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Nuclear Physics B 878 [FS] (2014) 169–185

www.elsevier.com/locate/nuclphysb

Generalized Penner model and the Gaussian beta ensemble

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Received 9 August 2013; received in revised form 31 October 2013; accepted 19 November 2013

Available online 21 November 2013

Abstract

In this paper, a new expression for the partition function of the generalized Penner model given by Goulden, Harer and Jackson is derived. The Penner and the orthogonal Penner partition functions are special cases of this formula. The parametrized Euler characteristic $\xi_g^s(\gamma)$ deduced from our expression of the partition function is shown to exhibit a contribution from the orbifold Euler characteristic of the moduli space of Riemann surfaces of genus g , with s punctures, for all parameters γ and g odd. The other contributions for g even are linear combinations of the Bernoulli polynomials at rational arguments. It turns out that the free energy coefficients of the generalized Penner model in the continuum limit, are identical to those coefficients in the large N expansion of the Gaussian β -ensemble. Moreover, the duality enjoyed by the generalized Penner model, is also the duality symmetry of the Gaussian β -ensemble. Finally, a shift in the 't Hooft coupling constant required by the refined topological string, would leave the Gaussian β -ensemble duality intact. This duality is identified with the remarkable duality of the $c = 1$ string at radius $R = \beta$.

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1. Introduction

In their interesting paper, Goulden, Harer and Jackson [1] generalized the Penner matrix model [2], and obtained an expression for the parametrized Euler characteristic $\xi_g^s(\gamma)$. This polynomial in γ^{-1} gives when specializing the parameter, γ , to $\gamma = 1$ and $\gamma = 1/2$, the orbifold Euler characteristic of the moduli space of complex algebraic curves (Riemann surfaces) of genus g with s punctures and real algebraic curves (non-orientable surfaces) of genus g with s

Funded by SCOAP³.

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<http://dx.doi.org/10.1016/j.nuclphysb.2013.11.011>

punctures, respectively. It was shown explicitly that for odd g , $\xi_g^s(1/2)$ coincides with the orbifold Euler characteristic of the moduli space of complex algebraic curves [2,3]. On the other hand, if g is even, $\xi_g^s(1/2)$ corresponds to the orbifold Euler characteristic of the moduli space of real algebraic curves, also known as the orthogonal Penner model [4,5]. One must say that the Penner approach is more accessible to physicists since it uses Feynman diagrams and random matrices [6].

In this paper we give an alternative formula for the partition function from which $\xi_g^s(\gamma)$ can be computed. The simplicity of this formula is that the partition functions of the Penner and the orthogonal Penner models are transparent, and so this formula may be considered as a parametrized partition function for the generalized Penner model. This formula shows that for odd g , the parametrized Euler characteristic $\xi_g^s(\gamma)$ is a sum of two terms, the orbifold Euler characteristic of the moduli space of complex algebraic curves and a linear combination of the Bernoulli polynomials at rational arguments. For even g , $\xi_g^s(\gamma)$ is shown to coincide with the results obtained previously in [1].

We find that in the continuum limit (double scaling), both the Penner and the generalized Penner matrix models have the same critical points. The free energy of the latter model in this limit is related to the $c = 1$ string free energy at radius $R = \gamma$ [7]. This observation was made recently in connection with $\mathcal{N} = 2$ gauge theory in the Ω -background [8]. It is also interesting to note that the free energy in the continuum limit, and the nonperturbative terms¹ of the present model are given by the free energy of the Gaussian β -ensemble in the large N limit. This follows from the expression for the partition function of the generalized Penner matrix model, see Eq. (1), below. This can be understood from the fact that matrix models are considered as topological gauge theories, like the Chern–Simons gauge theory. In particular, the nonperturbative terms of the $SU(N)$ -Chern–Simons gauge theory, that is, the volume of the $SU(N)$ gauge group [10], may be reproduced by taking the double scaling limit of the perturbative $SU(N)$ -Chern–Simons gauge theory [11]. Note that the double scaled theory also corresponds to the $c = 1$ matrix model at self-dual radius, that is, the Penner model or equivalently, closed topological (B model) strings on S^3 deformation of the conifold [12]. The nonperturbative terms in the Penner model may be shown to be captured by the continuum limit of the Penner model itself. This was extended to SO/Sp -Chern–Simons gauge theories in [13].

These aforementioned observations show clearly that the Gaussian β -ensemble partition function may be considered as a volume of a certain gauge group of the generalized Penner matrix model. For example, the partition function for $\beta = 1$, gives the volume of the $SU(N)$ gauge group, while for $\beta = 1/2, 2$ the corresponding gauge groups are the $SO(N), Sp(N)$ respectively. Therefore the generalized Penner matrix model in the continuum limit may be considered as an alternative approach to carry out the large N -expansion of the Gaussian β -ensemble free energy [8,14–16]. In particular, the free energy in the continuum limit of this model is the asymptotic expansion of the Barnes double-Gamma function [14].

The explicit expression for the free energy of the generalized Penner model in the continuum limit (see Eq. (37)) bears the duality symmetry $\gamma \rightarrow 1/\gamma$ and $\mu \rightarrow -\gamma\mu$, where $\mu = N(1 - t)$ is a continuum parameter. This duality symmetry is similar to the one found earlier in [17], which may be considered as a natural generalization of the equivalence of $Sp(2N)$ and $SO(-2N)$ gauge theories [18]. As a result, the Gaussian β -ensemble free energy in the large N limit should

¹ Here, these terms arise from the volume of the gauge group only, that is, instanton corrections are turned-off. If, however, these corrections are taken into account, then the matrix model partition function would be defined in the complex plane with a suitable contour of integration [9].

be invariant under the duality transformation $\beta \rightarrow 1/\beta$ and $t \rightarrow -\beta t$, where t is the 't Hooft coupling. For $\beta = 1$, one recovers the $c = 1$ string at self-radius, i.e., the topological B -model string on the deformed conifold. This is known to be even in the string coupling constant g_s . Shifting the coupling constant t required by the refined topological string [19,20], the duality symmetry is kept intact, and coincides with the remarkable duality symmetry of the $c = 1$ string at radius $R = \beta$ [7].

In the following sections, we proceed as follows: in Section 2, we derive formulas for the partition function of the generalized Penner matrix model, and the parametrized Euler characteristic $\xi_g^s(\gamma)$. A brief review of the Gaussian β -ensemble and its connection to the generalized Penner model is given in Section 3. Here, the double-scaling limit is carried out and is shown to have the same critical points as in the usual Penner model. Also, it is shown that the generalized Penner model in the continuum limit reproduces the large N -expansion of the Gaussian β -ensemble free energy. The duality symmetry of the generalized Penner model is discussed in Section 4, and is shown to induce the same duality symmetry in the Gaussian β -ensemble, where it holds order by order. Finally, the conclusion of this work is drawn in Section 5.

