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Stiffness reduction of cracked general symmetric laminates using a variational approach

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ABSTRACT

In this paper, stiffness reduction of general symmetric laminates containing a uniform distribution of matrix cracks in a single orientation is analyzed. An admissible stress field is considered, which satisfies equilibrium and all the boundary and continuity conditions. This stress field has been used in conjunction with the principle of minimum complementary energy to get the effective stiffness matrix of a cracked general symmetric laminate. Natural boundary conditions have been derived from the variational principle to overcome the limitations of the existing variational methods on the analysis of general symmetric laminates. Therefore, the capability of analyzing cracked symmetric laminates using the variational approach has been enhanced significantly. It has been shown that the method provides a rigorous lower bound for the stiffness matrix of a cracked laminate, which is very important for practical applications. Results derived from the developed method for the properties of the cracked laminates showed an excellent agreement with experimental data and with those obtained from McCartney's stress transfer model. The differences of the developed model with McCartney's model are discussed in detail. It can be emphasized that the current approach is simpler than McCartney's model, which needs an averaging procedure to obtain the governing equations. Moreover, it has been shown that the existing variational models are special cases of the current formulation.

1. Introduction

Matrix cracking has long been recognized as the first damage mode observed in composite laminates under static and fatigue tensile loading. Matrix cracks, also known as intralaminar cracks, do not usually cause the final failure of a laminate, but may significantly impair the effective properties of the composite and serve as a source for other damage modes initiation, such as delamination and microcracking in the adjacent plies.

Matrix cracks and their effects on material properties degradation have gained much attention both experimentally, numerically and analytically due to their practical importance, see reviews (Nairn, 2000a; Berthelot, 2003; Kashtalyan and Soutis, 2005; Kaddour et al., 2013). Although most research works have been done on cross-ply laminates, the problems of stiffness degradation and damage accumulation in more general lay-ups have also been addressed. Generally, approaches to the problem include different modifications of shear lag model (Nairn and Mendels, 2001; Yokozeki and Aoki, 2005) (to mention a few of them), stress-based variational model (Hashin, 1985), stress transfer model (McCartney, 1992), displacement-based variational model (Zhang and Minnetyan, 2006), continuum damage mechanics approach (Talreja, 1985; Barbero et al., 2013a; Jalalvand et al., 2013), discrete damage mechanic (Barbero et al., 2011), synergistic damage mechanic (Singh and Talreja, 2008), numerical methods such as finite element (Yuan and Selek, 1993) and finite strip method (Li et al., 1994), etc.

Among the approximate analytical and numerical models, the stress-based variational approach (Hashin, 1985, 1986, 1987), stress transfer model of McCartney (McCartney, 1992, 2000; McCartney and Pierse, 1997; Katerelos et al., 2006) and discrete damage mechanics of Barbero (Barbero et al., 2011, 2014; Barbero and Cosso, 2014) have shown to accurately predict stiffness reduction and crack evolution of symmetric laminates (Nairn, 2000b; Vinogradov and Hashin, 2005; Barbero et al., 2013b). These approaches are interesting because the material properties of the damaged laminate depend exclusively on the crack density and no additional parameters or functions are required (Barbero and Cortes, 2010). Nevertheless, the stress analysis of cracked laminates needed in stiffness reduction and damage evolution is generally a complex task (Singh and Talreja, 2010).

Although, McCartney (2000, 2005) and Barbero (Barbero and Cortes, 2010; Barbero et al., 2013b) have extended their models to analyze stiffness reduction of general symmetric laminates with

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arbitrary stacking sequence and even containing cracks in more than one orientation, the variational approach has mostly been used for treating either cross-ply laminates (Hashin, 1985, 1986, 1987; Nairn, 1989; Nairn and Hu, 1992; Varna and Berglund, 1992; Rebiere et al., 2001; Kuriakose and Talreja, 2004) or other symmetric laminates have been reduced to cross-ply by averaging out the off-axis plies (Joffe and Varna, 1999; Li and Lim, 2005). Recently, Vinogradov and Hashin (2010) have extended the capability of the variational approach to analyze stiffness reduction of $[\theta_m{}^{(1)}/\theta_n{}^{(2)}]_s$ laminates containing matrix cracks in the middle ply. In addition, it should be noted that the mathematical model for all mentioned variational works involves effectively only two layers, one cracked and one un-cracked, representing a three-layered laminate after applying symmetry considerations. Therefore, these models do not have the capability of analyzing stiffness reduction for the symmetric laminates with arbitrary stacking sequence and multiple cracked and un-cracked layers, due to the lack of boundary conditions for un-cracked layers. Li and Hafeez (2009) have overcome this drawback for cross ply laminates by introducing some boundary conditions as an outcome of variational procedure and translational symmetry (Li et al., 2009), called natural boundary conditions, in the terminology of variational calculus. It is noted that their model (Li and Hafeez, 2009) has only considered cross ply laminates under axial loading due to the assumed admissible stress field. More recently, Hajikazemi and Sadr (2014) have developed a variational model to analyze the stress field of cracked symmetric laminates under general in-plane loading. As a result, the applicability of the variational approach has been extended fundamentally for considering multiple layers symmetric laminates with arbitrary stacking sequence. However, they only compared their developed stress field with the available stress results obtained from other variational approaches, for the case of cross-ply laminates.

In the current research work, as distinct from the latter publication by the same authors (Hajikazemi and Sadr, 2014), stiffness reduction of cracked general symmetric laminates with arbitrary stacking sequence and multiple cracked and un-cracked layers is analyzed. Therefore, the recently developed stress field (Hajikazemi and Sadr, 2014) has been used in conjunction with the principle of minimum complementary energy to get the effective stiffness matrix of a cracked general symmetric laminate. In this regard, it has been revealed that the present method provides a rigorous lower bound for the stiffness matrix of a general symmetric cracked laminate, which is very important for practical applications. Moreover, a systematic way of evaluating governing equations is developed, which is completely analytical. Therefore, the model could enjoy the advantages of ply refinement technique where each layer of the laminate is subdivided into plies having the same properties in order that important through the thickness variations of the stress components could be taken into account. Stiffness reduction of symmetric laminates obtained by the suggested approach is in excellent agreement with experimental data available in the literature and with those obtained from McCartney's stress transfer model. The differences of the developed model with McCartney's model are discussed in detail. The study of the results has revealed that the McCartney's stress transfer solution can also be derived using a variational method that is not a Reissner method. Moreover, it can be emphasized that the current approach is simpler than McCartney's model, which needs an averaging procedure to obtain the governing equations. It has been shown that the existing variational models are special cases of the current formulation. Finally, it is worth mentioning that the assumed stress filed satisfies equilibrium and all the boundary and continuity conditions and here, the principle of minimum complementary energy is implemented to get the effective stiffness matrix. Therefore, the current model is the most complete and versatile variational model developed so far based upon the single fundamental assumption that the in-plane stresses in each ply element are independent from the through-thickness direction.

2. Admissible stress field construction

Consider a symmetric multilayered laminate including 2 N perfectly bonded layers, which can have any combination of orientations while the symmetry about the mid-plane of the laminate is preserved. As laminate symmetry is assumed, it is better to consider only the upper set of N layers as shown in Fig. 1. A global set of rectangular Cartesian coordinates is chosen having the origin at the center of the laminate as shown in Fig. 1. The x-direction defines the longitudinal or axial direction, the y-direction defines the in-plane transverse direction and the z-direction defines the direction through the thickness. The locations of the N-1 interfaces of the first half of the laminate ($z > 0$) are specified by $z = z_i$; $i = 1, 2 \dots N - 1$. The mid-plane of the laminate is specified by $z = z_0 = 0$ and the external surface is demonstrated by $z = z_N = h$, where $2h$ is the total thickness of the laminate. The thickness of the ith layer is denoted by $h_i = z_i - z_{i-1}$. The orientation of the ith layer is specified by the angle θ_i (measured clockwise) between the x-axis and the fiber direction of this layer. The laminate must be such that the orientation of fibers in at least one set of plies is aligned in y-direction. This assumption is not a limitation because general in-plane loading conditions are considered so that if cracks form in another single orientation, the laminate can rotate so that the crack planes are parallel to the y-axis and the applied stresses transform to appropriate values for the new orientation. The stress and strain components and also material properties associated with the ith layer are denoted by a superscript or subscript i. Some layers might have similar properties, and therefore be modeled perfectly through the thickness variations in the stress fields. We assume that the laminate can be infinitely extended in both x and y directions (see Fig. 1), and consecutively the effect of the edges is neglected.

The laminate is subject to external uniform membrane loads of N_{xx}, N_{yy}, and N_{xy} in the coordinate system of xyz associated with cracks. In an un-cracked laminate, the only nonzero components of the stress tensor defined in the coordinate system associated with cracks are $\sigma_{xx}^{0(i)}, \sigma_{yy}^{0(i)}, \sigma_{xy}^{0(i)}$, where the superscript 0 denotes the undamaged state and the superscript (i), $i = 1,2, \dots, N$, denotes the number of the layer. The stresses are spatially uniform within each layer and linear functions of the applied loads of N_{xx}, N_{yy}, and N_{xy}.

It is assumed that the ply crack distribution in damaged 90° plies is uniform, having a separation $2a$, and the cracks in each damaged 90° ply of the laminate are in the same plane. The cracked laminate can be seen as a sequence of laminate fragments, bounded by pairs of adjacent cracks (see Fig. 1). Further, it will be shown that each fragment has the same 'admissible' traction boundary conditions in the crack planes and hence can be treated separately. Fig. 2 indicates a fragment of length $2a$, which would be served as an elementary cell for constructing an admissible stress field. This fragment is enclosed by the mid-surface and top-surface of the laminate, the surfaces of two consecutive cracks, and a unit length along the parallel cracks. The origin of the coordinate system is located in the mid length of the fragment as shown in Fig. 2. The geometry of the fragment is then symmetrical with respect to the xy and yz planes.

