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# Qualitative behaviour of numerical approximations to Volterra integro-differential equations

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**Abstract**

In this paper, we investigate the qualitative behaviour of numerical approximations to a nonlinear Volterra integro-differential equation with unbounded delay. We consider the simple single-species growth model

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t)=\lambda N(t)\left(1-c^{-1}\int_{-\infty}^{t}k(t-s)N(s)\,\mathrm{d}s\right).$$

We apply the (composite) $\theta$-rule as a quadrature to discretize the equation. We are particularly concerned with the way in which the long-term qualitative properties of the analytical solution can be preserved in the numerical approximation. Using results in (S.N. Elaydi and S. Murakami, J. Differ. Equations Appl. 2 (1996) 401; Y. Song and C.T.H. Baker, J. Differ. Equations Appl. 10 (2004) 379) for Volterra difference equations, we show that, for a small bounded initial function and a small step size, the corresponding numerical solutions display the same qualitative properties as found in the original problem.

In the final section of this paper, we discuss how the analysis can be extended to a wider class of Volterra integral equations and Volterra integro-differential equations with fading memory.


*Keywords:* Volterra integro-differential equations; Numerical stability; Volterra difference equations

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## 1. Introduction

A variety of mathematical models are used in the study of population dynamics. One model that has been studied extensively [7,13,15] is the well-known simple single-species growth model, a nonlinear Volterra integro-differential equations with unbounded delay given by

$$\frac{\mathrm{d}}{\mathrm{d}t}N(t) = \lambda N(t)\left(1 - c^{-1}\int_{-\infty}^{t} k(t-s)N(s)\,\mathrm{d}s\right), \quad c, \lambda > 0, \tag{1.1}$$

where

$$k(t) = t\mathrm{e}^{-t}. \tag{1.2}$$

In this model the maximum growth-rate response occurs at one unit of time earlier. This model displays surprisingly rich dynamical behaviour for the solutions. The qualitative behaviour of the equilibrium solution $N = c$ varies with the parameter $\lambda$. In addition, it is known that for the so-called strong generic delay kernel (1.2), Eq. (1.1) can be transformed into a system of three first-order equations, and the Hopf bifurcation theorem can be applied to give the result of a one-sided bifurcation. For the details we recommend [7].

In this paper, we investigate qualitative behaviour of numerical approximations to a nonlinear Volterra integro-differential equation with unbounded delay. We apply the general $\theta$-rule to solve Eq. (1.1) numerically. We are particularly concerned with the way in which the long-term qualitative properties of the analytical solution can be preserved in the numerical approximation. Using results in [10,18] for Volterra difference equations, we show that, for a small bounded initial function and a small step size, the numerical solutions display the same qualitative properties as found in the corresponding solutions of the original problem.

For discussion of the numerical solutions of Volterra integro-differential equations with unbounded delay, we recommend (cf., e.g., [1–5,11,14]). To the best of our knowledge, rather little work has appeared in the research journals on the numerics of integro-differential equations, with unbounded delay, undergoing bifurcation.

In the following sections, we study the long-term qualitative properties of numerical solutions of form (1.1) related the value of $\lambda$. In Section 2, we outline the known stability behaviour and derive the values of $\lambda$ at which the true solution bifurcates. We give a simple criterion to determine that all roots of a $n$th order polynomial with real coefficients lying in the region $|z| \geqslant 1$ in Section 3. In Section 4, we discretize the nonlinear Volterra integro-differential equation (1.1) by using the quadrature rules known as the $\theta$ methods. In Section 5, we review basic results in [10,18] about Volterra difference equations. Finally, we prove our main results Theorem 6.1 and Theorem 6.2 and discuss how the analysis can be extended to a wider class of Volterra integral equations and Volterra integro-differential equations with fading memory in Section 6.

## 2. Theoretical background

Eq. (1.1) has two equilibrium solutions: one is $N = 0$, another is $N = c$. In the usual way set

$$N(t) = c + x(t) \tag{2.1}$$

and substitution of (2.1) into (1.1) leads to an equation in $x(t)$, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\lambda(1 + c^{-1}x(t))\int_{-\infty}^{t}(t-s)\mathrm{e}^{-(t-s)}x(s)\,\mathrm{d}s. \tag{2.2}$$

It has been shown by Cushing [6] that, for kernel (1.2), the local stability behaviour of the zero solution of (2.2) is equivalent to the stability behaviour of the linear integro-differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\lambda\int_{0}^{t}(t-s)\mathrm{e}^{-(t-s)}x(s)\,\mathrm{d}s. \tag{2.3}$$

Miller [15] also proved that the null solution of (2.3) is asymptotically stable to small disturbances for a specific value of $\lambda$ if and only if

$$D_\lambda(z) = z + \lambda\int_0^\infty t\mathrm{e}^{-t}\mathrm{e}^{-zt}\,\mathrm{d}t = \frac{z^3+2z^2+z+\lambda}{(1+z)^2} \neq 0 \quad \text{for } \mathrm{Re}(z) \geqslant 0. \tag{2.4}$$

This equation has no zeros in the right half of the complex plane for $\lambda < 2$. At $\lambda = 2$ the cubic in $z$ has the following zeros: $z = -2$, $z = \pm\mathrm{i}$ ($\mathrm{i}^2 = -1$). For all $\lambda$ slightly greater than $\lambda = 2$, (2.4) has zeros with $\mathrm{Re}(z) > 0$. For details we recommend [13]. Therefore, we have the following result.

