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The scrambling index of symmetric primitive matrices ☆

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ABSTRACT

A nonnegative square matrix A is primitive if some power $A^k > 0$ (that is, A^k is entrywise positive). The least such k is called the exponent of A. In [2], Akelbek and Kirkland defined the scrambling index of a primitive matrix A, which is the smallest positive integer k such that any two rows of A^k have at least one positive element in a coincident position. In this paper, we give a relation between the scrambling index and the exponent for symmetric primitive matrices, and determine the scrambling index set for the class of symmetric primitive matrices. We also characterize completely the symmetric primitive matrices in this class such that the scrambling index is equal to the maximum value.

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1. Introduction

A nonnegative square matrix A is primitive if some power $A^k > 0$ (that is, A^k is entrywise positive). The least such k is called the exponent of A, denoted by $\exp(A)$. In [2], by using Seneta's [9] definition of coefficients of ergodicity, Akelbek and Kirkland provided an attainable upper bound on the second largest moduli of eigenvalues of a primitive matrix that makes use of the so-called scrambling index. The scrambling index of a primitive matrix A is the smallest positive integer k such that any two rows of A^k have at least one positive element in a coincident position, and denoted by $k(A)$.

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In the study of exponents, the maximum index problem (MIP), the extremal matrix problem (EMP) and the index set problem (ISP) are the three main problems. For surveys on the exponents of various classes of primitive matrices, see [7].

The scrambling index gives another characterization of primitivity. For a primitive matrix A, by the definition of the scrambling index and the exponent it is easy to see that $k(A) \leqslant \exp(A)$. It is natural that we consider the relation between $\exp(A)$ and $k(A)$, the estimation and evaluation of $k(A)$, and the corresponding MIP, EMP and ISP for scrambling indices for various classes of primitive matrices.

It is well known that graph theoretical methods are often useful in the study of the powers of square matrices, so we now introduce some graph theoretical concepts.

Let $D = (V, E)$ denote a digraph on n vertices. Loops are permitted, but no multiple arcs. A $u \to v$ walk in D is a sequence of vertices $u, u_1, \ldots, u_p = v$ and a sequence of arcs $(u, u_1), (u_1, u_2), \ldots, (u_{p-1}, v)$, where the vertices and the arcs are not necessarily distinct. A closed walk is a $u \to v$ walk, where $u = v$. A path is a walk with distinct vertices. A cycle is a closed $u \to v$ walk with distinct vertices except for $u = v$. The length of a walk W is the number of arcs in W, and denoted by $|W|$. The length of a shortest cycle in D is called the girth of D. The notation $u \xrightarrow{k} v$ (resp. $u \overset{k}{\nrightarrow} v$) is used to indicate that there is a $u \to v$ walk (resp. no $u \to v$ walk) of length k. The distance from vertex u to vertex v in D, is the length of a shortest walk from u to v, and denoted by $d(u, v)$. The diameter of D is $\max\{d(u, v) | u, v \in V(D)\}$. A p-cycle is a cycle of length p, denoted by C_p.

A digraph D is strongly connected if there is a $u \to v$ path for each pair u, v of vertices of D. A digraph D is primitive if there exists some positive integer k such that $u \xrightarrow{k} v$ for every vertex u and vertex v (not necessarily distinct) of D. The smallest such k is called the exponent of D, denoted by $\exp(D)$. For any pair of vertices u and v (not necessarily distinct) of D, the local exponent from u to v, denoted by $\exp_D(u, v)$, is the least integer k such that $u \xrightarrow{m} v$ for every $m \geqslant k$. It is clear that

$$\exp(D) = \max\{\exp_D(u, v) | \, u, \ v \in V(D)\}. \tag{1.1}$$

It is well known (see, e.g. [3]) that a digraph D is primitive if and only if D is strongly connected and the greatest common divisor of the lengths of its cycles is 1. The scrambling index of a primitive digraph D is the smallest positive integer k such that for every pair of vertices u and v, there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D, it is denoted by $k(D)$. For two distinct vertices $u, v \in V(D)$, the local scrambling index of u and v is the number

$$k_{u,v}(D) = \min\{k | \, u \xrightarrow{k} w \text{ and } v \xrightarrow{k} w, \text{ for some } w \in V\}.$$

Clearly,

$$k(D) = \max\{k_{u,v}(D) | u, v \in V(D), u \neq v\}. \tag{1.2}$$

It is easy to see a nonnegative square matrix A is primitive if and only if its associated digraph $D(A)$ is primitive, and in this case we have

$$\exp(A) = \exp(D(A)) \quad \text{and} \quad k(A) = k(D(A)).$$

Akelbek and Kirkland's definition of the scrambling index is the same as Cho and Kim's [5] definition of the competition index in the case of primitive digraphs. The two research groups started from different point, but got the results almost at the same time. Their achievements are widely applied to stochastic matrices and food webs. For details, see, e.g. [1,2,5,6].

In [2], Akelbek and Kirkland gave the best upper bound of scrambling index in terms of the order n and the girth s of a primitive digraph, and settled the MIP and EMP for the scrambling index for the class of all primitive matrices of order n. In [1], Akelbek and Kirkland characterized all the primitive matrices in the class of all primitive matrices whose associated digraphs having n vertices and girth s, such that the scrambling index is equal to the upper bound. Namely, Akelbek and Kirkland settled the corresponding EMP.

In [10], Shao settled the ISP and EMP for the exponent for the class of symmetric primitive matrices with order n. In [8], Liu et al. settled the ISP and EMP for the exponent for the class of symmetric primitive matrices with order n and zero trace. The associated digraph of a symmetric matrix is a symmetric digraph, namely, a digraph such that for any vertices u and v, (u, v) is an arc if and only if (v, u) is an arc. An undirected graph (possibly with loops) can be regarded as a symmetric digraph. It is well known (see, e.g. [4,10]) that an undirected graph G is primitive if and only if G is connected and has at least one odd cycle; namely, G is a connected nonbipartite graph.

The partitions of the set of all symmetric primitive matrices with order n are of two types:

(i) those in which the associated graphs of the symmetric primitive matrices have a cycle of length r but no cycle of any odd length less than r, where $1 \leqslant r \leqslant n$ and $r \equiv 1 (\text{mod } 2)$.

(ii) those in which the symmetric primitive matrices have exactly l positive diagonal entries, namely, the associated graphs of the symmetric primitive matrices have exactly l loops, where $0 \leqslant l \leqslant n$.

In this paper, we investigate the scrambling index of symmetric primitive matrices. Noting the correspondence between symmetric primitive matrices and primitive graphs, we will establish our results using graph theory.