Following the approach developed by Hashin (1985), the stresses in the cracked material are represented as a superposition of the stresses in the un-cracked material and some yet unknown perturbation stresses caused by the presence of the cracks.

$$\tilde{\sigma}_{mn}^{(i)}(X) = \sigma_{mn}^{0(i)} + \sigma_{mn}^{(i)}(X) \tag{2.1}$$

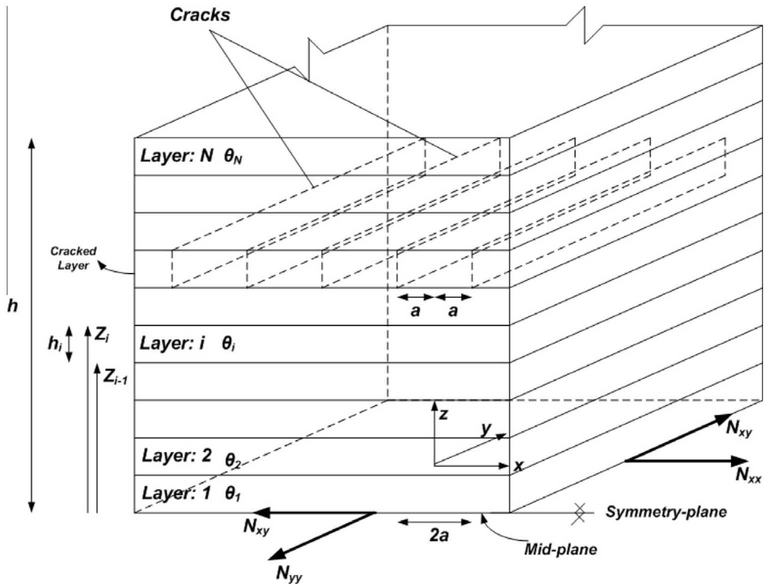

Fig. 1. Geometry of an arbitrary symmetric laminate containing cracks in 90° layers (only the upper set of N layers ($z > 0$) is shown).

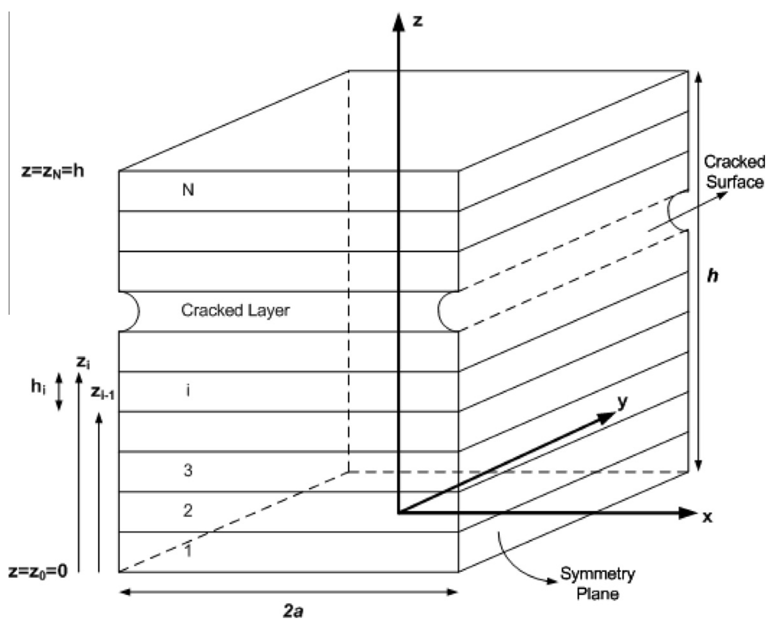

Fig. 2. Schematic of an elementary cell containing a cracked layer for the construction of admissible stress field.

where m, $n = x, y, z$. The second term in Eq. (2.1) is the stress in the ith ply of undamaged laminate, which can be obtained from a simple analysis using classic laminate theory. The last term in Eq. (2.1) is the perturbation stress in the ith ply, which, in contrast to the stresses in the undamaged laminate, is a function of location. For a general symmetric laminate, all the components of the perturbation stress tensor are expected to be nonzero, even when some of the external membrane forces are not applied. It is necessary to find an admissible stress field that satisfies the equilibrium equations $\tilde{\sigma}^{(i)}_{mn,n}(X) = 0$, traction boundary and continuity conditions, namely:

(1) Zero traction condition on the external surfaces $z = \pm h$: $\tilde{\sigma}^{(N)}_{xz} = \tilde{\sigma}^{(N)}_{yz} = \tilde{\sigma}^{(N)}_{zz} = 0$.
(2) Continuity condition at the interface between the plies at $z = z_i$, $i = 1, 2, \ldots, N-1$: $\tilde{\sigma}^{(i)}_{xz} = \tilde{\sigma}^{(i+1)}_{xz}$, $\tilde{\sigma}^{(i)}_{yz} = \tilde{\sigma}^{(i+1)}_{yz}$, $\tilde{\sigma}^{(i)}_{zz} = \tilde{\sigma}^{(i+1)}_{zz}$.
(3) Zero traction condition on the crack surfaces at $x = \pm a$: $\tilde{\sigma}^{(c)}_{xx} = \tilde{\sigma}^{(c)}_{xz} = \tilde{\sigma}^{(c)}_{xy} = 0$, where superscript ($c$) represents cracked plies. In addition, the stress field should balance the membrane forces applied to the laminate.

It should be mentioned that in the coordinate system where the crack planes are parallel to the y-axis, the stress fields are independent of the y coordinate due to this fact that cracks extend across the entire width. Moreover, we assume that the membrane perturbation stresses of each layer vary merely along the x direction, normal to the crack surfaces, and are denoted as:

$$\sigma^{(i)}_{xx}(X) = -\varphi_i(x)/h_i, \quad \sigma^{(i)}_{xy}(X) = -\psi_i(x)/h_i, \quad \sigma^{(i)}_{yy}(X) = -\eta_i(x)/h_i \tag{2.2}$$

where $\varphi_i(x)$, $\psi_i(x)$ and $\eta_i(x)$ are unknown functions of x coordinate. The other components of the stress tensor are allowed to depend on both x and z coordinates. Using Eq. (2.2), the equilibrium equations $\tilde{\sigma}^{(i)}_{mn,n}(\mathrm{X}) = 0$ reduce to:

$$-\varphi_i'(x)/h_i + \sigma^{(i)}_{xz,z} = 0, \quad -\psi_i'(x)/h_i + \sigma^{(i)}_{yz,z} = 0, \quad \sigma^{(i)}_{xz,x} + \sigma^{(i)}_{zz,z} = 0 \tag{2.3}$$

The solution of the equilibrium equations can be written in the form of:

$$\sigma^{(i)}_{xz}(x,z) = \varphi_i'(x)(z - z_i)/h_i + f_i(x) \tag{2.4}$$

$$\sigma^{(i)}_{yz}(x,z) = \psi_i'(x)(z - z_i)/h_i + g_i(x) \tag{2.5}$$

$$\sigma^{(i)}_{zz}(x,z) = -\frac{1}{2h_i}\varphi_i''(x)(z - z_i)^2 - z f_i'(x) + j_i(x) \tag{2.6}$$

where $f_i(x)$, $g_i(x)$ and $j_i(x)$, $i = 1, 2, \ldots N$ are unknown functions, which will be determined later using continuity and traction condition. Also, the primes denote derivatives with respect to x.

The external membrane forces applied to the laminate should be balanced. The load is applied so that N_{xx}, N_{yy}, and N_{xy} remain constant as cracks appear in the laminate. Therefore, for an undamaged laminate:

$$N_{xx} = \int_{-h}^{h} \sigma^0_{xx} dz = 2\sum_{i=1}^{N} \sigma^{0(i)}_{xx} h_i, \quad N_{yy} = \int_{-h}^{h} \sigma^0_{yy} dz = 2\sum_{i=1}^{N} \sigma^{0(i)}_{yy} h_i,$$
$$N_{xy} = \int_{-h}^{h} \sigma^0_{xy} dz = 2\sum_{i=1}^{N} \sigma^{0(i)}_{xy} h_i \tag{2.7}$$

And for a cracked laminates, we have:

$$N_{xx} = \int_{-h}^{h} \tilde{\sigma}_{xx} dz = 2\sum_{i=1}^{N} (\sigma^{0(i)}_{xx} h_i - \varphi_i(x)) \tag{2.8}$$

$$N_{xy} = \int_{-h}^{h} \tilde{\sigma}_{xy} dz = 2\sum_{i=1}^{N} \left(\sigma^{0(i)}_{xy} h_i - \psi_i(x)\right) \tag{2.9}$$

$$N_{yy} = \int_{-h}^{h} \tilde{\sigma}_{yy} dz = 2\sum_{i=1}^{N} \left(\sigma^{0(i)}_{yy} h_i - \eta_i(x)\right) \tag{2.10}$$

Consequently, the following relations are valid:

$$\sum_{i=1}^{N} \varphi_i(x) = 0, \quad \sum_{i=1}^{N} \psi_i(x) = 0, \quad \sum_{i=1}^{N} \eta_i(x) = 0 \tag{2.11}$$

It can be seen that equilibrium of external forces provides three relationships between the perturbation functions as shown in Eq. (2.11).

