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Conformal and Grassmann structures

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Abstract: The main results on the theory of conformal and almost Grassmann structures are presented. The common properties of these structures and also the differences between them are outlined. In particular, the structure groups of these structures and their differential prolongations are found. A complete system of geometric objects of the almost Grassmann structure totally defining its geometric structure is determined. The vanishing of these objects determines a locally Grassmann manifold. It is proved that the integrable almost Grassmann structures are locally Grassmann. The criteria of semiintegrability of almost Grassmann structures is proved in invariant form.

Keywords: Conformal structure, Grassmann structure, almost Grassmann structure, structure group, geometric object, semiintegrable almost Grassmann structure, integrable almost Grassmann structure.

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0. Introduction

The theory of conformal structures arose in studying those properties of Riemannian and pseudo-Riemannian manifolds that remain invariant under conformal transformations of the metric. This theory was studied by many authors (see, for example, the paper [30] by Weyl, who first defined the tensor of conformal curvature of a Riemannian manifold, and the paper [15] by Cartan, who introduced an n-dimensional space with a conformal connection).

Almost Grassmann manifolds were introduced by Hangan [23] as a generalization of the Grassmannian $G(m, n)$. Hangan [23, 24] and Ishihara [26] studied mostly some special almost Grassmann manifolds, especially locally Grassmann manifolds. Later the almost Grassmann manifolds were studied by Goldberg [20], Mikhailov [27], and Akivis [2] in connection with the construction of the theory of multidimensional webs. Goncharov [22] considered the almost Grassmann manifolds as generalized conformal structures.

Baston [11] constructed a theory of a general class of structures, called almost Hermitian symmetric (AHS) structures, which include conformal, projective, almost Grassmann, and

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quaternionic structures and for which the construction of the Cartan normal connection is possible. He constructed a tensor invariant for them and proved that its vanishing is equivalent to the structure being locally that of a Hermitian symmetric space. In [22], the AHS structures have been studied from the point of view of cone structures. Bailey and Eastwood [10] extended the theory of local twistors, which was known for four-dimensional conformal structures, to the almost Grassmann structures (they called them the paraconformal structures). Dhooghe [16] considered the almost Grassmann structures (he called them Grassmannian structures) as subbundles of the second-order frame bundle and constructed a canonical normal connection for these structures. The structure equations derived in [16] are very close to the structure equations of the almost Grassmann structures considered in the present paper.

Note that in the current paper we consider the real theory of conformal and almost Grassmann structures while in [22, 11, 10] their complex theory was studied.

Although some of the authors who studied almost Grassmann structures proved that an almost Grassmann structure is a G-structure of finite type two (see [25, 27]), no one of the authors went further than the development of the first structure tensor.

In the current paper we present the main results on the theory of conformal and almost Grassmann structures. The presentation is of survey nature, and as a rule, it does not contain complete proofs (most of them can be found in the papers [2, 3, 4, 5, 7, 20, 25, 27], and in our book [6]). However, the paper contains some new results, mostly related to the almost Grassmann structures. In particular, in a third-order neighborhood, we construct a complete system of geometric objects of the almost Grassmann structure totally defining its geometric structure. The vanishing of these objects determines a locally Grassmann manifold. As for conformal structures, the complete object of the almost Grassmann structure is determined in a fourth-order differential neighborhood.

In the theory of almost Grassmann structures, integrable and semiintegrable structures play an important role. The integrable almost Grassmann structures are locally Grassmann. The condition of semiintegrability of almost Grassmann structures was first found in [27]. However, the proof in [27] has been done in a certain reduced frame bundle. Unlike the proof in [27], our proof in Section 4 is given in invariant form.

In the paper, we consider simultaneously the proper conformal structure $CO(n)$ and the pseudoconformal structures $CO(p, q)$ of signature (p, q).

We find the common properties of conformal and almost Grassmann structures and also the differences between them. In particular, we find the structure groups of these structures and their differential prolongations. The structure group G of the conformal structure $CO(p, q)$ is represented in the form $\mathbf{SO}(p, q) \times \mathbf{H}$, and the structure group of the almost Grassmann structure in the form $\mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}$, where $\mathbf{SO}(p, q)$ is the special orthogonal group of signature (p, q); $\mathbf{SL}(p)$, $\mathbf{SL}(q)$ are the special linear groups of order p and q, respectively, and $\mathbf{H}$ is the group of homotheties. For both structures, the prolonged group G' is isomorphic to the semidirect product $G \ltimes \mathbf{T}(n)$, where $\mathbf{T}(n)$ is the n-dimensional group of parallel translations acting on M, and the $\ltimes$ is the symbol of the semidirect product, but $n = p + q$ for the conformal structure and $n = pq$ for the almost Grassmann structure.

The almost Grassmann structure defines on the manifold M two fiber bundles E_α and E_β. For the conformal structure, these fiber bundles arise only if $p = q = 2$. For the general almost Grassmann structure and the four-dimensional conformal structure, the first nonvanishing structure tensor splits into two subtensors, which are the structure tensors of fiber bundles E_α and E_β. For four-dimensional conformal and almost Grassmann structures the vanishing of any of these subtensors leads to integrability of the corresponding fiber bundle.

1. Conformal structures

1. It is known that at any point x of a pseudo-Euclidean space R^n_q of signature (p, q), $p+q = n$, there is an isotropic cone $C_x(p, q)$ of second order with vertex at the point x. It is also known that by compactification of the space R^n_q one can construct a pseudoconformal space C^n_q of the same signature. The compactification mentioned above is the enlargement of the space R^n_q by the point at infinity, $y = \infty$, and by the isotropic cone C_y with vertex at this point:

$$C^n_q = R^n_q \cup \{C_y\}.$$

The space C^n_q is a homogeneous space with the fundamental group

$$\mathbf{PO}(n+2, q+1) \cong \begin{cases} \mathbf{SO}(n+2, q+1) & \text{if } n \text{ is odd,} \\ \mathbf{O}(n+2, q+1)/\mathbb{Z}_2 & \text{if } n \text{ is even.} \end{cases}$$

Applying Darboux mapping, we can realize a pseudoconformal space C^n_q on a hyperquadric Q^n_q in a projective space P^{n+1}. After reduction to a sum of squares, the left-hand side of an equation of this hyperquadric will have $p+1$ positive and $q+1$ negative squares. Under the Darboux mapping, to the isotropic cones of the space C^n_q there correspond the asymptotic cones of the hyperquadric Q^n_q which are the intersections of the hyperquadric Q^n_q with its tangent subspaces:

$$C_x(p, q) = Q^n_q \cap T_x(Q^n_q), \quad x \in Q^n_q.$$

Note that for $q = 0$, the cone C_x is imaginary, and for its consideration one must complexify the tangent space C_x, i.e., to enlarge it to the space $\mathbb{C}T_x = T_x \otimes \mathbb{C}$.

Consider a family of moving frames in the space C^n_q, each of which is made up of two points $A_0 = x$ and A_{n+1} of general position and n independent hyperspheres A_i, $i = 1, \dots, n$, passing through these points. Under the Darboux mapping, this frame will pass into a point frame of a projective space P^{n+1} such that its points A_0 and A_{n+1} lie on the hyperquadric Q^n_q, and the points A_i, form a basis of the $(n-1)$-dimensional plane of intersection of two n-dimensional planes which are tangent to Q^n_q at the points A_0 and A_{n+1}. If we denote by $(\cdot\,,\cdot)$ the scalar product of points in the space P^{n+1}, then the frame elements of the frames we have chosen satisfy the following analytical conditions:

$$(A_0, A_0) = 0, \quad (A_{n+1}, A_{n+1}) = 0, \quad (A_0, A_i) = 0, \quad (A_{n+1}, A_i) = 0. \tag{1.1}$$

In addition, we normalize the points A_0 and A_{n+1} by the condition

$$(A_0, A_{n+1}) = -1. \tag{1.2}$$

We will not assume that the hyperspheres A_i are orthogonal and will write their scalar products in the form

$$(A_i, A_j) = g_{ij}, \quad i, j = 1, \dots, n \tag{1.3}$$

(cf. [15] and [31]). In the space P^{n+1}, the hyperquadric Q^n_q has the following equation with respect to the chosen frame:

$$g_{ij} x^i x^j - 2x^0 x^{n+1} = 0. \tag{1.4}$$

The quadratic form $g_{ij} x^i x^j$ is of signature (p, q), i.e., its canonical form contains p positive and q negative squares.

We will write the equations of infinitesimal displacement of the moving frames in the form

$$dA_u = \omega_u^v A_v, \quad u, v = 0, 1, \dots, n+1, \tag{1.5}$$

where ω_u^v are differential 1-forms satisfying the structure equations of the space P^{n+1}:

$$d\omega_u^v = \omega_u^w \wedge \omega_w^v, \tag{1.6}$$

which are the integrability conditions of equations (1.5). If we differentiate conditions (1.1)–(1.3), we obtain the following equations which the forms ω_u^v satisfy:

$$\begin{aligned} &\omega_0^{n+1} = \omega_{n+1}^0 = 0, \qquad \omega_0^0 + \omega_{n+1}^{n+1} = 0, \\ &\omega_i^{n+1} - g_{ij}\omega_0^j = 0, \qquad \omega_i^0 - g_{ij}\omega_{n+1}^j = 0 \end{aligned} \tag{1.7}$$

and

$$dg_{ij} = g_{ik}\omega_j^k + g_{kj}\omega_i^k. \tag{1.8}$$

By equations (1.7), only the forms ω_0^0, ω_0^i, ω_i^0 and ω_i^j are linearly independent. In addition, the forms ω_i^j are connected with the fundamental tensor g_{ij} by equations (1.8).

The family of frames we have constructed in the space C^n_q depends on $(n+1)^2$ parameters but this number does not coincide with the number r of parameters on which the group of conformal transformations of the space C^n_q depends. The latter number is equal to $(n+1)^2 - \frac{1}{2}n(n+1)$ where the subtrahend is equal to the number of independent among equations (1.8), i.e., $r = \frac{1}{2}(n+1)(n+2)$.

Thus, the structure equations of the conformal space C^n_q can be written as

$$\begin{aligned} &\nabla g = 0, \\ &d\omega = \kappa \wedge \omega - \theta \wedge \omega, \\ &d\kappa = -\varphi \wedge \omega, \\ &d\theta = \varphi \wedge \omega - \theta \wedge \theta + (g\omega) \wedge (\varphi g^{-1}), \\ &d\varphi = \varphi \wedge \kappa - \varphi \wedge \theta, \end{aligned} \tag{1.9}$$

where $g = (g_{ij})$, $\nabla g = (dg_{ij} - g_{ik}\omega_j^k - g_{kj}\omega_i^k)$, $\omega = (\omega^i)$ (where $\omega^i = \omega_0^i$), $\kappa = \omega_0^0$, $\theta = (\omega_j^i)$, $\varphi = (\omega_i^0)$, d is the operator of exterior differentiation, and $\wedge$ is the symbol of exterior matrix multiplication. Note that in all exterior products of 1-forms occurring in equations (1.9) multiplication is performed row by column: for example, a detailed writing of second equation (1.9) has the following form:

$$d\omega^i = \omega_0^0 \wedge \omega^i - \omega_j^i \wedge \omega^j.$$

2. A generalization of a conformal space is a conformal structure $CO(p, q)$ defined on a differentiable manifold M of dimension n.

