})g(a_{n-1}, b_{n-1})\|y_{\alpha,n-1} - x_{\alpha,n-1}\| \\ &\le \cdots \left(\prod_{j=0}^{n-1} f(a_j)g(a_j, b_j)\right)\|y_{\alpha,0} - x_{\alpha,0}\| \\ &\le \left(\prod_{j=0}^{n-1} f(a_j)g(a_j, b_j)\right)\eta \end{aligned} \tag{23}$$

and

$$\begin{aligned} \|x_{\alpha,m+n} - x_{\alpha,m}\| &\le \|x_{\alpha,m+n} - x_{\alpha,m+n-1}\| + \cdots \|x_{\alpha,m+1} - x_{\alpha,m}\| \\ &\le \frac{2 - a_{m+n-1} + (\alpha - 1)a_{m+n-1}^2}{2(1 - a_{m+n-1})}\|y_{\alpha,m+n-1} - x_{\alpha,m+n-1}\| + \cdots \\ &\quad + \frac{2 - a_m + (\alpha - 1)a_m^2}{2(1 - a_m)}\|y_{\alpha,m} - x_{\alpha,m}\| \\ &\le \frac{2 - a_m + (\alpha - 1)a_m^2}{2(1 - a_m)}\left[\prod_{j=0}^{m+n-2} f(a_j)g(a_j, b_j) + \cdots\cdots + \prod_{j=0}^{m-1} f(a_j)g(a_j, b_j)\right]\eta. \end{aligned} \tag{24}$$

Now, for $a_0 = r_0$, we have $b_0 = \Phi_p(a_0) = 0$. Hence, from Lemma 2, we obtain $f(a_0)^2 g(a_0, b_0) = 1$, $a_n = a_{n-1} = \cdots = a_0$ and $b_n = b_{n-1} = \cdots = b_0 = 0$. This gives

$$\|y_{\alpha,n} - x_{\alpha,n}\| \le (f(a_0)g(a_0, b_0))^n \|y_{\alpha,0} - x_{\alpha,0}\| = \Delta^n \eta$$

and

$$\|x_{\alpha,m+n} - x_{\alpha,m}\| \le \frac{(2 - a_0 + (\alpha - 1)a_0^2)\Delta^m}{2(1 - a_0)} \left(\frac{1 - \Delta^n}{1 - \Delta}\right) \eta. \tag{25}$$

Hence, if we take $m = 0$, we get

$$\|x_{\alpha,n} - x_{\alpha,0}\| \le \frac{(2 - a_0 + (\alpha - 1)a_0^2)}{2(1 - a_0)} \left(\frac{1 - \Delta^n}{1 - \Delta}\right) \eta. \tag{26}$$

Thus, $x_{\alpha,n} \in \overline{\mathcal{B}}(x_{\alpha,0}, R\eta)$. Similarly we can prove that $y_{\alpha,n} \in \overline{\mathcal{B}}(x_{\alpha,0}, R\eta)$. Also, we can conclude that $\{x_{\alpha,n}\}$ is a Cauchy sequence. For $0 < a_0 < r_0$ and $0 < b_0 < \Phi_p(a_0)$ and from (14), we get on using Lemma 3(iii), for $n \ge 1$,

$$\|y_{\alpha,n} - x_{\alpha,n}\| \le \left(\prod_{j=0}^{n-1} f(a_j)g(a_j, b_j)\right) \eta \le \prod_{j=0}^{n-1} (\gamma^{(2+p)^j} \Delta)\eta = \gamma^{((2+p)^n - 1)/(1+p)} \Delta^n.$$