The boundary conditions for each cracked ply (c) on the crack surfaces $x = \pm a$ become:

$$\sigma^{(c)}_{xx}(\pm a, z) = -\sigma^{0(c)}_{xx}, \quad \sigma^{(c)}_{xy}(\pm a, z) = -\sigma^{0(c)}_{xy}, \quad \sigma^{(c)}_{xz}(\pm a, z) = 0 \tag{2.12}$$

The boundary conditions on the external surface of $z = z_N = h$ are:

$$\sigma^{(N)}_{xz}(x, z = h) = 0, \quad \sigma^{(N)}_{yz}(x, z = h) = 0, \quad \sigma^{(N)}_{zz}(x, z = h) = 0 \tag{2.13}$$

Symmetry with respect to xy plane (mid-plane) requires:

$$\sigma^{(1)}_{xz}(x, z = 0) = 0, \quad \sigma^{(1)}_{yz}(x, z = 0) = 0 \tag{2.14}$$

As stated above, traction continuity at $N - 1$ interfaces ($z = z_i, i = 1,2,\ldots N - 1$) requires:

$$\sigma^{(i)}_{xz}(x, z = z_i) = \sigma^{(i+1)}_{xz}(x, z = z_i) \tag{2.15}$$

$$\sigma^{(i)}_{yz}(x, z = z_i) = \sigma^{(i+1)}_{yz}(x, z = z_i) \tag{2.16}$$

$$\sigma^{(i)}_{zz}(x, z = z_i) = \sigma^{(i+1)}_{zz}(x, z = z_i) \tag{2.17}$$

Substituting (2.14) into (2.4) and (2.5), it could be concluded that $f_1(x) = \varphi_1'(x)$ and $g_1(x) = \psi_1'(x)$. Moreover, (2.15) and (2.16) provide enough recursive relations to find $f_i(x), g_i(x), i = 2, \ldots N$, respectively. Therefore, substituting (2.15) and (2.16) into (2.4) and (2.5) yields to:

$$f_i(x) = \sum_{j=1}^{i} \varphi_j'(x), \quad g_i(x) = \sum_{j=1}^{i} \psi_j'(x) \tag{2.18}$$

It can be easily verified that the traction boundary conditions in $(2.13)^1$ and $(2.13)^2$ are automatically satisfied (Hajikazemi and Sadr, 2014) considering $(2.11)^1$ and $(2.11)^2$, respectively.

Substituting $(2.13)^3$ into (2.6) and using (2.18), one concludes that

$$j_N(x) = h\sum_{j=1}^{N} \varphi_j'' \tag{2.19}$$

Moreover, Eq. (2.17) provides enough recursive relations to find $j_i(x), i = 1, 2 \ldots N - 1$. Therefore, substituting (2.17) into (2.6), it can be concluded that

$$j_i(x) = \sum_{j=1}^{N} h\varphi_j''(x) - \frac{1}{2}\sum_{j=i+1}^{N} (z_j + z_{j-1})\varphi_j''(x) \tag{2.20}$$

Finally, the admissible stress field that satisfies all equilibrium equations, tractions and continuity boundary conditions can be summarized as (Hajikazemi and Sadr, 2014):

$$\tilde{\sigma}^{(i)}_{xx}(x) = \sigma^{0(i)}_{xx} - \varphi_i(x)/h_i \tag{2.21}$$
$$\tilde{\sigma}^{(i)}_{yy}(x) = \sigma^{0(i)}_{yy} - \eta_i(x)/h_i \tag{2.22}$$
$$\tilde{\sigma}^{(i)}_{xy}(x) = \sigma^{0(i)}_{xy} - \psi_i(x)/h_i \tag{2.23}$$

$$\tilde{\sigma}^{(i)}_{xz}(x,z) = \varphi_i'(x)(z - z_i)/h_i + \sum_{j=1}^{i} \varphi_j'(x) \tag{2.24}$$

$$\tilde{\sigma}^{(i)}_{yz}(x,z) = \psi_i'(x)(z - z_i)/h_i + \sum_{j=1}^{i} \psi_j'(x) \tag{2.25}$$

$$\tilde{\sigma}^{(i)}_{zz}(x,z) = -\frac{1}{2h_i}\varphi_i''(x)(z - z_i)^2 - \sum_{j=1}^{i} \varphi_j''(x) + h\sum_{j=1}^{N} \varphi_j''(x) - \frac{1}{2}\sum_{j=i+1}^{N} (z_j + z_{j-1})\varphi_j''(x) \tag{2.26}$$

where $\varphi_i(x)$, $\psi_i(x)$ and $\eta_i(x)$, $i = 1, 2, \ldots N$, are unknown perturbation functions yet to be determined.

Notice that the expressions for the admissible stress field are valid for any in-plane loading conditions. The influence of the actual loads under consideration comes through the constant field in the boundary conditions at cracked surfaces of $x = \pm a$ (Eq. (2.12)) for the unknown functions that will be discussed in detail.

3. Variational formulation

We will evaluate the optimal functions that minimize the complementary energy of the cracked laminate. The total complementary energy $\tilde{U}_C$ associated with the admissible stresses in a laminate subject to traction boundary conditions is defined as follows:

$$\tilde{U}_C = \frac{1}{2}\int_V \tilde{\sigma} S \tilde{\sigma} dV = \frac{1}{2}\int_V (\sigma + \sigma^0) S (\sigma + \sigma^0) dV = U_C^0 + U_C + 2U_m \tag{3.1}$$

$$U_C^0 = \frac{1}{2}\int_V \sigma^0 S \sigma^0 dV = \frac{1}{2}\overline{\sigma} S^0 \overline{\sigma} V, \quad U_C = \frac{1}{2}\int_V \sigma S \sigma dV,$$
$$U_m = \frac{1}{2}\int_V \sigma S \sigma^0 dV$$

where S is the local compliance matrix, V is the volume of the cracked laminate, S^0 is the compliance matrix of un-cracked laminate, σ^0 is the stress tensor in un-cracked laminate, $\overline{\sigma}$ is the average stress tensor equal to the applied stresses and σ denotes the perturbation stresses. Hashin (1985) went through a lengthy proof to demonstrate that U_m vanishes. This is in fact a direct consequence of the virtual work principle. Thus, we have:

$$\tilde{U}_C = U_C^0 + U_C \tag{3.2}$$

where U_C^0 is the total complementary potential energy before cracking, which does not contribute to the variation.

Using the principle of minimum complementary energy one can write

$$\tilde{U}_C = \frac{1}{2}\overline{\sigma} S^* \overline{\sigma} V \le U_{Ca} = \frac{1}{2}\overline{\sigma} S^0 \overline{\sigma} V + \frac{1}{2}\int_V \sigma S \sigma dV \tag{3.3}$$

where S^* is the effective compliance matrix of the cracked laminate. The left-hand side of the inequality represents the true complementary energy and the right-hand side of the inequality represents the complementary energy computed with an admissible stress field. Thus, minimization of the functional in the right-hand side of (3.3) with respect to the perturbation stresses would lead to the upper bound for the effective compliance matrix and correspondingly a lower bound for effective stiffness matrix of cracked laminate.

For a cracked laminate, we would minimize the functional over the volume of a fragment of length $2a$ bounded by two adjacent transverse cracks, such that $|x| \le a$ and $|z| \le h$. Due to the symmetry with respect to mid-plane $z = 0$, only the region $0 \le z \le h$ can be considered.

Consequently:

$$U_C = 2\sum_{i=1}^{N}\left(\int_{-a}^{a}\int_{z_{i-1}}^{z_i} W^{(i)} dz dx\right) \quad i = 1, 2, \ldots, N \tag{3.4}$$

$$W^{(i)} = \frac{1}{2}\{\sigma^{(i)}\}^T \left[S^{(i)}\right]\{\sigma^{(i)}\}, \quad \{\sigma^{(i)}\}^T = \left\{\sigma_{xx}^{(i)}, \sigma_{yy}^{(i)}, \sigma_{zz}^{(i)}, \sigma_{yz}^{(i)}, \sigma_{xz}^{(i)}, \sigma_{xy}^{(i)}\right\}$$

where $W^{(i)}$ is the perturbation stress energy density of ply (i) and $\left[S^{(i)}\right]$ is the compliance matrix of ply (i) in the coordinate system associated with cracks. In order to compute $\left[S^{(i)}\right]$, the compliance matrix of the unidirectional fiber composite material should be rotated to the corresponding angles of the plies θ_i, e.g. 90° for any cracked plies. Substituting the expressions for the perturbation stresses (Eqs. (2.2), and 2.24, 2.25, 2.26) and inserting the rotated compliance matrices into (3.4), we can perform the integration over z. It should be noted that all the unknown functions are not independent and one out of any n unknowns of $\varphi_i, \psi_i, \eta_i, i = 1, 2, \ldots, N$ can be eliminated using Eq. (2.11), thus, the number of unknown perturbation functions, which should be determined, is actually $3N - 3$. In order to have a systematic way for developing the formulation, we choose the nearest un-cracked ply to the upper surface of the laminate (assigned its ply number to the integer variable m) and eliminate its perturbation functions based on (2.11). Thus, the result of integration in Eq. (3.4) based on the independent unknown functions can be written in the following form:

$$U_C = \int_{-a}^{a} F(x, \{\varphi\}, \{\varphi'\}, \{\varphi''\}, \{\psi\}, \{\psi'\}, \{\eta\}) dx \tag{3.5}$$

where

$$\begin{aligned} & F(x, \{\varphi\}, \{\varphi'\}, \{\varphi''\}, \{\psi\}, \{\psi'\}, \{\eta\}) \\ & \quad = \{\varphi\}^T \left[A_{11}^{00}\right]\{\varphi\} + \{\psi\}^T \left[A_{22}^{00}\right]\{\psi\} + \{\eta\}^T \left[A_{33}^{00}\right]\{\eta\} \\ & \qquad + \{\varphi\}^T \left[A_{12}^{00}\right]\{\psi\} + \{\varphi\}^T \left[A_{13}^{00}\right]\{\eta\} + \{\psi\}^T \left[A_{23}^{00}\right]\{\eta\} \\ & \qquad + \{\varphi'\}^T \left[A_{11}^{11}\right]\{\varphi'\} + \{\psi'\}^T \left[A_{22}^{11}\right]\{\psi'\} + \{\varphi'\}^T \left[A_{12}^{11}\right]\{\psi'\} \\ & \qquad + \{\varphi''\}^T \left[A_{11}^{20}\right]\{\varphi\} + \{\varphi''\}^T \left[A_{12}^{20}\right]\{\psi\} + \{\varphi''\}^T [A_{13}^{20}]\{\eta\} \\ & \qquad + \{\varphi''\}^T \left[A_{11}^{22}\right]\{\varphi''\} \end{aligned} \tag{3.6}$$