Let M be a differentiable manifold of dimension $n = p + q$ defined over reals $\mathbb{R}$. Consider the frame bundle consisting of vectorial frames $\{e_i\}$ and the conjugate coframe bundle $\{\omega^i\}$ consisting of 1-forms on M such that

$$\omega^i(e_j) = \delta_j^i.$$

Let g be a nondegenerate quadratic form of signature (p, q) (where $p + q = n$):

$$g = g_{ij}\omega^i\omega^j, \quad i, j, k = 1, \dots, n. \tag{1.10}$$

The form g defines a *Riemannian metric* on M, and the tensor $g_{ij} = g_{ji}$ is its *metric tensor.* The 1-form $\omega = \{\omega^i\}$ is a vectorial form with its values in $T_x(M)$. This 1-form is defined in the first-order frame bundle over M.

A pair (M, g) is called a *Riemannian manifold.*

Two Riemannian metrics g and $\bar{g}$ given on the manifold M are *conformally equivalent* if there exists a function $\sigma(x) \neq 0, \ x \in M$, such that

$$\bar{g} = \sigma g.$$

Definition 1.1. A *conformal structure* on the manifold M is the collection of all conformally equivalent Riemannian metrics given on M.

We will denote such a structure by $CO(p, q)$. Note that on the conformal structure $CO(p, q)$, the form (1.10) is relatively invariant, and on the Riemannian manifold (M, g), it is absolutely invariant.

Note also that if $\sigma(x) > 0$, then the forms g and $\bar{g}$ have the same signature, and if $\sigma(x) < 0$, then the form $\bar{g}$ is of signature (q, p). Thus, *the structures $CO(p, q)$ and $CO(q, p)$ are equivalent*: $CO(p, q) \sim CO(q, p)$.

The equation $g = 0$ defines in $\mathbb{C}T_x(M) = T_x(M) \otimes \mathbb{C}$ a cone C_x of second order and of signature (p, q) which is called the *isotropic cone*:

$$C_x = \{\xi \in \mathbb{C}T_x(M) \mid g(\xi, \xi) = 0\}.$$

Conversely, it is easy to see that the structure $CO(p, q)$ *is defined on a real manifold M by a fibration of cones C_x of second order and of signature* (p, q). Thus,

$$CO(p, q) = \Big(M, \bigcup_{x\in M} C_x(p, q) \mid C_x \subset \mathbb{C}T_x(M)\Big).$$

The structure group G of the structure $CO(p, q)$ is locally isomorphic to the subgroup of $\mathbf{GL}(n, \mathbb{R})$ leaving invariant the cone C_x:

$$G \cong \mathbf{SO}(p, q) \times \mathbf{H}, \quad p + q = n,$$

where $\mathbf{SO}(p, q)$ is the special n-dimensional pseudoorthogonal group of signature (p, q) (the connected component of the unity of the pseudoorthogonal group $\mathbf{O}(p, q)$), $\mathbf{H}$ is the group of homotheties, and $\cong$ is the symbol of local isomorphism.

In $\mathbb{C}T_x(M)$, the action of the group G is as follows:

$$G = \{\gamma \in \mathbf{SO}(p, q) \times \mathbf{H}, \xi \in \mathbb{C}T_x(M) \mid \gamma T_x = T_x, \gamma(\bar{\xi}) = \overline{\gamma(\xi)}, \gamma C_x = C_x\}.$$

It follows that *the structure $CO(p, q)$ is a G-structure of first order on M with the structure group G.*

3. Let us consider some particular cases and examples of $CO(p, q)$-structures.

First, if $p = n$ and $q = 0$, then we have the *proper conformal structure* $CO(n, 0) = CO(n)$. In this case, the cone C_x is imaginary, and structural group $G \cong \mathbf{O}(n) \times \mathbf{H}$. If $0 < q < n$, then we have a pseudoconformal structure. For this structure, the cone C_x is real. In particular, if $p = 1$ and $q = n - 1$, then we obtain a pseudoconformal structure of *Lorentzian type*, and if $2 \leqslant p \leqslant n - 2$, the $CO(p, q)$-structure is called *ultrahyperbolic*.

Example 1.2. Since there is an isotropic cone C_x of signature (p, q) at any point x of the pseudoconformal space C_q^n defined in Subsection 1, this space carries a $CO(p, q)$-structure. We will call the $CO(p, q)$-structure associated with the space C_q^n *conformally flat.*

It is obvious that the quadric Q_q^n, which arises if one apply the Darboux mapping to the space C_q^n, carries also a conformally flat $CO(p, q)$-structure defined by the asymptotic cones of Q_q^n ($C_x = Q_q^n \cap T_x(Q_q^n)$) (see Subsection 1).

Example 1.3. Suppose that $n = 4$. Then there are three $CO(p, q)$-structures: $CO(4, 0)$- (or $CO(4)$-), $CO(1, 3)$- and $CO(2, 2)$-structures.

On the $CO(2, 2)$-structure, the cones C_x carry two families of real plane generators, α- and β-planes, that form the *isotropic distributions* E_α and E_β on M. Thus, $G \cong \mathbf{SL}(2) \times \mathbf{SL}(2) \times \mathbf{H}$.

On the $CO(1, 3)$-structure, the cones C_x carry a family of real rectilinear generators and two families of complex plane generators; $E_\beta = \overline{E}_\alpha$; $G \cong \mathbf{SL}(2, \mathbb{C}) \times \mathbf{H} \cong \overline{\mathbf{SL}(2, \mathbb{C})} \times \mathbf{H}$, and the groups $\mathbf{SL}(2, \mathbb{C})$ and $\overline{\mathbf{SL}(2, \mathbb{C})}$ act concordantly on E_α and E_β. Note that in general relativity, C_x are light cones.

On the $CO(4)$-structure, the cones C_x are pure imaginary, E_α and E_β are self-conjugate: $\overline{E}_\alpha = E_\alpha$; $\overline{E}_\beta = E_\beta$, and $G \cong \mathbf{SU}(2) \times \mathbf{SU}(2) \times \mathbf{H}$, where $\mathbf{SU}(2)$ is the special two-dimensional unitary group.

4. The 1-form $\omega = \{\omega^i\}$ is defined in the bundle $\mathcal{R}^1(M)$ of frames of first order. In addition to this form, one can invariantly define a matrix 1-form $\theta = (\omega_j^i)$ and a scalar form κ in the bundle $\mathcal{R}^2(M)$ of frames of second order, and a covector 1-form $\varphi = (\omega_i)$ in the bundle $\mathcal{R}^3(M)$

of frames of third order (see [7]). All these forms satisfy the following structure equations:

$$\begin{aligned} &\nabla g = 0, \\ &d\omega = \kappa \wedge \omega - \theta \wedge \omega, \\ &d\kappa = -\varphi \wedge \omega, \\ &d\theta = \varphi \wedge \omega - \theta \wedge \theta + (g\omega) \wedge (\varphi g^{-1}) + \Theta, \\ &d\varphi = \varphi \wedge \kappa - \varphi \wedge \theta + \Phi, \end{aligned} \tag{1.11}$$

where, as in formulas (1.9), $\nabla g = (dg_{ij} - g_{ik}\omega_j^k - g_{kj}\omega_i^k)$, and the 2-forms $\Theta = (\Theta^i_j)$ and $\Phi = (\Phi_i)$ are the curvature 2-forms. Note that for $\Theta = \Phi = 0$, the structure equations (1.11) coincide with the structure equations (1.9) of the pseudoconformal space C^n_q. Thus, *the space C^n_q carries the curvature-free $CO(p, q)$-structure, i.e., the space C^n_q is conformally flat.*

We will now clarify the geometric meaning for the 1-forms θ, κ, and φ.

First, we note that the first of equations (1.11) is the condition for the cone C_x to be invariant. As to other equations of (1.11), if $\omega^i = 0$, then they become the structure equations of a pseudoconformal space C^n_q in which also $\omega^i = 0$ being set (i.e., a point A_0 is fixed):

$$d\kappa = 0, \quad d\theta = -\theta \wedge \theta, \quad d\varphi = \varphi \wedge \kappa - \varphi \wedge \theta. \tag{1.12}$$

From equations (1.12) it follows that

(1) The form κ is an invariant form of the group $\mathbf{H}$ of homotheties acting in the space $T_x(M)$.

(2) The form θ is an invariant forms of the special pseudoorthogonal group $\mathbf{SO}(p, q)$ leaving the points x and y invariant.

(3) The forms κ and θ are invariant forms of the structural group G of the pseudoconformal structure $CO(p, q)$ whose transformations leave cone $C_x(p, q)$ invariant. The group G is isomorphic to the direct product $\mathbf{SO}(p, q) \times \mathbf{H}$:

$$G \cong \mathbf{SO}(p, q) \times \mathbf{H}. \tag{1.13}$$

(4) The 1-forms θ, κ, and φ are invariant forms of the group G' which is a differential prolongation of G. The group G' is the group of motions of the space $(C^n_q)_x$ which is the compactification of the tangent space $T_x(M)$: $(C^n_q)_x = T_x(M) \cup C_y$, and $(C^n_q)_x$ is referred to a frame $\{x, y, a_i\}$ where a_i are hyperspheres passing through the points x and y.

(5) The covector φ is an invariant form of the group of translations $\mathbf{T}(p + q)$ of the pseudo-Euclidean space $R^n_q = C^n_q \backslash C_x$ whose transformations move the point y.

Thus, the group G' is isomorphic to the semidirect product $G \ltimes \mathbf{T}(p + q)$, i.e.,

$$G' \cong G \ltimes \mathbf{T}(p+q) \cong (\mathbf{SO}(p, q) \times \mathbf{H}) \ltimes \mathbf{T}(p+q), \tag{1.14}$$

and is the group of motions of the space R^n_q.

Since the group G' does not admit further prolongations, *a pseudoconformal structure $CO(p, q)$ is a G-structure of finite type two* (see [29] for definition of finite type).

The structure equations of the $CO(p, q)$-structure in the form (1.11) can be found in [15, 18, 5, 7]. Cartan [15] called equations (1.11) the structure equations of the normal conformal connection associated with the quadratic form g. Note also that while in [15] only proper conformal structures were considered, in [13, 14] the pseudoconformal structures were studied as well.

5. The forms

$$\Theta^i_j = b^i_{jkl}\omega^k \wedge \omega^l, \qquad \Phi_i = c_{ijk}\omega^j \wedge \omega^k \tag{1.15}$$

appearing in the last two equations of (1.11) are called the *curvature forms* of the $CO(p, q)$-structure. The quantities b^i_{jkl} are the components of the *tensor of conformal curvature* (the Weyl tensor) $b = \{b^i_{jkl}\}$ of the $CO(p, q)$-structure, and the quantities c_{ijk} together with b^i_{jkl} constitute a homogeneous geometric object $(b, c) = \{b^i_{jkl}, c_{ijk}\}$. The tensor b^i_{jkl} is defined in a third-order differential neighborhood of a point $x \in M$. It satisfies all conditions which the curvature tensor of a Riemannian manifold satisfies, and in addition, the tensor b^i_{jkl} is trace-free:

$$b^i_{jki} = 0 \tag{1.16}$$

(see, for example, the book [17]).