2. The generalized Penner partition function and the parametrized Euler characteristic

The partition function for the generalized Penner model [1] can be written as

$$W_\gamma(N, t) = \frac{\int_{\mathbb{R}^N} |\Delta(\lambda)|^{2\gamma} \prod_{j=1}^N e^{-i\gamma\lambda_j/\sqrt{t}} e^{-\frac{\gamma}{t} \log(1-i\sqrt{t}\lambda_j)} d\lambda_j}{\int_{\mathbb{R}^N} |\Delta(\lambda)|^{2\gamma} \prod_{j=1}^N e^{-\gamma \sum_{i=1}^N \lambda_i^2/2} d\lambda_j}, \tag{1}$$

where $\Delta(\lambda) = \prod_{1 \leq i < j \leq N} (\lambda_j - \lambda_i)$ is the Vandermonde determinant. If we set $\gamma = 1$, then $W_1(N, t)$ is the Penner model partition function [2], and so this generalized model may be considered as a deformed Penner model; the deformation parameter being γ . In this model, the partition function of the Gaussian β -ensemble that appears in the denominator plays the role of the volume of certain gauge group. This corresponds to the nonperturbative terms in the generalized Penner model.

In the next section, we will check that the free energy for this model in the continuum limit is the large N expansion for the free energy of the Gaussian β -ensemble. The parametrized Euler characteristic [1] was shown to be connected to the partition function $W_\gamma(N, t)$ through the following expression

$$\xi_g^s(\gamma) = s!(-1)^s [N^s t^{s+s-1}] \frac{1}{\gamma} \log W_\gamma(N, t), \tag{2}$$

where $[X]Y$ is a short notation for the coefficient of X in the expansion of Y , and

$$W_\gamma(N, t) = \left(\sqrt{2\pi} e^{-\gamma/t} \left(\frac{\gamma}{t} \right)^{\frac{\gamma}{t} - \frac{1}{2}((N-1)\gamma+1)} \right)^N \prod_{j=0}^{N-1} \frac{1}{\Gamma(\frac{\gamma}{t} - \gamma j)},$$

is the partition function obtained from Eq. (1) using the Selberg integration formula.

To motivate our method in obtaining $\xi_g^s(\gamma)$, let us consider the case in which $\gamma = 1/2$. To that end, we use the Legendre duplication formula to show that

$$\prod_{j=0}^{N-1} \Gamma\left(\frac{1}{2t} - \frac{j}{2}\right) = \prod_{j=0}^{N/2-1} \frac{\sqrt{\pi}}{2^{1/t-(N-(2j+1))}} \Gamma\left(\frac{1}{t} - (N - (2j + 1))\right), \tag{3}$$

and from the identity

$$\Gamma\left(\frac{1}{t} - (N - (2j + 1))\right) = \frac{t^{N-(2j+1)}\Gamma(\frac{1}{t})}{\prod_{p=1}^{N-(2j+1)}(1 - pt)}, \tag{4}$$

we get

$$W_{1/2}(N, t) = \left(\frac{\sqrt{2\pi t} (et)^{-1/t}}{\Gamma(\frac{1}{t})}\right)^{N/2} \prod_{j=0}^{N/2-1} \prod_{p=1}^{N-(2j+1)} (1 - pt). \tag{5}$$

By setting $\gamma = 1/q$, q being an integer and $N = qK$, Goulden, Harer and Jackson have derived the following formula

$$W_{\frac{1}{q}}(qK, t) = \left(\frac{\sqrt{2\pi t}}{\Gamma(\frac{1}{t})(et)^{\frac{1}{t}}}\right)^K \frac{\prod_{l=1}^K \prod_{j=1}^{ql} (1 - jt)}{\prod_{j=1}^K (1 - tqj)}, \tag{6}$$

and in particular,

$$W_{1/2}(N, t) = \left(\frac{\sqrt{2\pi t} (et)^{-1/t}}{\Gamma(\frac{1}{t})}\right)^{N/2} \frac{\prod_{l=1}^{N/2} \prod_{j=1}^{2l} (1 - jt)}{\prod_{j=1}^{N/2} (1 - 2tj)}. \tag{7}$$

As a consequence, we deduce the following identity

$$\begin{aligned} \frac{\prod_{l=1}^{N/2} \prod_{j=1}^{2l} (1 - jt)}{\prod_{j=1}^{N/2} (1 - 2tj)} &= \prod_{j=0}^{N/2-1} \prod_{p=1}^{N-(2j+1)} (1 - pt) \\ &= \prod_{p=1}^{N/2} (1 - (2p - 1)t)^{N/2-p+1} (1 - (2p)t)^{N/2-p}, \end{aligned} \tag{8}$$

where N is assumed to be even. Note that the last equality follows from the identity

$$\prod_{p=1}^{N-(2j+1)} (1 - pt) = \prod_{p=1}^{N/2-j} (1 - (2p - 1)t) \prod_{p=1}^{N/2-1-j} (1 - 2pt).$$

As a result, the partition function $W_{1/2}(N, t)$ is reduced to a single product as follows

$$W_{1/2}(N, t) = \left(\frac{\sqrt{2\pi t} (et)^{-1/t}}{\Gamma(\frac{1}{t})}\right)^{N/2} \prod_{p=1}^{N/2} (1 - (2p - 1)t)^{N/2-p+1} (1 - (2p)t)^{N/2-p}. \tag{9}$$

Therefore, the free energy in this case reads

$$\begin{aligned} &2 \log W_{1/2}(N, t) \\ &= \log\left(\left(\frac{\sqrt{2\pi t} (et)^{-1/t}}{\Gamma(\frac{1}{t})}\right)^N \prod_{p=1}^N (1 - pt)^{N-p}\right) + \log \prod_{p=1}^{N/2} (1 - (2p - 1)t). \end{aligned} \tag{10}$$

This is exactly the free energy of the orthogonal Penner model [5], where the first term represents the Penner free energy.

If we use the following identity, the formula for the partition function of the generalized Penner model for all q 's, would involve a single product

$$\begin{aligned} \frac{\prod_{l=1}^K \prod_{j=1}^{ql} (1 - jt)}{\prod_{j=1}^K (1 - tqj)} &= \prod_{j=0}^{N/q-1} \prod_{p=1}^{N-(qj+1)} (1 - pt) \\ &= \prod_{p=1}^{N/q} (1 - (qp - (q - 1))t)^{N/q-p+1} (1 - (qp - (q - 2))t)^{N/q-p+1} \\ &\quad \times (1 - (qp - (q - 3))t)^{N/q-p+1} \dots (1 - (qp)t)^{N/q-p}. \end{aligned} \tag{11}$$

The products on the right-hand side are taken over all non-congruent and congruent to q , of which $q - 1$ products are non-congruent to q , and N being a multiple of q . This identity which was first proposed and checked for several values of N , is a natural generalization of Eq. (8). The derivation of this identity is given in [Appendix A](#). Using this identity, one can write the partition function of the generalized Penner model in terms of a single product as

$$\begin{aligned} W_{\frac{1}{q}}(N, t) &= \left(\frac{\sqrt{2\pi t}}{\Gamma(\frac{1}{t})(et)^{\frac{1}{t}}} \right)^{N/q} \prod_{p=1}^{N/q} (1 - (qp - (q - 1))t)^{N/q-p+1} \\ &\quad \times (1 - (qp - (q - 2))t)^{N/q-p+1} \\ &\quad \times (1 - (qp - (q - 3))t)^{N/q-p+1} \dots (1 - (qp)t)^{N/q-p}. \end{aligned} \tag{12}$$