**Theorem 2.1.** *The zero solution of* (2.3) (*or equivalently* $N = c$ *of* (1.1)) *is asymptotically stable for* $\lambda < 2$; *no longer asymptotically stable for* $\lambda \geqslant 2$.

## 3. The location of roots of real polynomial equations

In some cases, the long-term behaviour of the solution of certain difference equations is related to the location of roots of an associated *stability polynomial*. We recall that one of the conditions we should then discuss is the condition

$$P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n \neq 0 \quad \text{for } |z| \geqslant 1, \tag{3.1}$$

where $P(z)$ is an $n$th order polynomial with real coefficients $a_i$, $i = 0, 1, \ldots, n$. The main purpose in this section is to give a criterion for (3.1). Clearly, (3.1) implies that all roots of the $n$th order algebraic equation

$$P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0 \tag{3.2}$$

are located in the region $|z| < 1$ of the complex $z$-plane. Therefore, if we can find a criterion to determine that all roots of (3.2) are located in the region $|z| < 1$ of the complex $z$-plane, this criterion can be used for problem (3.1). To this end, we assume that $P(1) \neq 0$. Consider the transformation

$$z = \frac{1+w}{w-1} \quad (w \neq 1) \text{ or equivalently } w = \frac{1+z}{z-1} \quad (z \neq 1). \tag{3.3}$$

Transformation (3.3) transforms the region $|z| \geqslant 1$ in the $z$-plane into the region $\mathrm{Re}(w) \geqslant 0$ in the $w$-plane and vice versa. Substituting (3.3) into (3.2), we have

$$a_0\left(\frac{1+w}{w-1}\right)^n + a_1\left(\frac{1+w}{w-1}\right)^{n-1} + \cdots + a_{n-1}\left(\frac{1+w}{w-1}\right) + a_n = 0,$$

or equivalently,

$$c_0w^n + c_1w^{n-1} + \cdots + c_{n-1}w + c_n = 0, \tag{3.4}$$

where each $c_i$, $i = 0, 1, \ldots, n$, is a linear combination of $a_i$, $i = 0, 1, \ldots, n$. Then, (3.4) is also a real $n$th order algebraic equation. The condition that $z = 1$ is not a root of (3.2) guarantees that all roots of (3.2) have been transformed into Eq. (3.4). Denote, by $T_i$, $i = 0, 1, \ldots, n$, the values

$$T_0 = c_0, \quad T_1 = c_1, \quad T_2 = \begin{vmatrix} c_1 & c_0 \\ c_3 & c_2 \end{vmatrix}, \quad T_3 = \begin{vmatrix} c_1 & c_0 & 0 \\ c_3 & c_2 & c_1 \\ c_5 & c_4 & c_3 \end{vmatrix}, \quad T_4 = \begin{vmatrix} c_1 & c_0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 \\ c_5 & c_4 & c_3 & c_2 \\ c_7 & c_6 & c_5 & c_4 \end{vmatrix}, \ldots . \tag{3.5}$$

Using the Routh–Hurwitz criterion [12, p. 17], we obtain the following known lemma.

**Lemma 3.1.** *Suppose that $z = 1$ is not a root of* (3.2). *Then* (3.1) *holds if and only if $T_i > 0$, $i = 0, 1, \ldots, n$.*

**Proof.** The condition that $z = 1$ is not a root of (3.2) guarantees that all roots of (3.2) have been transformed into Eq. (3.4). By the Routh–Hurwitz criterion (see, e.g., [12, p. 17]), all roots of (3.4) have negative real parts if and only if $T_i > 0$, $i = 0, 1, \ldots, n$. Note that if $\Re(w) < 0$, then

$$|z| = \frac{|1 + w|}{|w - 1|} < 1.$$

Thus, if all roots of (3.4) have negative real parts, then the absolute value of each root of (3.2) is less than 1. Since the degree of (3.2) and (3.4) is equal, all roots of (3.2) are in the region $|z| < 1$. Lemma 3.1 is proved. □

## 4. Construction of difference analogues

Consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\lambda(1 + c^{-1}x(t)) \int_{-\infty}^{t} (t - s)\mathrm{e}^{-(t-s)}x(s)\,\mathrm{d}s, \quad t \geqslant 0,$$
$$x(t) = \phi(t), \quad t \leqslant 0, \tag{4.1}$$

where $\phi(t)$ is any continuous initial function. Let us write

$$\psi(t) = \int_{-\infty}^{0} (t - s)\mathrm{e}^{-(t-s)}\phi(s)\,\mathrm{d}s. \tag{4.2}$$

Invoking the function $\psi(t)$, (4.1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = -\lambda(1 + c^{-1}x(t)) \left(\psi(t) + \int_{0}^{t} (t - s)\mathrm{e}^{-(t-s)}x(s)\right) \mathrm{d}s, \quad t \geqslant 0. \tag{4.3}$$

We assume that $\psi(t)$ is known exactly. If this is not true, then the term $\psi(t)$ has to be discretized by suitable quadrature approximations.