The geometric object (b, c) is defined in a fourth differential neighborhood of a point $x \in M$. If $n \geqslant 4$, the quantities c_{ijk} are expressed linearly in terms of the covariant derivatives of the tensor b^i_{jkl}:

$$c_{ijk} = -\frac{1}{n-3} b^m_{ijk,m}. \tag{1.17}$$

Thus, if $n \geqslant 4$ and $b = 0$, then $c = 0$, and the $CO(p, q)$-structure is locally flat. Note that if $n = 3$, then $b \equiv 0$, and the condition $c = 0$ is the condition for the $CO(p, q)$-structure to be locally flat.

Since the second equation of (1.11) does not have an exterior quadratic form of type (1.15), a *$CO(p, q)$-structure is torsion-free.*

It is proved in [4] (see also [7]) that for the $CO(2, 2)$-structure, the tensor b of conformal curvature splits into two subtensors b_α and b_β: $b = b_\alpha \dotplus b_\beta$, and b_α and b_β are the curvature tensors of the fibre bundles E_α and E_β (see Subsection 3). Each of the subtensors b_α and b_β has five independent components.

Since for the $CO(1, 3)$-structure, the fiber bundles E_α and E_β are complex conjugates, its curvature tensor admits two complex conjugate representations b_α and b_β, $b_\beta = \bar{b}_\alpha$, which themselves are the curvature tensors of the fiber bundles E_α and E_β. For the $CO(4)$-structure, we again have the splitting $b = b_\alpha \dotplus b_\beta$, and the subtensors b_α and b_β are self-conjugates: $b_\alpha = \bar{b}_\alpha$ and $b_\beta = \bar{b}_\beta$.

For $p + q = 4$, a conformal $CO(p, q)$-structure for which the tensor b_α or b_β vanishes is called *conformally semiflat*. If both these tensors vanish, the structure is called *conformally flat*. The conditions which the tensors b_α and b_β satisfy show that *only the $CO(2, 2)$- and $CO(4)$-structures can be conformally semiflat*, and that *the $CO(1, 3)$-structure cannot be conformally semiflat without being conformally flat.*

It is easy to see that *the conformally flat $CO(2, 2)$-structure is locally isomorphic to the structure of the pseudoconformal space C^4_2.*

Finally note that for the $CO(4)$-structure the notion of local semiflatness is connected with the notions of self-duality and anti-self-duality introduced in [9].

2. Grassmann structures

1. Let P^n be an n-dimensional projective space. The set of m-dimensional subspaces $P^m \subset P^n$ is called the *Grassmann manifold*, or the *Grassmannian*, and is denoted by the symbol $G(m, n)$. It is well-known that the Grassmannian is a differentiable manifold, and that its dimension is equal to $\rho = (m+1)(n-m)$. It will be convenient for us to set $p = m + 1$ and $q = n - m$. Then we have $n = p + q - 1$.

Let a subspace $P^m = x$ be an element of the Grassmannian $G(m, n)$. With any subspace x, we associate a family of projective point frames $\{A_u\}$, $u = 0, 1, \dots, n$, such that the vertices A_α, $\alpha = 0, 1, \dots, m$, of its frames lie in the plane P^m, and the points A_i, $i = m+1, \dots, n$, lie outside P^m and together with the points A_α make up the frame $\{A_u\}$ of the space P^n.

We will write the equations of infinitesimal displacement of the moving frames we have chosen in the form:

$$dA_u = \theta_u^v A_v, \qquad u, v = 0, \dots, n. \tag{2.1}$$

Since the fundamental group of the space P^n is locally isomorphic to the group $\mathbf{SL}(n+1)$, the forms θ_u^v are connected by the relation

$$\theta_u^u = 0. \tag{2.2}$$

The structure equations of the space P^n have the form

$$d\theta_u^v = \theta_u^w \wedge \theta_w^v. \tag{2.3}$$

By (2.3), the exterior differential of the left-hand side of equations (2.1) is identically equal to 0, and hence the system of equations (2.1) is completely integrable.

By (2.1), we have

$$dA_\alpha = \theta_\alpha^\beta A_\beta + \theta_\alpha^i A_i.$$

It follows that the 1-forms θ_α^i are basis forms of the Grassmannian. These forms are linearly independent, and their number is equal to $\rho = (m+1)(n-m) = pq$, i.e., it equals the dimension of the Grassmannian $G(m, n)$. We will assume that the integers p and q satisfy the inequalities $p \geqslant 2$ and $q \geqslant 2$, since for $p = 1$, we have $m = 0$, and the Grassmannian $G(0, n)$ is the projective space P^n, and for $q = 1$, we have $m = n - 1$, and the Grassmannian $G(n-1, n)$ is isomorphic to the dual projective space $(P^n)^*$.

Let us rename the basis forms by setting $\theta_\alpha^i = \omega_\alpha^i$ and find their exterior differentials:

$$d\omega_\alpha^i = \theta_\alpha^\beta \wedge \omega_\beta^i + \omega_\alpha^j \wedge \theta_j^i. \tag{2.4}$$

Define the trace-free forms

$$\omega_\alpha^\beta = \theta_\alpha^\beta - \frac{1}{p}\delta_\alpha^\beta \theta_\gamma^\gamma, \quad \omega_j^i = \theta_j^i - \frac{1}{q}\delta_i^j \theta_k^k, \tag{2.5}$$

satisfying the conditions

$$\omega_\alpha^\alpha = 0, \qquad \omega_i^i = 0. \tag{2.6}$$

Eliminating the forms θ_α^β and θ_i^j from equations (2.4), we find that

$$d\omega_\alpha^i = \omega_\alpha^\beta \wedge \omega_\beta^i + \omega_\alpha^j \wedge \omega_j^i + \kappa \wedge \omega_\alpha^i, \tag{2.7}$$

where $\kappa = (1/p)\,\theta_\gamma^\gamma - (1/q)\,\theta_k^k$, or by (2.2),

$$\kappa = \left(\frac{1}{p} + \frac{1}{q}\right)\theta_\gamma^\gamma. \tag{2.8}$$

Setting

$$\omega_i^\alpha = -\left(\frac{1}{p} + \frac{1}{q}\right)\theta_i^\alpha \tag{2.9}$$

and taking the exterior derivatives of equations (2.7) and (2.8), we obtain

$$\begin{aligned} d\omega_\alpha^\beta &= \omega_\alpha^\gamma \wedge \omega_\gamma^\beta + \frac{q}{p+q}\omega_\gamma^k \wedge (\delta_\alpha^\beta \omega_k^\gamma - p\delta_\alpha^\gamma \omega_k^\beta), \\ d\omega_j^i &= \omega_j^k \wedge \omega_k^i + \frac{p}{p+q}(\delta_j^i \omega_k^\gamma - q\delta_k^i \omega_j^\gamma) \wedge \omega_\gamma^k, \end{aligned} \tag{2.10}$$

and

$$d\kappa = \omega_i^\alpha \wedge \omega_\alpha^i. \tag{2.11}$$

Exterior differentiation of equations (2.9) gives

$$d\omega_i^\alpha = \omega_i^j \wedge \omega_j^\alpha + \omega_i^\beta \wedge \omega_\beta^\alpha + \omega_i^\alpha \wedge \kappa. \tag{2.12}$$

Finally, exterior differentiation of equations (2.12) leads to identities.

Thus, the structure equations of the Grassmannian $G(m, n)$ take the form (2.7), (2.10), (2.11), and (2.12). This system of differential equations is *closed* in the sense that its further exterior differentiation leads to identities.

If we fix a subspace $x = P^m \subset P^n$, then we obtain $\omega_\alpha^i = 0$, and equations (2.10) and (2.11) become

$$d\pi_\alpha^\beta = \pi_\alpha^\gamma \wedge \pi_\gamma^\beta, \quad d\pi_j^i = \pi_j^k \wedge \pi_k^i, \quad d\pi = 0, \tag{2.13}$$

where $\pi = \kappa(\delta)$, $\pi_\alpha^\beta = \omega_\alpha^\beta(\delta)$, $\pi_j^i = \omega_j^i(\delta)$, and δ is the operator of differentiation with respect to the fiber parameters of the second-order frame bundle associated with the Grassmannian $G(m, n)$. Moreover, the forms π_α^β and π_j^i satisfy equations similar to equations (2.6), i.e., these forms are trace-free. The forms π_α^β are invariant forms of the group $\mathbf{SL}(p)$ which is locally isomorphic to the group of projective transformations of the subspace P^m. The forms π_j^i are invariant forms of the group $\mathbf{SL}(q)$ which is locally isomorphic to the group of projective transformations of the bundle of $(m+1)$-dimensional subspaces of the space P^n containing P^m. The form π is an invariant form of the group $\mathbf{H} = \mathbb{R}^* \otimes \mathrm{Id}$ of the space P^n with center at P^m; here $\mathbb{R}^*$ is the multiplicative group of real numbers.

The direct product of these three groups is the structural group G of the Grassmann manifold $G(m, n)$:

$$G = \mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}. \tag{2.14}$$

Finally, the forms $\pi_i^\alpha = \omega_i^\alpha(\delta)$, which by (2.12) satisfy the structure equations

$$d\pi_i^\alpha = \pi_i^j \wedge \pi_j^\alpha + \pi_i^\beta \wedge \pi_\beta^\alpha + \pi_i^\alpha \wedge \pi, \tag{2.15}$$

are also fiber forms on the Grassmannian $G(m, n)$ but unlike the forms π_α^β, π_i^j and π, they are connected with the third-order frame bundle of the Grassmannian $G(m, n)$.

The forms π_α^β, π_j^i, π, and π_i^α, satisfying the structure equations (2.13) and (2.15), are invariant forms of the group

$$G' = G \ltimes \mathbf{T}(pq) \tag{2.16}$$

arising under the differential prolongation of the structure group G of the Grassmannian $G(m, n)$. The group G' is the group of motions of an $(n - m - 1)$-quasiaffine space A^n_{n-m-1} (see [28]) which is a projective space P^n with a fixed m-dimensional subspace $P^m = A_0 \wedge A_1 \wedge \ldots \wedge A_m$ and the generating element $P^{n-m-1} = A_{m+1} \wedge \ldots \wedge A_n$. The dimension of the space A^n_{n-m-1} coincides with the dimension of the Grassmannian $G(n-m-1, n)$, and this dimension is the same as the dimension of the Grassmannian $G(m, n)$: $\rho = (m+1)(n-m)$. The forms π_i^α are invariant forms of the group $\mathbf{T}(pq)$ of parallel translations of the space A^n_{n-m-1}, and the group G is the stationary subgroup of its element P^{n-m-1}.

In the index-free notation, the structure equations (2.7) and (2.10)–(2.12) of the Grassmannian $G(m, n)$ can be written as follows:

$$\begin{aligned}
&d\omega = \kappa \wedge \omega - \omega \wedge \theta_\alpha - \theta_\beta \wedge \omega,\\
&d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = \frac{q}{p+q}[-I_\alpha \operatorname{tr}(\varphi \wedge \omega) + p\varphi \wedge \omega],\\
&d\theta_\beta + \theta_\beta \wedge \theta_\beta = \frac{p}{p+q}[-I_\beta \operatorname{tr}(\varphi \wedge \omega) + q\omega \wedge \varphi],\\
&d\kappa = \operatorname{tr}(\varphi \wedge \omega),\\
&d\varphi + \theta_\alpha \wedge \varphi + \varphi \wedge \theta_\beta + \kappa \wedge \varphi = -(a\varphi) \wedge \omega,
\end{aligned} \tag{2.17}$$

where $\omega = (\omega_\alpha^i)$ is the matrix 1-form defined in the first-order fiber bundle; κ is a scalar 1-form; $\theta_\alpha = (\omega_\beta^\alpha)$ and $\theta_\beta = (\omega_j^i)$ are the matrix 1-forms defined in a second-order fiber bundle for which

$$\operatorname{tr}\theta_\alpha = 0, \qquad \operatorname{tr}\theta_\beta = 0;$$

$\varphi = (\omega_i^\alpha)$ is a matrix 1-form defined in a third-order fiber bundle; and $I_\alpha = (\delta_\alpha^\beta)$ and $I_\beta = (\delta_i^j)$ are the unit tensors of orders p and q, respectively.