Finally, the expression for the free energy reads

$$\begin{aligned} q \log W_{\frac{1}{q}}(N, t) &= \log \left(\frac{\sqrt{2\pi t}}{\Gamma(\frac{1}{t})(et)^{\frac{1}{t}}} \right)^N + \sum_{p=1}^N (N - p) \log(1 - pt) \\ &\quad + \sum_{p=1}^N \log(1 - pt) - \sum_{p=1}^{N/q} \log(1 - qpt) + \sum_{p=1}^{N/q} \log(1 - (qp - (q - 2))t) \\ &\quad + 2 \sum_{p=1}^{N/q} \log(1 - (qp - (q - 3))t) + \dots + (q - 2) \sum_{p=1}^{N/q} \log(1 - (qp - 1)t). \end{aligned} \tag{13}$$

The first line in this formula is nothing but the free energy of the Penner model. This computes the orbifold Euler characteristic of the moduli space of Riemann surfaces of genus g with s punctures $\chi(\mathcal{M}_g^s)$ [2,3]. As a consequence, the parametrized Euler characteristic $\xi_g^s(\gamma)$ for any $q \geq 2$, contains a contribution coming from the orbifold Euler characteristic of the moduli space of complex algebraic curves given by

$$\chi(\mathcal{M}_g^s) = (-1)^s \frac{(g + s - 2)!}{(g + 1)(g - 1)!} B_{g+1}, \tag{14}$$

for odd g , and B_g is the g th Bernoulli number. One should note that the third and the fourth lines in Eq. (13) do contribute to $\xi_g^s(\gamma)$ only for $q \geq 3$. The free energy in Eq. (13) gives the well known results for $q = 1, q = 2$.

Next, we derive a suitable expression for the free energy $q \log W_{\frac{1}{q}}(N, t)$ that computes $\xi_g^s(\gamma)$, such that the first line which generates $\chi(\mathcal{M}_g^s)$ is omitted. For this purpose, $q \log W_{\frac{1}{q}}(N, t)$ is considered to be the free energy. Expanding the latter term, one has

$$\begin{aligned}
 q \log W_{\frac{1}{q}}(N, t) &= - \sum_{m \geq 1} \frac{t^m}{m} \left(\sum_{p=1}^N p^m - \sum_{p=1}^{N/q} (qp)^m \right) - \sum_{m \geq 1} \frac{t^m}{m} \sum_{j \geq 0}^m (-1)^j \binom{m}{j} \sum_{p \geq 1}^{N/q} q^m p^{m-j} \\
 &\quad \times \left(\left(1 - \frac{2}{q}\right)^j + 2\left(1 - \frac{3}{q}\right)^j + \dots + (q-2)\left(\frac{1}{q}\right)^j \right). \tag{15}
 \end{aligned}$$

Using the power sum formula

$$\sum_{j=1}^n j^k = \frac{1}{k+1} \sum_{r=1}^{k+1} \binom{k+1}{r} B_{k+1-r} (-1)^{k+1-r} n^r,$$

the free energy reads

$$\begin{aligned}
 q \log W_{\frac{1}{q}}(N, t) &= - \sum_{m \geq 1} \frac{t^m}{m} \left(\frac{1}{m+1} \sum_{l=1}^{m+1} \binom{m+1}{l} (-1)^{m+1-l} B_{m+1-l} N^l \right) \\
 &\quad + \sum_{m \geq 1} \frac{t^m}{m} \left(\frac{q^m}{m+1} \sum_{l=1}^{m+1} \binom{m+1}{l} (-1)^{m+1-l} B_{m+1-l} q^{-l} N^l \right) \\
 &\quad - \sum_{m \geq 1} \frac{t^m}{m} \left(\sum_{j=0}^m (-1)^j \binom{m}{j} \frac{q^m}{m-j+1} \right) \\
 &\quad \times \left(\sum_{l=1}^{m-j+1} \binom{m-j+1}{l} (-1)^{m-j+1-l} B_{m-j+1-l} q^{-l} N^l \right) \\
 &\quad \times \left(\left(1 - \frac{2}{q}\right)^j + 2\left(1 - \frac{3}{q}\right)^j + \dots + (q-2)\left(\frac{1}{q}\right)^j \right). \tag{16}
 \end{aligned}$$

If we extract the coefficient $\xi_g^s(\gamma)$ of $s!(-1)^s N^s t^{g+s-1}$ from Eq. (16), we get (with $m = g + s - 1$ and $l = s$)

$$\begin{aligned}
 \xi_g^s(\gamma) &= s!(-1)^s [N^s t^{g+s-1}] q \log W_{\frac{1}{q}}(N, t) \\
 &= (-1)^{s+1} \frac{(g+s-2)!}{g!} (-1)^g (1 - q^{g-1}) B_g \\
 &\quad + (-1)^{s+1} \frac{(g+s-2)!}{g!} (-1)^g q^{g-1} \sum_{j=0}^g \binom{g}{j} B_{g-j} \\
 &\quad \times \left(\left(1 - \frac{2}{q}\right)^j + 2\left(1 - \frac{3}{q}\right)^j + \dots + (q-2)\left(\frac{1}{q}\right)^j \right). \tag{17}
 \end{aligned}$$

For even g , the explicit computation for $\xi_g^s(\gamma)$ which is carried out in [Appendix B](#) leads to the following closed form formula

$$\begin{aligned} \xi_g^s(\gamma) &= (-1)^{s+1} \frac{(g+s-2)!}{g!} (1-q^{g-1}) B_g \\ &\quad + (-1)^{s+1} \frac{(g+s-2)!}{g!} q^{g-1} \left(\frac{q-2}{2}\right) \left(\frac{1}{q^{g-1}} - 1\right) B_g \\ &= (-1)^s \frac{(g+s-2)!}{g!2} (q^g - q) B_g. \end{aligned} \tag{18}$$

This result is indeed in complete agreement with the expression obtained by Goulden, Harer, and Jackson [\[1\]](#).

For odd g , the first line given in [Eq. \(13\)](#) generates the orbifold Euler characteristic for the moduli space of complex algebraic curves of genus g with s punctures $\chi(\mathcal{M}_g^s)$. The other contributions for $\xi_g^s(\gamma)$ do come only from the last term of [Eq. \(17\)](#) since the first term of latter equation does not contribute for odd g and $g > 1$. Hence,

$$\begin{aligned} \xi_g^s(\gamma) &= (-1)^s \frac{(g+s-2)!}{(g+1)(g-1)!} B_{g+1} \\ &\quad + (-1)^s \frac{(g+s-2)!}{g!} q^{g-1} \\ &\quad \times \left(B_g \left(1 - \frac{2}{q}\right) + 2B_g \left(1 - \frac{3}{q}\right) + \dots + (q-2) B_g \left(\frac{1}{q}\right) \right), \end{aligned} \tag{19}$$

where $B_n(x) = \sum_{k=0}^n \binom{n}{k} B_k x^{n-k}$ is the Bernoulli polynomial of degree n . Making use of the symmetry $B_n(1-x) = -B_n(x)$ for odd n , the sum on the right-hand side for odd q can be written as

$$\begin{aligned} &B_g \left(1 - \frac{2}{q}\right) + 2B_g \left(1 - \frac{3}{q}\right) + \dots + (q-2) B_g \left(\frac{1}{q}\right) \\ &= (q-2) B_g \left(\frac{1}{q}\right) + (q-4) B_g \left(\frac{2}{q}\right) + \dots + B_g \left(\frac{q-1}{2q}\right). \end{aligned} \tag{20}$$

However, if q is even, [Eq. \(20\)](#) still holds except that the last term is replaced by $2B_g \left(\frac{q-2}{2q}\right)$.

