Lower bounds for codes correcting moderate-density bursts of fixed length with Lee weight consideration

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Abstract

Lee weight is more appropriate for some practical situations than Hamming weight as it takes into account magnitude of each digit of the word. In this paper, considering Lee weight, we obtain necessary lower bound over the number of parity checks to correct bursts of length $b$ (fixed) whose weight lies between certain limits. We also obtain Lee weight bound for such type of moderate-density bursts with limited intensity. © 2006 Elsevier Inc. All rights reserved.

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1. Introduction

It is well known that during the process of transmission errors occur predominantly in the form of bursts. Burst error correcting codes are developed to protect clustered errors over a particular length. However, it does not generally happen that all the digits inside any burst length get corrupted. Also, when burst length is large, the actual number of errors inside the burst length is not very less. This requires the study of codes which could deal with bursts having moderate density. Such bursts may be termed as moderate-density bursts. Codes developed to detect and
correct such errors have been studied by many authors. However, most of the studies in this direction have been made with respect to Hamming weight/distance [1] of a code. A generalized distance introduced by Lee [6] is more suitable for phase modulation [1] as it takes into account the magnitude of the changes. Also, in the literature, various kinds of burst errors have been studied viz. open loop bursts [5, 7, 8], closed loop bursts [2], CT bursts [3], low-density bursts [9]. One important kind of burst errors which has not drawn much attention is burst of specified length $b$ (fixed) [4]. In this paper, we derive necessary bound for linear codes that correct moderate-density bursts of length $b$ (fixed) ($b \geq 1$, an integer) with Lee weight consideration.

In what follows, we consider the following:

Let $Z_q$ be the ring of integers modulo $q$. Let $V_q^n$ be the set of all $n$-tuples over $Z_q$. Then $V_q^n$ is a module over $Z_q$. Let $V$ be a submodule of the module $V_q^n$ over $Z_q$. For $q$ prime, $Z_q$ becomes a field and correspondingly $V_q^n$ and $V$ become the vector space and subspace respectively over the field $Z_q$. Also, we define the modular value $|a|$ of an element $a \in Z_q$ by

$$|a| = \begin{cases} 
  a & \text{if } 0 \leq a \leq q/2, \\
  q - a & \text{if } q/2 < a \leq q - 1,
\end{cases}$$

and then for a given vector $u = (a_0, a_1, \ldots, a_{n-1}), a_i \in Z_q$, the Lee weight $w_L(u)$ of $u$ is given by

$$w_L(u) = \sum_{i=0}^{n-1} |a_i|.$$ 

In determining the Lee weight of a vector, a nonzero entry $a$ has a contribution $|a|$ which is obtained by two different entries $a$ and $q - a$ provided \{q is odd\} or \{q is even and $a \neq q/2$, i.e.,

$$|a| = |q - a|$$

if $q$ is odd or

$$|a| = |q - a|$$

if $q$ is even and $a \neq q/2$.

If $q$ is even and $a = q/2$ or if $a = 0$, then $|a|$ is obtained in only one way viz. $|a| = a$.

Thus, for the Lee weight, there may be one or two equivalent values of $|a|$ which we shall refer to as repetitive equivalent values of $a$. The number of repetitive equivalent values of $a$ will be denoted by $e_a$, where

$$e_a = \begin{cases} 
  1 & \text{if } \{q \text{ is even and } a = q/2 \} \text{ or } \{a = 0\}, \\
  2 & \text{if } \{q \text{ is odd and } a \neq 0 \} \text{ or } \{q \text{ is even, } a \neq 0 \text{ and } a \neq q/2\}.
\end{cases}$$

We shall denote $\{x\}$ as the smallest integer not less than $x$ and $[x]$ as the largest integer not greater than $x$.

2. Bound for moderate-density burst error correction

Fire [5] gave the idea of open loop bursts defined as follows:

**Definition 2.1.** An open loop burst of length $b$ is a vector all of whose nonzero entries are confined to some $b$ consecutive positions, the first and the last of which are nonzero.