We use the composite version of the quadrature rule known as the $\theta$-rule to approximate the integral

$$\int_0^t (t-s)\mathrm{e}^{-(t-s)}x(s)\,\mathrm{d}s = \int_0^t k(t-s)x(s)\,\mathrm{d}s, \quad t \geqslant 0. \tag{4.4}$$

We divide the interval $[0,t]$ into $n$ intervals of fixed length $h>0$, $t=t_n$ and $t_j = jh$, $j=0,\ldots,n$. For simplicity, we set $c=1$, $\phi_j=\phi(jh)$, $\psi_j=\psi(jh)$, $\mathrm{k}(j)=k(jh)$, where $h$ is the fixed step length for our discretized scheme, and function (4.4) takes the form

$$\int_0^{t_n} (t_n-s)\mathrm{e}^{-(t_n-s)}x(s)\,\mathrm{d}s = \sum_{j=0}^{n-1}\int_{t_j}^{t_{j+1}} k(t_n-s)x(s)\,\mathrm{d}s \approx h\sum_{j=0}^{n} w_j^{(n)}\mathrm{k}(n-j)\mathrm{x}(j),$$

where $\mathrm{x}(j)$ denotes a numerical approximation to $x(t_j)$, the general composite $\theta$-rule has weights

$$\{w_0^{(n)}, w_1^{(n)},\ldots,w_{n-1}^{(n)},w_n^{(n)}\} = \{\theta,\ldots,1,1-\theta\}, \quad 0\leqslant\theta\leqslant 1$$

and $\sum_{j=0}^{n} w_j^{(n)} = n$, $n\geqslant 0$. Using the right-hand difference derivative $(\mathrm{x}(n+1)-\mathrm{x}(n))/h$ for the approximation of $x'(t)$ at the point $t_n=nh$, we obtain the difference scheme for (4.3) (an explicit scheme)

$$\mathrm{x}(n+1) = \mathrm{x}(n) - \lambda h(1+\mathrm{x}(n))\left(\psi_n + h\sum_{j=0}^{n} w_j^{(n)}\mathrm{k}(n-j)\mathrm{x}(j)\right),$$
$$\mathrm{x}(0)=\phi(0). \tag{4.5}$$

Note that $\mathrm{k}(0)=0$.

## 5. Volterra difference equations

To analyse (4.5), we need some basic results for Volterra difference equations. Consider the following scalar Volterra system:

$$z(n+1) = Az(n) + \sum_{j=0}^{n} b(n-j)z(j) + g(n,z(n)) + q(z)(n) + f(n),$$
$$z(0)=z_0, \tag{5.1}$$

where $0\leqslant n<\infty$, $A$ is $d\times d$ constant matrix, $b(n)$ are $d\times d$ kernel functions from $\mathbb{Z}^+=\{0,1,2,\ldots\}$ to $\mathbb{R}^d$, $z(n)$, $f(n)$ and $g$ are vectors in $\mathbb{R}^d$, $q(\phi)$ is a "small" nonlinear functional. A example of the "small" functional is given by the following.

$$q(\xi)(n) = \xi(n)\sum_{j=0}^{n} b(n-j)\xi(j), \quad \xi=\{\xi(n)\}_{n\geqslant 0}, \quad n\geqslant 0. \tag{5.2}$$

System (5.1) can be regarded as a perturbation of the linear system

$$y(n+1) = Ay(n) + \sum_{j=0}^{n} b(n-j)y(j), \quad y(0) = z_0. \tag{5.3}$$

Let $\{r(n)\}_{n\geqslant 0}$ denote the resolvent (see, e.g., [10]) of (5.3) which satisfies

$$r(n+1) = Ar(n) + \sum_{j=0}^{n} b(n-j)r(j), \quad n \geqslant 0,$$

$$r(0) = I.$$

System (5.1) has been studied in [16,18]. The basic result [18, Theorem 3.16] states that if the zero solution (for $z_0 = 0$) of (5.3) is uniformly asymptotically stable, then the zero solution (for $f(n) \equiv 0$) of (5.1) is also asymptotically totally stable (see the definition in [10]) under certain conditions. We summarize here the results in [10, Theorem 2, 18, Theorem 3.16] required for our purpose. Let $\widehat{b}(z)$ denote the $Z$-transform (see, [8,9]) of $\{b(n)\}_{n\geqslant 0}$.

**Theorem 5.1** (Elaydi and Murakami [10, Theorem 2]). *Suppose that* $\{b(n)\}_{n\geqslant 0} \in \ell^1$, *namely*, $\sum_{n=0}^{\infty} |b(n)| < \infty$. *For Eq.* (5.3) *the statements*

(i) $\det(zI - A - \widehat{b}(z)) \neq 0$ *for* $|z| \geqslant 1$,
(ii) *The zero solution of* (5.3) (*for* $z_0 = 0$) *is uniformly asymptotically stable*,
(iii) $\{r(n)\}_{n\geqslant 0} \in \ell^1$

*are equivalent*.

For the perturbed Volterra difference equations (5.1), we have following result which includes the result (ii) of Theorem 5.1 as a special case.

Recall that

$$\ell^\infty = \left\{ \xi \mid \xi = \{\xi(n)\}_{n\geqslant 0} \text{ with } \|\xi\|_\infty = \|\{\xi(n)\}\|_\infty = \sup_{n\geqslant 0} |\xi(n)| < \infty \right\}.$$

**Theorem 5.2** (Song and Baker [18, Theorem 3.16]). *Suppose that* $\{b(n)\}_{n\geqslant 0} \in \ell^1$, $g(n,\cdot)$ *and* $q(\cdot)$ *in* (5.1) *satisfy the following assumptions.*

(H1) $g(n,0) \equiv 0$ *for each* $n \geqslant 0$ *and for each* $\tau > 0$, *there exists* $\eta > 0$ *such that*

$$|g(n,x) - g(n,y)| \leqslant \tau|x-y|$$

*uniformly for* $n \geqslant 0$ *whenever* $|x|, |y| \leqslant \eta$.

