The structure of strong arc-locally semicomplete digraphs

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Abstract

Arc-locally semicomplete digraphs were introduced in (Preprint, No. 10, 1993, Department of Mathematics and Computer Science, University of Southern Denmark) as a common generalization of semicomplete digraphs and semicomplete bipartite digraphs. In this note, we prove that the structure of these digraphs is very closely related to that of semicomplete and semicomplete bipartite digraphs. In fact, we show that if a strong arc-locally semicomplete digraph is neither semicomplete nor semicomplete bipartite, then it is obtained from a directed cycle by substituting independent sets of vertices for each vertex of the cycle. We also identify a new class of digraphs for which the hamiltonian cycle problem seems tractable and non-trivial. As it is the case for arc-locally semicomplete digraphs, this new class of digraphs contains all semicomplete digraphs and all semicomplete bipartite digraphs.

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1. Introduction

In [2], Bang-Jensen introduced locally semicomplete digraphs as a generalization of semicomplete digraphs. These are the digraphs in which the set of in-neighbours and the set of out-neighbours of a vertex induces a semicomplete digraph. In [5], Bang-Jensen and Huang introduced quasi-transitive digraphs. These are the digraphs in which the presence of arcs from \( x \) to \( y \) and from \( y \) to \( z \) implies that \( x \) and \( z \) must be adjacent. Locally semicomplete digraphs and quasi-transitive digraphs both form two very nice classes of digraphs which are much larger than the class of semicomplete digraphs but still have a lot of structure. For a number of results on these classes of digraphs see [2–6,8]. By definition, a digraph is locally semicomplete if it contains none of the digraphs \( D_1, \ldots, D_5 \) of Fig. 1 as induced subdigraphs and a digraph is quasi-transitive if it does not contain any of the digraphs \( D_3, \ldots, D_6 \) as an induced subdigraph.

In [1], Bang-Jensen tried to extend the idea, of only requiring a special local structure, to include bipartite digraphs. Namely, we studied the digraphs with the property that if \( x, y \) are adjacent vertices, then every in-neighbour (out-neighbour) \( z \) of \( x \) is adjacent to every in-neighbour (out-neighbour) \( w \) of \( y \) (provided \( z \neq w \)). In [1] these digraphs were called arc-local tournament digraphs, but since we allow directed 2-cycles the name arc-locally semicomplete digraphs seems more appropriate. To see the analogy to the definition of locally semicomplete digraphs it is helpful to note that the definition above is equivalent to requiring that if there are two vertices \( z, w \) in \( V(D) - \{x, y\} \) such that \( x \to z, y \to w, w \to z \) or \( z \to x, z \to w, w \to y \) are all arcs then \( x \) and \( y \) must be adjacent or the same vertex. See Fig. 2. A digraph is arc-locally semicomplete if it contains no induced subdigraph from any of the classes \( \mathcal{H}_1, \mathcal{H}_2 \).

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We demonstrated in [1] that results on hamiltonian paths and cycles in semicomplete bipartite digraphs, including the polynomial solvability of these problems, can be extended to arc-locally semicomplete digraphs. In this note we prove that the class of strong arc-locally semicomplete digraphs is very close to semicomplete and semicomplete bipartite digraphs. We also prove that every strong digraph containing no induced subdigraph from the class $\mathcal{H}_i$ is either semicomplete or semicomplete bipartite. Finally we conjecture that for digraphs which are $\mathcal{H}_i$-free where $i$ is either 1, 2 or 4 one can solve the hamiltonian path and cycle problems in polynomial time.

2. Terminology and preliminaries

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [4] for terminology not defined here. For any digraph $D$, we denote by $UG(D)$, the underlying undirected graph of $D$. We say that $D$ is connected if $UG(D)$ is a connected graph. We shall always assume that the digraphs in this note are connected. We use $n(m)$ to denote the number of vertices (arcs) of the digraph studied.

Let $D$ be a digraph. If there is an arc from a vertex $x$ to a vertex $y$ in $D$ we say that $x$ dominates $y$ and use the notation $x \to y$ to denote this. If $A$ and $B$ are disjoint subsets of vertices of $D$ we use the notation $x \Rightarrow y$ to denote that $x \to y$ for any choice of $x \in A$ and $y \in B$ and there is no arc from $B$ to $A$. For each $x \in V(D)$ the sets $N^+(x)$ ($N^-(x)$) denote the set of those vertices $y \in V(D)$ such that $x \to y$ ($y \to x$). With this notation a digraph is arc-locally semicomplete if and only if, for each edge $xy$ in $UG(D)$, every vertex $z \in N^-(x)$ is adjacent to every vertex $w \in N^-(y)$ for which $w \neq z$ and every vertex $z \in N^+(x)$ is adjacent to every vertex $w \in N^+(y)$ for which $w \neq z$. A semicomplete bipartite digraph is a bipartite digraph whose underlying undirected graph is complete bipartite.

Cycles and paths are always directed. A $k$-cycle is a cycle with $k$ vertices and we use the notation $C_k$ for a $k$-cycle. An extended cycle is a digraph which can be obtained from a cycle $C_k$, for some $k \geq 2$, by substituting an independent set for each vertex of $C_k$. We write $D = C_k[E_{n_1}, E_{n_2}, \ldots, E_{n_k}]$ where $E_{n_i}$ denotes the independent set of $n_i \geq 1$ vertices which we substitute for the $i$'th vertex of $C_k$. That is, we have $E_{n_1} \Rightarrow E_{n_2} \Rightarrow \cdots \Rightarrow E_{n_k} \Rightarrow E_{n_1}$ and there are no other arcs in $D$. See Fig. 3 for an example.