There is another definition of a burst due to Chien and Tang [3] which reads as:

**Definition 2.2.** A CT burst of length $b$ is a sequence of $b$ digits, the first of which is nonzero.
The definition of CT burst due to Chien and Tang [3] has been further modified by Dass [4] and reads as:

**Definition 2.3.** A burst of length \( b \) (fixed) is an \( n \)-tuple whose only nonzero components are confined to any \( b \) consecutive positions, the first of which is nonzero and the number of its starting position is the first \((n - b + 1)\) positions.

This definition of burst of length \( b \) (fixed) is useful for channels not producing bursts near the end of a code word. Another advantage of considering such bursts is that these burst patterns of length \( b \) (fixed) include several open loop burst patterns of length \( b \) or less in an obvious way. Moreover, these are twice in number than the open loop burst patterns of the same length in the binary case and over the field \( Z_q \) (q prime) these are \( \frac{q}{q-1} \) times the number of open loop bursts.

We now define moderate-density burst of length \( b \) (fixed).

**Definition 2.4.** A moderate-density burst of length \( b \) (fixed) over \( Z_q \) is a burst of length \( b \) (fixed) with Lee weight lying between \( w_1 \) and \( w_2 \) \((1 \leq w_1 \leq w_2 \leq b[q/2])\).

We now obtain bound for codes correcting all moderate-density bursts of length \( b \) (fixed) with Lee weight lying between \( w_1 \) and \( w_2 \) \((1 \leq w_1 \leq w_2 \leq b[q/2])\). The problem here is to find all vectors of Lee weight \( w_L \) \((1 \leq w_1 \leq w_L \leq w_2 \leq b[q/2])\) which are bursts of length \( b \) (fixed). We obtain the number of these patterns in the following lemma.

**Lemma 2.1.** If \( B^*(b, w_1, w_2) \) denotes the total number of moderate-density bursts of length \( b \) (fixed) with Lee weight lying between \( w_1 \) and \( w_2 \) \((1 \leq w_1 \leq w_2 \leq b[q/2])\) over \( Z_q \), then

\[
B^*(b, w_1, w_2) = \begin{cases} 
(n - b + 1) \sum_{\eta=w_1}^{w_2} \sum_{\lambda=1}^{L} P^*_b(\eta) & \text{for } b \geq 2, \\
n \sum_{i=w_1}^{w_2} e_i & \text{for } b = 1,
\end{cases}
\]

where

\[
P^*_b(\eta) = e^\lambda \sum_{N=M_1}^{M_2} \sum_{r_0, r_1, \ldots, r_N} \frac{(b - 1)!}{N!} \prod_{i=0}^{N} r_i^{e_i!}
\]

and \( r_0, r_1, \ldots, r_N \) being integers such that

\[
\sum_{i=0}^{N} r_i = b - 1, \quad r_N \geq 1, \quad r_i \geq 0, \quad i \neq N, \quad N \geq 0,
\]

\[
\sum_{i=1}^{N} i r_i = \eta - \lambda,
\]

\[
L = \min(\eta, [q/2]),
\]

\[
M_1 = \left\{ \frac{\eta - \lambda}{b - 1} \right\}, \quad M_2 = \min(\eta - \lambda, [q/2]).
\]
Proof. There are two cases: (i) when $b = 1$, (ii) when $b \geq 2$.