(H2) $q : \ell^\infty \to \ell^\infty$, $q(0) \equiv 0$ *and for each* $\tau > 0$, *there exists* $\eta > 0$ *such that*

$$|q(\xi)(n) - q(\zeta)(n)| \leqslant \tau\|\xi - \zeta\|_\infty \quad (n \geqslant 0)$$

*whenever* $\|\{\xi(n)\}\|_\infty < \eta, \|\{\zeta(n)\}\|_\infty < \eta$ *and*

$$\lim_{n\to\infty} q(\xi)(n) = 0 \quad \textit{for each } \xi = \{\xi(n)\}_{n\geqslant 0}$$

*with* $\lim \xi(n) = 0$ *as* $n \to \infty$.

*Then there exists* $\varepsilon_0 > 0$ *such that, for any* $\varepsilon$ $(0 < \varepsilon \leqslant \varepsilon_0)$ *there exists a value* $\delta > 0$ *such that if* $|z_0| \leqslant \delta$ *and* $\|\{f(n)\}\|_\infty \leqslant \delta$ *with* $\lim_{n\to\infty} f(n) = 0$, *then system* (5.1) *has a unique solution* $\{z(n)\}_{n\geqslant 0}$ *satisfying* $\lim_{n\to\infty} z(n) = 0$ *and* $\|\{z(n)\}\|_\infty \leqslant \varepsilon$.

## 6. Qualitative behaviour of numerical solutions

To apply Theorem 5.1 to our discrete equations, we study (4.5) for the explicit Euler formula $(\theta = 1)$. In this case, (4.5) takes the form

$$\mathrm{x}(n+1) = \mathrm{x}(n) - \lambda h(1+\mathrm{x}(n))\left(\psi_n + h\sum_{j=0}^{n}\mathrm{k}(n-j)\mathrm{x}(j)\right),$$

$$\mathrm{x}(0) = \phi(0), \tag{6.1}$$

or equivalently,

$$\mathrm{x}(n+1) = \mathrm{x}(n) - \lambda h^2\sum_{j=0}^{n}\mathrm{k}(n-j)\mathrm{x}(j) - \lambda h\psi_n - \lambda h\psi_n\mathrm{x}(n)$$
$$- \lambda h^2\mathrm{x}(n)\sum_{j=0}^{n}\mathrm{k}(n-j)\mathrm{x}(j),$$

$$\mathrm{x}(0) = \phi(0). \tag{6.2}$$

Let $b_h^\lambda(n) = -\lambda h^2\mathrm{k}(n)$, $g_h(n,\mathrm{x}(n)) = -\lambda h\psi_n\mathrm{x}(n)$, $f_h(n) = -\lambda h\psi_n$ and $q_h(\mathrm{x})(n) = \mathrm{x}(n)\sum_{j=0}^{n} b_h^\lambda(n-j)\mathrm{x}(j)$. Then (6.2) takes the standard form

$$\mathrm{x}(n+1) = \mathrm{x}(n) + \sum_{j=0}^{n} b_h^\lambda(n-j)\mathrm{x}(j) + f_h(n) + g_h(n,\mathrm{x}(n)) + q_h(\mathrm{x})(n),$$

$$\mathrm{x}(0) = \phi(0). \tag{6.3}$$

It is obvious that $\{b_h^\lambda(n)\}_{n\geqslant 0} \in \ell^1$ for any fixed $\lambda$ and $h > 0$. The corresponding linear Volterra difference equations are

$$y(n+1) = y(n) + \sum_{j=0}^{n} b_h^\lambda(n-j)y(j), \quad n \geqslant 0,$$

$$y(0) = \phi(0) \tag{6.4}$$

Note that (6.4) is the equations when one applies the same numerical scheme to (2.3). By Theory 5.2, the long-term stability behaviour of (6.3) is locally determined by that of (6.4). Hence, let us first consider (6.4).

Applying Theorem 5.1 to (6.4), we can discover conditions under which the zero solution (for $y(0) = 0$) of (6.4) is uniformly asymptotically stable. In fact, we have the following result.

**Theorem 6.1.** *For small step size $h>0$, the zero solution of the Volterra difference equations* (6.4) (*for* $y(0)=0$), *which is the numerical analogue of Volterra integro-differential equation* (2.3), *is uniformly asymptotically stable if* $0<\lambda<2$ *but is no longer uniformly asymptotically stable if* $\lambda>2$.

**Proof.** Condition (i) of Theorem 5.1 for (6.4) is

$$z-1-\widehat{b_h^\lambda}(z)\neq 0 \quad \text{for } |z|\geqslant 1, \tag{6.5}$$

where $\widehat{b_h^\lambda}(z)$ is the $Z$-transform of $\{b_h^\lambda(n)\}_{n\geqslant 0}$ defined by

$$\widehat{b_h^\lambda}(z)=\sum_{n=0}^{\infty}\frac{b_h^\lambda(n)}{z^n}=-\lambda h^2\sum_{n=0}^{\infty}\frac{\mathrm{k}(n)}{z^n}. \tag{6.6}$$

Since $\mathrm{k}(n)=nh\mathrm{e}^{-nh}=hn(\mathrm{e}^{-h})^n$, we have

$$\widehat{b_h^\lambda}(z)=-\lambda h^3\sum_{n=0}^{\infty}\frac{n(\mathrm{e}^{-h})^n}{z^n}=-\lambda h^3\frac{z\mathrm{e}^{-h}}{(z-\mathrm{e}^{-h})^2}.$$