And the independent unknown functions are as follows:

$$\begin{aligned} & \{\varphi\}^T = \{\varphi_1, \varphi_2, \ldots, \varphi_{m-1}, \varphi_{m+1}, \ldots, \varphi_N\}_{1\times N-1}, \\ & \{\varphi'\}^T = \{\varphi'_1, \varphi'_2, \ldots, \varphi'_{m-1}, \varphi'_{m+1}, \ldots, \quad \varphi'_N\}_{1\times N-1} \end{aligned} \tag{3.7}$$

$$\begin{aligned} & \{\varphi''\}^T = \{\varphi''_1, \varphi''_2, \ldots, \varphi''_{m-1}, \varphi''_{m+1}, \ldots, \quad \varphi''_N\}_{1\times N-1}, \ldots, \\ & \{\eta\}^T = \{\eta_1, \eta_2, \ldots, \eta_{m-1}, \eta_{m+1}, \ldots, \eta_N\}_{1\times N-1} \end{aligned}$$

where m is one of the un-cracked plies (we chose the nearest un-cracked ply to the upper surface of the laminate), which is assumed as the dependent ply. In addition, the coefficient matrices of $\left[A_{11}^{00}\right]_{(N-1)\times(N-1)}$, etc., with the superscripts corresponding to the order of derivatives and subscripts corresponding to independent unknown functions involved (e.g. $[A_{13}^{20}]$ is associated with $\{\varphi''\}$ and $\{\eta\}$ or $[A_{12}^{11}]$ is associated with $\{\varphi'\}$ and $\{\psi'\}$), can be evaluated analytically as given in (Hajikazemi and Sadr, 2014).

The independent unknown functions $\{\varphi\}_{N-1,1}, \{\psi\}_{N-1,1}$ and $\{\eta\}_{N-1,1}$ are now to be determined from minimization of the complementary energy functional. It is known that when a variation is taken for the total complementary potential energy, the stationary value condition, $d\tilde{U}_C = 0$, leads to Euler–Lagrange equations as the governing equations of the problem as follows:

$$\begin{aligned} \frac{\partial F}{\partial\{\varphi\}} - \frac{d}{dx}\left(\frac{\partial F}{\partial\{\varphi'\}}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial\{\varphi''\}}\right) = 0 \Rightarrow & [B_1]\{\varphi''''\} + [B_2]\{\varphi''\} \\ & + [B_3]\{\varphi\} + [B_4]\{\psi''\} \\ & + [B_5]\{\psi\} + [B_6]\{\eta''\} \\ & + [B_7]\{\eta\} = 0 \end{aligned} \tag{3.8}$$

$$\begin{aligned} \frac{\partial F}{\partial\{\psi\}} - \frac{d}{dx}\left(\frac{\partial F}{\partial\{\psi'\}}\right) = 0 \Rightarrow & [B_4]^T\{\varphi''\} + [B_5]^T\{\varphi\} + [B_8]\{\psi''\} \\ & + [B_9]\{\psi\} + [B_{10}]\{\eta\} = 0 \end{aligned} \tag{3.9}$$

$$\frac{\partial F}{\partial\{\eta\}} = 0 \Rightarrow [B_6]^T\{\varphi''\} + [B_7]^T\{\varphi\} + [B_{10}]^T\{\psi\} + [B_{11}]\{\eta\} = 0 \tag{3.10}$$

where

$$\begin{aligned} & [B_1] = \left[A_{11}^{22}\right] + \left[A_{11}^{22}\right]^T, \quad [B_2] = \left[A_{11}^{20}\right] + \left[A_{11}^{20}\right]^T - \left[A_{11}^{11}\right] - \left[A_{11}^{11}\right]^T, \\ & [B_3] = [A_{11}^{00}] + [A_{11}^{00}]^T, \quad [B_4] = -\left[A_{12}^{11}\right] + \left[A_{12}^{20}\right], \quad [B_5] = \left[A_{12}^{00}\right], \\ & [B_6] = \left[A_{13}^{20}\right], \quad [B_7] = \left[A_{13}^{00}\right], \quad [B_8] = -\left[A_{22}^{11}\right] - \left[A_{22}^{11}\right]^T, \\ & [B_9] = \left[A_{22}^{00}\right] + \left[A_{22}^{00}\right]^T, \quad [B_{10}] = \left[A_{23}^{00}\right], \quad [B_{11}] = \left[A_{33}^{00}\right] + \left[A_{33}^{00}\right]^T \end{aligned} \tag{3.11}$$

Eq. (3.10) define $\{\eta\}$ in terms of the other two unknown functions:

$$\{\eta\} = -[B_{11}]^{-1}\left([B_6]^T\{\varphi''\} + [B_7]^T\{\varphi\} + [B_{10}]^T\{\psi\}\right) \tag{3.12}$$

Substituting the above expression back to the functional in (3.6), we can rewrite it in terms of two sets of independent unknown functions of $\{\varphi\}$ and $\{\psi\}$:

$$\begin{aligned} F(x,\{\varphi\},\{\varphi'\},\{\varphi''\},\{\psi\},\{\psi'\}) \\ = \{\varphi\}^T[P_{11}^{00}]\{\varphi\} + \{\psi\}^T[P_{22}^{00}]\{\psi\} + \{\varphi\}^T[P_{12}^{00}]\{\psi\} \\ + \{\varphi'\}^T[P_{11}^{11}]\{\varphi'\} + \{\psi'\}^T[P_{22}^{11}]\{\psi'\} + \{\varphi'\}^T[P_{12}^{11}]\{\psi'\} \\ + \{\varphi''\}^T[P_{11}^{20}]\{\varphi\} + \{\varphi''\}^T[P_{12}^{20}]\{\psi\} + \{\varphi''\}^T[P_{11}^{22}]\{\varphi''\} \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &\left[P_{11}^{00}\right] = \left[A_{11}^{00}\right] - \frac{1}{4}\left[A_{13}^{00}\right]\left[A_{33}^{00}\right]^{-1}\left[A_{13}^{00}\right]^T, \\ &\left[P_{22}^{00}\right] = \left[A_{22}^{00}\right] - \frac{1}{4}\left[A_{23}^{00}\right]\left[A_{33}^{00}\right]^{-1}\left[A_{23}^{00}\right]^T, \\ &\left[P_{12}^{00}\right] = \left[A_{12}^{00}\right] - \frac{1}{2}\left[A_{13}^{00}\right]\left[A_{33}^{00}\right]^{-1}\left[A_{23}^{00}\right]^T, \quad \left[P_{11}^{11}\right] = \left[A_{11}^{11}\right], \\ &\left[P_{22}^{11}\right] = \left[A_{22}^{11}\right], \quad \left[P_{12}^{11}\right] = \left[A_{12}^{11}\right], \\ &\left[P_{11}^{20}\right] = \left[A_{11}^{20}\right] - \frac{1}{2}\left[A_{13}^{20}\right]\left[A_{33}^{00}\right]^{-1}\left[A_{13}^{00}\right]^T, \\ &\left[P_{12}^{20}\right] = \left[A_{12}^{20}\right] - \frac{1}{2}\left[A_{13}^{20}\right]\left[A_{33}^{00}\right]^{-1}\left[A_{23}^{00}\right]^T, \\ &\left[P_{11}^{22}\right] = \left[A_{11}^{22}\right] - \frac{1}{4}\left[A_{13}^{20}\right]\left[A_{33}^{00}\right]^{-1}\left[A_{13}^{20}\right]^T. \end{aligned} \tag{3.14}$$

Again, the Euler–Lagrange equations as the governing equations for the problem based on functional in (3.13) are as follows:

$$[T_1]\{\varphi''''\} + [T_2]\{\varphi''\} + [T_3]\{\varphi\} + [T_4]\{\psi''\} + [T_5]\{\psi\} = 0 \tag{3.15}$$

$$[T_4]^T\{\varphi''\} + [T_5]^T\{\varphi\} + [T_6]\{\psi''\} + [T_7]\{\psi\} = 0 \tag{3.16}$$

where

$$\begin{aligned} &[T_1] = [P_{11}^{22}] + [P_{11}^{22}]^T, \quad [T_2] = [P_{11}^{20}] + [P_{11}^{20}]^T - [P_{11}^{11}] - [P_{11}^{11}]^T, \\ &[T_3] = [P_{11}^{00}] + [P_{11}^{00}]^T, \quad [T_4] = -[P_{12}^{11}] + [P_{12}^{20}], \quad [T_5] = [P_{12}^{00}], \\ &[T_6] = -[P_{22}^{11}] - [P_{22}^{11}]^T, \quad [T_7] = [P_{22}^{00}] + [P_{22}^{00}]^T. \end{aligned} \tag{3.17}$$

Eqs. (3.15) and (3.16) are an extension of the research work done by Vinogradov and Hashin (2010) for two-layer angle ply laminates containing matrix cracks in the middle ply, into the more complex and applicable case of general symmetric laminates with multiple-layer cracked and un-cracked plies. Here, it should be noted that for a cross-ply laminates $[T_4] = [T_5] = 0$ and Eqs. (3.15) and (3.16) become uncoupled.

$$[T_1]\{\varphi''''\} + [T_2]\{\varphi''\} + [T_3]\{\varphi\} = 0 \tag{3.18}$$

$$[T_6]\{\psi''\} + [T_7]\{\psi\} = 0 \tag{3.19}$$

which is the case considered by Hashin (1985) for a two-layer cross ply; with only one (usually negligible for a cross-ply) difference that here the perturbation of σ_{yy} has not been neglected.