Along with the Grassmannian $G(m, n)$, in the space P^n one can consider the dual manifold $G(n - m - 1, n)$. Its base forms are the forms ω_i^α, and its geometry is identical to that of the Grassmannian $G(m, n)$.

2. With the help of Grassmann coordinates, the Grassmannian $G(m, n)$ can be mapped onto a smooth algebraic variety $\Omega(m, n)$ of dimension $\rho = pq$ embedded into a projective space P^N of dimension $N = \binom{p+q}{p} - 1$.

Suppose that $x = A_0 \wedge A_1 \wedge \ldots \wedge A_m$ is a point of the variety $\Omega(m, n)$. Then

$$dx = \tau x + \omega_\alpha^i e_i^\alpha, \tag{2.18}$$

where $\tau = \omega_0^0 + \cdots + \omega_m^m$ and

$$e_i^\alpha = A_0 \wedge \ldots \wedge A_{\alpha-1} \wedge A_i \wedge A_{\alpha+1} \wedge \ldots \wedge A_m,$$

and the points e_i^α together with the point x determine a basis in the tangent subspace $T_x(\Omega)$. The second differential of the point x satisfies the relation

$$d^2x \equiv \sum_{\alpha<\beta, i<j} (\omega_\alpha^i \omega_\beta^j - \omega_\alpha^j \omega_\beta^i) e_{ij}^{\alpha\beta} \quad (\text{mod } T_x(\Omega)), \tag{2.19}$$

where

$$e_{ij}^{\alpha\beta} = A_0 \wedge \ldots \wedge A_{\alpha-1} \wedge A_i \wedge A_{\alpha+1} \wedge \ldots \wedge A_{\beta-1} \wedge A_j \wedge A_{\beta+1} \wedge \ldots \wedge A_m$$

are points of the space P^N lying on the variety $\Omega(m, n)$. The quadratic forms

$$\omega_{\alpha\beta}^{ij} = \omega_\alpha^i \omega_\beta^j - \omega_\alpha^j \omega_\beta^i \tag{2.20}$$

are the second fundamental forms of the variety $\Omega \subset P^N$.

The equations $\omega_{\alpha\beta}^{ij} = 0$ determine the cone of asymptotic directions of the variety Ω at a point $x \in \Omega$. The equations of this cone can be written as follows:

$$\operatorname{rank}(\omega_\alpha^i) = 1. \tag{2.21}$$

In view of (2.21), parametric equations of this cone have the form

$$\omega_\alpha^i = t_\alpha s^i, \quad \alpha = 0, 1, \ldots, m; \quad i = m+1, \ldots, n. \tag{2.22}$$

If we consider a projectivization of this cone, then t_α and s^i can be taken as homogeneous coordinates of projective spaces P^m and P^{n-m-1}. Thus such a projectivization is an embedding of the direct product $P^{p-1} \times P^{q-1}$ into a projective space $P^{\rho-1}$ of dimension $\rho - 1$. Such an embedding is called the *Segre variety* and is denoted by $S(p-1, q-1)$. This is the reason that the cone of asymptotic directions of the variety Ω determined by equations (2.21) is called the *Segre cone*. This cone is denoted by $SC_x(p, q)$ since it carries two families of plane generators of dimensions p and q. Plane generators from different families of the cone $SC_x(p, q)$ have a common straight line. It is possible to prove that the cone $SC_x(p, q)$ is the intersection of the tangent subspace $T_x(\Omega)$ and the variety Ω:

$$C_x(p, q) = T_x(\Omega) \cap \Omega.$$

The differential geometry of Grassmannians was studied in detail in the paper [3].

3. Almost Grassmann structures

1. Now we can define the notion of an almost Grassmann structure.

Definition 3.1. Let M be a differentiable manifold of dimension pq, and let $SC(p,q)$ be a differentiable fibration of Segre cones with the base M such that $SC_x(M) \subset T_x(M)$, $x \in M$. The pair $(M, SC(p,q))$ is said to be an *almost Grassmann structure* and is denoted by $AG(p-1, p+q-1)$. The manifold M endowed with such a structure is said to be an *almost Grassmann manifold.*

As was the case for Grassmann structures, the almost Grassmann structure $AG(p-1, p+q-1)$ is equivalent to the structure $AG(q-1, p+q-1)$ since both of these structures are generated on the manifold M by a differentiable family of Segre cones $SC_x(p,q)$.

Let us consider some examples.

Example 3.2. The main example of an almost Grassmann structure is the almost Grassmann structure associated with the Grassmannian $G(m,n)$. As we saw, there is a field of Segre cones $SC_x(p,q) = T_x(\Omega) \cap \Omega$, $x \in \Omega$, where $p = m+1$ and $q = n-m$, which defines an almost Grassmann structure.

Example 3.3. Consider a pseudoconformal $CO(2,2)$-structure on a four-dimensional manifold M. The isotropic cones C_x of this structure carry two families of plane generators. Hence, these cones are Segre cones $SC_x(2,2)$. Therefore, a pseudoconformal $CO(2,2)$-structure is an almost Grassmann structure $AG(1,3)$.

If we complexify the four-dimensional tangent subspace $T_x(M^4)$ and consider Segre cones with complex generators, then conformal $CO(1,3)$- and $CO(4,0)$-structures can also be considered as complex almost Grassmann structures of the same type $AG(1,3)$. However, in this paper, we will consider only real almost Grassmann structures.

Almost Grassmann structures arise also in the study of multidimensional webs (see [8] and [21]).

Example 3.4. Consider a three-web formed on a manifold M^{2q} of dimension $2q$ by three foliations λ_u, $u = 1, 2, 3$, of codimension q which are in general position (see [1] and [8]).

Through any point $x \in M^{2q}$, there pass three leaves $\mathcal{F}_u$ belonging to the foliations λ_u. In the tangent subspace $T_x(M^{2q})$, we consider three subspaces $T_x(\mathcal{F}_u)$ which are tangent to $\mathcal{F}_u$ at the point x. If we take the projectivization of this configuration with center at the point x, then we obtain a projective space P^{2q-1} of dimension $2q-1$ containing three subspaces of dimension $q-1$ which are in general position. These three subspaces determine a Segre variety $S(1, q-1)$, and the latter variety is the directrix for a Segre cone $SC_x(2,q) \subset T_x(M^{2q})$. Thus, on M^{2q}, a field of Segre cones arises, and this field determines an almost Grassmann structure on M^{2q}.

The structural group of the web $W(3,2,q)$ is smaller than that of the induced almost Grassmann structure, since transformations of this group must keep invariant the subspaces $T_x(\mathcal{F}_u)$. Thus, the structural group of the three-web is the group $\mathbf{GL}(q)$.

Example 3.5. Consider a $(p+1)$-web $W(p+1, p, q) = (M; \lambda_1, \ldots, \lambda_{p+1})$ formed on a differentiable manifold M of dimension pq by $p+1$ foliations λ_u, $u = 1, \ldots, p+1$, of dimension q which are in general position on M (see [19] or [21]).

As in Example 3.4, the tangent spaces $T_x(\mathcal{F}_u)$ define the cone $SC_x(p, q) \supset T_x(\mathcal{F}_u)$, and the field of these cones defines an almost Grassmann structure $AG(p-1, p+q-1)$ on M.

The structural group of the web $W(p+1, p, q)$ is the same group $G = \mathbf{GL}(q)$ as for the web $W(3, 2, q)$, and this group does depend on p.

2. The structural group of the almost Grassmann structure is a subgroup of the general linear group $\mathbf{GL}(pq)$ of transformations of the space $T_x(M)$, which leave the cone $SC_x(p, q) \subset T_x(M)$ invariant. We denote this group by $G = \mathbf{GL}(p, q)$.

To clarify the structure of this group, in the tangent space $T_x(M)$, we consider a family of frames $\{e_i^\alpha\}$, $\alpha = 1, \ldots, p$; $i = p+1, \ldots, p+q$, such that for any fixed i, the vectors e_i^α belong to a p-dimensional generator ξ of the Segre cone $SC_x(p, q)$, and for any fixed α, the vectors e_i^α belong to a q-dimensional generator η of $SC_x(p, q)$. In such a frame, the equations of the cone $SC_x(p, q)$ can be written as follows:

$$z_\alpha^i = t_\alpha s^i, \quad \alpha = 1, \ldots, p, \quad i = p+1, \ldots, p+q, \tag{3.1}$$

where z_α^i are the coordinates of a vector $z = z_\alpha^i e_i^\alpha \subset T_x(M)$, and t_α and s^i are parameters on which a vector $z \subset SC_x(M)$ depends.

The family of frames $\{e_i^\alpha\}$ attached to the cone $SC_x(p, q)$ admits a transformation of the form

$$'e_i^\alpha = A_\beta^\alpha A_i^j e_j^\beta, \tag{3.2}$$

where (A_β^α) and (A_j^i) are nonsingular square matrices of orders p and q, respectively. These matrices are not defined uniquely since they admit a multiplication by reciprocal scalars. However, they can be made unique by restricting to unimodular matrices (A_β^α) or (A_i^j): $\det(A_\beta^\alpha) = 1$ or $\det(A_i^j) = 1$. Thus the structural group of the almost Grassmann structure defined by equations (3.2), can be represented in the form

$$G = \mathbf{SL}(p) \times \mathbf{GL}(q) \cong \mathbf{GL}(p) \times \mathbf{SL}(q), \tag{3.3}$$

where $\mathbf{SL}(p)$ and $\mathbf{SL}(q)$ are special linear groups of dimension p and q, respectively. Such a representation has been used by Hangan [23, 24, 25], Goldberg [20] (see also the book [21, Ch. 2], and Mikhailov [27]. Unlike this approach, we will assume that both matrices (A_β^α) and (A_i^j) are unimodular but the right-hand side of equation (3.2) admits a multiplication by a scalar factor. As a result, we obtain a more symmetric representation of the group G:

$$G = \mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}, \tag{3.4}$$

where $\mathbf{H} = \mathbb{R}^* \otimes \mathrm{Id}$ is the group of homotheties of the $T_x(M)$.

It follows that *an almost Grassmann structure $AG(m, n)$ is a G-structure of first order.*

It follows from condition (3.1) that p-dimensional plane generators ξ of the Segre cone $SC_x(p, q)$ are determined by values of the parameters s^i, and t_α are coordinates of points of a generator ξ. But a plane generator ξ is not changed if we multiply the parameters s^i by the same number. Thus, the family of plane generators ξ depends on $q - 1$ parameters.

Similarly, q-dimensional plane generators η of the Segre cone $SC_x(p, q)$ are determined by values of the parameters t_α, and s^i are coordinates of points of a generator η. But a plane generator η is not changed if we multiply the parameters t_α by the same number. Thus, the family of plane generators η depends on $p - 1$ parameters.