Thus, (6.5) takes the form

$$z-1-\widehat{b_h^\lambda}(z)=\frac{z^3-(2\mathrm{e}^{-h}+1)z^2+(\mathrm{e}^{-2h}+2\mathrm{e}^{-h}+\lambda h^3\mathrm{e}^{-h})z-\mathrm{e}^{-2h}}{(z-\mathrm{e}^{-h})^2}$$

$$\neq 0 \quad \text{for } |z|\geqslant 1. \tag{6.7}$$

Since the denominator has no zeros with $|z|\geqslant 1$ for any $h>0$, we only need to be concerned with the zeros of the cubic equation in $z$ which is in the numerator, namely,

$$z^3+c_1z^2+c_2z+c_3=0, \tag{6.8}$$

where

$$c_1=-(2\mathrm{e}^{-h}+1),\quad c_2=\mathrm{e}^{-2h}+2\mathrm{e}^{-h}+\lambda h^3\mathrm{e}^{-h},\quad c_3=-\mathrm{e}^{-2h}.$$

It is obvious that if (6.8) has no roots in the region $|z|\geqslant 1$, then condition (6.5) is satisfied. To confirm this, let

$$z=\frac{1+w}{w-1}\quad\text{or equivalently}\quad w=\frac{1+z}{z-1}. \tag{6.9}$$

Since $z=1$ is not a root of (6.8) for $\lambda\neq 0$ and $h>0$, (6.9) is well defined. Substituting (6.9) into (6.8), we have

$$\left(\frac{1+w}{w-1}\right)^3+c_1\left(\frac{1+w}{w-1}\right)^2+c_2\left(\frac{1+w}{w-1}\right)+c_3=0,$$

or equivalently,

$$a_0w^3+a_1w^2+a_2w+a_3=0, \tag{6.10}$$

where

$$a_0 = (1 + c_1 + c_2 + c_3) = \lambda h^3 e^{-h},$$

$$a_1 = (3 + c_1 - c_2 - 3c_3) = 2 - 4e^{-h} + 2e^{-2h} - \lambda h^3 e^{-h},$$

$$a_2 = (3 - c_1 - c_2 + 3c_3) = 4 - 4e^{-2h} - \lambda h^3 e^{-h},$$

$$a_3 = (1 - c_1 + c_2 - c_3) = 2 + 4e^{-h} + 2e^{-2h} + \lambda h^3 e^{-h}.$$

It follows from Lemma 3.1 that all roots of (6.10) have negative parts if and only if $T_i > 0,\ i=0,1,2$, where

$$T_0 = a_0, \quad T_1 = a_1, \quad T_2 = a_1 a_2 - a_0 a_3.$$

Notice that

$$e^{-h} \approx 1 - h + \frac{h^2}{2}, \quad e^{-2h} \approx 1 - 2h + 2h^2$$

for small $h > 0$. Then, for finite $\lambda > 0$ and small $h > 0$,

$$T_0 = a_0 = \lambda h^3 e^{-h} \approx \lambda h^3 > 0, \quad a_0 = \lambda h^3 + \mathcal{O}(h^4)$$

To estimate $T_1$, we note that

$$T_1 = a_1 \approx 2 - 4\left(1 - h + \frac{h^2}{2}\right) + 2(1 - 2h + 2h^2) - \lambda h^3 e^{-h} = 2h^2 - \lambda h^3 e^{-h}.$$

Thus,

$$a_1 \approx 2h^2 > 0 \quad \text{and} \quad a_1 = 2h^2 + \mathcal{O}(h^3).$$

Similarly,

$$a_2 \approx 4 - 4(1 - 2h + 2h^2) - \lambda h^3 e^{-h} \approx 8h, \quad a_2 = 8h + \mathcal{O}(h^2)$$

and

$$a_3 \approx 8, \quad a_3 = 8 + \mathcal{O}(h).$$

Thus,

$$a_1 a_2 = 16h^3 + \mathcal{O}(h^4) \quad \text{and} \quad a_0 a_3 = 8\lambda h^3 + \mathcal{O}(h^4).$$

Finally, we have

$$T_2 = a_1 a_2 - a_0 a_3 = 16h^3 - 8\lambda h^3 + \mathcal{O}(h^4) \tag{6.11}$$

and

$$\left.\begin{aligned} T_2 > 0 \quad \text{if } \lambda < 2 \\ T_2 < 0 \quad \text{if } \lambda > 2 \end{aligned}\right\} \text{ for corresponding sufficiently small } h > 0.$$

We conclude that for $0 < \lambda < 2$ and correspondingly small $h > 0$ Eq. (6.10) has no roots in the region $\Re(w) \geqslant 0$. Equivalently, Eq. (6.8) has no roots in the region $|z| \geqslant 1$. Thus, (6.5) holds for $0 < \lambda < 2$ and small $h > 0$. By Theorem 5.1, the zero solution of (6.4) is uniformly asymptotically stable. Further, for small $h>0$ and $\lambda>2$ (6.10) has at least one root in the region $\mathrm{Re}(w\geqslant 0)$. Equivalently, Eq. (6.8) has at least one root in the region $|z|\geqslant 1$, which implies that (6.5) does not hold. Thus, the zero solution of (6.4) is no longer uniformly asymptotically stable by Theorem 5.1. The proof is completed. □

Combining Theorem 5.2 and Theorem 6.1, we obtain the following.

**Theorem 6.2.** *Suppose that the zero solution of* (6.5) *is uniformly asymptotically stable for fixed small step size* $h>0$ *and* $0<\lambda<2$. *Then there exists* $\varepsilon_0>0$ *such that for any* $0<\varepsilon<\varepsilon_0$, *there is* $\delta>0$ *such that for any bounded initial function* $\phi(t)$ *satisfying* $\sup_{-\infty<t\leqslant 0}|\phi(t)|<\delta$, *the solution* $\{x(n)\}_{n\geqslant 0}$ *of* (6.1), *which is the numerical solution of Volterra integro-differential equation* (2.2), *satisfies* $\sup_{n\geqslant 0}|x(n)|\leqslant\varepsilon$ *and* $\lim_{n\to\infty}x(n)=0$.