Also from (24), we get

$$\begin{aligned}
\|x_{\alpha,m+n} - x_{\alpha,m}\| &\le \frac{2 - a_m + (\alpha - 1)a_m^2}{2(1 - a_m)} \left[\prod_{j=0}^{m+n-2} f(a_j)g(a_j, b_j) + \cdots + \prod_{j=0}^{m-1} f(a_j)g(a_j, b_j)\right] \eta \\
&< \frac{2 - a_m + (\alpha - 1)a_m^2}{2(1 - a_m)} \left[\gamma^{((2+p)^{m+n-1} - 1)/(1+p)} \Delta^{m+n-1} + \cdots\cdots + \gamma^{((2+p)^m - 1)/(1+p)} \Delta^m\right] \\
&< \frac{(2 - a_m + (\alpha - 1)a_m^2)\Delta^m}{2(1 - a_m)} \left[\gamma^{((2+p)^{m+n-1} - 1)/(1+p)} \Delta^{n-1} + \cdots + \gamma^{((2+p)^m - 1)/(1+p)}\right] \eta \\
&< \frac{(2 - a_m \gamma^{((2+p)^m - 1)/(1+p)} + (\alpha - 1)a_m^2 \gamma^{((2+p)^m - 1)/(1+p)})\Delta^m}{2(1 - a_m \gamma^{((2+p)^m - 1)/(1+p)})} \\
&\quad \times \gamma^{((2+p)^m - 1)/(1+p)} \left[\gamma^{(2+p)^m [(2+p)^{n-1} - 1]/(1+p)} \Delta^{n-1} + \cdots + \gamma^{(2+p)^m [(2+p) - 1]/(1+p)} \Delta^{+} 1\right] \eta.
\end{aligned}$$

By Bernoulli's inequality, we get

$$\|x_{\alpha,m+n} - x_{\alpha,m}\| < \frac{(2 - a_m \gamma^{((2+p)^m - 1)/(1+p)} + (\alpha - 1)a_m^2 \gamma^{((2+p)^m - 1)/(1+p)})\Delta^m}{2(1 - a_m \gamma^{((2+p)^m - 1)/(1+p)})} \left(\frac{1 - \gamma^{(2+p)^m n} \Delta^n}{1 - \gamma^{(2+p)^m} \Delta}\right) \eta. \tag{27}$$

Thus, for $m = 0$, we get

$$\|x_{\alpha,n} - x_{\alpha,0}\| < \frac{2 - a_0 + (\alpha - 1)a_0^2}{2(1 - a_0)} \frac{(1 - \gamma^n \Delta^n)}{(1 - \gamma \Delta)} \eta. \tag{28}$$

Hence, $x_{\alpha,n} \in \mathcal{B}(x_{\alpha,0}, R\eta)$. Also $y_{\alpha,n} \in \mathcal{B}(x_{\alpha,0}, R\eta)$ follows from

$$\begin{aligned}
\|y_{\alpha,n} - x_{\alpha,0}\| &\le \|y_{\alpha,n+1} - x_{\alpha,n+1}\| + \|x_{\alpha,n+1} - x_{\alpha,n}\| + \cdots + \|x_{\alpha,1} - x_{\alpha,0}\| \\
&\le \|y_{\alpha,n+1} - x_{\alpha,n+1}\| + \frac{2 - a_n + (\alpha - 1)a_n^2}{2(1 - a_n)} \|y_{\alpha,n} - x_{\alpha,n}\| + \cdots \\
&\quad + \frac{2 - a_0 + (\alpha - 1)a_0^2}{2(1 - a_0)} \|y_{\alpha,0} - x_{\alpha,0}\| \\
&< \cdots < \frac{2 - a_0 + (\alpha - 1)a_0^2}{2(1 - a_0)} \frac{1 - \gamma^{n+2} \Delta^{n+2}}{1 - \gamma \Delta} \eta < R\eta.
\end{aligned}$$

On taking the limit as $n \to \infty$ in (25) and (27), we get $x^* \in \overline{\mathcal{B}}(x_{\alpha,0}, R\eta)$. To show that x^* is a solution of $F(x) = 0$. We have that $\|F(x_{\alpha,n})\| \le \|F'(x_{\alpha,n})\| \, \|\Gamma_{\alpha,n} F(x_{\alpha,n})\|$ and the sequence $\{\|F'(x_{\alpha,n})\|\}$ is bounded as

$$\|F'(x_{\alpha,n})\| \le \|F'(x_{\alpha,0})\| + M\|x_{\alpha,n} - x_{\alpha,0}\| < \|F'(x_{\alpha,0})\| + MR\eta.$$

Since F is continuous, by taking limit as $n \to \infty$ and $\alpha \in [0, 1]$ we get $F(x^*) = 0$. To prove the uniqueness of the solution, if y^* be the another solution of (1) in $\mathcal{B}(x_{\alpha,0}, \frac{2}{M\beta} - R\eta) \cap \Omega$ then we have