Let n, l and r be integers with $n \geqslant 2$, $0 \leqslant l \leqslant n$, $1 \leqslant r \leqslant n$ and $r \equiv 1 (\text{mod } 2)$. Let $S_n(r)$ denote the set of all primitive graphs of order n having a cycle of length r but no cycle of any odd length less than r, and let $H_n(l)$ denote the set of all primitive graphs of order n having l loops. In Section 2, we give a relation between the exponent and the scrambling index for primitive symmetric digraphs. In Sections 3 and 4, we settle the MIP, EMP and ISP for the scrambling index for $S_n(r)$ and $H_n(l)$ respectively.

2. The scrambling index and the exponent

In this section we investigate the relation between $k(D)$ and $\exp(D)$ for a primitive symmetric digraph D.

Theorem 2.1. *Let D be any primitive symmetric digraph of order $n \geqslant 2$, and let u, v be any pair of vertices of D. Then*

$$k_{u,v}(D) \leqslant \left\lceil \frac{\exp_D(u, v)}{2} \right\rceil, \tag{2.1}$$

and

$$k(D) = \left\lceil \frac{\exp(D)}{2} \right\rceil, \tag{2.2}$$

where $\lceil a \rceil$ denotes the smallest integer not less than a.

Proof. By the definition of $\exp_D(u, v)$, we have

$$u \xrightarrow{\exp_D(u,v)} v \quad \text{and} \quad u \xrightarrow{\exp_D(u,v)+1} v.$$

If $\exp_D(u, v)$ is even, then there is a vertex w of D such that

$$u \xrightarrow{\frac{\exp_D(u,v)}{2}} w \xrightarrow{\frac{\exp_D(u,v)}{2}} v.$$

Hence

$$u \xrightarrow{\frac{\exp_D(u,v)}{2}} w \quad \text{and} \quad v \xrightarrow{\frac{\exp_D(u,v)}{2}} w.$$

By the definition of $k_{u,v}(D)$, we have

$$k_{u,v}(D) \leqslant \frac{\exp_D(u, v)}{2}.$$

If $\exp_D(u, v)$ is odd, then $\exp_D(u, v) + 1$ is even, and there is a vertex w of D such that

$$u \xrightarrow{\frac{\exp_D(u,v)+1}{2}} w \quad \text{and} \quad v \xrightarrow{\frac{\exp_D(u,v)+1}{2}} w.$$

Hence

$$k_{u,v}(D) \leqslant \frac{\exp_D(u,v)+1}{2}.$$

Therefore (2.1) holds.

We now prove (2.2). If $n = 2$, then it is not difficult to verify that $k(D) = 1$ and $\exp(D) = 1$ or 2. So (2.2) holds. Therefore in the following we assume that $n \geqslant 3$.

By (1.1), (1.2) and (2.1), we have

$$\begin{aligned} k(D) =\ & \max\{k_{u,v}(D) | u, v \in V(D), u \neq v\} \\ & \leqslant \max\left\{\left\lceil \frac{\exp_D(u,v)}{2} \right\rceil | u, v \in V(D), u \neq v\right\} \\ & \leqslant \max\left\{\left\lceil \frac{\exp_D(x,y)}{2} \right\rceil | x, y \in V(D)\right\} \\ & = \left\lceil \frac{\max\{\exp_D(x,y) | x, y \in V(D)\}}{2} \right\rceil = \left\lceil \frac{\exp(D)}{2} \right\rceil. \end{aligned} \tag{2.3}$$

We now show that

$$k(D) \geqslant \left\lceil \frac{\exp(D)}{2} \right\rceil. \tag{2.4}$$

By the definition of $\exp(D)$, we know that there exist $x, y \in V(D)$ (perhaps $x = y$) such that

$$x \overset{\exp(D)-1}{\not\to} y \tag{2.5}$$

Suppose that

$$k(D) < \left\lceil \frac{\exp(D)}{2} \right\rceil.$$

If $\exp(D)$ is odd, then clearly $x \neq y$ and $\frac{\exp(D)-1}{2} \geqslant k(D) \geqslant k_{x,y}(D)$. Thus

$$x \xrightarrow{2k_{x,y}(D)} y \xrightarrow{\left(\frac{\exp(D)-1}{2} - k_{x,y}(D)\right)2} y,$$

and hence $x \xrightarrow{\exp(D)-1} y$. This contradicts (2.5).

If $\exp(D)$ is even, then $\frac{\exp(D)}{2} - k(D) - 1 \geqslant 0$. Let $w \in V(D)\backslash\{x, y\}$ be a neighbor of x or y (which exists, since $n \geqslant 3$ and D is connected). If w is a neighbor of y, then $w \xrightarrow{1} y$. Since

$$\frac{\exp(D)}{2} - k_{x,w}(D) - 1 \geqslant \frac{\exp(D)}{2} - k(D) - 1 \geqslant 0,$$

we have

$$x \xrightarrow{2k_{x,w}(D)} w \xrightarrow{\left(\frac{\exp(D)}{2} - k_{x,w}(D) - 1\right)2} w.$$

Hence

$$x \xrightarrow{\exp(D)-2} w \xrightarrow{1} y,$$

and hence $x \xrightarrow{\exp(D)-1} y$, contradicting (2.5). Similarly we can get a contradiction if w is a neighbor of x.

Therefore in any case we have $k(D) \geqslant \left\lceil \frac{\exp(D)}{2} \right\rceil$, and (2.4) is proved. Combining (2.3) and (2.4), we obtain (2.2).

This completes the proof of the theorem. □

The next theorem will be used in the subsequent two sections, which is contained in [8].

Theorem 2.2 [8]. *Let D be a primitive symmetric digraph, and let u and v be any pair of vertices of D. If $u \xrightarrow{k_1} v$ and $u \xrightarrow{k_2} v$, where $k_1 - k_2 \equiv 1 (\mathrm{mod}\ 2)$, then*

$$\exp_D(u, v) \leqslant \max\{k_1, k_2\} - 1.$$

3. The scrambling index of $S_n(r)$

In this section we investigate the scrambling index of $S_n(r)$. Let $D = (V, E)$ be a strongly connected digraph. For a vertex $u \in V$ and a set $X \subseteq V$, let $d(u, X) = \min\{d(u, x) | \ x \in X\}$. For $u \in X$, we define $d(u, X) = 0$. We first establish the following lemmas.