The p-dimensional subspaces ξ form a fiber bundle on the manifold M. The base of this bundle is the manifold M, and its fiber attached to a point $x \in M$ is the set of all p-dimensional plane generators ξ of the Segre cone $SC_x(p, q)$. The dimension of a fiber is $q - 1$, and it is parametrized by means of a projective space P_α, $\dim P_\alpha = q - 1$. We will denote this fiber bundle of p-subspaces by $E_\alpha = (M, P_\alpha)$.

In a similar manner, q-dimensional plane generators η of the Segre cone $SC_x(p, q)$ form on M the fiber bundle $E_\beta = (M, P_\beta)$ with the base M and fibers of dimension $p - 1 = \dim P_\beta$. The fibers are q-dimensional plane generators η of the Segre cone $SC_x(p, q)$.

Consider the manifold $M_\alpha = M \times P_\alpha$ of dimension $pq + q - 1$. The fiber bundle E_α induces on M_α the distribution Δ_α of plane elements ξ_α of dimension q. In a similar manner, on the manifold $M_\beta = M \times P_\beta$ the fiber bundle E_β induces the distribution Δ_β of plane elements η_β of dimension p.

Definition 3.6. An almost Grassmann structure $AG(p - 1, p + q - 1)$ is said to be *α-semiintegrable* if the distribution Δ_α is integrable on this structure. Similarly, an almost Grassmann structure $AG(p - 1, p + q - 1)$ is said to be *β-semiintegrable* if the distribution Δ_β is integrable on this structure. A structure $AG(p - 1, p + q - 1)$ is called *integrable* if it is both α- and β-semiintegrable.

Integral manifolds $\widetilde{V}_\alpha$ of the distribution Δ_α of an α-semiintegrable almost Grassmann structure are of dimension p. They are projected on the original manifold M in the form of a submanifold V_α of the same dimension p, which, at any of its points, is tangent to the p-subspace ξ_α of the fiber bundle E_β. Through each point $x \in M$, there passes a $(q - 1)$-parameter family of submanifolds V_α.

Similarly, integral manifolds $\widetilde{V}_\beta$ of the distribution Δ_β of a β-semiintegrable almost Grassmann structure are of dimension q. They are projected on the original manifold M in the form of a submanifold V_α of the same dimension q, which, at any of its points, is tangent to the q-subspace η_β of the fiber bundle E_α. Through each point $x \in M$, there passes a $(p-1)$-parameter family of submanifolds V_β.

If an almost Grassmann structure on M is integrable, then through each point $x \in M$, there pass a $(q - 1)$-parameter family of submanifolds V_α and a $(p - 1)$-parameter family of submanifolds V_β which were described above.

The Grassmann structure $G(m, n)$ is an integrable almost Grassmann structure $AG(m, n)$ since through any point $x \in \Omega(m, n)$, onto which the manifold $G(m, n)$ is mapped bijectively under the Grassmann mapping, there pass a $(q - 1)$-parameter family of p-dimensional plane generators (which are the submanifolds V_α) and a $(p - 1)$-parameter family of q-dimensional plane generators (which are the submanifolds V_β). In the projective space P^n, to submanifolds

V_α there corresponds a family of m-dimensional subspaces belonging to a subspace of dimension $m+1$, and to submanifolds V_β there corresponds a family of m-dimensional subspaces passing through a subspace of dimension $m-1$.

3. We will now write the structure equations which the forms ω_α^i satisfy. These structure equations differ from equations (2.7) only by the fact that they contain an additional term with the product of the basis forms:

$$d\omega_\alpha^i = \omega_\alpha^\beta \wedge \omega_\beta^i + \omega_\alpha^j \wedge \omega_j^i + \kappa \wedge \omega_\alpha^i + u_{\alpha jk}^{i\beta\gamma} \omega_\beta^j \wedge \omega_\gamma^k, \tag{3.5}$$

where $u_{\alpha jk}^{i\beta\gamma} = -u_{\alpha kj}^{i\gamma\beta}$, and as earlier (see conditions (2.6)), we have

$$\omega_\gamma^\gamma = 0, \quad \omega_k^k = 0. \tag{3.6}$$

If we prolong equations (3.5), we can see that the quantities $u_{\alpha jk}^{i\beta\gamma}$ as well as the quantities $u_{\alpha k}^{\beta\gamma} = u_{\alpha ik}^{i\beta\gamma}$ and $u_{jk}^{i\gamma} = u_{\alpha jk}^{i\alpha\gamma}$ form geometric objects defined in a second-order neighborhood of the almost Grassmann structure $AG(p-1, p+q-1)$.

The following lemma can be proved (see [6, Section 7.2]):

Lemma 3.7. *By a reduction of third-order frames of the almost Grassmann structure $AG(p-1, p+q-1)$, the geometric objects $u_{\alpha k}^{\beta\gamma}$ and $u_{jk}^{i\gamma}$ can be reduced to* 0:

$$u_{\alpha k}^{\beta\gamma} = 0, \quad u_{jk}^{i\gamma} = 0.$$

The reduction indicated in Lemma 3.7 can be carried out by means of the fiber forms $\pi_{\alpha k}^{\beta\gamma}$ and $\pi_{jk}^{i\gamma}$ which appear if one finds exterior differentials of the forms ω_α^β and ω_j^i.

If we denote by $a_{\alpha jk}^{i\beta\gamma}$ the values of the quantities $u_{\alpha jk}^{i\beta\gamma}$ in a reduced third-order frame, then the quantities $a_{\alpha jk}^{i\beta\gamma}$ satisfy the conditions

$$a_{\alpha jk}^{i\alpha\gamma} = 0, \quad a_{\alpha ik}^{i\beta\gamma} = 0, \tag{3.7}$$

$$a_{\alpha jk}^{i\beta\gamma} = -a_{\alpha kj}^{i\gamma\beta}, \tag{3.8}$$

and

$$\nabla_\delta a_{\alpha jk}^{i\beta\gamma} + a_{\alpha jk}^{i\beta\gamma}\pi = 0, \tag{3.9}$$

where $\nabla_\delta a_{\alpha jk}^{i\beta\gamma} = \delta a_{\alpha jk}^{i\beta\gamma} - a_{\sigma jk}^{i\beta\gamma}\pi_\alpha^\sigma - a_{\alpha sk}^{i\beta\gamma}\pi_j^s - a_{\alpha js}^{i\beta\gamma}\pi_k^s + a_{\alpha jk}^{s\beta\gamma}\pi_s^i + a_{\alpha jk}^{i\sigma\gamma}\pi_\sigma^\beta + a_{\alpha jk}^{i\beta\sigma}\pi_\sigma^\gamma$, $\pi_m^\varepsilon = \omega_m^\varepsilon(\delta)$, $\pi_\alpha^\beta = \omega_\alpha^\beta(\delta)$, $\pi_i^j = \omega_i^j(\delta)$. This implies the following theorem.

Theorem 3.8. *The quantities $a_{\alpha jk}^{i\beta\gamma}$, defined in a second-order neighborhood by the reduction of third-order frames indicated above, form a relative tensor of weight* -1 *and satisfy conditions* (3.7) *and* (3.8).

Definition 3.9. The tensor $a = \{a_{\alpha jk}^{i\beta\gamma}\}$ is said to be the *first structure tensor*, or the *torsion tensor*, of an almost Grassmann manifold $AG(p-1, p+q-1)$.

After the reduction of third-order frames has been made, the first structure equations (3.5) become

$$d\omega_\alpha^i = \omega_\alpha^j \wedge \omega_j^i + \omega_\alpha^\beta \wedge \omega_\beta^i + \kappa \wedge \omega_\alpha^i + a_{\alpha jk}^{i\beta\gamma} \omega_\beta^j \wedge \omega_\gamma^k. \tag{3.10}$$

The expressions of the components of the tensor $a = \{a_{\alpha jk}^{i\beta\gamma}\}$ in the general (not reduced) third-order frame was found in [20] (see also [21, Section 2.2]). These expressions are:

$$\begin{aligned} a_{\alpha jk}^{i\beta\gamma} = u_{\alpha jk}^{i\beta\gamma} &- \frac{2}{q^2-1}\delta_{[j}^i\left(q u_{|\alpha|k]}^{[\beta\gamma]} + u_{|\alpha|k]}^{[\gamma\beta]}\right) - \frac{2}{p^2-1}\delta_\alpha^{[\beta}\left(p u_{[jk]}^{|i|\gamma]} + u_{[kj]}^{|i|\gamma]}\right) \\ &+ \frac{2}{(p^2-1)(q^2-1)}\Big[(pq-1)\left(\delta_{[j}^i \delta_{|\alpha|}^{[\beta} u_{k]}^{\gamma]} + \delta_{[k}^i \delta_{|\alpha|}^{[\beta} \widetilde{u}_{j]}^{\gamma]}\right) \\ &\qquad + (q-p)\left(\delta_{[j}^i \delta_{|\alpha|}^{[\beta} \widetilde{u}_{k]}^{\gamma]} + \delta_{[k}^i \delta_{|\alpha|}^{[\beta} u_{j]}^{\gamma]}\right)\Big], \end{aligned} \tag{3.11}$$

where the alternation is carried out with respect to the pairs of indices $\binom{\beta}{j}, \binom{\gamma}{k}$ or $\binom{\beta}{k}, \binom{\gamma}{j}$, and

$$u_k^\gamma = u_{lk}^{l\gamma} = u_{\sigma lk}^{l\sigma\gamma} = u_{\sigma k}^{\sigma\gamma}, \quad \widetilde{u}_k^\gamma = u_{kl}^{l\gamma} = u_{\sigma kl}^{l\sigma\gamma} = -u_{\sigma lk}^{l\gamma\sigma} = -u_{\sigma k}^{\gamma\sigma}.$$

If we prolong equations (3.10) and make a reduction of fourth-order frames of the almost Grassmann structure $AG(p-1, p+q-1)$, we will find the following remaining structure equations of $AG(p-1, p+q-1)$ which the forms ω_α^β, ω_j^i and κ satisfy:

$$\begin{aligned} d\omega_\alpha^\beta - \omega_\alpha^\gamma \wedge \omega_\gamma^\beta &= \frac{q}{p+q}\left(\delta_\alpha^\beta \omega_\gamma^k \wedge \omega_k^\gamma - p\omega_\alpha^k \wedge \omega_k^\beta\right) + b_{\alpha kl}^{\beta\gamma\delta} \omega_\gamma^k \wedge \omega_\delta^l, \\ d\omega_j^i - \omega_j^k \wedge \omega_k^i &= \frac{p}{p+q}\left(\delta_j^i \omega_k^\gamma \wedge \omega_\gamma^k - q\omega_j^\gamma \wedge \omega_\gamma^i\right) + b_{jkl}^{i\gamma\delta} \omega_\gamma^k \wedge \omega_\delta^l, \\ d\kappa &= \omega_i^\alpha \wedge \omega_\alpha^i, \end{aligned} \tag{3.12}$$

where the quantities $b_{\alpha lm}^{\beta\delta\varepsilon}$ and $b_{jlm}^{i\delta\varepsilon}$ are defined in a third-order neighborhood and satisfy the conditions

$$\begin{aligned} &b_{\alpha lm}^{\beta\delta\varepsilon} = -b_{\alpha ml}^{\beta\varepsilon\delta}, && b_{jlm}^{i\delta\varepsilon} = -b_{jml}^{i\varepsilon\delta}, \\ &b_{\sigma lm}^{\sigma\delta\varepsilon} = 0, && b_{klm}^{k\delta\varepsilon} = 0, \\ &b_{\alpha kl}^{\gamma\alpha\delta} - b_{kil}^{i\gamma\delta} + b_{\alpha lk}^{\delta\alpha\gamma} - b_{lik}^{i\delta\gamma} = 0. \end{aligned} \tag{3.13}$$

Note that the last conditions in (3.12) and (3.13) are the result of reduction of fourth-order frames mentioned above.