Eqs. (3.15) and (3.16) are a pair of coupled systems of simultaneous linear ordinary differential equations with constant coefficients. There are numerous standard treatments (Simmons, 1972) in order to solve these equations, which could not be described here. The reader can also refer to Hajikazemi and Sadr (2014) to find more details about solving the governing equations. Moreover, the equations can be solved numerically using mathematical software like MAPLE.

Once the governing equations (Eqs. (3.15) and (3.16)) are solved and general solutions are achieved, the last step would be finding arbitrary constants of the solution using boundary conditions. Indeed, boundary conditions are required in order to determine the solution. It could be easily seen that one system (Eq. (3.15)) is fourth order in terms of the variables φ and second order in terms of the variables ψ and the other system (Eq. (3.16)) is second order both in terms of φ and the ψ. Thus, the first system requires $4(N-1)$ boundary conditions and the second needs $2(N-1)$ boundary conditions, making a total of $6(N-1)$ boundary conditions. Suppose that there are N_c cracked and N_u un-cracked plies. Then clearly, we have:

$$N_c + N_u = N \tag{3.20}$$

Eq. (2.12) should be satisfied for each of the cracked plies, and these equations can be written in terms of unknown functions as follows:

$$\begin{aligned} &\varphi_i(a) = \varphi_i(-a) = \sigma_{xx}^{0(i)}h_i, \quad \psi_i(a) = \psi_i(-a) \\ &\quad = \sigma_{xy}^{0(i)}h_i, \quad \varphi_i'(a) = \varphi_i'(-a) = 0 \end{aligned} \tag{3.21}$$

There are clearly $6N_c$ of these boundary conditions.

For an un-cracked ply, the physical conditions available in the problem can only produce four boundary conditions. We took the in-plane shear stress σ_{xy}, axial stress σ_{xx} and transverse shear stress $\sigma_{xz}$from the free body diagram as shown in Fig. 3, to start with. The continuity consideration leads to

$$\begin{aligned} &\sigma_{xy1} = \sigma_{xy2} \quad \text{and} \quad \sigma_{xy3} = \sigma_{xy4} \\ &\sigma_{xx1} = \sigma_{xx2} \quad \text{and} \quad \sigma_{xx3} = \sigma_{xx4} \\ &\sigma_{xz1} = \sigma_{xz2} \quad \text{and} \quad \sigma_{xz3} = \sigma_{xz4} \end{aligned} \tag{3.22}$$

While the periodic condition or translational symmetry requires

$$\begin{aligned} &\sigma_{xy1} = \sigma_{xy3} \quad \text{and} \quad \sigma_{xy2} = \sigma_{xy4} \\ &\sigma_{xx1} = \sigma_{xx3} \quad \text{and} \quad \sigma_{xx2} = \sigma_{xx4} \\ &\sigma_{xz1} = \sigma_{xz3} \quad \text{and} \quad \sigma_{xz2} = \sigma_{xz4} \end{aligned} \tag{3.23}$$

Consequently:

$$\sigma_{xy2} = \sigma_{xy3} \to \sigma_{xy}(a,z) = \sigma_{xy}(-a,z) \Rightarrow \psi_i(a) = \psi_i(-a) \tag{3.24}$$

$$\sigma_{xx2} = \sigma_{xx3} \to \sigma_{xx}(a,z) = \sigma_{xx}(-a,z) \Rightarrow \varphi_i(a) = \varphi_i(-a) \tag{3.25}$$

$$\sigma_{xz2} = \sigma_{xz3} \to \sigma_{xz}(a,z) = \sigma_{xz}(-a,z) \Rightarrow \varphi_i'(a) = \varphi_i'(-a) \tag{3.26}$$

The rotational symmetry about the vertical central axis can result in another boundary condition. This symmetry is always present in the laminate, cracked or not, in general (Li and Reid, 1992). The rotational symmetry on σ_{xy} and σ_{xx} yields to the same condition as in Eqs. (3.24) and (3.25), respectively. However, for σ_{xz}, as it is anti-symmetric under this particular symmetry transformation, this symmetry requires

$$\sigma_{xz2} = -\sigma_{xz3} \to \sigma_{xz}(a,z) = -\sigma_{xz}(-a,z) \Rightarrow \varphi_i'(a) = -\varphi_i'(-a) \tag{3.27}$$

which together with Eq. (3.26), the boundary conditions associated with transverse shear can be given as

$$\sigma_{xz}(a,z) = \sigma_{xz}(-a,z) = 0 \Rightarrow \varphi_i'(a) = \varphi_i'(-a) = 0 \tag{3.28}$$

Finally, Eqs. (3.24), (3.25), and (3.28), which belong to independent un-cracked plies, clearly provide $4(N_u-1)$ boundary conditions.

The physical construction of the problem itself does not offer any more boundary conditions without resorting to displacements. As a result, there will not be sufficient boundary conditions directly from the physical conditions. McCartney (1992, 2000) has introduced a displacement boundary condition. In a stress-based approach, as is the case here, displacements are not involved, and one has to find extra boundary conditions in terms of stresses for each independent un-cracked lamina before the solution can be determined. Li and Hafeez (2009) have shown that for any cross ply laminate having more than two un-cracked plies, an extension

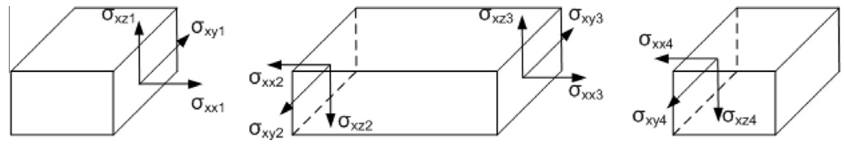

Fig. 3. Axial stress, in-plane shear stress, and transverse shear stress at the boundary of an un-cracked lamina.

is required in terms of boundary conditions. There is a shortfall of two boundary conditions for each independent un-cracked ply ($2(N_u-1)$ in total). Hajikazemi and Sadr (2014) have shown that the fifth and sixth boundary conditions for each independent un-cracked ply of a general symmetric laminate can be obtained mathematically from the variational calculus itself in terms of natural boundary conditions. As Euler–Lagrange's equations are derived using variational calculus, terms also emerge which take their values at boundaries resulting from steps of integration by parts. For the variation of the functional to vanish so that the functional takes its stationary value, these terms must also vanish. This leads to boundary conditions, called natural boundary conditions in variational principles, which are as follows:

$$\left[\left(\frac{\partial F}{\partial\{\varphi'\}}-\frac{d}{dx}\frac{\partial F}{\partial\{\varphi''\}}\right)^T\{\partial\varphi\}\right]_{x=-a}^{x=a}=0 \tag{3.29}$$

$$\left[\left(\frac{\partial F}{\partial\{\psi'\}}\right)^T\{\partial\psi\}\right]_{x=-a}^{x=a}=0 \tag{3.30}$$

The above equations can be simplified as:

$$[T_1](\{\varphi'''(-a)\}-\{\varphi'''(+a)\})+[T_4](\{\psi'(-a)\}-\{\psi'(+a)\})=0 \tag{3.31}$$

$$[T_6](\{\psi'(-a)\}-\{\psi'(+a)\})=0 \tag{3.32}$$

To obtain Eqs. (3.31) and (3.32), physical boundary conditions of Eq. (3.28) have been considered. Apparently, Eqs. (3.31) and (3.32) can only be applied for independent un-cracked plies and thus they provide $2(N_u-1)$ boundary conditions. Here, it should be noted that for a cross-ply laminates $[T_4]=0$ and Eqs. (3.31) and (3.32) become uncoupled.

$$[T_1](\{\varphi'''(-a)\}-\{\varphi'''(+a)\})=0 \tag{3.33}$$

$$[T_6](\{\psi'(-a)\}-\{\psi'(+a)\})=0 \tag{3.34}$$

which is the case considered by Li and Hafeez (2009) for cross ply laminates; with only two differences where here the perturbation of σ_{yy} has not been neglected and also the laminate is under general in-plane loading.

As mentioned above, Eq. (3.21) makes $6N_c$, Eqs. (3.24), (3.25), and (3.28) make $4(N_u-1)$ and Eqs. (3.31) and (3.32) prepare $2(N_u-1)$ boundary conditions which provide

$$6N_c+4(N_u-1)+2(N_u-1)=6(N_c+N_u)-6=6(N-1) \tag{3.35}$$

boundary conditions in total, which is as it was required.