**Proof.** Obviously, we only need to consider equation (6.3), namely,

$$\mathrm{x}(n+1)=\mathrm{x}(n)+\sum_{j=0}^{n}b_h^{\lambda}(n-j)\mathrm{x}(j)+f_h(n)+g_h(n,\mathrm{x}(n))+q_h(\mathrm{x})(n),$$

$$\mathrm{x}(0)=\phi(0),$$

where $b_h^{\lambda}(n)=-\lambda h^2\mathrm{k}(n)$, $g_h(n,(\mathrm{x}(n)))=-\lambda h\psi_n\mathrm{x}(n)$, $f_h(n)=-\lambda h\psi_n$ and $q_h(\mathrm{x})(n)=\mathrm{x}(n)\sum_{j=0}^{n}b_h^{\lambda}(n-j)\mathrm{x}(j)$. For any bounded initial function $\phi(t)$, $t\leqslant 0$, denote $\|\phi\|=\sup_{t\leqslant 0}|\phi(t)|$. We obtain the estimation of $\psi(t)$ in (4.2) as

$$|\psi(t)|\leqslant\int_{-\infty}^{0}(t-s)\mathrm{e}^{-(t-s)}|\phi(s)|\,\mathrm{d}s\leqslant\|\phi\|(t+1)\mathrm{e}^{-t}\leqslant\|\phi\|,\quad t\geqslant 0. \tag{6.12}$$

We confine our discussion on the set $\Omega$ of initial functions.

$$\Omega=\{\phi(t)\colon\ \|\phi\|\leqslant 1,\quad t\leqslant 0\}.$$

Then for any $x,y\in\Omega$ we have

$$|g_h(n,x)-g_h(n,y)|\leqslant\lambda h|\psi_n(x-y)|\leqslant\lambda h|x-y|,$$

which implies that $g_h(n,\cdot)$ satisfies assumption (H1) on $\Omega$. From $b_h^{\lambda}(n)=-\lambda h^2\mathrm{k}(n)=-\lambda h^2nh\mathrm{e}^{-nh}$, it follows that $\{b_h^{\lambda}(n)\}_{n\geqslant 0}\in\ell^1$ for any fix $\lambda>0$ and $h>0$. Hence, one can readily show that $q_h(\xi)(n)=\xi(n)\sum_{j=0}^{n}b_h^{\lambda}(n-j)\xi(j)$ satisfies assumption (H2). In addition, it follows from (6.12) that

$$|f_h(n)|=\lambda h|\psi_n|\leqslant\lambda h\|\phi\|(nh+1)\mathrm{e}^{-nh}\leqslant\lambda h\|\phi\|.$$

Then, $|f_h(n)|<\tau$ $(\tau>0)$ if $\|\phi\|<\tau/(\lambda h)$ and $\lim_{n\to\infty}f_h(n)=0$ for any bounded initial function $\phi(t)$.

Now by Theorem 5.2, there exists $\varepsilon_0>0$ such that for any $\varepsilon$ $(0<\varepsilon\leqslant\varepsilon_0)$, there exists a value $\tau>0$ such that for any initial function $\phi(t)$ satisfying $\|\phi\|<\delta=\min\{1,\tau,\tau/(\lambda h)\}$ (in this case, $|\mathrm{x}(0)|=|\phi(0)|\leqslant\tau$ and $\|\{f_h(n)\}\|\leqslant\tau$), the solution $\{\mathrm{x}(n)\}_{n\geqslant 0}$ of (6.1) satisfies $\sup_{n\geqslant 0}|\mathrm{x}(n)|\leqslant\varepsilon$ and $\lim_{n\to\infty}\mathrm{x}(n)=0$. □