If we prolong equations (3.12) and set $\omega^i = 0$ in the resulting equations, we find that the quantities $b_{\alpha lm}^{\beta\delta\varepsilon}$ and $b_{jlm}^{i\delta\varepsilon}$ satisfy the following equations:

$$\begin{aligned} &\nabla_\delta b_{\alpha kl}^{\beta\gamma\delta} + 2b_{\alpha kl}^{\beta\gamma\delta}\pi - \frac{pq}{2(p+q)}\left(2\delta_\varepsilon^\beta a_{\alpha kl}^{m\gamma\delta} - \delta_\alpha^\gamma a_{\varepsilon kl}^{m\beta\delta} + \delta_\alpha^\delta a_{\varepsilon lk}^{m\beta\gamma}\right)\pi_m^\varepsilon = 0, \\ &\nabla_\delta b_{jkl}^{i\gamma\delta} + 2b_{jkl}^{i\gamma\delta}\pi + \frac{pq}{2(p+q)}\left(2\delta_j^m a_{\varepsilon kl}^{i\gamma\delta} - \delta_k^i a_{\varepsilon jl}^{m\gamma\delta} + \delta_l^i a_{\varepsilon jk}^{i\delta\gamma}\right)\pi_m^\varepsilon = 0, \end{aligned} \tag{3.14}$$

where the operator ∇_δ is defined in the same way as in formula (3.9), and δ is the symbol of differentiation with respect to the fiber parameters. It follows from equations (3.14) that the quantities $\{b^{\beta\gamma\delta}_{\alpha lm}\}$ and $\{b^{i\alpha\beta}_{jlm}\}$ do not form tensors or even homogeneous geometric objects, but the quantities $\{b^{\beta\gamma\delta}_{\alpha lm}, a^{i\beta\gamma}_{\alpha jk}\}$ as well as the quantities $\{b^{i\alpha\beta}_{jlm}, a^{i\beta\gamma}_{\alpha jk}\}$ form linear homogeneous objects. They represent two subobjects of the *second structure object* (or the *torsion-curvature object*) $\{a^{i\beta\gamma}_{\alpha jk}, b^{\beta\gamma\delta}_{\alpha lm}, b^{i\alpha\beta}_{jlm}\}$ of the almost Grassmann structure $AG(p-1, p+q-1)$.

Moreover, the prolongation of equations (3.12) leads to the following structure equations:

$$d\omega^\alpha_i - \omega^\beta_i \wedge \omega^\alpha_\beta - \omega^j_i \wedge \omega^\alpha_j + \kappa \wedge \omega^\alpha_i = c^{\alpha\beta\gamma}_{ijk}\omega^k_\gamma \wedge \omega^j_\beta - a^{k\alpha\beta}_{\gamma ij}\omega^\gamma_k \wedge \omega^j_\beta, \tag{3.15}$$

where $c^{\alpha\beta\gamma}_{ijk} = -c^{\alpha\gamma\beta}_{ikj}$.

In addition, taking exterior derivatives of the last equation of (3.12) and applying (3.10) and (3.15), we find the following condition for the quantities $c^{\alpha\beta\gamma}_{ijk}$:

$$c^{[\alpha\beta\gamma]}_{[ijk]} = 0. \tag{3.16}$$

Finally, if we prolong equations (3.15), we find that

$$\begin{aligned} &\big[\nabla c^{\alpha\beta\gamma}_{ijk} + 3c^{\alpha\beta\gamma}_{ijk}\kappa - b^{\alpha\gamma\beta}_{\sigma kj}\omega^\sigma_i + b^{l\gamma\beta}_{ikj}\omega^\alpha_l + (a^{l\alpha\beta}_{\delta ij}a^{m\delta\gamma}_{\varepsilon lk} + a^{m\alpha\delta}_{\varepsilon il}a^{l\gamma\beta}_{\delta kj})\omega^\varepsilon_m \\ &\quad + (2c^{\alpha\varepsilon\delta}_{iml}a^{l\gamma\beta}_{\beta kj}\omega^m_\varepsilon - c^{\delta\beta\gamma}_{ljk}a^{l\alpha\varepsilon}_{\delta im})\omega^m_\varepsilon\big] \wedge \omega^k_\gamma \wedge \omega^j_\beta = 0, \end{aligned} \tag{3.17}$$

where

$$\nabla c^{\alpha\beta\gamma}_{ijk} = dc^{\alpha\beta\gamma}_{ijk} - c^{\alpha\beta\gamma}_{ljk}\omega^l_i - c^{\alpha\beta\gamma}_{ilk}\omega^l_j - c^{\alpha\beta\gamma}_{ijl}\omega^l_k + c^{\delta\beta\gamma}_{ijk}\omega^\alpha_\delta + c^{\alpha\delta\gamma}_{ijk}\omega^\beta_\delta + c^{\alpha\beta\delta}_{ijk}\omega^\gamma_\delta.$$

For $\omega^i_\alpha = 0$, it follows from equation (3.17) that

$$\nabla_\delta c^{\alpha\delta\gamma}_{ilk} + 3c^{\alpha\beta\gamma}_{ilk}\pi - b^{\alpha\gamma\delta}_{\sigma kl}\pi^\sigma_i + b^{j\gamma\delta}_{ikl}\pi^\alpha_j + (a^{j\alpha\delta}_{\beta il}a^{m\beta\gamma}_{\varepsilon jk} + a^{m\alpha\beta}_{\varepsilon ij}a^{j\gamma\delta}_{\beta kl})\pi^\varepsilon_m = 0. \tag{3.18}$$

Equations (3.9), (3.13), and (3.18) prove that the quantities $\{a^{i\beta\gamma}_{\alpha jk}, b^{\beta\gamma\delta}_{\alpha km}, b^{l\beta\gamma}_{ijk}, c^{\alpha\beta\gamma}_{ijk}\}$ also form a linear homogeneous object which is called the *third structure object* of the almost Grassmann structure $AG(m, n)$. Note that the torsion tensor $\{a^{i\beta\gamma}_{\alpha jk}\}$ is defined in a second-order differential neighborhood of a point $x \in M$, the second structure object $\{a^{i\beta\gamma}_{\alpha jk}, b^{\beta\gamma\delta}_{\alpha km}, b^{i\alpha\beta}_{jlm}\}$ is defined in a third-order differential neighborhood of a point $x \in M$, and the third structure object $\{a^{i\beta\gamma}_{\alpha jk}, b^{\beta\gamma\delta}_{\alpha km}, b^{l\beta\gamma}_{ijk}, c^{\alpha\beta\gamma}_{ijk}\}$ is defined in a fourth-order differential neighborhood of a point $x \in M$.

The structure equations (3.10), (3.12), and (3.15) can be written in the index-free notation as follows:

$$\begin{aligned} &d\omega = \kappa \wedge \omega - \omega \wedge \theta_\alpha - \theta_\beta \wedge \omega + \Omega, \\ &d\theta_\alpha + \theta_\alpha \wedge \theta_\alpha = \frac{q}{p+q}\Big[-I_\alpha \operatorname{tr}(\varphi \wedge \omega) + p\varphi \wedge \omega\Big] + \Theta_\alpha, \\ &d\theta_\beta + \theta_\beta \wedge \theta_\beta = \frac{p}{p+q}\Big[-I_\beta \operatorname{tr}(\varphi \wedge \omega) + q\omega \wedge \varphi\Big] + \Theta_\beta, \\ &d\kappa = \operatorname{tr}(\varphi \wedge \omega), \\ &d\varphi + \theta_\alpha \wedge \varphi + \varphi \wedge \theta_\beta + \kappa \wedge \varphi = -(a\varphi) \wedge \omega + \Phi, \end{aligned} \tag{3.19}$$

where $\omega = (\omega^i_\alpha)$ is a matrix 1-form defined in the first-order frame bundle; κ is a scalar 1-form, $\theta_\alpha = (\omega^\alpha_\beta)$ and $\theta_\beta = (\omega^i_j)$ are the matrix 1-forms defined in the second-order frame bundle for which

$$\operatorname{tr}\theta_\alpha = 0, \qquad \operatorname{tr}\theta_\beta = 0;$$

$\varphi = (\omega_i^\alpha)$ is a matrix 1-form defined in a third-order fiber bundle; $I_\alpha = (\delta_\alpha^\beta)$ and $I_\beta = (\delta_i^j)$ are the unit tensors of orders p and q, respectively; the 2-form $\Omega = (\Omega_\alpha^i)$ is the torsion form; and the 2-forms $\Theta_\alpha = (\Theta_\beta^\alpha)$, $\Theta_\beta = (\Theta_j^i)$, and $\Phi = (\Phi_i^\alpha)$ are the *curvature 2-forms* of the $AG(p-1, p+q-1)$-structure. The components of 2-forms Ω, Θ_α, Θ_β and Φ are

$$\begin{aligned} \Omega_\alpha^i &= a_{\alpha jk}^{i\beta\gamma}\omega_\gamma^k \wedge \omega_\delta^l, \quad \Theta_\beta^\alpha = b_{\beta kl}^{\alpha\gamma\delta}\omega_\gamma^k \wedge \omega_\delta^l, \\ \Theta_j^i &= b_{jkl}^{i\gamma\delta}\omega_\gamma^k \wedge \omega_\delta^l, \quad \Phi_i^\alpha = c_{ikl}^{\alpha\gamma\delta}\omega_\delta^l \wedge \omega_\gamma^k. \end{aligned} \tag{3.20}$$

4. The restrictions of equations (3.19) to a fiber frame bundle, i.e., for $\omega = 0$, have the form

$$\begin{aligned} &d\kappa = 0, \\ &d\theta_\alpha = -\theta_\alpha \wedge \theta_\alpha, \quad d\theta_\beta = -\theta_\beta \wedge \theta_\beta, \\ &d\varphi = -\kappa \wedge \varphi - \theta_\alpha \wedge \varphi - \varphi \wedge \theta_\beta. \end{aligned} \tag{3.21}$$

From equations (3.21) it follows that

(1) The form κ is an invariant form of the group $\mathbf{H}$ of homotheties acting in the space $T_x(M)$.

(2) The forms θ_α and θ_β are invariant forms of the special linear groups $\mathbf{SL}(p)$ and $\mathbf{SL}(q)$, respectively.