As a consequence, and for odd g and q , the expression for the parametrized Euler characteristic becomes

$$\begin{aligned} \xi_g^s(\gamma) &= (-1)^s \frac{(2g+s-3)!}{(2g)(2g-2)!} B_{2g} \\ &\quad + (-1)^s \frac{(2g+s-3)!}{(2g-1)!} q^{2g-2} \left(\sum_{i=1}^{(q-1)/2} (q-2i) B_{2g-1} \left(\frac{i}{q}\right) \right), \end{aligned} \tag{21}$$

while for even q , the maximum value of i in the sum is $(q-2)/2$. Therefore, the parametrized Euler characteristic $\xi_g^s(\gamma)$ exhibits a contribution from the orbifold Euler characteristic of the moduli space of complex algebraic curves $\chi(\mathcal{M}_g^s)$. The other contributions are linear combinations of the Bernoulli polynomials at rational arguments. Note that for the real algebraic curves case ($q = 2$), the parametrized Euler characteristic is equal to $\chi(\mathcal{M}_g^s)$ for odd g .

Recall that the parametrized Euler characteristic $\xi_g^s(\gamma)$, is coefficient of $s!(-1)^s N^s t^{g+s-1}$ in the expansion of $\frac{1}{\gamma} \log W_\gamma(N, t)$. In terms of the generating series for the number of embedded

graphs in a surface, where an edge and its end are distinguished, these kind of graphs are called rooted maps. The parametrized generating series is given by

$$M_\gamma(\mathbf{y}, x, z) = \sum_{\mathbf{I}, j, n} m_\gamma(\mathbf{I}, x^j, z^n) \mathbf{y}^{\mathbf{I}} x^j z^n,$$

where $\mathbf{I} = (i_1, i_2, \dots)$, $\mathbf{y} = (y_1, y_2, \dots)$, $\mathbf{y}^{\mathbf{I}} = \prod_{k \geq 1} y_k^{i_k}$, and $m_\gamma(\mathbf{I}, x^j, z^n)$ is the number of rooted maps in a surface, with n edges, j faces and i_k vertices of valence (degree) k . It was shown in [1], that $\xi_g^s(\gamma)$ and $M_\gamma(\mathbf{y}, x, z)$ are related through the following formula

$$\xi_g^s(\gamma) = s!(-1)^s [N^s t^{g+s-1}] \Psi M_\gamma,$$

Ψ is the operator defined by

$$\Psi f(\mathbf{y}, x, z) = \frac{2}{2} \int_0^1 f((u(t)), x, z) \frac{dz}{z},$$

where $\mathbf{u}(\mathbf{t}) = (u_1, u_2, \dots)$, $u_1 = u_2 = 0$, $u_k = -i(\sqrt{t})^{k-2}$, $k \geq 3$, physically means that we respectively discard the tadpole and the self-energy insertions.

These results show that $\xi_g^s(\gamma)$ may be expressed as coefficients in the generalized Penner free energy, or equivalently as an alternating summation for the number of rooted maps; $m_\gamma(\mathbf{I}, x^j, z^n)$. It was conjectured that the number of rooted maps is a polynomial in $1/\gamma$ with integer coefficients [1]. It is also possible to obtain $\xi_g^s(\gamma)$ through the action of the puncture operator $\frac{1}{s!} \frac{\partial^s}{\partial \mu^s}$ ($\frac{1}{s!} \frac{\partial^s}{\partial t^s}$) on the expression for the free energy in the continuum limit of the generalized Penner model (the Gaussian β -ensemble), respectively. The last statement is a consequence of the Penner model because differentiating the free energy n -times with respect to the continuum variable μ , bring back the punctures to the Riemann surface.

Now, identifying our results for odd- g $\xi_g^s(\gamma)$ with those in [1], one reaches the following equality

$$\begin{aligned} \xi_g^s(\gamma) &= (-1)^s \frac{(g+s-2)!}{(g+1)(g-1)!} B_{g+1} \\ &\quad + (-1)^s \frac{(g+s-2)!}{g!} q^{g-1} \left(\sum_{i=1}^{(q-1)/2} (q-2i) B_g \left(\frac{i}{q} \right) \right) \\ &= \frac{(g+s-2)!(-1)^{s+1}}{(g+1)!} \left\{ (g+1) B_g q^g + \sum_{r=0}^{g+1} \binom{g+1}{r} B_{g+1-r} B_r q^r \right\}. \end{aligned} \tag{22}$$

The last expression in the above equation corresponds to the parametrized Euler characteristic derived in [1]. If $g = 1$, the following formula is deduced

$$\sum_{i=1}^{(q-1)/2} (q-2i) B_1 \left(\frac{i}{q} \right) = - \left(\frac{1}{12} q^2 - \frac{1}{4} q + \frac{1}{6} \right). \tag{23}$$

On the other hand, and for odd g ($g > 1$)

$$\sum_{i=1}^{(q-1)/2} (q-2i) B_g \left(\frac{i}{q} \right) = -q^{1-g} \left(B_{g+1} + \frac{1}{g+1} \sum_{r=1}^{g+1} \binom{g+1}{r} B_{g+1-r} B_r q^r \right), \tag{24}$$

from which the following interesting identity² is obtained

$$\begin{aligned} \sum_{r=1}^{2g} \binom{2g}{r} B_{2g-r} B_r q^r &= \sum_{r=1}^g \binom{2g}{2r} B_{2g-2r} B_{2r} q^{2r} \\ &= (1 - 2g) B_{2g} - (2g) q^{2g-2} \sum_{i=1}^{(q-1)/2} (q - 2i) B_{2g-1} \left(\frac{i}{q}\right). \end{aligned} \tag{25}$$

Since the second sum given in Eq. (21) has contributions for $q \geq 3$ only, we should have

$$\sum_{r=1}^{2g} \binom{2g}{2r} B_{2g-2r} B_{2r} = \sum_{r=1}^g \binom{2g}{2r} B_{2g-2r} B_{2r} 2^{2r} = (1 - 2g) B_{2g}. \tag{26}$$

These are well known formulae for Bernoulli numbers. The consistency of the formulas given by Eqs. (23) and (24) can be checked through the following simple examples. The first formula for $q = 3$ and $q = 4$ gives $B_1(1/3) = -1/6$ and $B_1(1/4) = -1/4$, respectively. Setting $g = 3$, $q = 3$ and $q = 4$, $B_3(1/3) = 1/27$ and $B_3(1/4) = 3/64$, respectively. This is in agreement with the direct evaluation of the Bernoulli polynomials at these rational values.