Case (i). When $b = 1$, then with a given starting position, number of moderate-density bursts of length 1 (fixed) with Lee weight lying between $w_{1}$ and $w_{2}$ ($1 \leq w_{1} \leq w_{2} \leq \lfloor q/2 \rfloor$) is given by \( \sum_{i=w_{1}}^{w_{2}} e_{i} \). Note that if there does not exist an element of $Z_{q}$ having equivalent value $i$ ($w_{1} \leq i \leq w_{2}$), then $e_{i}$ will have value zero. Since the number of starting positions for a burst of length 1 (fixed) can be $n$, therefore, total number of moderate-density bursts of length 1 (fixed) with Lee weight lying between $w_{1}$ and $w_{2}$ ($1 \leq w_{1} \leq w_{2} \leq \lfloor q/2 \rfloor$) is given by $n \sum_{i=w_{1}}^{w_{2}} e_{i}$.

Case (ii). When $b \geq 2$ Consider a burst of length $b$ (fixed) ($b \geq 2$) in which the first entry takes the equivalent value $\lambda ( \lambda \neq 0)$. If we want to make its Lee weight $\eta$ then we are required to make up a sum $\eta - \lambda$ with $b - 1$ entries drawn from $Z_{q}$ taking their equivalent values from the set \{0, 1, \ldots, \lfloor q/2 \rfloor\}. One of these entries will have the greatest equivalent value, say $N$ ($N \geq 0$). If $r_{i}$ is the number of times $i$ or an entry equivalent to $i$ occurs in the partition of the integer $\eta - \lambda$ where $r_{i} \geq 0$, $i \neq N$ and $r_{N} \geq 1$, then, number of vectors of length $b - 1$ that can be formed by filling $b - 1$ positions from the integers 0, 1, \ldots, $N$ is given by

$$
\frac{(b - 1)\prod_{i=0}^{N} r_{i}}{\prod_{i=1}^{N} i!}
$$

Therefore, the number of bursts of length $b$ (fixed) with a given starting position and $N$ as the largest equivalent entry, the first entry equivalent to $\lambda$ and total Lee weight $\eta$ is given by

$$
P_{b}^{\lambda}(\eta) = e_{\lambda} \left[ \sum_{N=M_{1}}^{M_{2}} \sum_{r_{0}, r_{1}, \ldots, r_{N}} \frac{(b - 1)!}{\prod_{i=0}^{N} r_{i}!} \prod_{i=1}^{N} e_{i}^{r_{i}} \right].
$$

(5)

where summation in (5) is over all values of $r_{0}, r_{1}, \ldots, r_{N}$ satisfying

$$
\sum_{i=0}^{N} r_{i} = b - 1, \quad r_{N} \geq 1, \quad r_{i} \geq 0, \quad i \neq N, \quad N \geq 0,
$$

and for $N$ from $M_{1}$ to $M_{2}$ given by

$$
M_{1} = \left\{ \frac{\eta - \lambda}{b - 1} \right\}, \quad M_{2} = \min(\eta - \lambda, \lfloor q/2 \rfloor).
$$

Now, summing (5) for different values of $\lambda$ varying from 1 to $L$ where $L = \min(\eta, \lfloor q/2 \rfloor)$, we get the total number of bursts of length $b$ (fixed) and Lee weight $\eta$ with a specific starting position and is given by

$$
\sum_{\lambda=1}^{L} P_{b}^{\lambda}(\eta) = \sum_{\lambda=1}^{L} e_{\lambda} \left[ \sum_{N=M_{1}}^{M_{2}} \sum_{r_{0}, r_{1}, \ldots, r_{N}} \frac{(b - 1)!}{\prod_{i=0}^{N} r_{i}!} \prod_{i=1}^{N} e_{i}^{r_{i}} \right].
$$