Lemma 3.1. *Let D be a primitive symmetric digraph, and let C_r be a cycle of odd length r in D. Let u, v be any pair of vertices of D. Then*

$$k_{u,v}(D) \leqslant \max\{d(u, V(C_r)), d(v, V(C_r))\} + \frac{r-1}{2}. \tag{3.1}$$

Proof. Let $w_1, w_2 \in V(C_r)$ such that $d(u, w_1) = d(u, V(C_r))$ and $d(v, w_2) = d(v, V(C_r))$(perhaps $w_1 = w_2$). Then w_1, w_2 divide C_r into two parts C', C''. So

$$u \xrightarrow{d(u,w_1)+|C'|+d(w_2,v)} v \quad \text{and} \quad u \xrightarrow{d(u,w_1)+|C''|+d(w_2,v)} v.$$

Set $m = \max\{d(u, w_1) + |C'| + d(w_2, v), d(u, w_1) + |C''| + d(w_2, v)\}$. Then clearly,

$$\begin{aligned} m &\leqslant d(u, w_1) + d(w_2, v) + |C| \\ &\leqslant 2\max\{d(u, V(C_r)), d(v, V(C_r))\} + r. \end{aligned}$$

Note that $|C'|$ and $|C''|$ have different parity since $|C| = r$ is odd. We have by Theorem 2.2 that

$$\exp_D(u, v) \leqslant m - 1 \leqslant 2\max\{d(u, V(C_r)), d(v, V(C_r))\} + r - 1.$$

Thus by (2.1) of Theorem 2.1 we obtain

$$k_{u,v}(D) \leqslant \left\lceil \frac{\exp_D(u, v)}{2} \right\rceil \leqslant \max\{d(u, V(C_r)), d(v, V(C_r))\} + \frac{r-1}{2},$$

as desired. □

Lemma 3.2. *Let n and r be integers with $r \equiv 1 (\mathrm{mod}\ 2)$ and $3 \leqslant r \leqslant n$. Let G be any primitive graph in $S_n(r)$, and let u, v be two distinct vertices of an odd cycle C_r of G. Then*

(i) *If $d(u, v) \equiv 0 (\mathrm{mod}\ 2)$, then $k_{u,v}(G) = \frac{d(u,v)}{2}$.*

(ii) *If $d(u, v) \equiv 1 (\mathrm{mod}\ 2)$, then $k_{u,v}(G) = \frac{r-d(u,v)}{2}$.*

Proof. Let u, v divide C_r into two parts C', C''. Without loss of generality, we may assume $|C'| < |C''|$. Then $|C'| = d(u, v)$ and $|C''| = r - |C'| = r - d(u, v)$, since C_r is a shortest odd length cycle of G. Let $w \in V(G)$ such that

$$u \xrightarrow{k_{u,v}(G)} w \quad \text{and} \quad v \xrightarrow{k_{u,v}(G)} w.$$

Then

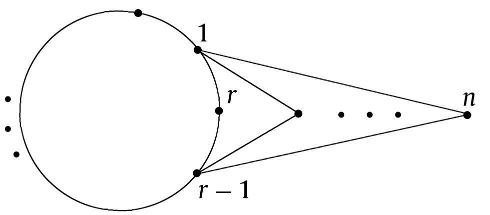

Fig. 1. $G^0_{n,r}$, $r \equiv 1(\mathrm{mod}\ 2)$ and $3 \leqslant r \leqslant n$.

$$u \xrightarrow{k_{u,v}(G)} w \xrightarrow{k_{u,v}(G)} v.$$

(i) Since C' is a shortest walk between u and v of even length $d(u,v)$, we can find a vertex $x \in V(C')$ such that

$$u \xrightarrow{\frac{d(u,v)}{2}} x \quad \text{and} \quad v \xrightarrow{\frac{d(u,v)}{2}} x.$$

Hence

$$k_{u,v}(G) \leqslant \frac{d(u,v)}{2}.$$

If $k_{u,v}(G) < \frac{d(u,v)}{2}$, then there is a walk between u and v with length $2k_{u,v}(G) < d(u,v)$, a contradiction. Thus $k_{u,v}(G) = \frac{d(u,v)}{2}$.

(ii) Since C'' is a walk between u and v of even length $r - d(u,v)$, we can find a vertex $x \in V(C'')$ such that

$$u \xrightarrow{\frac{r-d(u,v)}{2}} x \quad \text{and} \quad v \xrightarrow{\frac{r-d(u,v)}{2}} x.$$

Hence

$$k_{u,v}(G) \leqslant \frac{r-d(u,v)}{2}.$$

If $k_{u,v}(G) < \frac{r-d(u,v)}{2}$, then there is a closed walk $u \to w \to v \to u$ with odd length $2k_{u,v}(G) + d(u,v) < r$. This implies G has an odd cycle of length less than r, a contradiction. Thus $k_{u,v}(G) = \frac{r-d(u,v)}{2}$. $\square$

Lemma 3.3. *Let n and r be integers with $r \equiv 1(\mathrm{mod}\ 2)$ and $3 \leqslant r \leqslant n$. Let $G^0_{n,r}$ be the primitive graph in $S_n(r)$ as shown in Fig. 1. Then*

$$k(G^0_{n,r}) = \frac{r-1}{2}.$$

Proof. Let i,j be any pair of vertices of $G^0_{n,r}$. If $i,j \in \{r, r+1, \ldots, n\}$, then

$$k_{i,j}(G^0_{n,r}) = 1.$$

If $i \notin \{r, r+1, \ldots, n\}$ or $j \notin \{r, r+1, \ldots, n\}$, then there is an odd cycle C_r of length r such that $i,j \in V(C_r)$. By Lemma 3.2 we have

$$k_{i,j}(G^0_{n,r}) \leqslant k_{1,2}(G^0_{n,r}) = \frac{r-1}{2}.$$

Hence

$$k(G^0_{n,r}) = \max_{i,j \in V(G^0_{n,r})} \{k_{i,j}(G^0_{n,r})\} = \frac{r-1}{2}. \quad \square$$

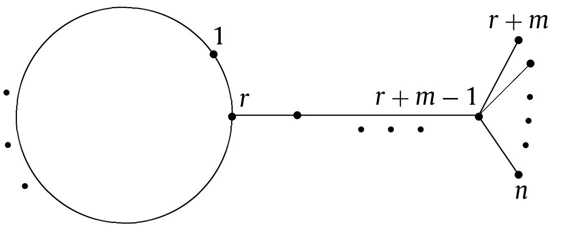

Fig. 2. $G^m_{n,r}$, $r \equiv 1(\mathrm{mod}\ 2)$, $1 \leqslant r \leqslant n-1$ and $1 \leqslant m \leqslant n-r$.