Finally, if one recalls the Almkvist–Meurman theorem [21], which states that the product $q^g B_g(i/q)$ for odd g ($g > 1$) and $0 \leq i \leq q$ is an integer,

$$2q^{g-1} \sum_{i=1}^{(q-1)/2} i B_g\left(\frac{i}{q}\right) - \left(B_{g+1} + \frac{1}{g+1} \sum_{r=1}^{g+1} \binom{g+1}{r} B_{g+1-r} B_r q^r \right), \tag{27}$$

must be an integer.

3. The generalized Penner model and the Gaussian β -ensemble

We have seen in the last section that the parametrized Euler characteristic $\xi_g^s(\gamma)$ for odd g exhibits a contribution from the Euler characteristic of moduli spaces of complex algebraic curves, thus giving strong evidence for an underlying geometrical meaning. It was suggested in [1] that $\xi_g^s(\gamma)$ may be considered as the virtual Euler characteristic of some moduli spaces, as yet unidentified. Since the generalized Penner model is a γ -deformation of the Penner model itself, one would expect that in the continuum limit (double-scaling), the free energy that computes the parametrized Euler characteristic $\xi_g^s(\gamma)$ is related to the $c = 1$ string theory at radius $R = \gamma$ [7]. Also, in the continuum limit, the free energy of this model corresponds to the large N asymptotic expansion of the Gaussian β -ensemble free energy. Before taking the continuum limit of the generalized Penner model we first review briefly the Gaussian β -ensemble and then proceed by showing, in detail, how this model and the generalized Penner model are related to each other.

3.1. The Gaussian β -ensemble

The Gaussian β -ensemble is defined by the following partition function [15,22]

$$Z = \frac{1}{N!(2\pi)^N} \int \prod_{i=1}^N d\lambda_i |\Delta(\lambda)|^{2\beta} e^{-\frac{\beta}{gs} \sum_{i=1}^N \lambda_i^2}, \tag{28}$$

² We very recently proved this identity. The work is in progress.

where g_s is the perturbative expansion parameter. This partition function is a deformation of the Gaussian ensemble partition function by the parameter β . For finite N , the above matrix integral can be evaluated using Mehta’s formula [23]

$$\int \prod_{i=1}^N d\lambda_i |\Delta(\lambda)|^{2\beta} e^{-\frac{1}{2} \sum_{i=1}^N \lambda_i^2} = (2\pi)^{N/2} \prod_{k=1}^N \frac{\Gamma(1 + \beta k)}{\Gamma(1 + \beta)}. \tag{29}$$

Setting $\beta = 1$, $Z \sim \prod_{k=1}^{N-1} k! \sim 1/\text{vol}(U(N))$. Here, $\text{vol}(U(N))$ is the volume of the unitary gauge group for the partition function of the Gaussian ensemble,

$$Z = \frac{1}{\text{vol}(U(N))} \int dM e^{-\frac{1}{g_s} \text{Tr} M^2},$$

where the integration is over $N \times N$ Hermitian matrix M . The expression $\prod_{k=1}^{N-1} k!$ is the Barnes gamma function $G_2(z)$ defined by $G_2(N + 1) = \prod_{k=1}^{N-1} (N - k)!$, known in the large N expansion to reproduce all the genera contributions of the B -model on the conifold [10]. This is also the Penner model in the continuum limit. The special values $\beta = 2, 1/2$ compute the volume of the gauge groups $Sp(N)$ and $SO(N)$ respectively. In the large N limit, these volumes give rise to the $Sp(N)/SO(N)$ Penner models in the continuum limit [13].

For $\beta = 2$, the Gaussian partition function can be written as

$$Z \sim \frac{1}{N!} \prod_{k=1}^N \frac{(2k)!}{2} = \prod_{k=1}^N (2k - 1)! \sim \frac{1}{\text{vol}(Sp(2N))},$$

while for $\beta = 1/2$ and using the Legendre duplication formula, the partition function reads

$$Z \sim \frac{1}{N!} \prod_{k=1}^N \frac{\Gamma(1 + k/2)}{1/2\sqrt{\pi}} \sim (N - 2)!(N - 4)! \dots 6!4!2! \sim \frac{1}{\text{vol}(Sp(N - 1))},$$

where N is assumed to be even. It was shown [13] that $\text{vol}(Sp(2N - 1))$ and $\text{vol}(SO(2N))$ are equivalent, thus, $Z \sim \frac{1}{\text{vol}(SO(N))}$ for $\beta = 1/2$.

This very close relationship between the Gaussian β -ensemble and the generalized Penner model is expected though. The partition function for the Gaussian β -ensemble given by Eq. (28) plays the role of the volume for certain gauge group of the generalized Penner model partition function given by Eq. (1). Here, we used the fact that the matrix models are gauge theories like the Chern–Simons gauge theories, and nonperturbative terms in such models are captured by the volume of the corresponding gauge groups. One should also point out that the nonperturbative terms which are reproduced in the double scaling limit of the Chern–Simons gauge theory may be extended to matrix models as well.

3.2. The double scaling limit of the generalized Penner model

We will show that in the continuum limit, this model reproduces the generating function for the parametrized Euler characteristic without punctures, and has the same critical points as the Penner model [24,26]. To that end, let us write the free energy for the generalized Penner model as

$$F_q(N, t) = \frac{1}{q} \sum_{g,s} \frac{(-1)^s}{s!} \xi_g^s(\gamma) N^s t^{g+s-1}, \tag{30}$$

where the natural scaling $t \rightarrow t/N$ is used. The parametrized Euler characteristic $\xi_g^s(\gamma)$ splits into two sectors of even and odd g . Therefore, $F_q(N, t) = F_q^e(N, t) + F_q^o(N, t)$, where $F_q^e(N, t)$ and $F_q^o(N, t)$ refer to the even and odd g contributions to the free energy, respectively. For even g , the contribution to the free energy reads

$$\begin{aligned}
 F_q^e(N, t) &= \sum_{g,s} \frac{(2g+s-2)!}{s!2} (q^{2g-1} - 1) \frac{B_{2g}}{(2g)!} N^{1-2g} t^{2g-1+s} \\
 &= \frac{N}{2} \sum_{s=2} \frac{t^{s-1}}{s(s-1)} \left(\frac{1}{q} - 1\right) \\
 &\quad + \frac{1}{2} \sum_{g \geq 1} \sum_{s \geq 0} \frac{(2g+s-2)!}{s!} (q^{2g-1} - 1) \frac{B_{2g}}{(2g)!} N^{1-2g} t^{2g-1+s}. \tag{31}
 \end{aligned}$$

The sum over punctures is carried out and gives

$$\begin{aligned}
 F_q^e(N, t) &= \frac{N}{2} \left(1 + \left(\frac{1-t}{t}\right) \log(1-t)\right) \left(\frac{1}{q} - 1\right) \\
 &\quad + \frac{1}{2} \sum_{g \geq 1} \left(\frac{N(1-t)}{t}\right)^{1-2g} (q^{2g-1} - 1) \frac{B_{2g}}{(2g)(2g-1)}. \tag{32}
 \end{aligned}$$

To obtain the continuum limit for the free energy $F_q(N, t)$, one sets $\mu = N(1-t)$, $N \rightarrow \infty$ and $t \rightarrow 1$, such that μ is kept fixed (double scaling limit) and gets

$$F_q^e(\mu) = \frac{\mu}{2} \log \mu \left(\frac{1}{q} - 1\right) + \frac{1}{2} \sum_{g \geq 1} \mu^{1-2g} (q^{2g-1} - 1) \frac{B_{2g}}{(2g)(2g-1)}. \tag{33}$$

This is the generalization of the orthogonal free energy Penner model in the continuum limit ($q = 2$) [4,5], with the same critical points as the Penner model [24,26].