Since a burst of length $b$ (fixed) over a vector of length $n$ can have first $(n - b + 1)$ positions as the starting positions, therefore, total number of bursts of length $b$ (fixed) having Lee weight $\eta$ is

$$
= (n - b + 1) \sum_{\lambda=1}^{L} P_{b}^{\lambda}(\eta)
$$

(6)
Finally, $B^*(b, w_1, w_2)$ i.e. total number of moderate-density bursts of length $b$ (fixed) with Lee weight lying between $w_1$ and $w_2$ ($1 \leq w_1 \leq w_2 \leq b[q/2]$) is obtained by summing (6) over $\eta$ from $w_1$ to $w_2$ and is given by

$$B^*(b, w_1, w_2) = \sum_{\eta=w_1}^{w_2} (n - b + 1) \sum_{\lambda=1}^{L} P^\lambda_b(\eta)$$

where

$$P^\lambda_b(\eta) = e_\lambda \left[ \sum_{N=M_1, r_0, r_1, \ldots, r_N} \frac{(b - 1)!}{\prod_{i=0}^{N} r_i!} \prod_{i=1}^{N} e^{r_i} \right].$$

Combining the cases (i) and (ii), we get the result. □

**Theorem 2.1.** An $(n, k)$ linear code over $\mathbb{Z}_q$ that corrects all moderate-density bursts of length $b$ (fixed) with Lee weight lying between $w_1$ and $w_2$ ($1 \leq w_1 \leq w_2 \leq b[q/2]$) should have at least

$$\log_q (1 + B^*(b, w_1, w_2))$$

parity check digits.

**Proof.** By Lemma 2.1, total number of moderate-density bursts of length $b$ (fixed) with Lee weight lying between $w_1$ and $w_2$ ($1 \leq w_1 \leq w_2 \leq b[q/2]$) including the vector of all zeros is given by

$$1 + B^*(b, w_1, w_2).$$

Since, number of available cosets $= q^{n-k}$, therefore, in order to correct all moderate-density bursts of length $b$ (fixed) with Lee weight lying between $w_1$ and $w_2$ ($1 \leq w_1 \leq w_2 \leq b[q/2]$), the code must satisfy

$$q^{n-k} \geq 1 + B^*(b, w_1, w_2)$$

or

$$n - k \geq \log_q [1 + B^*(b, w_1, w_2)].$$

Hence the theorem. □

**Particular case**

The weight constraint over the burst can be removed by taking $w_1 = 1$ and $w_2 = b[q/2]$. The result in that case reduces to the following corollary.

**Corollary 2.1.** A linear Lee weight code of length $n$ that corrects all bursts of length $b$ (fixed) should have at least
\[
\log_q[1 + B^*(b, 1, b[q/2])]
\]  
(8)

parity check digits.

3. Equality of Lee weight bound with the corresponding Hamming weight bound over \(\mathbb{Z}_2\) and \(\mathbb{Z}_3\)

We know that Hamming weight of a vector \(u = (a_0, a_1, \ldots, a_{n-1})\) is the number of nonzero entries in it and is denoted by \(w(u)\) i.e.,

\[
w(u) = \sum_{i=0}^{n-1} |a_i|,
\]

where

\[
|a_i| = \begin{cases} 
1 & \text{if } a_i \neq 0, \\
0 & \text{if } a_i = 0.
\end{cases}
\]

Also for \(q = 2, 3\) we know that Hamming and Lee weight coincide. Now, we show that for \(q = 2, 3\), the bound obtained in Theorem 2.1 coincides with the corresponding bound with Hamming metric. To show this it suffices to show that for \(q = 2, 3\) the number \(B^*(b, w_1, w_2)\) obtained in Lemma 2.1 coincides with the number \(BH(b, w_1, w_2)\) where \(BH(b, w_1, w_2)\) is the number of moderate-density bursts of length \(b\) (fixed) with Hamming weight lying between \(w_1\) and \(w_2\) \((1 \leq w_1 \leq w_2 \leq b)\).