Lemma 3.4. *Let n, r and m be integers with $r \equiv 1(\mathrm{mod}\ 2)$, $1 \leqslant r \leqslant n-1$ and $1 \leqslant m \leqslant n-r$. Let $G^m_{n,r}$ be the primitive graph in $S_n(r)$ as shown in Fig. 2. Then*

$$k(G^m_{n,r}) = m + \frac{r-1}{2}. \tag{3.2}$$

Proof. Let W be a walk of odd length from the vertex n to n. Then W must contain an odd cycle of $G^m_{n,r}$. Note that C_r is the unique odd cycle of $G^m_{n,r}$, and $d(n, V(C_r)) = m$. Then $|W| \geqslant r + 2m$. This shows there is no walk of length $2m + r - 2$ from n to n, so

$$\exp(G^m_{n,r}) \geqslant 2m + r - 1.$$

Hence by (2.2) of Theorem 2.1 we have

$$k(G^m_{n,r}) = \left\lceil \frac{\exp(G^m_{n,r})}{2} \right\rceil \geqslant m + \frac{r-1}{2}.$$

On the other hand, since $\max_{u \in V(G^m_{n,r})}\{d(u, V(C_r))\} = m$, we have by (3.1) of Lemma 3.1 that

$$k_{u,v}(G^m_{n,r}) \leqslant m + \frac{r-1}{2}$$

for any pair of vertices u, v of $G^m_{n,r}$. So

$$k(G^m_{n,r}) \leqslant m + \frac{r-1}{2}.$$

Combining the above two relations, we obtain (3.2), as desired. □

Theorem 3.1. *Let n and r be integers with $n \geqslant 2$, $r \equiv 1(\mathrm{mod}\ 2)$ and $1 \leqslant r \leqslant n$, and let*

$$\delta_r = \begin{cases} 1 & \text{for } r = 1, \\ \frac{r-1}{2} & \text{for } r \equiv 1(\mathrm{mod}\ 2) \text{ and } r \geqslant 3. \end{cases}$$

Let G be any primitive graph in $S_n(r)$. Then

$$k(G) \geqslant \delta_r,$$

and this bound can be attained.

Proof. By definition we know that $k(G) \geqslant 1$. If $r = 1$, then we can use Lemma 3.4 to obtain for $m = 1$ that

$$k(G^1_{n,1}) = 1.$$

If $r \equiv 1(\mathrm{mod}\ 2)$ and $r \geqslant 3$, then by Lemma 3.3 we have that

$$k(G^0_{n,r}) = \frac{r-1}{2}.$$

Now let C_r be an r-cycle of G. Then there are two vertices $u, v \in V(C_r)$ with $d(u, v) = 1$. Hence by (ii) of Lemma 3.2 we have

$$k(G) \geqslant k_{u,v}(G) = \frac{r-1}{2}.$$

The theorem now follows. □

Theorem 3.2. *Let n be an integer with $n \geqslant 2$, and let r be an odd integer with $1 \leqslant r \leqslant n$. Let G be any primitive graph in $S_n(r)$. Then*

$$k(G) \leqslant n - \frac{r+1}{2}.$$

Equality holds if and only if

(i) $n \geqslant 3$ and G is isomorphic to $G_{n,r}^{n-r}$, or

(ii) $n = 2$ (and so $r = 1$) and either G is isomorphic to $G_{2,1}^1$ or G is isomorphic to the graph $G'_{2,1}$ obtained from $G_{2,1}^1$ by adding a loop to its other vertex.

Proof. Let C_r be an r-cycle of G. Then $\max_{i \in V(G)} d(i, C_r) \leqslant n - r$. It follows from Lemma 3.1 that

$$k(G) = \max_{i,j \in V(G)} k_{i,j}(G) \leqslant n - r + \frac{r-1}{2} = n - \frac{r+1}{2}.$$

If $1 \leqslant r \leqslant n-1$ and G is isomorphic to $G_{n,r}^{n-r}$, then by Lemma 3.4 we have that

$$k(G) = k(G_{n,r}^{n-r}) = n - \frac{r+1}{2}.$$

If $r = n \equiv 1 (\mathrm{mod}\ 2)$ and G is isomorphic to $G_{n,n}^0$, then by Lemma 3.3 we have that

$$k(G) = k(G_{n,n}^0) = \frac{n-1}{2} = n - \frac{n+1}{2}.$$

If $n = 2$ and G is the graph $G'_{2,1}$ in (ii), then it is easy to see that

$$k(G) = 1 = 2 - \frac{1+1}{2}.$$

Now let G be a graph in $S_n(r)$ and assume that $k(G) = n - \frac{r+1}{2}$. First suppose that $n \geqslant 3$. Let C_r be an r-cycle of G. Then

$$\max_{i \in V(G)} \{d(i, C_r)\} \leqslant n - r.$$

Also by Lemma 3.1 we have that

$$n - \frac{r+1}{2} = k(G) = \max_{u,v \in V(G)} k_{u,v}(G) \leqslant \max_{i \in V(G)} \{d(i, C_r)\} + \frac{r-1}{2}.$$

Hence

$$\max_{i \in V(G)} \{d(i, C_r)\} = n - r,$$

and hence G contains a spanning subgraph G^* isomorphic to $G_{n,r}^{n-r}$.

We now show that G^* equals G. If $r = n \equiv 1 (\mathrm{mod}\ 2)$, then G^* is a cycle C_n of odd length n. Notice that C_n is a shortest odd length cycle of G, so $G = G^*$ and G is isomorphic to $G_{n,n}^0$. If $r \leqslant n-1$, then there exists a vertex u of G and a vertex v of C_r such that $d(u, v) = d(u, C_r) = n - r \geqslant 1$. Let P be a shortest path between u and v. Then $|P| = n - r$ and P is an induced subgraph of G. Since C_r is a shortest odd length cycle of G, C_r is also an induced subgraph of G. Suppose that there is an edge e of G, but not of G^*, which joins a vertex x of P and a vertex y of C_r, where $x \neq y$. Then clearly, $d(x,v) = 1$ and $y \neq v$. Let the distinct two vertices y and v divide C_r into two parts C' and C''. Then $|C'|$ and $|C''|$ have different parity. Without loss of generality, we may assume that $|C'|$ is odd. Then $1 \leqslant |C'| \leqslant r-2$. Hence, there is an odd cycle C of G with length $|C| \leqslant (r-2)+1+1$, which contains the three distinct vertices v, x and y. Since G has no cycle of odd length less than r, we conclude $|C| = r$. Thus

$$\max_{i \in G}\{d(i,C)\} \leqslant \max\{1, n-r-1\}.$$

If $\max_{i\in G}\{d(i,C)\} = n-r-1 \geqslant 1$, then by Lemma 3.1 we obtain the contradiction

$$k(G) = \max_{i,j\in V(G)} k_{i,j}(G) \leqslant n-r-1+\frac{r-1}{2} < n-\frac{r+1}{2}.$$