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))dt(y^* - x^*).$$

Clearly, $y^* = x^*$, if $\int_0^1 F'(x^* + t(y^* - x^*))dt$ is invertible. This follows from

$$\begin{aligned} \|\Gamma_0\| \left\| \int_0^1 [F'(x^* + t(y^* - x^*)) - F'(x_{\alpha,0})]dt \right\| &\le M\beta \int_0^1 \|x^* + t(y^* - x^*) - x_{\alpha,0}\|dt \\ &\le M\beta \int_0^1 (1-t)\|x^* - x_{\alpha,0}\| + t\|y^* - x_{\alpha,0}\|dt \\ &\le \frac{M\beta}{2}\left(R\eta + \frac{2}{M\beta} - R\eta\right) = 1 \end{aligned}$$

and by Banach Lemma. Thus, $y^* = x^*$. $\square$

5. Numerical examples

Example 1. Let $\mathbb{X}$ be the space of all continuous functions on [0, 1] and consider the integral equation $F(x) = 0$ where

$$F(x)(s) = x(s) - f(s) - \lambda \int_0^1 \frac{s}{s+t} x(t)^{2+p} dt \tag{29}$$

with $s \in [0, 1]$,$x, f \in \Omega \subset \mathbb{X}$, $p \in (0, 1]$ and λ is a real number. The given integral equation is called Fredholm-type integral equations. Here norm is taken as the sup-norm. Now, it is easy to find that first and second Frëchet derivatives of F as

$$F'(x)u(s) = u(s) - \lambda(2+p)\int_0^1 \frac{s}{s+t} x(t)^{1+p} u(t)dt, \quad u \in \Omega$$

$$F''(x)uv(s) = -\lambda(2+p)(1+p)\int_0^1 \frac{s}{s+t} x(t)^p uv(t)dt, \quad u, v \in \Omega.$$

Clearly, F'' does not satisfy the Lipschitz continuity condition as for $p \in (0, 1)$ and for all $x, y \in \Omega$

$$\begin{aligned} \|F''(x) - F''(y)\| &= \|\lambda(2+p)(1+p)\int_0^1 \frac{s}{s+t}[x(t)^p - y(t)^p]dt\| \\ &\le |\lambda|(2+p)(1+p) \max_{s\in[0,1]} \left|\int_0^1 \frac{s}{s+t}dt\right| \|x(t)^p - y(t)^p\| \\ &\le |\lambda|(2+p)(1+p)\log 2\|x-y\|^p. \end{aligned}$$

However, it satisfies the Hölder continuity condition for $p \in (0, 1]$ and $N = |\lambda|(2+p)(1+p)\log 2$. To obtain a bound for $\Gamma_{\alpha,0}$, we find

$$\|F(x_{\alpha,0})\| \le \|x_{\alpha,0} - f\| + |\lambda| \log 2\|x_{\alpha,0}\|^p$$

and

$$\|I - F'(x_{\alpha,0})\| \le |\lambda|(2+p)\log 2\|x_{\alpha,0}\|^{1+p}.$$

Now, if $|\lambda|(2+p)\log 2\|x_{\alpha,0}\|^{1+p} < 1$, then by Banach Lemma, we get

$$\|\Gamma_{\alpha,0}\| = \|F'(x_{\alpha,0})^{-1}\| \le \frac{1}{1 - |\lambda|(2+p)\log 2\|x_{\alpha,0}\|^{1+p}} = \beta.$$

Also,

$$\|F''(x)\| \le |\lambda|(2+p)(1+p)\log 2\|x\|^p.$$

Hence,

$$\|\Gamma_{\alpha,0}F(x_{\alpha,0})\| \le \frac{\|x_{\alpha,0} - f\| + |\lambda|\log 2\|x_{\alpha,0}\|^p}{1 - |\lambda|(2+p)\log 2\|x_{\alpha,0}\|^{1+p}}.$$

Now, for $\lambda = 1/4$, $p = 1/5$, $f(s) = 1$ and $x_{\alpha,0} = x_0(s) = 1$, we get $\|\Gamma_{\alpha,0}\| \le \beta = 1.61611$, $\|\Gamma_{\alpha,0}F(x_{\alpha,0})\| \le \eta = 0.280051$, $N = 0.457477$, and $b_0 = N\beta\eta^{1+p} = 0.160518$. The conditions of Theorem 1 requires to find the values of a parameter