4. Effective compliance matrix of a cracked symmetric laminate

Having optimal stress field as a function of loading case, it is possible to calculate perturbation energy using Eqs. (3.5) and (3.13). In order to obtain effective compliance matrix of cracked laminate S^*, six cases of loadings are applied in the coordinate system associated with cracks:

$$\begin{aligned}[N_{xx}/2h, N_{yy}/2h, N_{xy}/2h]=[\overline{\sigma_{xx}},\overline{\sigma_{yy}},\overline{\sigma_{xy}}]=[1\ \ 0\ \ 0]^a,\ \ [0\ \ 1\ \ 0]^b,\\ [0\ \ 0\ \ 1]^c,\ \ [1\ \ 1\ \ 0]^d,\ \ [1\ \ 0\ \ 1]^e,\\ [0\ \ 1\ \ 1]^f\end{aligned} \tag{4.1}$$

where using Eq. (3.3) and Eq. (3.6), the effective compliance matrix terms can be written as follows:

$$S^*_{11}\le S^0_{11}+\frac{1}{2ah}\left(\int_{-a}^{a}F dx\right)_{[\overline{\sigma_{xx}},\overline{\sigma_{yy}},\overline{\sigma_{xy}}]=[1,0,0]} \tag{4.2}$$

$$S^*_{22}\le S^0_{22}+\frac{1}{2ah}\left(\int_{-a}^{a}F dx\right)_{[\overline{\sigma_{xx}},\overline{\sigma_{yy}},\overline{\sigma_{xy}}]=[0,1,0]} \tag{4.3}$$

$$S^*_{33}\leqslant S^0_{33}+\frac{1}{2ah}\left(\int_{-a}^{a}F dx\right)_{[\overline{\sigma_{xx}},\overline{\sigma_{yy}},\overline{\sigma_{xy}}]=[0,0,1]} \tag{4.4}$$

$$\begin{aligned}S^*_{12}\leqslant\frac{1}{2}\Big(2S^0_{12}+\left(S^0_{11}-S^*_{11}\right)+\left(S^0_{22}-S^*_{22}\right)\\ +\frac{1}{2ah}\left(\int_{-a}^{a}F dx\right)_{[\overline{\sigma_{xx}},\overline{\sigma_{yy}},\overline{\sigma_{xy}}]=[1,1,0]}\Big)\end{aligned} \tag{4.5}$$

$$\begin{aligned}S^*_{13}\leqslant\frac{1}{2}\Big(2S^0_{13}+\left(S^0_{11}-S^*_{11}\right)+\left(S^0_{33}-S^*_{33}\right)\\ +\frac{1}{2ah}\left(\int_{-a}^{a}F dx\right)_{[\overline{\sigma_{xx}},\overline{\sigma_{yy}},\overline{\sigma_{xy}}]=[1,0,1]}\Big)\end{aligned} \tag{4.6}$$

$$\begin{aligned}S^*_{23}\le\frac{1}{2}\Big(2S^0_{23}+\left(S^0_{22}-S^*_{22}\right)+\left(S^0_{33}-S^*_{33}\right)\\ +\frac{1}{2ah}\left(\int_{-a}^{a}F dx\right)_{[\overline{\sigma_{xx}},\overline{\sigma_{yy}},\overline{\sigma_{xy}}]=[0,1,1]}\Big)\end{aligned} \tag{4.7}$$

It can be easily seen that the above equations give an upper bound for the effective compliance matrix and correspondingly a lower bound for effective stiffness matrix of cracked laminate. Finally, the effective engineering constants of cracked laminates are also defined as follows:

$$E_x=\frac{1}{S^*_{11}},\quad E_y=\frac{1}{S^*_{22}},\quad G_{xy}=\frac{1}{S^*_{33}},\quad \upsilon_{xy}=-\frac{S^*_{12}}{S^*_{11}} \tag{4.8}$$

5. Results and discussion

Firstly, in order to verify the developed approach, the elastic properties of $[-\theta/+\theta/90_4]_s$ laminates ($\theta=0°, 15°$ and $30°$) made of glass-fiber/epoxy (GF/EP) material are computed as a function of crack density in 90° ply. They are compared to the experimental results reported in Joffe et al. (2001). The unidirectional ply properties of glass-fiber/epoxy (GF/EP) material, taken from Joffe et al. (2001), are listed as follows:

$$\begin{aligned}&E_1=44.73\text{ GPa},\quad E_2=12.76\text{ GPa}\quad G_{12}=5.8\text{ GPa}\\ &G_{23}=4.49\text{ GPa},\quad \upsilon_{12}=0.297,\quad \upsilon_{23}=0.42,\\ &\text{Plythickness}=0.144\text{ mm}\end{aligned} \tag{5.1}$$

Predictions of axial modulus and Poisson's ratio presented in Figs. 4–7 are in good agreement with experimental results. Moreover, it is noted that all experimental points fell on or slightly above the prediction curve. This is expected in view of the fact that the present model provides a lower bound for the effective stiffness of a cracked laminate. It is worth mentioning that the existing variational models are not able to analyze stiffness reduction of $[-\theta/+\theta/90_4]_s$ laminates without using an averaging procedure (Joffe and Varna, 1999). It should be noted that the averaging procedure reduces $[-\theta/+\theta/90_4]_s$ laminate to a two-layer cross-ply laminate.

In the next step, the axial stiffness of $[0/45]_s$ laminates made of glass fiber reinforced plastic (GFRP) material are computed as a function of crack density in 45° ply (see Fig. 8). It is compared to the experimental results reported in Katerelos et al. (2006) and with those obtained by Vinogradov and Hashin (2010). The unidirectional ply properties of GFRP are listed as follows:

$$E_1 = 43\ \text{GPa} \quad E_2 = 13\ \text{GPa} \quad G_{12} = 3.4\ \text{GPa}$$
$$G_{23} = 4.58\ \text{GPa}, \quad \upsilon_{12} = 0.3, \quad \upsilon_{23} = 0.42,$$
$$\text{Plythickness} = 0.61\ \text{mm} \tag{5.2}$$

Again, a very good agreement with the experimental data is observed and all experimental points fell on or slightly above the prediction curve. Moreover, a perfect match is observed between the results obtained from the present model and with those obtained from Vinogradov and Hashin (2010). This was expected since the formulation presented here reduces to theirs for a two-layer angle ply laminate containing matrix cracks in the middle ply, which means that their formulation is an especial case of the current formulation.

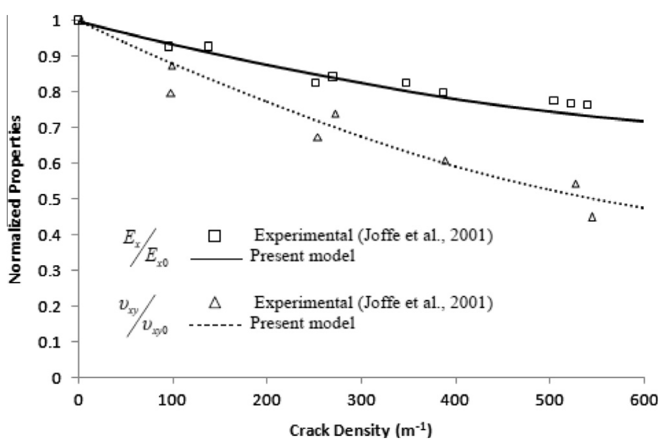

Fig. 4. Normalized properties of a $[0_2/90_4]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

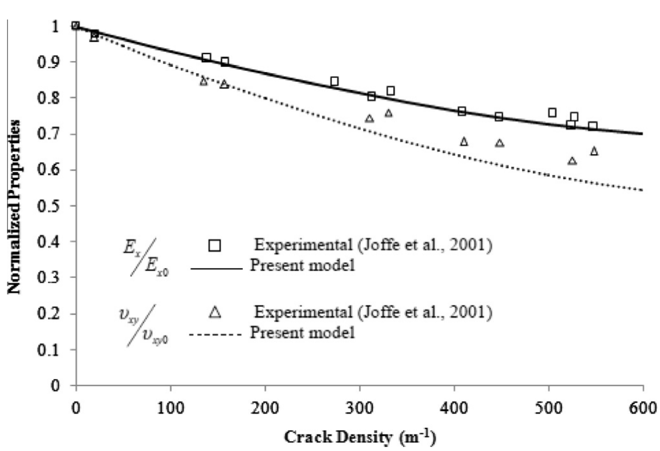

Fig. 5. Normalized properties of a $[-15/15/90_4]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

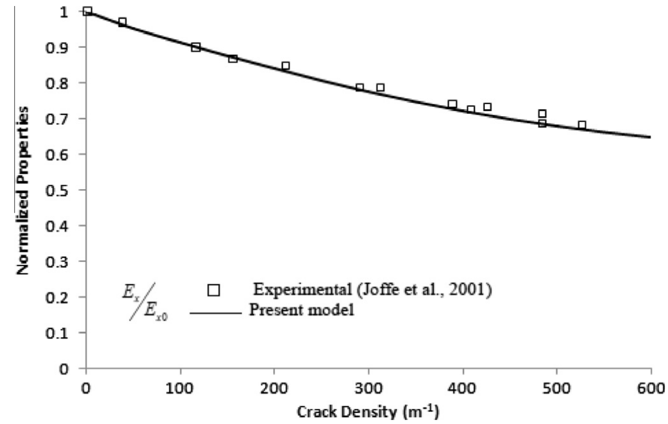

Fig. 6. Normalized axial stiffness of a $[-30/30/90_4]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

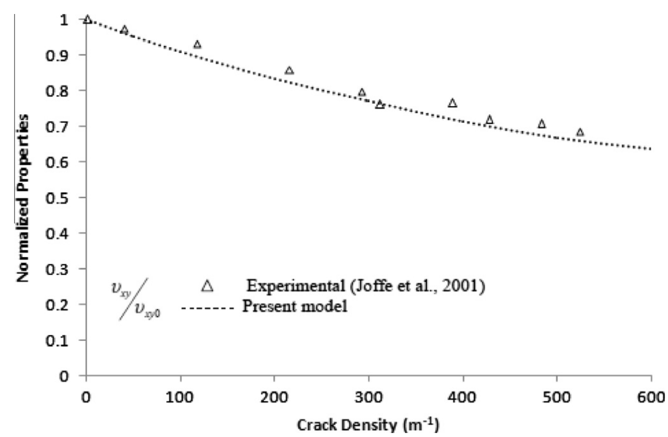

Fig. 7. Normalized Poisson's ratio of a $[-30/30/90_4]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

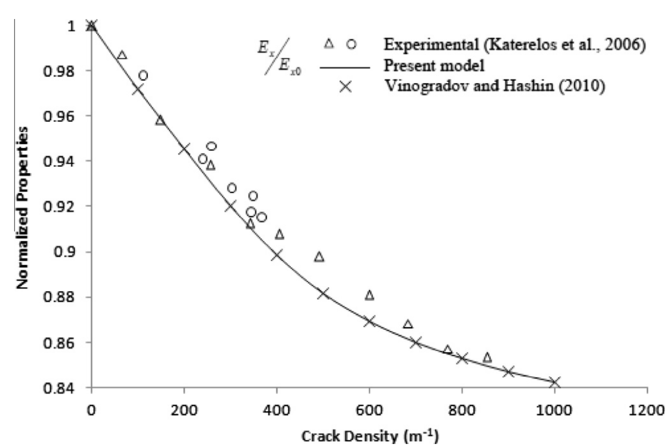

Fig. 8. Normalized axial stiffness of a $[0/45]_s$ glass fiber reinforced plastic (GFRP) laminate as a function of crack density in 45° ply (without ply refinement).