If $\max_{i\in G}\{d(i,C)\} = 1 > n-r-1$, then $n = r+1$, and G contains a spanning subgraph G' isomorphic to $G^0_{n,n-1}$. Hence by Lemma 3.3 we obtain the contradiction

$$k(G) \leqslant k(G') = \frac{(n-1)-1}{2} < n - \frac{(n-1)+1}{2}.$$

Finally, suppose that there is a loop of G which is not an edge of G^*. Then $r = 1$ and it follows that $|C_r| = 1$ implying that there is at least one vertex $w \in V(G)\backslash\{v\}$ with a loop. Hence

$$\max_{i\in V(G)\backslash\{v,w\}}\{d(i,v), d(i,w)\} \leqslant n-2.$$

Applying Lemma 3.1 we have that

$$k_{i,j}(G) \leqslant n-2 \text{ for } i, j \in V(G)\backslash\{v,w\},$$
$$k_{i,v}(G) \leqslant n-2 \text{ for } i \in V(G)\backslash\{v,w\},$$

and

$$k_{i,w}(G) \leqslant n-2 \text{ for } i \in V(G)\backslash\{v,w\}.$$

Also we have $k_{v,w}(G) \leqslant \left\lceil \frac{d(v,w)}{2} \right\rceil \leqslant \left\lceil \frac{n-1}{2} \right\rceil \leqslant n-2$ since $n \geqslant 3$. Thus

$$k(G) = \max_{i,j\in V(G)}\{k_{i,j}(G)\} \leqslant n-2 < n-\frac{1+1}{2},$$

a contradiction because $r = 1$. Therefore $G = G^*$ and G is isomorphic to $G^{n-r}_{n,r}$.

We now suppose that $n = 2$ and hence that $r = 1$. Then there are exactly two graphs $G^1_{2,1}$ and $G'_{2,1}$ in $S_2(1)$. Since $k(G^1_{2,1}) = k(G'_{2,1}) = 1$, G satisfies (ii) of the theorem. The theorem now follows. □

Theorem 3.3. *Let n be an integer with $n \geqslant 2$, and let r be an odd integer with $1 \leqslant r \leqslant n$. Let $K(n,r) = \{k(G) | G \in S_n(r)\}$ be the scrambling index set for the class $S_n(r)$. Then*

$$K(n,r) = \{\delta_r, \delta_r+1, \ldots, n-\frac{r+1}{2}\},$$

where the expression for δ_r is given in Theorem 3.1.

Proof. Take an integer m with $1 \leqslant m \leqslant n-r$, and let $G = G^m_{n,r}$ as in Fig. 2. Then by Lemma 3.4 we have that

$$m + \frac{r-1}{2} \in K(n,r).$$

So

$$\left\{\frac{r-1}{2}+1, \frac{r-1}{2}+2, \ldots, \frac{r-1}{2}+n-r\right\} \subseteq K(n,r).$$

From this and the fact $\delta_r \in K(n,r)$, it now follows that

$$\{\delta_r, \delta_r + 1, \ldots, n - \frac{r+1}{2}\} \subseteq K(n,r).$$

On the other hand, by Theorem 3.1 and Theorem 3.2 we obviously have

$$K(n,r) \subseteq \{\delta_r, \delta_r + 1, \ldots, n - \frac{r+1}{2}\}.$$

Therefore, $K(n,r) = \left\{\delta_r, \delta_r + 1, \ldots, n - \frac{r+1}{2}\right\}$. This completes the proof of the theorem. □

4. The scrambling index of $H_n(l)$

In this section we continue with the notation of the previous section but now we investigate the scrambling index of $H_n(l)$. Since the case $n = 2$ and the case $0 \leqslant l \leqslant 1$ were already settled in Section 3, we will only consider the remaining case $n \geqslant 3$ and $2 \leqslant l \leqslant n$.

Lemma 4.1. *Let G be any primitive graph in $H_n(l)$, and let u and v be any pair of vertices of G. Let P be a shortest walk between u and v, and let w be a vertex with a loop such that*

$$d(w, V(P)) = \min\{d(x, V(P)) \mid x \text{ is a vertex with a loop, } x \in V(G)\}.$$

Then

(i) *If $d(w, V(P)) = 0$, then*

$$k_{u,v}(G) \leqslant \left\lceil \frac{|P|}{2} \right\rceil \leqslant \left\lceil \frac{n-1}{2} \right\rceil.$$

(ii) *If $d(w, V(P)) > 0$, then*

$$k_{u,v}(G) \leqslant d(w, V(P)) + \left\lceil \frac{|P|}{2} \right\rceil \leqslant n - l - |P| + \left\lceil \frac{|P|}{2} \right\rceil \leqslant n - l.$$

Proof. (i) If $d(w, V(P)) = 0$, then

$$u \xrightarrow{|P|} v \quad \text{and} \quad u \xrightarrow{|P|+1} v.$$

It follows from Theorem 2.2 that

$$\exp_G(u,v) \leqslant |P|,$$

and hence by (2.1) of Theorem 2.1 that

$$k_{u,v}(G) \leqslant \left\lceil \frac{\exp_D(u,v)}{2} \right\rceil \leqslant \left\lceil \frac{|P|}{2} \right\rceil \leqslant \left\lceil \frac{n-1}{2} \right\rceil.$$

(ii) If $d(w, V(P)) > 0$, then $d(w, V(P)) \leqslant n - l - |P|$, and

$$u \xrightarrow{|P|+2d(w,V(P))} v \quad \text{and} \quad u \xrightarrow{|P|+2d(w,V(P))+1} v.$$

It follows from Theorem 2.2 that

$$\exp_G(u,v) \leqslant |P| + 2d(w, V(P)).$$

Hence by (2.1) of Theorem 2.1 we have

$$k_{u,v}(G) \leqslant d(w, V(P)) + \left\lceil \frac{|P|}{2} \right\rceil \leqslant n - l - |P| + \left\lceil \frac{|P|}{2} \right\rceil \leqslant n - l.$$

The lemma now follows. □

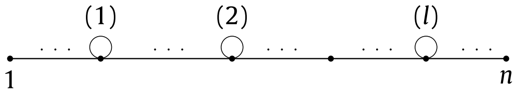

Fig. 3. $G \in H_n(l, n-1), n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$.

Now we construct some subsets of $H_n(l)$ as follows.