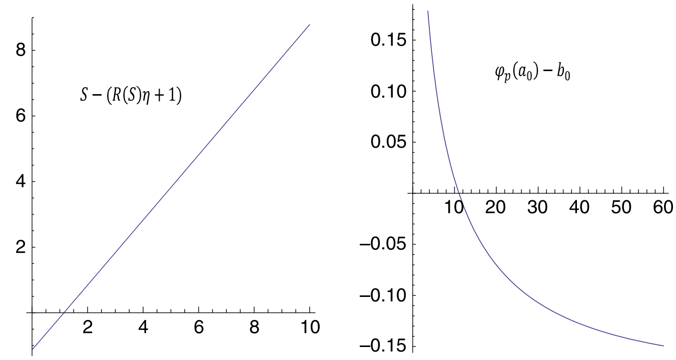

Fig. 1. Conditions on the parameters S.

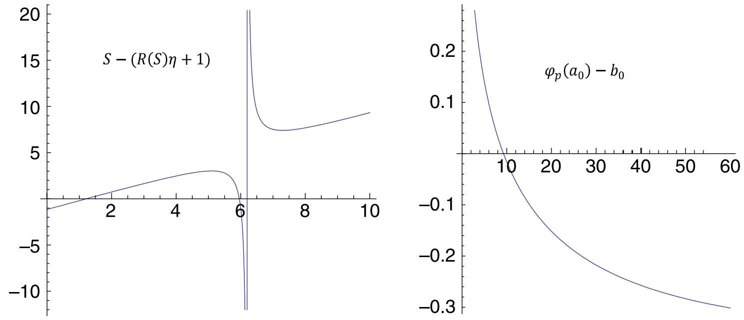

Fig. 2. Conditions of the parameter S.

S such that $S \in \overline{\mathcal{B}}(x_{\alpha,0}, R\eta) \subseteq \Omega$. The values of $S \in (1.25574, 11.0458)$ are obtained from Fig. 1 for $\alpha = 0$ such that $S-(R(S)\eta+1) > 0$ and $\Phi_p(a_0(S)) - b_0 \le 0$. Also, $a_0(S) \le r_0 = 0.380778$ if and only if $S < 21.0365$. From $M = M(S) = 0.457477S^p$, $a_0 = a_0(S) = M(S)\beta\eta = 0.207051S^p$ for $S = 9$, we get $\Omega = \mathcal{B}(1, 9)$, $M = 0.709934$, $a_0 = 0.321311$ and $b_0 = 0.160518 < 0.0284957 \le \Phi_p(a_0)$. Thus, the conditions of Theorem 1 are satisfied. Hence, a solution of (29) exists in $\overline{\mathcal{B}}(1, 0.735778) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(1, 1.0074) \cap \Omega$. Now, for $\alpha = 1$,we get $S \in (1.2117, 9.26314)$ from Fig. 2. Again taking $S = 9$ then $a_0 = 0.321311 < 0.380778$ and $b_0 < \Phi_p(a_0)$,we find that a solution of (29) exists in $\overline{\mathcal{B}}(1, 0.581361) \subseteq \Omega$ and the solution is unique in the ball $\mathcal{B}(1, 1.16181) \cap \Omega$.

On the other hand, working with majorizing sequences for $\alpha \in [0, 1]$, we get the solution exists in the ball $\overline{\mathcal{B}}(1, 0.355213) \subseteq \Omega$ and unique in the ball $\mathcal{B}(1, 1.15516)$. On comparing these results, we see that for $\alpha = 0$, our existence region for the solution is improved but not the uniqueness region. However, for $\alpha = 1$, both the existence and uniqueness regions are improved.

Example 2. Consider the boundary value problem given by

$$y'' + y' - y^3 = 0, \qquad y(0) = y(1) = 0. \tag{30}$$

To find the solution, we divided the interval $[0, 1]$ into n subintervals by taking stepsize $h = \frac{1}{n}$. Let $\{z_k\}$ be the points of the subdivision with

$$0 = z_0 < z_1 < z_2 < \cdots < z_n = 1$$

and corresponding values of the function

$$y_0 = y(z_0), y_1 = y(z_1), \dots, y_n = y(z_n).$$

Using approximations for the first and second derivatives given by

$$y_i' = (y_{i+1} - y_i)/h, \qquad y_i'' = (y_{i-1} - 2y_i + y_{i+1})/h^2, \quad i = 1, 2, \dots, n-1$$