Moreover, in order to verify the developed approach for a more complex lay-up, the elastic properties of $[0/45/30/90_3]_s$ made of carbon-epoxy are computed as a function of crack density in 90° ply. They are compared to the results, obtained by McCartney's stress transfer model. The unidirectional ply properties of carbon-epoxy material, taken from McCartney (1996), are listed as follows:

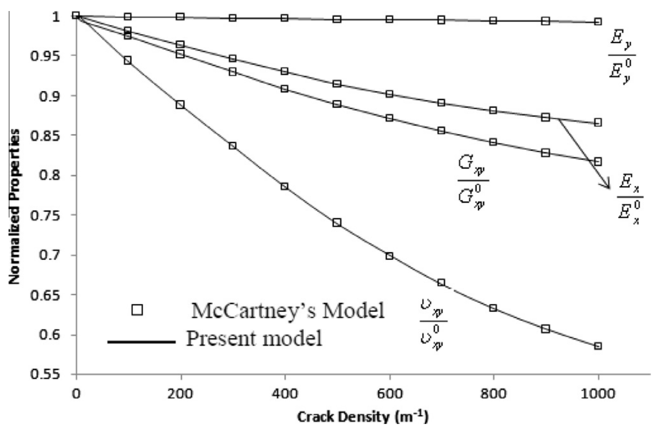

Fig. 9. Normalized properties of a $[0/45/30/90_3]_s$ carbon-epoxy laminate as a function of crack density in 90° ply. The square points represent the prediction based on McCartney's stress transfer model computed without ply refinement (data was kindly provided by Prof. N. McCartney), see McCartney (2000) and references therein for details about the McCartney model.

$$\begin{aligned} &E_1 = 136.6\ \text{GPa} \quad E_2 = 9.79\ \text{GPa} \quad G_{12} = 6.474\ \text{GPa} \\ &G_{23} = 3.364\ \text{GPa} \quad \upsilon_{12} = 0.286 \quad \upsilon_{23} = 0.455, \\ &\text{Plythickness} = 0.125\ \text{mm} \end{aligned} \tag{5.3}$$

The elastic properties of cracked laminate without ply refinement (i.e. $E_x, E_y, G_{xy}, \upsilon_{xy}$), normalized with respect to the elastic properties of the un-cracked laminate are plotted in Fig. 9.

There is an excellent agreement between the results obtained from the present model with those obtained from McCartney's stress transfer model (McCartney, 2000). This agreement shows that the McCartney's stress transfer solution can also be derived using a variational method that is not a Reissner method (McCartney, 2003). The excellent agreement between the overall elastic properties states that the stress field obtained from the present model should be also in excellent agreement with those obtained from McCartney's model. It should be noted that the McCartney's admissible stress field is very similar to the assumed one in the current work. Both methods assume that σ_{xx} and σ_{xy} are constant through the thickness of each ply. The only one difference is that we have assumed a perturbation for σ_{yy}, however, McCartney has considered a generalized plane strain assumption to determine σ_{yy}. Nevertheless, the procedure of determining the assumed admissible stress field is completely different for each method. While the current method is based on the minimization of complementary energy, McCartney's model uses 2D elasticity equations in order to obtain assumed admissible stress field. The McCartney's model satisfies the equilibrium equations, the interface continuities and the boundary conditions. However, some of the stress–strain relations and boundary conditions are satisfied in an average sense. In the current approach equilibrium equations, the interface continuities and the boundary conditions are also satisfied exactly; moreover, the obtained stress field minimizes the complementary energy. It is also noted that both models could enjoy the advantages of ply refinement technique in order to take into account the important through the thickness variations of the stress. It is noteworthy to mention that the present model is a stress based variational model, which does not obtain the displacement field, however, the stress transfer model of McCartney is a 2D analysis, which considers both the stress and displacement components based on the generalized plane strain assumptions. The McCartney's technique is basically analytical, but because of the resulting complexity, the analysis must be handled numerically in some steps while making predictions of the behavior of laminate. As the current method only considers the stress field and does not include any averaging procedure to obtain the governing equations, one may consider that the present model is simpler than McCartney's stress transfer model, which needs an averaging procedure to obtain the governing equations.

Finally, in order to investigate the effects of ply angle on stiffness reduction of un-balanced cracked laminates, a set of $[\theta/90_2/\theta_{0.5}]_s$ ($\theta = 0°, 30°, 45°, 60°$) laminates made of glass-fiber/epoxy (GF/EP) is chosen and stiffness reduction of cracked laminates are shown in Figs. 10–13 as a function of crack density in 90° ply.

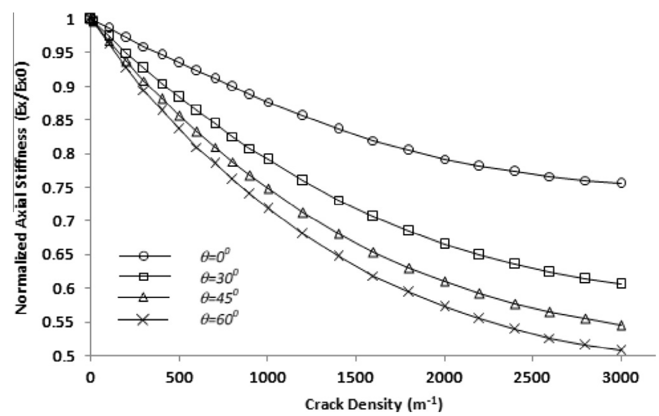

Fig. 10. Normalized axial stiffness (E_x/E_{x0}) of a $[\theta/90_2/\theta_{0.5}]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

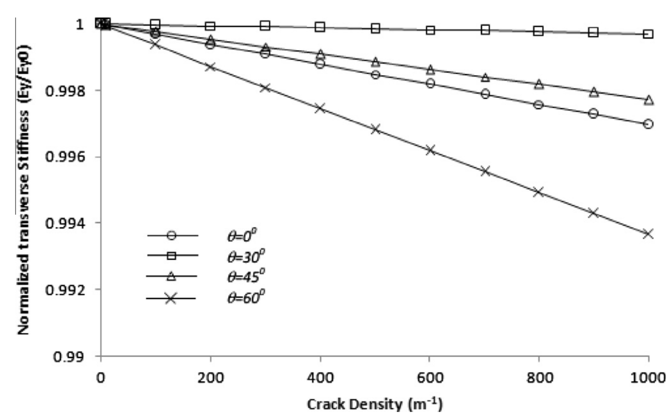

Fig. 11. Normalized transverse stiffness (E_y/E_{y0}) of a $[\theta/90_2/\theta_{0.5}]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

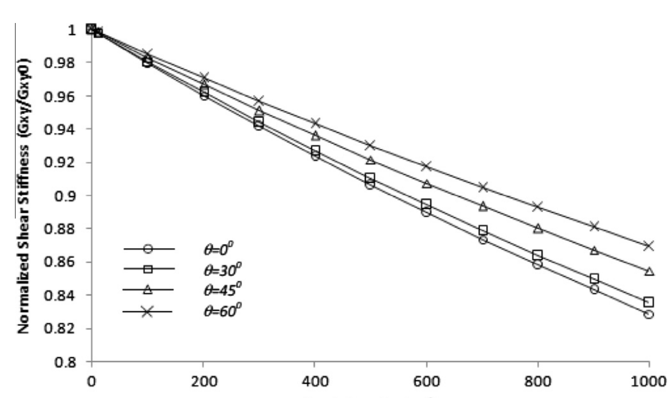

Fig. 12. Normalized shear stiffness (G_{xy}/G_{xy0}) of a $[\theta/90_2/\theta_{0.5}]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

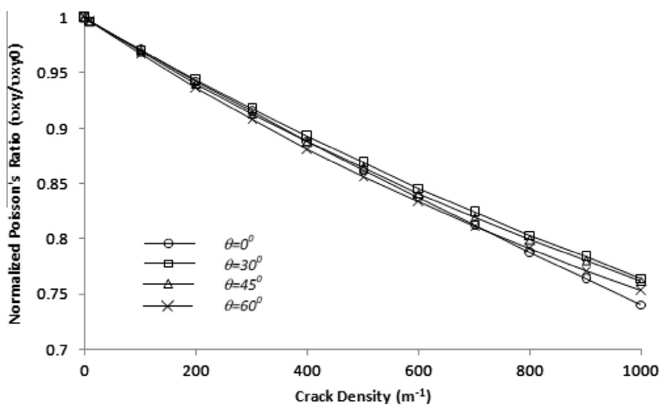

Fig. 13. Normalized Poisson's ratio ($\upsilon_{xy}/\upsilon_{xy0}$) of a $[\theta/90_2/\theta_{0.5}]_s$ glass fiber-epoxy (GF/EP) laminate as a function of crack density in 90° ply (without ply refinement).