(3) The 1-forms κ, θ_α, and θ_β are invariant forms of the structural group G of the almost Grassmann structure $AG(m, n)$ whose transformations leave cone $SC_x(p, q)$ invariant. As noted earlier, the group G is isomorphic to the direct product $\mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}$:

$$G \cong \mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}. \tag{3.22}$$

(4) The 1-forms $\kappa, \theta_\alpha, \theta_\beta$, and φ are invariant forms of the group G' which is a differential prolongation of G. The group G' is isomorphic to the group $G \ltimes \mathbf{T}(pq)$ whose subgroup $\mathbf{T}(pq)$ is defined by the invariant forms ω_i^α. Thus,

$$G' \cong (\mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}) \ltimes \mathbf{T}(pq). \tag{3.23}$$

Since the group G' does not admit further prolongations, *an almost Grassmann structure $AG(m, n)$ is a G-structure of finite type two.*

In order to describe the group G' geometrically, we compactify the tangent subspace $T_x(M)$ by enlarging it by the point at infinity, $y = \infty$, and the Segre cone $SC_y(p, q)$ with its vertex at this point. Then the manifold $T_x(M) \cap SC_y(p, q)$ is equivalent to the algebraic variety $\Omega(p, q)$. Since the point x, at which the variety $\Omega(p, q)$ is tangent to the manifold M, is fixed, the geometry defined by the group G' on $\Omega(p, q)$ is equivalent to that of the space obtained as a projection of the variety $\Omega(p, q)$ from the point x onto a flat space of dimension pq. The group G' is the group of motions of this space, its subgroup G is the isotropy group of this space, and the subgroup $\mathbf{T}(pq)$ is the subgroup of parallel translations.

The group G' can be also represented as the group of motions of a projective space P^n leaving invariant a fixed subspace $P^m = x$. The subgroup $\mathbf{SL}(p)$ of G' is locally isomorphic to the group of projective transformations of the subspace P^m; the subgroup $\mathbf{SL}(q)$ of G' is locally isomorphic to the group of transformations in the bundle of $(m+1)$-dimensional subspaces of P^n passing through x; the subgroup $\mathbf{H}$ of G' is the group of homotheties with its center at x; and the subgroup $\mathbf{T}(pq)$ of G' is the group of translations of the subspaces $y = P^{n-m-1}$ which are complementary to $x = P^m$ in P^n.

5. As it was proved in [20] (see also [21] and [27]), the torsion tensor $a = \{a^{i\beta\gamma}_{\alpha jk}\}$ of an almost Grassmann structure $AG(p-1, p+q-1)$ satisfying conditions (3.7)–(3.8) decomposes into two subtensors:

$$a = a_\alpha \dotplus a_\beta, \tag{3.24}$$

where $a_\alpha = \{a^{i\beta\gamma}_{\alpha(jk)}\}$ and $a_\beta = \{a^{i(\beta\gamma)}_{\alpha jk}\}$. Note that $a^{i\beta\gamma}_{\alpha(jk)} = a^{i[\beta\gamma]}_{\alpha jk}$ and $a^{i(\beta\gamma)}_{\alpha jk} = a^{i\beta\gamma}_{\alpha[jk]}$.

It is easy to see that the components of the subtensors a_α and a_β satisfy the conditions similar to conditions (3.8).

In addition, it is easy to prove that:

(1) *If $p = 2$, then $a_\alpha = 0$.*

(2) *If $q = 2$, then $a_\beta = 0$.*

There are certain dependencies between three structure objects indicated above. Let us set $b^1 = \{b^{i\gamma\delta}_{jkl}\}$, $b^2 = \{b^{\alpha\gamma\delta}_{\beta kl}\}$ and $c = \{c^{\alpha\gamma\delta}_{ikl}\}$. It can be proved that

(1) *If $q > 2$, then the components of b^2 are expressed in terms of components of the tensor a and their Pfaffian derivatives, and the components of c are expressed in terms of components of the object (a, b^2) and their Pfaffian derivatives.*

(2) *If $p > 2$, then the components of b^1 are expressed in terms of components of the tensor a and their Pfaffian derivatives, and the components of c are expressed in terms of components of the (a, b^1) and their Pfaffian derivatives.*

(3) *If $p > 2$ and $q > 2$, then the components of b and c are expressed in terms of components of the tensor a and their Pfaffian derivatives.*

However, the tensor a itself is not arbitrary since if we substitute for the components b and c their expressions in terms of the tensor a and its Pfaffian derivatives into conditions of integrability of the structure equations (3.12) and (3.15), we obtain certain algebraic conditions for the components of the tensor a and its covariant derivatives. The latter conditions are analogues of the Bianchi equations in the theory of spaces with affine connection.

The structure object $S = \{a, b, c\}$ of the almost Grassmann structure $AG(p-1, p+q-1)$ is *complete* in the sense that if we prolong the structure equations (3.19) of $AG(p-1, p+q-1)$, then all newly arising objects are expressed in terms of the components of the object S and their Pfaffian derivatives. This follows from the fact that *the almost Grassmann structure $AG(p-1, p+q-1)$ is a G-structure of finite type two.*

Definition 3.10. An almost Grassmann structure $AG(p-1, p+q-1)$ is said to be *locally Grassmann* (or *locally flat*) if it is locally equivalent to a Grassmann structure.

This means that a locally flat almost Grassmann structure $AG(p-1, p+q-1)$ admits a mapping onto an open domain of the algebraic variety $\Omega(m, n)$ of a projective space P^N, where

$$N = \binom{n+1}{m+1} - 1, \quad m = p-1, \quad n = p+q-1,$$

under which the Segre cones of the structure $AG(p-1, p+q-1)$ correspond to the asymptotic cones of variety $\Omega(m, n)$.

From the equivalence theorem of É. Cartan (see [12] or [18]), it follows that in order for an almost Grassmann structure $AG(p-1, p+q-1)$ to be locally Grassmann, it is necessary and sufficient that its structure equations have the form (2.17). Comparing these equations with equations (3.19), we see that *an almost Grassmann structure $AG(p-1, p+q-1)$ is locally Grassmann if and only if its complete structure object $S = (a, b, c)$ vanishes.*

However, as was noted above, if $p > 2$ and $q > 2$, the components of b are expressed in terms of the components of the tensor a and their Pfaffian derivatives, and the components of c are expressed in terms of the components of the subobject (a, b) and their Pfaffian derivatives. Moreover, it can be proved that the vanishing of the tensor a on a manifold M carrying an almost Grassmann structure implies the vanishing of the components of b and c.

This implies the following result.

Theorem 3.11. *For $p > 2$ and $q > 2$, an almost Grassmann structure $AG(p-1, p+q-1)$ is locally Grassmann if and only if its first structure tensor a vanishes.*

4. Semiintegrability of almost Grassmann structures

1. Now we will prove the following necessary and sufficient conditions for the almost Grassmann structure $AG(p-1, p+q-1)$ to be α- or β-semiintegrable.

Theorem 4.1. (1) *If $p > 2$ and $q \geqslant 2$, then for an almost Grassmann structure $AG(p-1, p+q-1)$ to be α-semiintegrable, it is necessary and sufficient that the following condition holds*:

$$a_\alpha = b^1_\alpha = b^2_\alpha = 0.$$

(2) *If $p \geqslant 2$ and $q > 2$, then for an almost Grassmann structure $AG(p-1, p+q-1)$ to be β-semiintegrable, it is necessary and sufficient that the following condition holds*:

$$a_\beta = b^1_\beta = b^2_\beta = 0.$$

Proof. We will prove part (1) of theorem. The proof of part (2) is similar.

Suppose that θ_α, $\alpha = 1, \ldots, p$, are basis forms of the subvarieties V_α, $\dim V_\alpha = p$, indicated in Definition 3.6. Then

$$\omega^i_\alpha = s^i \theta_\alpha, \quad \alpha = 1, \ldots, p; \quad i = p+1, \ldots, p+q. \tag{4.1}$$

For the structure $AG(p-1, p+q-1)$ to be α-semiintegrable, it is necessary and sufficient that system (4.1) be completely integrable. Taking the exterior derivatives of equations (4.1) by means of structure equations (3.10), we find that

$$(ds^i + s^j\omega_j^i - s^i\omega)\wedge\theta_\alpha + s^i(d\theta_\alpha - \omega_\alpha^\beta\wedge\theta_\beta) = a_{\alpha jk}^{i\beta\gamma}s^j s^k\theta_\beta\wedge\theta_\gamma. \tag{4.2}$$

It follows from these equations that

$$d\theta_\alpha - \omega_\alpha^\beta\wedge\theta_\beta = \varphi_\alpha^\beta\wedge\theta_\beta, \tag{4.3}$$

where φ_α^β is a 1-form that is not expressed in terms of the basis forms θ_α.

For brevity, we set

$$\varphi^i = ds^i + s^j\omega_j^i - s^i\omega. \tag{4.4}$$

Then the exterior quadratic equation (4.2) takes the form:

$$(\delta_\alpha^\beta\varphi^i + s^i\varphi_\alpha^\beta)\wedge\theta_\beta = a_{\alpha jk}^{i\beta\gamma}s^j s^k\theta_\beta\wedge\theta_\gamma. \tag{4.5}$$

From (4.5) it follows that for $\theta_\alpha = 0$, the 1-form $\delta_\alpha^\beta\varphi^i + s^i\varphi_\alpha^\beta$ vanishes:

$$\delta_\alpha^\beta\varphi^i(\delta) + s^i\varphi_\alpha^\beta(\delta) = 0. \tag{4.6}$$

Contracting equation (4.6) with respect to the indices α and β, we find that

$$\varphi^i = -s^i\varphi(\delta), \quad \varphi_\alpha^\beta = \delta_\alpha^\beta\varphi(\delta), \tag{4.7}$$

where we set $\varphi(\delta) = \varphi_\gamma^\gamma/p$.

It follows from (4.7) that on the subvariety V_α, the 1-forms φ^i and φ_α^β can be written as follows:

$$\varphi^i = -s^i\varphi + s^{i\beta}\theta_\beta, \quad \varphi_\alpha^\beta = \delta_\alpha^\beta\varphi + \widehat{s}_\alpha^{\beta\gamma}\theta_\gamma. \tag{4.8}$$

Substituting these expressions into equations (4.3) and (4.4), we find that

$$d\theta_\alpha - \omega_\alpha^\beta\wedge\theta_\beta = s_\alpha^{\beta\gamma}\theta_\gamma\wedge\theta_\beta \tag{4.9}$$

where $s_\alpha^{\beta\gamma} = \widehat{s}_\alpha^{[\beta\gamma]}$, and

$$ds^i + s^j\omega_j^i - s^i\kappa = -s^i\varphi + s^{i\beta}\theta_\beta. \tag{4.10}$$

Substituting (4.9) and (4.10) into equation (4.2), we obtain

$$-s^i s_\alpha^{\beta\gamma} - \delta_\alpha^{[\beta}s^{|i|\gamma]} = a_{\alpha jk}^{i[\beta\gamma]}s^j s^k. \tag{4.11}$$

Contracting equation (4.11) with respect to the indices α and β, we find that

$$-2s^i s_\alpha^{\alpha\gamma} - ps^{i\gamma} + s^{i\gamma} = 0.$$

It follows that

$$s^{i\gamma} = s^i s^\gamma, \tag{4.12}$$

where we set $s^\gamma = -2s_\alpha^{\alpha\gamma}/(p-1)$. Substituting (4.12) into (4.11), we find that

$$s^i(\delta_\alpha^\gamma s^\beta - \delta_\alpha^\beta s^\gamma - 2s_\alpha^{\beta\gamma}) = 2a_{\alpha jk}^{i[\beta\gamma]}s^j s^k. \tag{4.13}$$

It follows that

$$\delta_\alpha^\gamma s^\beta - \delta_\alpha^\beta s^\gamma - 2s_\alpha^{\beta\gamma} = s_{\alpha j}^{\beta\gamma} s^j, \tag{4.14}$$

where $s_{\alpha j}^{\beta\gamma} = -s_{\alpha j}^{\gamma\beta}$. Substituting (4.14) into (4.13), we arrive at the equation

$$s_{\alpha(j}^{\beta\gamma}\delta_{k)}^i = a_{\alpha(jk)}^{i\beta\gamma}, \tag{4.15}$$

where the alternation sign in the right-hand side was dropped since $a_{\alpha(jk)}^{i\beta\gamma} = a_{\alpha jk}^{i[\beta\gamma]}$.