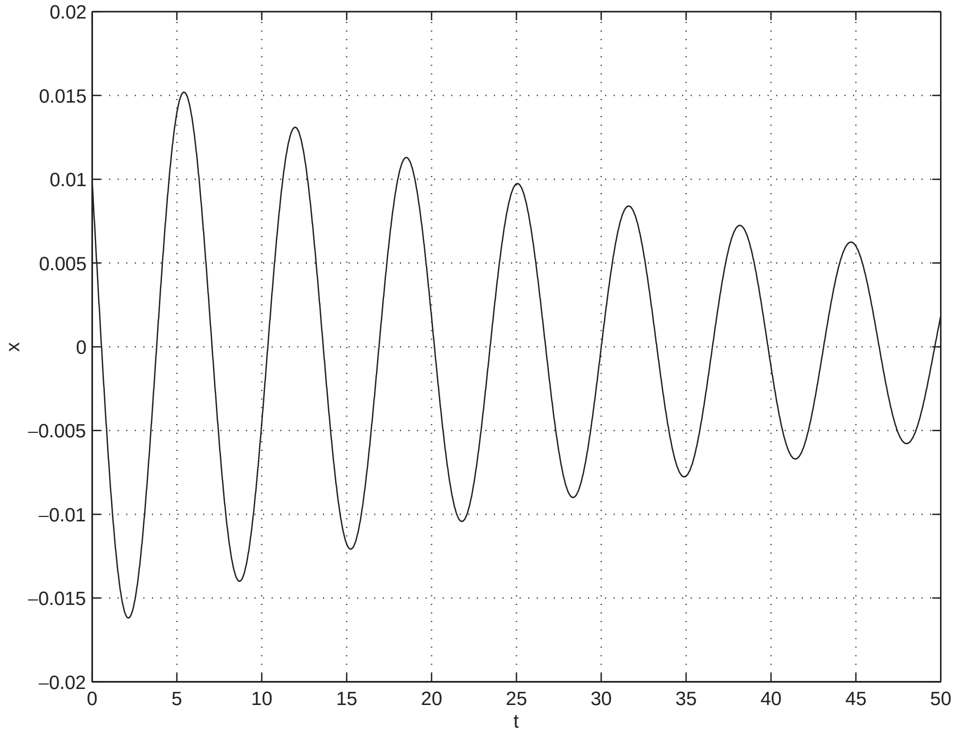


Fig. 1. Solution of (6.1) with $\lambda = 1.8$; $h = 0.01$; $\phi(n) = 0.01$ $(n \leqslant 0)$.

We illustrate our results with the following two numerical examples. Figs. 1 and 2 demonstrate the solutions of (6.1) with small step size $h = 0.01$ and small initial function $\phi(n) = 0.01$ $(n \leqslant 0)$ for $\lambda = 1.8$ and $\lambda = 2.2$, respectively. Fig. 1 with $\lambda = 1.8$ is exactly consistent with Theorem 6.2, while the Fig. 2 with $\lambda = 2.2$ shows that the solution of (6.1) tends to infinity as $n \to \infty$, which is determined by the corresponding linear Volterra difference equation (6.4).

**Remark 6.3.** If we apply a general $\theta$-rule to (4.1) and notice that $\mathrm{k}(0) = 0$, the corresponding Volterra difference equations are

$$\mathrm{x}(n+1) = \mathrm{x}(n) - \lambda h(1 + \mathrm{x}(n))\left(\psi_n + h(\theta - 1)\mathrm{k}(n)\mathrm{x}(0) + h\sum_{j=0}^{n}\mathrm{k}(n-j)\mathrm{x}(j)\right),$$

$$\mathrm{x}(0) = \phi(0),$$

or

$$\mathrm{x}(n+1) = \mathrm{x}(n) + \sum_{j=0}^{n} b_h^{\lambda}(n-j)\mathrm{x}(j) + f_h^*(n) + g_h^*(n, \mathrm{x}(n)) + q_h(\mathrm{x})(n),$$

$$\mathrm{x}(0) = \phi(0), \tag{6.13}$$

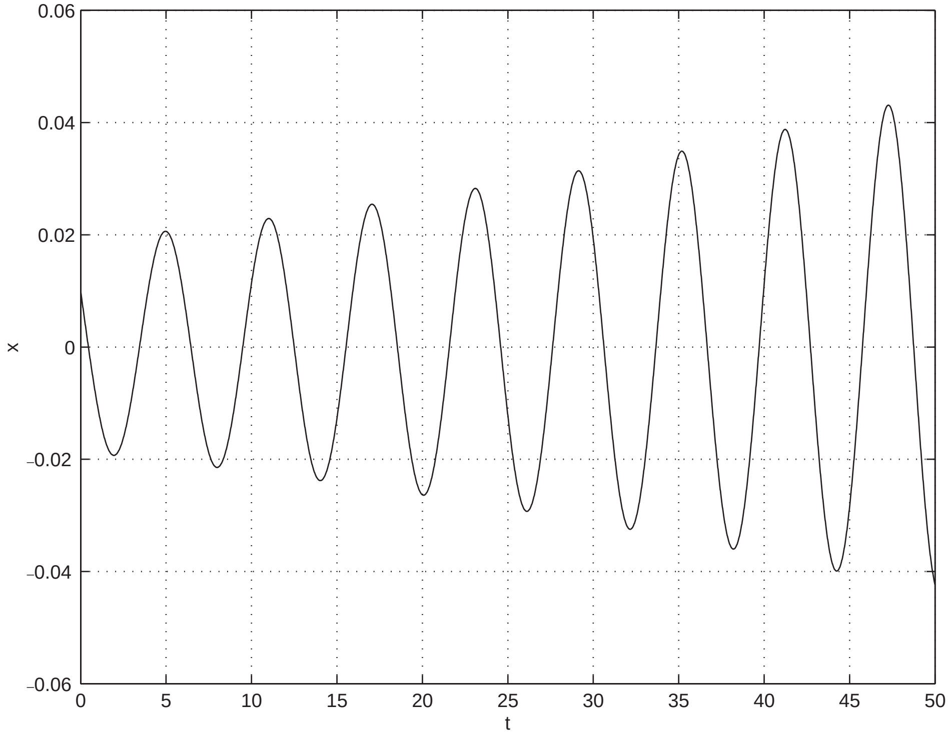


Fig. 2. Solution of (6.1) with $\lambda = 2.2$; $h = 0.01$; $\phi(n) = 0.01$ $(n \leqslant 0)$.

where $b_h^{\lambda}(n)$ and $q_h(\mathrm{x})(n)$ are those in Eq. (6.2), $g_h^*(n, (\mathrm{x}(n)) = -\lambda h(\psi_n + h(\theta - 1)\mathrm{k}(n)\phi(0))\mathrm{x}(n)$ and $f_h^*(n) = -\lambda h(\psi_n + h(\theta - 1)\mathrm{k}(n)\phi(0))$.

Observing that $g_h^*(n, \cdot)$ satisfies assumption (H1), $|f_h^*(n)| \leqslant \lambda h(\|\phi\|(nh + 1)\mathrm{e}^{-nh} + h|\theta - 1|(nh\mathrm{e}^{-nh}|\phi(0)|)$ and $\lim f_h^*(n) = 0$ as $n \to \infty$ for fixed $\lambda$ and $h > 0$. Thus, Theorem 6.2 holds for Eq. (6.13).