For odd g , the free energy contribution is

$$\begin{aligned}
 F_q^o(N, t) &= \frac{1}{q} \sum_{g,s} \frac{(2g+s-3)!}{s!(2g)!} (2g-1) B_{2g} N^{2-2g} t^{2g-2+s} \\
 &\quad + \sum_{g,s} \frac{(2g+s-3)!}{s!(2g-1)!} q^{2g-3} \left(\sum_{i=1}^{(q-1)/2} (q-2i) B_{2g-1} \left(\frac{i}{q}\right)\right) N^{2-2g} t^{2g-2+s}, \tag{34}
 \end{aligned}$$

where the first term is the free energy of the Penner model discussed in detail elsewhere [24], while the second term is the free energy contribution for $q \geq 3$. Summing the latter term over the punctures gives

$$\begin{aligned}
 &\sum_{g,s} \frac{(2g+s-3)!}{s!(2g-1)!} q^{2g-3} \left(\sum_{i=1}^{(q-1)/2} (q-2i) B_{2g-1} \left(\frac{i}{q}\right)\right) N^{2-2g} t^{2g-2+s} \\
 &= \log(1-t) \left(\frac{1}{12}q - \frac{1}{4} + \frac{1}{6q}\right) \\
 &\quad + \sum_{g \geq 2} \left(\frac{N(1-t)}{t}\right)^{2-2g} \frac{q^{2g-3}}{(2g-1)(2g-2)} \left(\sum_{i=1}^{(q-1)/2} (q-2i) B_{2g-1} \left(\frac{i}{q}\right)\right). \tag{35}
 \end{aligned}$$

Therefore, the contribution to the free energy reads

$$\begin{aligned}
 F_q^o(\mu) &= \frac{1}{2q} \mu^2 \log \mu - \frac{1}{12q} \log \mu + \frac{1}{q} \sum_{g \geq 2} \frac{1}{(2g-2)} \frac{B_{2g}}{2g} \mu^{2-2g} \\
 &+ \frac{1}{q} \left(\frac{1}{12} q^2 - \frac{q}{4} + \frac{1}{6} \right) \log \mu \\
 &+ \frac{1}{q} \sum_{g \geq 2} \frac{1}{(2g-1)(2g-2)} \left(\sum_{i=1}^{(q-1)/2} (q-2i) B_{2g-1} \left(\frac{i}{q} \right) \right) (\mu/q)^{2-2g}. \tag{36}
 \end{aligned}$$

If we set $q = 2$, we recover our previous results [5] on the orthogonal Penner model in which the orientable contribution part gives half the Penner free energy.

Our expression for the free energy $F_q(\mu)$ has two sectors; the even and the odd powers in μ . This is reminiscent of the large N asymptotic expansion of the Gaussian β -ensemble free energy [8,15,16]. Discarding the regular terms in μ and adding Eqs. (34)–(36), the total free energy is

$$\begin{aligned}
 \mathcal{F}_\gamma(\mu) &= \frac{\gamma \mu^2}{2} \log \mu + \frac{\gamma-1}{2} \mu \log \mu + \frac{1-3\gamma+\gamma^2}{12\gamma} \log \mu \\
 &+ \frac{1-\gamma}{24\gamma\mu} + \frac{1-5\gamma^2+\gamma^4}{720\gamma^3\mu^2} + \frac{1-\gamma^3}{720\gamma^3\mu^3} + \dots \\
 &+ \left(\frac{\gamma}{10080} - \frac{1}{3040\gamma} - \frac{1}{4370\gamma^3} - \frac{1}{3040\gamma^5} + \frac{1}{10080\gamma^7} \right) \frac{1}{\mu^6} \\
 &+ \frac{1}{2} \sum_{g \geq 3} \left(\frac{1}{\gamma^{2g-1}} - 1 \right) \frac{B_{2g}}{(2g)(2g-1)} \mu^{1-2g} \\
 &- \gamma \sum_{g \geq 5} \frac{1}{(2g-2)(2g-1)} \frac{B_{2g}}{2g} \mu^{2-2g} \\
 &- \gamma \sum_{g \geq 5} \frac{1}{(2g-2)(2g-1)2g} \left(\sum_{r=1}^{2g} \binom{2g}{r} B_{2g-r} B_r \left(\frac{1}{\gamma} \right)^r \right) \mu^{2-2g}. \tag{37}
 \end{aligned}$$

Let us now compare our results with those obtained in [15] for the free energy of the Gaussian β -ensemble in the large N limit. In [15], the partition function for the Gaussian β -ensemble for finite N is expressed in terms of the double Gamma Barnes function $\Gamma_2(x; a, b)$. Using the asymptotic expansion for $\log \Gamma_2(x; a, b)$ up to some additive terms [14], the expression for the free energy [15] reads

$$\begin{aligned}
 F &= -\log \Gamma_2(t; -g_s, g_s/\beta) \\
 &= \frac{\beta t^2}{2} \left(\log t - \frac{3}{2} \right) g_s^{-2} + \frac{\beta-1}{2} t (\log(\beta t) - 1) g_s^{-1} + \frac{1-3\beta+\beta^2}{12\beta} \log(\beta t) \\
 &+ \frac{1-\beta}{24\beta t} g_s + \frac{1-5\beta^2+\beta^4}{720\beta^3 t^2} g_s^2 + \frac{1-\beta^3}{720\beta^3 t^3} g_s^3 + \dots, \tag{38}
 \end{aligned}$$

where t, g_s are the 't Hooft coupling and the string coupling constant respectively. This expression shows clearly that if we let $\gamma = \beta$ and $\mu = \frac{t}{g_s}$, up to regular terms, the coefficients of $\mathcal{F}_\gamma(\mu)$ are exactly those of the Gaussian β -ensemble free energy for large N [15]. This also shows that

one may consider the partition function for the Gaussian β -ensemble as the volume of certain gauge group which we do not know in general. For $\beta = 2, 1/2$, the gauge groups are $Sp(N)$ and $SO(N)$, respectively. Note that the coefficients of even and odd powers of μ are separated in our formula from each other, unlike those of the Gaussian β -ensemble.

4. Duality symmetry of the generalized Penner model and the Gaussian β -ensemble

It was shown in [17], that the free energy of the generalized Penner model bears the duality $F_Y(N, t) = F_{\frac{1}{Y}}(-\gamma N, t)$. This duality first appeared in connection with the equivalence of $Sp(2N)$ and $SO(-2N)$ gauge theories [18]. It is interesting to note that such symmetry still holds in the continuum limit for all terms without logarithmic singularities albeit N is replaced by the parameter μ , that is, $\mathcal{F}_Y(\mu) = \mathcal{F}_{1/Y}(-\gamma\mu)$. Although the logarithmic singularities break this duality, their coefficients do enjoy it.