Now, \(BH(b, w_1, w_2)\) over \(\mathbb{Z}_q\) is given by

\[
BH(b, w_1, w_2) = (n - b + 1) \sum_{\eta = w_1}^{w_2} (q - 1) \left( \begin{array}{c} b - 1 \\ \eta - 1 \end{array} \right) (q - 1)^{\eta - 1}
\]

\[
= (n - b + 1) \sum_{\eta = w_1}^{w_2} (b - 1) \left( \begin{array}{c} \eta - 1 \\ \eta - 1 \end{array} \right) (q - 1)^{\eta}.
\]

Also for \(q = 2\)

\[
BH(b, w_1, w_2) = B^*(b, w_1, w_2) = (n - b + 1) \sum_{\eta = w_1}^{w_2} \left( \begin{array}{c} b - 1 \\ \eta - 1 \end{array} \right)
\]

\[
= (n - b + 1) \sum_{\eta = w_1}^{w_2} (b - 1) \left( \begin{array}{c} \eta - 1 \\ \eta - 1 \end{array} \right) \times 2^{\eta}.
\]

and for \(q = 3\)

\[
BH(b, w_1, w_2) = B^*(b, w_1, w_2) = (n - b + 1) \sum_{\eta = w_1}^{w_2} \left( \begin{array}{c} b - 1 \\ \eta - 1 \end{array} \right) \times 2^{\eta}.
\]

Hence for \(q = 2, 3\), Lee weight bound and Hamming weight bound for moderate-density burst of length \(b\) (fixed) coincides.

4. Bound for codes correcting moderate-density bursts with limited intensity

Here, we have the situation in which the effect of the noise on a single position is no greater than an intensity \(a(< [q/2])\) and the errors occur in the form of moderate-density bursts. In other words, our error patterns are moderate-density bursts with Lee weight lying between certain limits and no nonzero entry has an equivalent value greater than \(a\).
To obtain the bound for codes correcting moderate-density bursts with limited intensity, we count the number of moderate-density bursts of length \( b \) (fixed) with Lee weight lying between \( w_1 \) and \( w_2 \) \((1 \leq w_1 \leq w_2 \leq ba)\) such that no entry has an equivalent value exceeding \( a \). The number of such moderate-density burst error patterns is given by \( B^a(b, w_1, w_2) \) where \( B^a(b, w_1, w_2) \) denotes the restriction of \( B^*(b, w_1, w_2) \) for entries not exceeding \( a \) in equivalent values and is given by

\[
B^a(b, w_1, w_2) = \begin{cases} 
(n - b + 1) \sum_{\eta=w_1}^{w_2} L^\lambda \sum_{i=1}^{L} P^b_{\lambda,a}(\eta) & \text{for } b \geq 2, \\
\frac{n \sum_{i=w_1}^{w_2} e_i}{n} & \text{for } b = 1,
\end{cases}
\]

\[(12)\]

where

\[
P^b_{\lambda,a}(\eta) = e_\lambda \sum_{M_1=r_0, r_1, \ldots, r_N} \left( \sum_{i=0}^{N} r_i \right) \prod_{i=1}^{N} e_i^{r_i}
\]

and \( r_0, r_1, \ldots, r_N \) being integers such that

\[
\begin{align*}
\sum_{i=0}^{N} r_i &= b - 1, \\
\sum_{i=1}^{N} i r_i &= \eta - \lambda, \\
L &= \min(\eta, a), \\
M_1 &= \left\{ \frac{\eta - \lambda}{b - 1} \right\}, \\
M_2 &= \min(\eta - \lambda, a).
\end{align*}
\]

\[(13)\]

\[(14)\]

\[(15)\]

We have the following bound for moderate-density bursts with limited intensity:

**Theorem 4.1.** An \((n, k)\) linear code over \( \mathbb{Z}_q \) that corrects all moderate-density bursts of length \( b \) (fixed) with Lee weight lying between \( w_1 \) and \( w_2 \) \((1 \leq w_1 \leq w_2 \leq ba)\) and no entry exceeding \( a \) should have at least

\[
\log_q(1 + B^a(b, w_1, w_2)) \geq \log_q(1 + \log_q(1 + B^a(b, w_1, w_2)))
\]

parity check digits.

**Proof.** Proof is same as that of Theorem 2.1.  \(\square\)

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**References**