For $1 \leqslant d \leqslant n-1$, we define

$$H_n(l, d) = \{G|\text{ the diameter of } G \text{ is } d, G \in H_n(l)\}.$$

For a path P and a graph $G \in H_{m+1}(m)$, the graph $P + G$ is obtained by identifying one of the end-vertices of P with the vertex of G with no loop (Notice that the vertex of G with no loop is unique). We define

$$H_n^{n-l}(l, *) = \{P + G|\, |P| = n - l - 1,\ G \in H_{l+1}(l)\}.$$

For $l = n-2$, we define

$$\begin{aligned} H_n^0(n-2) = \{G|\, G \in H_n(n-2),\ &\text{the two vertices of } G \text{ with no loop} \\ &\text{are neighboring and have no common neighbor}\}. \end{aligned}$$

Lemma 4.2. *Let n and l be integers with $n \geqslant 3$ and $n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$, and let G be any primitive graph in $H_n(l, n-1)$ (see Fig. 3). Then*

$$k(G) = \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.1}$$

Proof. By Lemma 4.1 we have that

$$k_{u,v}(G) \leqslant \max\left\{n - l, \left\lceil \frac{n-1}{2} \right\rceil\right\}$$

for any pair of vertices u, v of G. Since $n - l \leqslant \left\lceil \frac{n-1}{2} \right\rceil$, it follows that

$$k(G) \leqslant \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.2}$$

On the other hand, it is obvious that $1 \overset{n-2}{\nrightarrow} n$. So

$$\exp(G) \geqslant \exp_G(1, n) \geqslant n - 1,$$

and so

$$k(G) = \left\lceil \frac{\exp(G)}{2} \right\rceil \geqslant \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain (4.1). □

Lemma 4.3. *Let n and l be integers with $n \geqslant 3, n \equiv 1 (\bmod 2)$ and $n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$. Let G be any primitive graph in $H_n(l, n-2)$ (see Fig. 4). Then*

$$k(G) = \left\lceil \frac{n-1}{2} \right\rceil = \frac{n-1}{2}. \tag{4.4}$$

Proof. Note that if $n \equiv 1 (\bmod 2)$, then $\left\lceil \frac{n-2}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil = \frac{n-1}{2}$. The proof of this lemma is similar to Lemma 4.2, we omit it. □

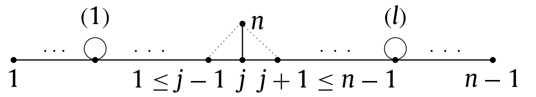

Fig. 4. $G \in H_n(l, n-2)$, n is odd and $n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$.

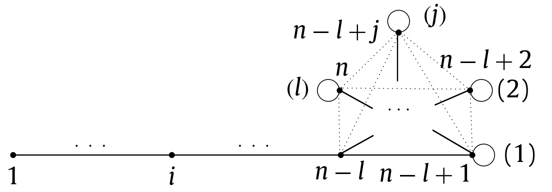

Fig. 5. $G \in H_n^{n-l}(l, *)$, $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil$.

Lemma 4.4. *Let n and l be integers with $n \geqslant 3$ and $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil$. Let G be any primitive graph in $H_n^{n-l}(l, *)$ (see Fig. 5). Then*

$$k(G) = n - l. \tag{4.5}$$

Proof. By Lemma 4.1 we have that

$$k_{u,v}(G) \leqslant \max\left\{n - l, \left\lceil \frac{n-1}{2} \right\rceil\right\}$$

for any pair of vertices u, v of G. Since $n - l \geqslant \left\lceil \frac{n-1}{2} \right\rceil$, it follows that

$$k(G) \leqslant n - l. \tag{4.6}$$

On the other hand, let W be any walk of odd length from the vertex 1 to 1. Then W must contain an odd cycle C of G. If $|C| = 1$, then $|W| \geqslant 2(n-l) + |C| = 2(n-l) + 1$. If $|C| \geqslant 3$, then $|W| \geqslant 2(n-l-1) + |C| \geqslant 2(n-l-1) + 3 = 2(n-l) + 1$. Thus in any case we have $|W| \geqslant 2(n-l) + 1$. This shows

$$1 \overset{2(n-l)-1}{\nrightarrow} 1.$$

Hence

$$\exp(G) \geqslant \exp_G(1, 1) \geqslant 2(n - l),$$

and hence

$$k(G) = \left\lceil \frac{\exp(G)}{2} \right\rceil \geqslant n - l. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain (4.5), as desired. $\square$

Lemma 4.5. *Let n and l be integers with $4 \leqslant n \leqslant 5$ and $l = n - 2$, and let G be any primitive graph in $H_n^0(n-2)$. Then*

$$k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n - l = 2. \tag{4.8}$$

Proof. By Lemma 4.1 we have that

$$k_{u,v}(G) \leqslant \max\left\{n - l, \left\lceil \frac{n-1}{2} \right\rceil\right\}$$

for any pair of vertices u, v of G. Since $4 \leqslant n \leqslant 5$ and $l = n - 2$, $\left\lceil \frac{n-1}{2} \right\rceil = n - l = 2$. So

$$k(G) \leqslant n - l = \left\lceil \frac{n-1}{2} \right\rceil = 2. \tag{4.9}$$

On the other hand, let x and y be the two vertices of G with no loop. By the definition of $H_n^0(n-2)$, it is obvious that

$$x \overset{2}{\nrightarrow} y.$$

Hence

$$\exp(G) \geqslant \exp_G(x, y) \geqslant 3,$$

and hence

$$k(G) = \left\lceil \frac{\exp(G)}{2} \right\rceil \geqslant 2. \tag{4.10}$$

Combining (4.9) and (4.10), we obtain (4.8). □

Theorem 4.1. *Let G be any primitive graph in $H_n(l)$. Then*

$$k(G) \leqslant \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil, & \text{if } n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n, \\ n - l, & \text{if } 2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil, \end{cases} \tag{4.11}$$

and the following hold:

(i) *If $l = n - \left\lceil \frac{n-1}{2} \right\rceil$, then $k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n - l$ if and only if $G \in H_n(l, n-1) \cup H_n^{n-l}(l, *)$, or $G \in H_n(l, n-2)$ and n is odd, or $G \in H_4^0(2)$, or $G \in H_5^0(3)$.*

(ii) *If $n - \left\lceil \frac{n-1}{2} \right\rceil + 1 \leqslant l \leqslant n$, then $k(G) = \left\lceil \frac{n-1}{2} \right\rceil$ if and only if $G \in H_n(l, n-1)$, or $G \in H_n(l, n-2)$ and n is odd.*

(iii) *If $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil - 1$, then $k(G) = n - l$ if and only if $G \in H_n^{n-l}(l, *)$.*

Proof. Let u, v be any pair of vertices of G. Then by Lemma 4.1 we have

$$\begin{aligned} k(G) &= \max_{u,v \in V(G)} \{k_{u,v}(G)\} \\ &\leqslant \max \left\{ n - l, \left\lceil \frac{n-1}{2} \right\rceil \right\} \\ &= \begin{cases} \left\lceil \frac{n-1}{2} \right\rceil, & \text{if } n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n, \\ n - l, & \text{if } 2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil. \end{cases} \end{aligned}$$

Hence (4.11) holds.