This duality manifests itself clearly in the odd sector, while for the even sector, the duality may be shown to hold by realizing that the last term but one in Eq. (37) is dual to the last term with $r = 2g$. The other terms that are dual to each other are those terms whose coefficients are $\binom{2g}{2r} B_{2g-2r} B_{2r} = \binom{2g}{2g-2r} B_{2r} B_{2g-2r}$, $B_r = 0$ for odd $r > 1$. There is also a self-dual term whose coefficient is $\binom{2g}{g} B_g^2$. As a consequence, this duality is also a duality symmetry of the Gaussian β -ensemble free energy for large N .

The duality transformation of the Gaussian β -ensemble should be $\beta \rightarrow \frac{1}{\beta}$ and $t \rightarrow -\beta t$. This fact can be easily checked using the non-logarithmic terms in Eq. (38). In order to test this duality term by term in the free energy, one needs to know the explicit expression for the terms in the free energy $F = -\log \Gamma_2(t; -g_s, g_s/\beta)$ that are neither constants nor logarithmic. Using the asymptotic expansion of the double Gamma Barnes function [14], these terms are

$$\sum_{n=1}^{\infty} (n-1)! e_n(a, b) t^{-n}, \tag{39}$$

where the coefficients $e_n(a, b)$ for $a = -g_s$ and $b = g_s/\beta$ are:

$$e_n(-g_s, g_s/\beta) = \frac{B_{n+1}}{2(n+1)!} \left((-1)^n + \frac{1}{\beta^n} \right) g_s^n - \frac{B_{n+2}}{(n+2)!} \left((-1)^n + \frac{1}{\beta^{n+2}} \right) \beta g_s^n - \sum_{j=0}^{n-2} (-1)^{n-j} \frac{B_{n-j} B_{j+2}}{(n-j)!(j+2)!} (1/\beta)^{j+1} g_s^n, \quad \text{for } n \geq 2. \tag{40}$$

At this point, it is not difficult to see that the first two terms in $(n-1)! e_n(-g_s, g_s/\beta) t^{-n}$ are invariant under the transformation $\beta \rightarrow \frac{1}{\beta}$ and $t \rightarrow -\beta t$. Under this transformation, the last term may be written as

$$- \sum_{j=0}^{n-2} (-1)^j \frac{B_{n-j} B_{j+2}}{(n-j)!(j+2)!} (\beta)^{j+1} g_s^n (\beta t)^{-n}.$$

A simple exercise shows that this is exactly the last term

$$- \sum_{j=0}^{n-2} (-1)^{n-j} \frac{B_{n-j} B_{j+2}}{(n-j)!(j+2)!} (1/\beta)^{j+1} g_s^n t^{-n}.$$

As in the previous derivation of the duality of the generalized Penner model, there is a self-dual term for $j = \frac{n-2}{2}$, for n even. If n is odd, the above sum is identically zero since $B_m = 0$, for odd $m \geq 3$. Therefore, we conclude that the duality symmetry of the Gaussian β -ensemble is a consequence of the duality symmetry of the generalized Penner model.

Note that the above results would have been obtained if we were to use the Schwinger integral [8]. In this representation, the free energy of the Gaussian β -ensemble can be written as

$$F^G \sim \int \frac{ds}{s} \frac{e^{-ts}}{(e^{\epsilon_1 s} - 1)(e^{\epsilon_2 s} - 1)} \sim \Phi^0(\beta) \log(t) + \sum_{n \geq 1} \frac{g_s^n}{t^n} \Phi^n(\beta),$$

where $\epsilon_1 = g_s \sqrt{\beta}$, $\epsilon_2 = -\frac{g_s}{\sqrt{\beta}}$. $\Phi^0(\beta)$ and $\Phi^n(\beta)$ are the β dependent coefficients that appeared in the double Gamma Barnes function. The above expression for the free energy is obviously symmetric under the exchange $\epsilon_1 \longleftrightarrow \epsilon_2$. However, the expression is generally not invariant under the exchange $(\epsilon_1, \epsilon_2) \rightarrow (-\epsilon_2, -\epsilon_1)$ except when $\epsilon_1 = -\epsilon_2$. Making a shift $t \rightarrow t + \frac{\epsilon_1 + \epsilon_2}{2}$, this symmetry is restored. The asymptotic expansion in this case is

$$F^G(t; g_s, \beta) = \int \frac{ds}{s} \frac{e^{-ts}}{(e^{\epsilon_1 s} - e^{-\epsilon_1 s})(e^{\epsilon_2 s} - e^{-\epsilon_2 s})} \sim \Psi^0(\beta) \log(t) + \sum_{n \geq 1} \frac{g_s^n}{t^n} \Psi^n(\beta),$$

where

$$\Psi^0(\beta) = -\frac{1}{24}(\beta + \beta^{-1}), \tag{41}$$

and for $n \geq 1$

$$\Psi^n(\beta) = (n - 1)! \sum_{j=0}^{n+2} (-1)^j \frac{B_{n+2-j} B_j}{(n + 2 - j)!(j)!} (2^{1-j} - 1)(2^{1-n-2+j} - 1)(\beta)^{j-n/2-1}. \tag{42}$$

The right-hand side is zero for n odd, and since $B_j \neq 0$ for j even, the free energy without logarithmic singularities reads

$$F^G(t; \beta, g_s) \sim \sum_{m=1}^{\infty} (2m - 1)! \sum_{j=0}^{m+1} \frac{B_{2m+2-2j} B_{2j}}{(2m + 2 - 2j)!(2j)!} \times (2^{1-2j} - 1)(2^{1-2m-2+2j} - 1)(\beta)^{2j-m-1} \frac{g_s^{2m}}{t^{2m}}. \tag{43}$$

The above equation shows that $F^G(t; \beta, g_s) = F^G(-t\beta; 1/\beta, g_s)$ under the duality transformation ($\beta \rightarrow \frac{1}{\beta}$ and $t \rightarrow -\beta t$) since the sum over j remains invariant. That is, the duality transformation is preserved when shifting the coupling; $t \rightarrow t + \frac{\epsilon_1 + \epsilon_2}{2}$. Under this shift, the powers of t in the expansion for the free energy are even, so the duality in this case is $\beta \rightarrow \frac{1}{\beta}$ and $t \rightarrow \beta t$. This is exactly the remarkable duality symmetry obtained for the $c = 1$ string at radius β [7].

5. Conclusion

In this work alternative new formulas for the partition function as well as the free energy that computes the parametrized Euler characteristic of the generalized Penner model are given. These formulas contain both the Penner and the orthogonal Penner models as special cases.

Furthermore, for odd g and for all the parameters γ , the parametrized Euler characteristic exhibits a contribution from the orbifold Euler characteristic of Riemann surfaces of genus g with s punctures.