**Remark 6.4.** Since the asymptotic stability of the zero solution of linear Volterra difference equations determines to a certain extent that of the corresponding nonlinear equations, for example, the perturbation equation, we show that the same method can be used to study the following general linear integro-differential equations.

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \alpha \int_0^t p(t-s)\mathrm{e}^{-\beta(t-s)}x(s)\,\mathrm{d}s, \quad t \geqslant 0, \tag{6.14}$$

where $\alpha$ is real number, $\beta > 0$, and $p(t) = a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ is a $m$th degree polynomial with real coefficients $a_i$, $i = 0, 1, \ldots, m$. To investigate asymptotic stability of the zero numerical solution, we should consider the equation

$$z - 1 - \left(\sum_{j=0}^{m} \widehat{b}_{jh}(z)\right) = 0, \tag{6.15}$$

where

$$\widehat{b}_{jh}(z)) = \alpha h^{2+j} a_j \sum_{n=0}^{\infty} \frac{n^j e^{-\beta hn}}{z^n}.$$

Let $v(n) = e^{-\beta hn}$. We have

$$\widehat{v}(z) = \mathscr{Z}\{v(n)\}(z) = \frac{z}{z - e^{-\beta h}}.$$

With the formula

$$\mathscr{Z}\{n^j v(n)\}(z) = \left(-z\frac{d}{dz}\right)^j \widehat{v}(z), \tag{6.16}$$

where $\mathscr{Z}\{v(n)\}(z)$ denotes the $Z$-transformation of the sequence $\{v(n)\}_{n \geqslant 0}$, we have

$$\widehat{b}_{jh}(z)) = \alpha h^{2+j} a_j \left(-z\frac{d}{dz}\right)^j \left(\frac{z}{z - e^{-\beta h}}\right).$$

The left-hand side of (6.15) is a fraction, the numerator of which is a $(m+2)$th degree polynomial denoted by $Q_{m+2}(z)$. Thus, (6.15) is equivalent to $Q_{m+2}(z)=0$. Using Lemma 3.1, we can determine readily whether or not all roots of $Q_{m+2}(z) = 0$ are in the region $|z| \geqslant 1$. If $Q_{m+2}(z) = 0$ has no root in the region $|z| \geqslant 1$, we can make the conclusion that the zero solution of the corresponding numerical scheme of (6.14) is uniformly asymptotically stable; if $Q_{m+2}(z) = 0$ has at least one root in the region $|z| \geqslant 1$, then the zero solution of (6.14) is no longer uniformly asymptotically stable.

In fact, we can replace the exponential function $e^{-\beta t}$ in (6.14) with any real function $f(t)$ such that

$$\widehat{f}_h(z) = \sum_{n=0}^{\infty} f(nh) z^{-n}$$

is a rational function of $z$. In this case, it follows from (6.16) that the left-hand side of (6.15) is also a rational function of $z$ and is equivalent to a polynomial equation. Applying Lemma 3.1 and Theorem 5.1, we can determine whether or not the zero solution of corresponding numerical Volterra difference equations is uniformly asymptotically stable.

**Remark 6.5.** The techniques used in this paper can also be applied to more general Volterra integral equations with fading memory. For example, consider a linear scalar equation

$$x(t) = \alpha \int_0^t p(t-s) e^{-\beta(t-s)} x(s)\, ds, \quad t \geqslant 0, \tag{6.17}$$

where $\alpha$ is a real number, $\beta > 0$, $p(t) = a_m t^m + a_{m-1} t^{m-1} + \cdots + a_1 t + a_0$ is a real $m$th degree polynomial and $f(t)$ is a forcing term. Applying a simple quadrature rule to approximate the (6.17) leads to a implicit form

$$\mathrm{x}(n) = \alpha h \sum_{j=0}^{n} w_j^n p((n-j)h) \mathrm{e}^{-\beta(n-j)h} \mathrm{x}(j), \tag{6.18}$$

where $h > 0$ is the step size, $\mathrm{x}(j) \sim x(jh)$, the weights $w_j^n$ are determined by the choice of quadrature rule as those in Section 4. Let $\mathrm{w}(n-j) = w_j^n$. By the discrete Paley–Wiener Theorem (see, e.g., [16] or [17]), the asymptotic behaviour of the zero solution of (6.18) is determined by the following condition:

$$(1 - \mathscr{Z}\{\alpha h \mathrm{w}(n) p(nh) \mathrm{e}^{-\beta hn}\}(z)) \neq 0 \quad \text{for} \quad |z| \geqslant 1. \tag{6.19}$$

Notice that the (6.19) is equivalent to condition (3.1). Thus, we can apply Lemma 3.1 to (6.19).

Worthy of mention is that the same method can be applied to nonscalar Volterra equations, in which the kernel function is a $d \times d$ matrix function. If we assume each entry function is an exponential function, then $\det(I - \mathscr{Z}\{\alpha h \mathrm{w}(n) p(nh) \mathrm{e}^{-\beta hn}\}(z))$ is a fraction. In this case, (6.19) is equivalent to the problem (3.1). Therefore, Lemma 3.1 can also be applied.

A further discussion of the relationship between the numerical method chosen, the step size $h > 0$ and the variation of $\alpha$ in (6.17) for which the numerical solution $\mathrm{x}(n)$ of (6.17) or corresponding perturbed equations will be the subject of a future paper.

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