It can be easily seen that depending on the angle, the cracks can have a weaker or stronger influence on the elastic modulus of an un-balanced symmetric laminate.

6. Conclusion

The applicability of the variational approach, which was first developed by Hashin (1985), is enhanced to analyze stiffness reduction of the cracked symmetric laminates with arbitrary stacking sequence and multiple cracked and un-cracked layers. In the current model, equilibrium equations, the interface continuities and the boundary conditions are satisfied exactly; moreover, the principle of minimum complementary energy is implemented to get the effective stiffness matrix. Results derived from the developed method for the elastic properties of the cracked symmetric laminates showed an excellent agreement with available experimental data and also with the results obtained from McCartney's stress transfer model. It has been shown that the method provides a rigorous lower bound for the stiffness matrix of a cracked laminate. The study of the results has revealed that the existing variational models are the especial cases of the current model. The study of the results has shown that the McCartney's stress transfer solution can also be derived using a variational method that is not a Reissner method. However, it can be clearly seen that the present model is simpler than McCartney's stress transfer model, which needs an averaging procedure to obtain the governing equations. Moreover, the study of the results has also revealed that depending on the angle, the cracks can have a weaker or stronger influence on the elastic modulus of an un-balanced symmetric laminate. Although not discussed explicitly, the described approach can readily be extended to deal with moisture and temperature effects. In the field of matrix cracks analysis for symmetric laminates, the present formulation is the most complete variational model developed so far.

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References

Barbero, E.J., Cortes, D.H., 2010. A mechanistic model for transverse damage initiation, evolution, and stiffness reduction in laminated composites. Composites Part B 41, 124–132.

Barbero, E.J., Cosso, F.A., 2014. Determination of material parameters for discrete damage mechanics analysis of carbon-epoxy laminates. Composites Part B 56, 638–646.

Barbero, E.J., Sgambitterra, G., Adumitroaie, A., Martinez, X., 2011. A discrete constitutive model for transverse and shear damage of symmetric laminates with arbitrary stacking sequence. Compos. Struct. 93, 1021–1030.

Barbero, E.J., Cosso, F.A., Roman, R., Weadon, T.L., 2013a. Determination of material parameters for Abaqus progressive damage analysis of E-glass epoxy laminates. Composites Part B 46, 211–220.

Barbero, E.J., Cosso, F.A., Campo, F.A., 2013b. Benchmark solution for degradation of elastic properties due to transverse matrix cracking in laminated composites. Compos. Struct. 98, 242–252.

Barbero, E.J., Cosso, F.A., Martinez, X., 2014. Identification of fracture toughness for discrete damage mechanics analysis of Glass-Epoxy laminates. Appl. Compos. Mater, in press. doi: 10.1007/s10443-013-9359-y.

Berthelot, J.-M., 2003. Transverse cracking and delamination in cross-ply glass-fiber and carbon-fiber reinforced plastic laminates: static and fatigue loading. Appl. Mech. Rev. 56, 111–147.

Hajikazemi, M., Sadr, M.H., 2014. A variational model for stress analysis in cracked laminates with arbitrary symmetric lay-up under general in-plane loading. Int. J. Solids Struct. 51, 516–529.

Hashin, Z., 1985. Analysis of cracked laminates: a variational approach. Mech. Mater. 4, 121–136.

Hashin, Z., 1986. Analysis of stiffness reduction of cracked cross-ply laminates. Eng. Fract. Mech. 25, 771–778.

Hashin, Z., 1987. Analysis of orthogonally cracked laminates under tension. J. Appl. Mech. 54, 872–879.

Jalalvand, M., Hosseini-Toudeshky, H., Mohammadi, B., 2013. Homogenization of diffuse delamination in composite laminates. Compos. Struct. 100, 113–120.

Joffe, R., Varna, J., 1999. Analytical modeling of stiffness reduction in symmetric and balanced laminates due to cracks in 90° layers. Compos. Sci. Technol. 59 (11), 1641–1652.

Joffe, R., Krasnikovs, A., Varna, J., 2001. COD-based simulation of transverse cracking and stiffness reduction in $[S/90_n]$ laminates. Compos. Sci. Technol. 61 (5), 637–656.

Kaddour, A.S., Hinton, M.J., Smith, P.A., Li, S., 2013. A comparison between the predictive capability of matrix cracking, damage and failure criteria for fibre reinforced composite laminates: part a of the third world-wide failure exercise. J. Compos. Mater. 47 (20–21), 2749–2779.

Kashtalyan, M., Soutis, C., 2005. Analysis of composite laminates with intra- and interlaminar damage. Prog. Aerosp. Sci. 41 (2), 152–173.

Katerelos, D., McCartney, L.N., Galiotis, C., 2006. Effect of off-axis matrix cracking on stiffness of symmetric angle-ply composite laminates. Int. J. Fract. 139, 529–536.

Kuriakose, S., Talreja, R., 2004. Variational solutions to stresses in cracked cross-ply laminates under bending. Int. J. Solids Struct. 41, 2331–2347.

Li, S., Hafeez, F., 2009. Variation-based cracked laminate analysis revisited and fundamentally extended. Int. J. Solids Struct. 46, 3505–3515.

Li, S., Lim, S.-H., 2005. Variational principles for generalised plane strain problems and their applications. Composites Part A 36, 353–365.

Li, S., Reid, S.R., 1992. On the symmetry conditions for laminated fibre-reinforced composite structures. Int. J. Solids Struct. 29 (23), 2867–2880.

Li, S., Reid, S.R., Soden, P.D., 1994. A finite strip analysis of cracked laminates. Mech. Mater. 18, 289–311.

Li, S., Singh, C.V., Talreja, R., 2009. A representative volume element based on translational symmetries for FE analysis of cracked laminates with two arrays of cracks. Int. J. Solids Struct. 46, 1793–1804.

McCartney, L.N., 1992. Theory of stress-transfer in a 0°–90°–0° cross-ply laminate containing a parallel array of transverse cracks. J. Mech. Phys. Solids 40, 27–68.

McCartney, L.N., 1996. Stress transfer mechanics for ply cracks in general symmetric laminates. NPL report CMMT (A) 50.

McCartney, L.N., 2000. Model to predict effects of triaxial loading on ply cracking in general symmetric laminates. Compos. Sci. Technol. 60, 2255–2279.

McCartney, N.L., 2003. Physically based damage models for laminated composites. In: Proc. Instn. Mech. Engrs. Part L: J. Materials: Design & Applications. 217(3), pp. 163–199.

McCartney, L.N., 2005. Energy-based prediction of progressive ply cracking and strength of general symmetric laminates using an homogenisation method. Composites Part A 36, 119–128.

McCartney, L.N., Pierse, S., 1997. Stress transfer mechanics for multiple ply laminates subject to bending. NPL report CMMT (A) 55.

Nairn, J.A., 1989. The strain energy release rate of composite microcracking: a variational approach. J. Compos. Mater. 23 (11), 1106–1129.

Nairn, J.A., 2000a. Matrix microcracking in composites. In: Kelly, A., Zweben, C., Talreja, R., Manson, J.-A. (Eds.), Polymer Matrix Composites. Comprehensive Composite Materials, Vol. 2. Elsevier, pp. 403–432.

Nairn, J.A., 2000b. Fracture mechanics of composites with residual stresses, imperfect interfaces, and traction-loaded cracks. Compos. Sci. Technol. 61, 2159–2167.

Nairn, J.A., Hu, S., 1992. The formation and effect of outer-ply microcracks in cross-ply laminates: a variational approach. Eng. Fract. Mech. 41 (2), 203–221.

Nairn, J.A., Mendels, D.A., 2001. On the use of planar shear-lag methods for stress-transfer analysis of multilayered composites. Mech. Mater. 33 (6), 335–362.

Rebiere, J.-L., Maatallah, M.-N., Gamby, D., 2001. Initiation and growth of transverse and longitudinal cracks in composite cross-ply laminates. Compos. Struct. 53, 173–187.

Simmons, G., 1972. Differential Equations with Applications and Historical Notes. McGraw-Hill, New York.

Singh, C.V., Talreja, R., 2008. Analysis of multiple off-axis ply cracks in composite laminates. Int. J. Solids Struct. 45, 4574–4589.
Singh, C.V., Talreja, R., 2010. Evolution of ply cracks in multidirectional composite laminates. Int. J. Solids Struct. 47, 1338–1349.
Talreja, R., 1985. Transverse cracking and stiffness reduction in composite laminates. J. Compos. Mater. 19 (4), 355–375.
Varna, J., Berglund, L.A., 1992. A model for prediction of the transverse cracking strain in cross-ply laminates. J. Reinf. Plast. Compos. 11, 708–728.
Vinogradov, V., Hashin, Z., 2005. Probabilistic energy based model for prediction of transverse cracking in cross-ply laminates. Int. J. Solids Struct. 42, 365–392.
Vinogradov, V., Hashin, Z., 2010. Variational analysis of cracked angle-ply laminates. Compos. Sci. Technol. 70, 638–646.
Yokozeki, T., Aoki, T., 2005. Overall thermoelastic properties of symmetric laminates containing obliquely crossed matrix cracks. Compos. Sci. Technol. 65 (11–12), 1647–1654.
Yuan, F.G., Selek, M.C., 1993. Transverse cracking and stiffness reduction in composite laminates. J. Reinf. Plast. Compos. 12, 987–1015.
Zhang, H., Minnetyan, L., 2006. Variational analysis of transverse cracking and local delamination in $[\theta_m/90_n]_s$ lamiantes. Int. J. Solids Struct. 43, 7061–7081.