Our explicit formula for the free energy of the generalized Penner model gives exactly the same coefficients as the Gaussian β -ensemble free energy in the large N limit. Also our formula shows clearly that the duality enjoyed by the generalized Penner model is preserved in the continuum limit, albeit N is replaced by the coupling μ , $\mathcal{F}_\gamma(\mu) = \mathcal{F}_{1/\gamma}(-\gamma\mu)$. This duality in turn induces the same duality for the Gaussian β -ensemble; $\beta \rightarrow \frac{1}{\beta}$ and $t \rightarrow -\beta t$. This duality symmetry survives the shift of the coupling constant t required by the refined topological string. The duality in this case coincides with that of the $c = 1$ string at radius β . In terms of the equivariant parameters $\epsilon_1 = \sqrt{\beta}g_s$, $\epsilon_2 = -\frac{g_s}{\sqrt{\beta}}$, $\beta = -\frac{\epsilon_1}{\epsilon_2}$, this duality may be written as $\epsilon_1 \leftrightarrow \epsilon_2$, and $t \rightarrow \frac{\epsilon_1}{\epsilon_2}t$.

We have recently shown that in the continuum limit both the SO Chern–Simons gauge theory [25] and the SO Penner model are equivalent [13]. Therefore, we may ask if there is a Chern–Simons gauge theory whose free energy in the continuum limit (the logarithm of the volume of the gauge group in the large N limit), is given by Eq. (37). This may correspond to a topological string on the quotient of the resolved conifold by the discrete group Z_γ .

Acknowledgements

I would like to thank G. Bonelli, S. Cecotti and K.S. Narain for discussions and reading the manuscript, J. Walcher for correspondence. Also, I would like to thank Martin O’Loughlin for useful comments and the Abdus Salam Centre for Theoretical Physics, Trieste, for the supports they give me.

Appendix A. Derivation of the proposed identity

We present an explicit derivation of the proposed identity given in Eq. (11). The multiple product of the left-hand side of Eq. (11) may be expanded to give

$$\frac{\prod_{l=1}^K \prod_{j=1}^{q^l} (1 - jt)}{\prod_{j=1}^K (1 - tqj)} = \prod_{j=1}^q (1 - jt) \prod_{j=1}^{2q} (1 - jt) \cdots \prod_{j=1}^{qK-q} (1 - jt) \prod_{j \nmid q}^{qK-1} (1 - jt),$$

where

$$\prod_{j \nmid q}^{qK-1} (1 - jt) = \frac{\prod_{j=1}^{qK} (1 - jt)}{\prod_{j=1}^K (1 - tqj)}$$

here, $j \nmid q$ means that j is not multiple of q , and $N = qK$. The above product is reminiscent of certain products connected with the Euler gas [27]. Explicitly, this product may be written as

$$\prod_{j \nmid q}^{qK-1} (1 - jt) = \prod_{j=1}^{q-1} (1 - jt)(1 - (q+1)t) \cdots (1 - (N - (q-1))t) \cdots (1 - (N-1)t),$$

by canceling all the terms in the product for which j is a multiple of q . Combining the $q - 1$ products $(1 - (N - (q - 1))t) \cdots (1 - (N - 1)t)$ with $\prod_{j=1}^{qK-q} (1 - jt)$ gives $\prod_{j=1}^{N-1} (1 - jt)$.

In the same way a term like $\prod_{j=1}^{N-q-1} (1 - jt)$ may be obtained by combining the $q - 1$ products $(1 - (N - (2q - 1)t)) \cdots (1 - (N - (q + 1)t))$ with $\prod_{j=1}^q K^{-2q} (1 - jt)$. Continuing this process, we obtain the following

$$\begin{aligned} & \frac{\prod_{l=1}^K \prod_{j=1}^{ql} (1 - jt)}{\prod_{j=1}^K (1 - tqj)} \\ &= \prod_{j=1}^{N-1} (1 - jt) \prod_{j=1}^{N-q-1} (1 - jt) \cdots \prod_{j=1}^{3q-1} (1 - jt) \prod_{j=1}^{2q-1} (1 - jt) \prod_{j=1}^{q-1} (1 - jt) \\ &= \prod_{j=0}^{N/q-1} \prod_{p=1}^{N-(qj+1)} (1 - pt). \end{aligned} \tag{44}$$

The last line in Eq. (11) is obtained by separating the product $\prod_{p=1}^{N-(qj+1)} (1 - pt)$ into a product for which p is congruent to q and a product for which p is not congruent to q . That is,

$$\begin{aligned} & \prod_{p=1}^{N-(qj+1)} (1 - pt) \\ &= \prod_{p=1}^{N/q-j} (1 - (qp - (q - 1))t) \prod_{p=1}^{N/q-j} (1 - (qp - (q - 2))t) \cdots \prod_{p=1}^{N/q-j-1} (1 - (qp)t). \end{aligned}$$

The product over j may be carried out by realizing that the term with $p = 1$ in all products with the coefficient of t being non-congruent to q appears N/q times, $p = 2$ appears $N/q - 1$ times and $p = i$ appears $N/q - i + 1$ times. The product over j for the last product in which the coefficient of t being congruent to q , shows that the term with $p = i$ appears $N/q - i$ times, q.e.d.

Appendix B. The parametrized Euler characteristic for g even

In this appendix we derive the expression for the parametrized Euler characteristic for even g as given in Eq. (18). The first line given in Eq. (13) does not contribute to the parametrized Euler characteristic $\xi_g^s(\gamma)$, while the contribution of Eq. (17) is

$$\begin{aligned} \xi_g^s(\gamma) &= (-1)^{s+1} \frac{(g + s - 2)!}{g!} (1 - q^{g-1}) B_g \\ &+ (-1)^{s+1} \frac{(g + s - 2)!}{g!} q^{g-1} \\ &\times \left(B_g \left(\frac{2}{q} \right) + 2B_g \left(\frac{3}{q} \right) + \cdots + (q - 2)B_g \left(\frac{1}{q} \right) \right). \end{aligned} \tag{45}$$

Using the symmetry $B_g(1 - x) = B_g(x)$, we may write

$$\begin{aligned} & B_g \left(\frac{2}{q} \right) + 2B_g \left(\frac{3}{q} \right) + \cdots + (q - 2)B_g \left(\frac{1}{q} \right) \\ &= \frac{q - 2}{2} \left(2B_g \left(\frac{1}{q} \right) + 2B_g \left(\frac{2}{q} \right) + \cdots + B_g \left(\frac{q/2}{q} \right) \right). \end{aligned}$$

The sum on the right-hand side can be written in a closed form by evaluating the multiplication formula for the Bernoulli polynomials

$$B_q(kx) = k^{q-1} \sum_{k=0}^{q-1} \binom{n}{k} B_q(x + j/k),$$

at $x = 0$. Then, a simple computation shows

$$2B_g\left(\frac{1}{q}\right) + 2B_g\left(\frac{2}{q}\right) + \dots + B_g\left(\frac{q/2}{q}\right) = \left(\frac{1}{q^{g-1}} - 1\right)B_g.$$

Therefore, if g is even, the parametrized Euler characteristic $\xi_g^s(\gamma)$ becomes

$$\begin{aligned} \xi_g^s(\gamma) &= (-1)^{s+1} \frac{(g+s-2)!}{g!} (1 - q^{g-1}) B_g \\ &\quad + (-1)^{s+1} \frac{(g+s-2)!}{g!} q^{g-1} \left(\frac{q-2}{2}\right) \left(\frac{1}{q^{g-1}} - 1\right) B_g \\ &= (-1)^s \frac{(g+s-2)!}{g!2} (q^g - q) B_g. \end{aligned} \tag{46}$$

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