and noting that $y_0 = 0 = y_n$, we define the operator $F : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$F(y) = G(y) + hJ(y) - 2h^2 g(y),$$

where,

$$G=\begin{pmatrix}-4 & 2 & 0 & \cdots & 0\\ 2 & -4 & 2 & \cdots & 0\\ 0 & 2 & -4 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & \cdots & \cdots & \cdots & -4\end{pmatrix},\qquad J=\begin{pmatrix}0 & 1 & 0 & \cdots & 0\\ -1 & 0 & 1 & \cdots & 0\\ 0 & -1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & \cdots & \cdots & \cdots & 0\end{pmatrix}$$

$$g(y)=\begin{pmatrix}y_1^3\\ y_2^3\\ \vdots\\ y_{n-1}^3\end{pmatrix},\qquad y=\begin{pmatrix}y_1\\ y_2\\ \vdots\\ y_{n-1}\end{pmatrix}.$$

Then, we get

$$F'(y)=G+hJ-6h^2\begin{pmatrix}y_1^2 & 0 & 0 & \cdots & 0\\ 0 & y_2^2 & 0 & \cdots & 0\\ 0 & 0 & y_3^2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & \cdots & \cdots & \cdots & y_{n-1}^2\end{pmatrix},$$

$$F''(y)=-12h^2\begin{pmatrix}y_1 & 0 & 0 & \cdots & 0\\ 0 & y_2 & 0 & \cdots & 0\\ 0 & 0 & y_3 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & \cdots & \cdots & \cdots & y_{n-1}\end{pmatrix}.$$

Let $x\in\mathbb{R}^{n-1}$, $A\in\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$, and define the norms of x and A by

$$\|x\|=\max_{1\le i\le n-1}|x_i|,\qquad \|A\|=\max_{1\le i\le n-1}|a_{ik}|.$$

Now, we get

$$\|F''(x)-F''(y)\|\le 0.12\|x-y\|.$$

This gives $\|\Gamma_{\alpha,0}\|\le\beta=6.11998638$, $\|\Gamma_{\alpha,0}F(x_{\alpha,0})\|\le\eta=0.168893$, $\|F''(x)\|\le M=0.0202824$, $N=0.12$, $a_0=M\beta\eta=0.02096435$, and $b_0=0.0209486$.

Also, for $\alpha=0$, we get $a_0<r_0=0.380778$, $b_0<\Phi_p(a_0)$. Hence, all the conditions of Theorem 1 are satisfied. Thus, the solution of Eq. (30) exists in the ball $\overline{\mathcal{B}}(1,0.171313)$ and unique in the ball $\mathcal{B}(1,15.9411)\cap\Omega$. Now for $\alpha=1$, a solution of Eq. (30) exists in the ball $\overline{\mathcal{B}}(1,0.171405)$ and unique in the ball $\mathcal{B}(1,15.941)\cap\Omega$. But, by using Majorizing sequences, we get the solution exists in $\overline{\mathcal{B}}(1,0.17017)$ for $\alpha\in[0,1]$ and unique in $\mathcal{B}(1,2.67306)\cap\Omega$. Comparing these results, one can easily conclude that the existence and uniqueness regions of solution for $\alpha=0$ and $\alpha=1$ are improved by our approach.

6. Conclusions

The semilocal convergence of a continuation method combining the Chebyshev method and the convex acceleration of Newton's method used for solving nonlinear equations in Banach spaces is established by using recurrence relations under the assumption that the second Frëchet derivative satisfies the Hölder continuity condition. This condition is mild and works for problems in which the second Frëchet derivative is either difficult to compute or fails to satisfy Lipschitz continuity condition. A new family of recurrence relations are defined based on the two constants, which depend on the operator. An existence–uniqueness regions along with a priori error bounds for the solution x^* is given. A closed form of the error bounds is also derived in terms of a real parameter $\alpha\in[0,1]$. Two numerical examples are worked out to demonstrate the efficacy of our approach. On comparing the existence and uniqueness regions for the solution obtained by our analysis with those obtained by using majorizing sequences under Hölder continuity condition, it is found that our analysis gives improved results. Further, we have observed that for particular values of α, our analysis reduces to those for the Chebyshev method ($\alpha=0$) and the convex acceleration of Newton's method ($\alpha=1$) respectively, with improved results.

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