Let $V(G) = \{1, 2, \ldots, n\}$, and let $1, i \in V(G)$ such that

$$k_{1,i}(G) = k(G).$$

Let $P_{1,i}$ be a shortest path between 1 and i, and let m be a vertex with a loop such that

$$d(m, V(P_{1,i})) = \min\{d(x, V(P_{1,i})) \mid x \text{ is a vertex with a loop, } x \in V(G)\}.$$

We now assume that $l = n - \left\lceil \frac{n-1}{2} \right\rceil$ and prove (i). First suppose that $k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n - l$. Then

$$k_{1,i}(G) = k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n - l. \tag{4.12}$$

We consider two cases:

Case 1: $d(m, V(P_{1,i})) = 0$. Then by (i) of Lemma 4.1 and (4.12) we have that

$$\left\lceil \frac{n-1}{2} \right\rceil = k_{1,i}(G) \leqslant \left\lceil \frac{|P_{1,i}|}{2} \right\rceil \leqslant \left\lceil \frac{n-1}{2} \right\rceil.$$

Hence $\left\lceil \frac{|P_{1,i}|}{2} \right\rceil = \left\lceil \frac{n-1}{2} \right\rceil$. Thus $|P_{1,i}| = n-1$, or $|P_{1,i}| = n-2$ and n is odd. Since $P_{1,i}$ is a shortest path between the vertex 1 and the vertex i, the diameter of G is not less than $|P_{1,i}|$. Therefore $G \in H_n(l, n-1)$, or $G \in H_n(l, n-2)$ and n is odd.

Case 2: $d(m, V(P_{1,i})) > 0$. Then by (ii) of Lemma 4.1 and (4.12) we have that

$$\begin{aligned} n-l = k_{1,i}(G) &\leqslant d(m, V(P_{1,i})) + \left\lceil \frac{|P_{1,i}|}{2} \right\rceil \\ &\leqslant n-l-|P_{1,i}| + \left\lceil \frac{|P_{1,i}|}{2} \right\rceil \leqslant n-l. \end{aligned}$$

Hence $|P_{1,i}| = \left\lceil \frac{|P_{1,i}|}{2} \right\rceil$ and $d(m, V(P_{1,i})) = n-l-|P_{1,i}|$. Thus

$$|P_{1,i}| = 1 \quad \text{and} \quad d(m, V(P_{1,i})) = n-l-1 = \left\lceil \frac{n-1}{2} \right\rceil - 1.$$

Without loss of generality, we may assume that

$$d(i,m) = d(m, V(P_{1,i})) = n-l-1 = \left\lceil \frac{n-1}{2} \right\rceil - 1, \; i = 2 \text{ and } m = n-l+1.$$

We consider two subcases:

Subcase 2.1: $d(m, V(P_{1,i})) = n-l-1 = 1$. Then $n = 4$ (and $l = 2$), or $n = 5$ (and $l = 3$). So G has a loop at each vertex in $V(G)\backslash\{1,2\}$, and there is no vertex in $\{1,2\}$ with loop. If there is a vertex j in $V(G)\backslash\{1,2\}$ such that j is a common neighbor of the vertices 1 and 2, then clearly $k_{1,2}(G) = 1$. This contradicts (4.12). Thus $G \in H_4^0(2)$ (when $n = 4$), or $G \in H_5^0(3)$ (when $n = 5$).

Subcase 2.2: $d(m, V(P_{1,i})) = n-l-1 \geqslant 2$. Then $\max\{1, n-l-2\} = n-l-2$. Let $P = 23\cdots(n-l)(n-l+1)$ be a shortest path between the vertex 2 and the vertex $n-l+1$. Then there is no vertex in $V(P)\backslash\{n-l+1\}$ with loop.

Let $P' = 123\cdots(n-l)$ is a path between the vertex 1 and the vertex $n-l$ (which exists, since $|P_{1,2}| = 1$). Then $|P'| = n-l-1$ and there is no vertex in $V(P')$ with loop. Suppose that there is an edge of G, but not of P', which joins two vertices of P'. Then $d(1, n-l) \leqslant |P'| - 1 = n-l-2$. So $d(1, n-l+1) \leqslant d(1, n-l) + 1 \leqslant n-l-1$, and so

$$k_{1,2}(G) \leqslant \max\{d(1, n-l+1),\, d(2, n-l+1)\} = n-l-1.$$

This contradicts (4.12). Thus P' is an induced subgraph of G.

Since G contains l loops, G has a loop at each vertex in $V(G)\backslash V(P') = \{n-l+1, n-l+2, \ldots, n\}$. Let G'_{l+1} be the subgraph of G induced by $\{n-l, n-l+1, \ldots, n\}$, and let j be any vertex of G'_{l+1}. Suppose that there is an edge of G, which joins j and a vertex in $V(P')\backslash\{n-l\}$. Then

$$d(j, V(P_{1,2})) \leqslant \max\{1, n-l-2\} = n-l-2.$$

Hence

$$\max\{d(1,j), d(2,j)\} \leqslant n-l-2+1 = n-l-1,$$

and hence $k_{1,2}(G) \leqslant n-l-1$. This contradicts (4.12). Thus we conclude that G'_{l+1} is connected since G is a primitive graph, and hence $G'_{l+1} \in H_{l+1}(l)$.

Therefore $G = P' + G'_{l+1} \in H_n^{n-l}(l, *)$.

If $G \in H_n(l, n-1)$, then by Lemma 4.2 we have $k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n-l$. If $G \in \cup H_n^{n-l}(l, *)$, then by Lemma 4.4 we have that $k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n-l$. If $G \in H_n(l, n-2)$ and n is odd, then by Lemma

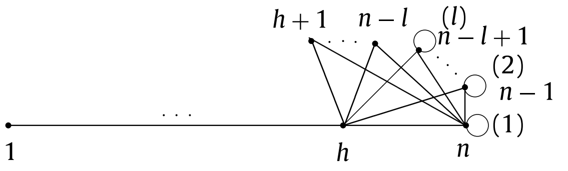

Fig. 6. $\Gamma^h_{n,l}$, $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil$ and $1 \leqslant h \leqslant n - l$.

4.3 we have that $k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n - l$. If $G \in H_4^0(2)$ or $G \in H_5^0(3)$, then by Lemma 4.5 we have that $k(G) = \left\lceil \frac{n-1}{2} \right\rceil = n - l = 2$. Therefore (i) holds.

We now assume that $n - \left\lceil \frac{n-1}{2} \right\rceil + 1 \leqslant l \leqslant n$ and prove (ii). Suppose that $k(G) = \left\lceil \frac{n-1}{2} \right\rceil$. Then

$$k_{1,i}(G) = k(G) = \left\lceil \frac{n-1}{2} \right\rceil. \tag{4.13}$$

If $d(m, V(P_{1,i})) > 0$, then by (ii) of Lemma 4.1 and (4.13) we have that

$$\left\lceil \frac{n-1}{2} \right\rceil = k(G) \leqslant n - l.$$

This contradicts the condition $n - \left\lceil \frac{n-1}{2} \right\rceil + 1 \leqslant l \leqslant n$. Thus $d(m, V(P_{1,i})) = 0$. The rest of proof is similar to (i) of this theorem, so we omit it.