Contracting (4.15) with respect to the indices i and j and taking into account equations (3.7) and (3.8), we obtain

$$s_{\alpha k}^{\beta\gamma} = 0. \tag{4.16}$$

By (4.15), it follows that

$$a_{\alpha(jk)}^{i\beta\gamma} = 0. \tag{4.17}$$

Thus, we proved that *if an almost Grassmann structure $AG(p-1, p+q-1)$ is α-semiintegrable, then its torsion tensor satisfies the condition* (4.17), *i.e.*, $a_\alpha = 0$.

Since as was noted earlier, for $p = 2$, the subtensor $a_\alpha = 0$, condition (4.17) is identically satisfied. Hence while proving sufficiency of this condition for α-semiintegrability, we must assume that $p > 2$.

Let us return to equations (4.9) and (4.10). Substitute into equation (4.10) the values $s^{i\beta}$ taken from (4.12) and set

$$\widetilde{\varphi} = \varphi - s^\beta\theta_\beta. \tag{4.18}$$

In addition, by (4.16), relations (4.14) imply that

$$s_\alpha^{\beta\gamma} = \delta_\alpha^{[\gamma} s^{\beta]}.$$

Then equations (4.9) and (4.10) take the form

$$d\theta_\alpha - (\omega_\alpha^\beta + \delta_\alpha^\beta\widetilde{\varphi}) \wedge \theta_\beta = 0 \tag{4.19}$$

and

$$ds^i + s^j\omega_j^i - s^i(\kappa - \widetilde{\varphi}) = 0. \tag{4.20}$$

Taking the exterior derivatives of (4.20), we obtain the following exterior quadratic equation:

$$s^i\Phi + b_{jkl}^{i\gamma\delta} s^j s^k s^l \theta_\gamma \wedge \theta_\delta = 0, \tag{4.21}$$

where

$$\Phi = d\widetilde{\varphi} - \frac{(p+1)q}{p+q} s^k\omega_k^\gamma \wedge \theta_\gamma.$$

Next, taking the exterior derivatives of (4.19), we find that

$$\Phi \wedge \theta_\alpha + b_{\alpha kl}^{\beta\gamma\delta} s^k s^l \theta_\beta \wedge \theta_\gamma \wedge \theta_\delta = 0. \tag{4.22}$$

Equation (4.21) shows that the 2-form Φ can be written as

$$\Phi = A_{kl}^{\gamma\delta} s^k s^l \theta_\gamma \wedge \theta_\delta, \tag{4.23}$$

where the coefficients $A_{kl}^{\gamma\delta}$ are symmetric with respect to the lower indices and skew-symmetric with respect to the upper indices. Substituting this value of the form Φ into equations (4.21) and (4.22), we arrive at the conditions:

$$b_{(jkl)}^{i[\gamma\delta]} + \delta_{(j}^{i} A_{kl)}^{\gamma\delta} = 0 \tag{4.24}$$

and

$$b_{\alpha(kl)}^{[\beta\gamma\delta]} + \delta_{\alpha}^{[\beta} A_{kl}^{\gamma\delta]} = 0. \tag{4.25}$$

Contracting equation (4.24) with respect to the indices i and j and equation (4.25) with respect to the indices α and β, we obtain

$$2(q+2)A_{kl}^{\gamma\delta} + b_{kli}^{i\gamma\delta} + b_{kil}^{i\gamma\delta} + b_{lik}^{i\gamma\delta} + b_{lki}^{i\gamma\delta} = 0 \tag{4.26}$$

and

$$2(p-2)A_{kl}^{\gamma\delta} + b_{\alpha kl}^{\gamma\delta\alpha} + b_{\alpha lk}^{\gamma\delta\alpha} + b_{\alpha kl}^{\delta\alpha\gamma} + b_{\alpha lk}^{\delta\alpha\gamma} = 0. \tag{4.27}$$

Note that for $p = 2$, equation (4.25) becomes an identity, and we shall not obtain equations (4.27).

If we add equations (4.26) and (4.27) and apply the last condition of (3.13), we find that

$$A_{kl}^{\gamma\delta} = 0. \tag{4.28}$$

As a result, equations (4.24) and (4.25) take the form

$$b_{(jkl)}^{i[\gamma\delta]} = 0, \qquad b_{\alpha(kl)}^{[\beta\gamma\delta]} = 0. \tag{4.29}$$

By the first conditions of (3.13), conditions (4.29) are equivalent to the conditions

$$b_{(jkl)}^{i\gamma\delta} = 0, \qquad b_{\alpha kl}^{[\beta\gamma\delta]} = 0. \tag{4.30}$$

It follows from equations (4.28) and (4.23) that

$$d\widetilde{\varphi} = \frac{(p+1)q}{p+q} s^k \omega_k^\gamma \wedge \theta_\gamma. \tag{4.31}$$

Finally, taking the exterior derivatives of equations (4.31) and applying (4.19), (4.20) and (3.15), we obtain the condition

$$c_{(ijk)}^{[\alpha\beta\gamma]} = 0. \tag{4.32}$$

This equation will not be trivial only if $p > 2$. It is easy to check that the last condition follows from integrability condition (3.16).

Thus, the system of Pfaffian equations (4.1), defining integral submanifolds of an α-semiintegrable almost Grassmann structure, together with Pfaffian equations (4.10) and (4.31) following from (4.1) is completely integrable if and only if conditions (4.17) and (4.30) are satisfied. This concludes the proof of part (1) of the theorem.

We introduce the following notation:

$$b^1_\alpha = \{b^{i\gamma\delta}_{(jkl)}\}, \qquad b^2_\alpha = \{b^{[\beta\gamma\delta]}_{\alpha kl}\}, \qquad c_\alpha = \{c^{[\alpha\beta\gamma]}_{(ijk)}\},$$
$$b^1_\beta = \{b^{i\gamma\delta}_{[jkl]}\}, \qquad b^2_\beta = \{b^{(\beta\gamma\delta)}_{\alpha kl}\}, \qquad c_\beta = \{c^{(\alpha\beta\gamma)}_{[ijk]}\}.$$

Note that for $p = 2$, we have $b^2_\alpha = 0$ and $c_\alpha = 0$; for $q = 2$, we have $b^1_\beta = 0$ and $c_\beta = 0$; for $p > 2$, we have $c_\alpha = 0$; and for $q > 2$, we have $c_\beta = 0$.

By equations (3.14), the quantities indicated above and the subtensors a_α and a_β form the following geometric objects:

$$(a_\alpha, b^1_\alpha), \qquad (a_\alpha, b^2_\alpha), \qquad S_\alpha = (a_\alpha, b^1_\alpha, b^2_\alpha),$$
$$(a_\beta, b^1_\beta), \qquad (a_\beta, b^2_\beta), \qquad S_\beta = (a_\beta, b^1_\beta, b^2_\beta),$$

which are subobjects of the second structural object and the complete structural object of the almost Grassmann structure.

Now we consider the cases $p = 2$ and $q = 2$. For definiteness, we take the case $p = 2$. As we have already seen, for $p = 2$, the tensor a_α as well as the quantities b^2_α and c_α vanish: $a_\alpha = b^2_\alpha = c_\alpha = 0$, and the object b^1_α becomes a tensor. Thus, the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure $AG(1, q+1)$ to be α-semiintegrable.

Hence we have proved the following result.

Theorem 4.2. (1) *If $p = 2$, then the structure subobject S_α consists only of the tensor b^1_α, and the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure $AG(1, q + 1)$ to be α-semiintegrable.*

(2) *If $q = 2$, then the structure subobject S_β consists only of the tensor b^2_β, and the vanishing of this tensor is necessary and sufficient for the almost Grassmann structure $AG(p - 1, p + 1)$ (which is equivalent to the structure $AG(1, p + 1)$) to be β-semiintegrable.*

(3) *If $p = q = 2$, then the complete structural object S consists only of the tensors b^1_α and b^2_β, and the vanishing of one of these tensors is necessary and sufficient for the almost Grassmann structure $AG(1, 3)$ to be α- and β-semiintegrable, respectively.*

We will make two more remarks:

1) The tensors b^1_α and b^2_β are defined in a third-order differential neighborhood of the almost Grassmann structure.

2) As was indicated earlier, for $p = q = 2$, the almost Grassmann structure $AG(1, 3)$ is equivalent to the conformal $CO(2, 2)$-structure. Thus, by results of Subsection 1.5, we have the following decomposition of its complete structural object: $S = b^1_\alpha \dotplus b^2_\beta$. This matches the splitting of the tensor of conformal curvature of the $CO(2, 2)$-structure: $b = b_\alpha \dotplus b_\beta$.

If $p = 2$ or $q = 2$, the conditions for an almost Grassmann structure to be semiflat or flat are connected with a differential neighborhood of third order and cannot be expressed in terms of the torsion tensor a. The reason for this is that if $q = 2$, then, as we noted earlier, the tensor a_β automatically vanishes, and if $p = 2$, then the same is true for the tensor a_α.

Finally, if $p = q = 2$, then for the almost Grassmann structure $AG(1, 3)$ we have $a = 0$, and the conditions for an $AG(1, 3)$-structure to be semiflat or flat are connected with a differential neighborhood of third order of the manifold.

In fact, if $p = q = 2$, then the almost Grassmann structure $AG(1,3)$ is equivalent to the $CO(2,2)$-structure. As equations (1.11) show, the torsion tensor of the latter structure vanishes, and the conditions for a $CO(2,2)$-structure to be semiintegrable or integrable are expressed in terms of the tensor of conformal curvature which is defined in a differential neighborhood of third order of the manifold.

2. The following table presents the similarities and differences between the conformal structures $CO(p,q)$ and almost Grassmann structures $AG(m,n)$:

#	Property	$CO(p,q)$	$AG(m,n)$
1.	$\dim M$	$n = p + q$	$n = p \cdot q$, $p = m + 1,\ q = n - m$
2.	Invariant construction in $T_x(M)$	2nd order cone $C_x(p,q)$	Segre cone $SC_x(p,q)$
3.	Order of G-structure	$s = 1$	$s = 1$
4.	Structure group	$G \cong \mathbf{SO}(p,q) \times \mathbf{H}$	$G \cong \mathbf{SL}(p) \times \mathbf{SL}(q) \times \mathbf{H}$
5.	Prolonged structure group	$G' \cong G \ltimes \mathbf{T}(p+q)$	$G' \cong G \ltimes \mathbf{T}(p \cdot q)$
6.	Type of G-structure	$t = 2$	$t = 2$
7.	Existence of torsion	torsion-free	torsion exists
8.	Complete structure object	$\{b, c\}$	$\{a, b, c\}$
9.	Local space	$(C_q^n)_x$	$G(m,n)_x$
10.	Locally flat structure	C_q^n	$G(m,n)$
11.	Existence of isotropic bundles	$p = q = 2$: $E_\alpha(M, \mathbf{SL}(2))$ and $E_\beta(M, \mathbf{SL}(2))$	for all p and q: $E_\alpha(M, \mathbf{SL}(p))$ and $E_\beta(M, \mathbf{SL}(q))$
12.	$p + q = p \cdot q \implies$ $p = q = 2$	$CO(2,2)$	$AG(1,3)$

Table 1

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