We now assume that $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil - 1$ and prove (iii). Suppose that $k(G) = n - l$. Then

$$k_{1,i}(G) = k(G) = n - l. \tag{4.14}$$

If $d(m, V(P_{1,i})) = 0$, then by (i) of Lemma 4.1 and (4.14) we have that

$$n - l = k(G) \leqslant \left\lceil \frac{n-1}{2} \right\rceil.$$

This contradicts the condition $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil - 1$. Thus $d(m, V(P_{1,i})) > 0$. The rest of proof is similar to (i) of this theorem, so we omit it (Note that in this case $n \geqslant 5$ and $n - l - 1 \geqslant \left\lceil \frac{n-1}{2} \right\rceil \geqslant 2$).

This completes the proof of the theorem. □

Lemma 4.6. *Let n, l and h be integers with $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil$ and $1 \leqslant h \leqslant n - l$. Let $\Gamma^h_{n,l}$ be the primitive graph in $H_n(l)$ as shown in Fig. 6. Then*

$$k(\Gamma^h_{n,l}) = h. \tag{4.15}$$

Proof. Let W be any walk of odd length from the vertex 1 to 1. Then W must contain an odd cycle C of $\Gamma^h_{n,l}$. If $|C| = 1$, then $|W| \geqslant 2h + |C| = 2h + 1$. If $|C| \geqslant 3$, then $|W| \geqslant 2(h-1) + |C| \geqslant 2h + 1$. Thus in any case we have $|W| \geqslant 2h + 1$. This shows

$$1 \overset{2h-1}{\nrightarrow} 1.$$

Hence

$$\exp(\Gamma^h_{n,l}) \geqslant \exp_{\Gamma^h_{n,l}}(1, 1) \geqslant 2h,$$

and hence

$$k(\Gamma^h_{n,l}) = \left\lceil \frac{\exp(\Gamma^h_{n,l})}{2} \right\rceil \geqslant h. \tag{4.16}$$

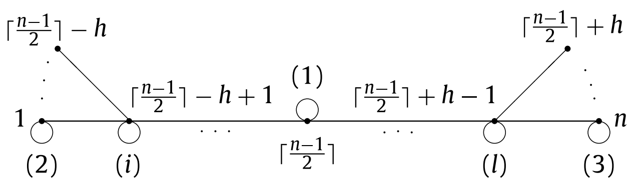

Fig. 7. $\Pi^h_{n,l}$, $n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$ and $1 \leqslant h \leqslant \left\lceil \frac{n-1}{2} \right\rceil$.

On the other hand, let u, v be any pair of vertices of $\Gamma^h_{n,l}$. Then

$$u \xrightarrow{d(u,n)} n \quad \text{and} \quad v \xrightarrow{d(v,n)} n.$$

Since $\max\{d(u,n),\ d(v,n)\} \leqslant h$ and there is a loop at the vertex n, we have

$$u \xrightarrow{h} n \quad \text{and} \quad v \xrightarrow{h} n.$$

So

$$k(\Gamma^h_{n,l}) = \max\{k_{u,v}(\Gamma^h_{n,l}) |\, u, v \in V(\Gamma^h_{n,l}), u \neq v\} \leqslant h. \tag{4.17}$$

Combining (4.16) and (4.17), we obtain (4.15). □

Lemma 4.7. *Let n, l and h be integers with $n \geqslant 3$, $n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$ and $1 \leqslant h \leqslant \left\lceil \frac{n-1}{2} \right\rceil$. Let $\Pi^h_{n,l}$ be the primitive graph in $H_n(l)$ as shown in Fig. 7. Then*

$$k(\Pi^h_{n,l}) = h. \tag{4.18}$$

Proof. Since $d(1, n) = 2h$, we have $\exp(\Pi^h_{n,l}) \geqslant 2h$. Hence

$$k(\Pi^h_{n,l}) = \left\lceil \frac{\exp(\Pi^h_{n,l})}{2} \right\rceil \geqslant h.$$

On the other hand, let u, v be any pair of vertices of $\Pi^h_{n,l}$. Note that there is a loop at the vertex $\left\lceil \frac{n-1}{2} \right\rceil$ and $\max\left\{d\left(u, \left\lceil \frac{n-1}{2} \right\rceil\right),\ d\left(v, \left\lceil \frac{n-1}{2} \right\rceil\right)\right\} \leqslant h$. We have

$$u \xrightarrow{h} \left\lceil \frac{n-1}{2} \right\rceil \quad \text{and} \quad v \xrightarrow{h} \left\lceil \frac{n-1}{2} \right\rceil.$$

So

$$k(\Pi^h_{n,l}) = \max\{k_{u,v}(\Pi^h_{n,l} |\, u, v \in V(\Pi^h_{n,l}), u \neq v\} \leqslant h.$$

Combining the above two relations, we obtain (4.18). □

Theorem 4.2. *Let n and l be integers with $n \geqslant 3$ and $2 \leqslant l \leqslant n$. Let $K^*(n,l) = \{k(G) |\, G \in H_n(l)\}$ be the scrambling index set for the class $H_n(l)$. Then*

(i) *If $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil$, then*

$$K^*(n,l) = \{1, 2, \ldots, n - l\}.$$

(ii) *If $n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$, then*

$$K^*(n,l) = \left\{1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\right\}.$$

Proof. We first assume that $2 \leqslant l \leqslant n - \left\lceil \frac{n-1}{2} \right\rceil$ and prove (i). Take an integer h with $1 \leqslant h \leqslant n-l$, and let $G = \Gamma_{n,l}^h$ as in Fig. 6. Then by Lemma 4.6 we have $h \in K^*(n,l)$. Hence

$$\{1, 2, \ldots, n-l\} \subseteq K^*(n,l).$$

On the other hand, by Theorem 4.1 we obviously have

$$K^*(n,l) \subseteq \{1, 2, \ldots, n-l\}.$$

Therefore $K^*(n,l) = \{1, 2, \ldots, n-l\}$.

We now assume that $n - \left\lceil \frac{n-1}{2} \right\rceil \leqslant l \leqslant n$ and prove (ii). Take an integer h with $1 \leqslant h \leqslant \left\lceil \frac{n-1}{2} \right\rceil$, and let $G = \Pi_{n,l}^h$ as in Fig. 7. Then by Lemma 4.7 we have $h \in K^*(n,l)$. Hence

$$\left\{1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\right\} \subseteq K^*(n,l).$$

On the other hand, by Theorem 4.1 we obviously have

$$K^*(n,l) \subseteq \left\{1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\right\}.$$

Therefore $K^*(n,l) = \left\{1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil\right\}$. The theorem now follows. $\square$

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