An $(x, y)$-path is a directed path from $x$ to $y$. A digraph $D$ is strongly connected (or just strong) if there exists an $(x, y)$-path and a $(y, x)$-path in $D$ for any choice of distinct vertices $x, y$ of $D$.

A cycle factor in a digraph $D$ is a collection of vertex disjoint cycles which cover $V(D)$. Gutin found the following very nice characterization of hamiltonian semicomplete bipartite digraphs.
Theorem 2.1 (Gutin [7]). A semicomplete bipartite digraph is hamiltonian if and only if it is strong and has a cycle factor.

The following easy fact will be very useful in our proofs.

Lemma 2.2. Let $D$ be a connected arc-locally semicomplete digraph and let $D'$ be any non-trivial strong subdigraph of $D$. Every vertex $x \in V(D) - V(D')$ is adjacent to some vertex in $V(D')$.

Proof. Suppose some vertex $x \in V(D) - V(D')$ is not adjacent to $D'$. Let $x = x_1x_2 \ldots x_n$, $n \geq 3$, $x_n \in V(D')$, be a shortest path between $x$ and $D'$ in $U(D)$. Let $u \in V(D')$ (or $w \in V(D')$) be some vertex which dominates (is dominated by) $x_n$ in $D'$. Now, depending on the orientation of the edge $x_n \rightarrow x_{n-1}$ in $D$, we conclude that $x_{n-2}$ is adjacent to $u$ or $w$, contradicting the minimality of the path above. □

The following result which can be derived from Lemma 2.2 shows that non-strong arc-locally semicomplete digraphs are semicomplete or semicomplete bipartite unless some vertex is not on any directed cycle.

Lemma 2.3 (Bang-Jensen [1]). Let $D$ be a non-strong arc-locally semicomplete digraph. If every vertex of $D$ is on some cycle, then $D$ is semicomplete or semicomplete bipartite.

Note that the underlying graph of an arc-locally semicomplete digraph may be any bipartite digraph, since the digraph obtained from a bipartite graph by orienting all edges from one colour class to the other is trivially an arc-locally semicomplete digraph.

3. Strong arc-locally semicomplete digraphs

Lemma 3.1. Let $D$ be a strong arc-locally semicomplete digraph. If $D$ contains an induced cycle of length at least 5, then $D$ is an extension of a cycle.

Proof. Let $C$ be an induced cycle of length at least 5 in $D$. We claim that $D = C_k[E_1, \ldots, E_n]$, where $k$ is the length of $C$. Recall that by Lemma 2.2 every vertex $x$ not on $C$ is adjacent to $C$. Let $C = u_1u_2 \ldots u_ku_1$ and suppose w.l.o.g. that $u_1 \rightarrow x$. Then $x$ is adjacent to $u_1$. If $u_1 \rightarrow x$, then $u_k$ and $u_1$ must be adjacent contradicting the fact that $C$ is induced and has at least 5 vertices. So $x \rightarrow u_1$. We claim that $x$ is not adjacent to any other vertices of $C$. Suppose first that $x$ is adjacent to $u_2$. If $u_2 \rightarrow x$, then this arc and the arcs $u_k \rightarrow u_1$, $u_1 \rightarrow x$ imply an arc between $u_2$ and $u_k$, contradicting the fact that $C$ is induced. So $x \rightarrow u_2$. Considering the arcs $x \rightarrow u_2$, $x \rightarrow u_3$ and $u_1 \rightarrow u_4$, we see that $u_2$ and $u_4$ must be adjacent, a contradiction. Suppose now that $x$ and $u_4$ are adjacent for some $i \in \{4, 5, \ldots, k\}$. Now using the presence of the two arcs $u_1 \rightarrow x$, $x \rightarrow u_4$ it is easy to show $u_2$ must be adjacent to $u_4$, contradicting the fact that $C$ is induced. Hence, we have shown that every $x \in V(D) - V(C)$ has precisely two neighbours on $C$, namely, they have the form $u_j \rightarrow x \rightarrow u_{j+2}$ modulo $k$. Now let $x, x' \in V(D) - V(C)$ such that $x' \rightarrow x$ and let $i, j$ satisfy that $u_i \rightarrow x \rightarrow u_{j+2}$ and $u_i \rightarrow x' \rightarrow u_{j+2}$. Then, using the observations above it is easy to show that we must have $i = j - 1$. Now it follows that $D = C_k[E_1, \ldots, E_n]$. □

![Fig. 3. An extended 3-cycle $C_3[E_1, E_2, E_3]$.](Image 202x567 to 339x673)
Lemma 3.2. Let $D$ be a strong non-bipartite arc-locally semicomplete digraph on $n \geq 4$ vertices such that $D - s$ is semicomplete bipartite for some vertex $s$. Then $D = C_3(E_{n_1}, E_{n_2}, E_1)$, where $E_1 = \{s\}$. In particular, $D - s$ is not strong.

Proof. Let $X, Y$ be the bipartition of $D - s$. Suppose $D$ contains a 3-cycle $xyx$, where $x \in X$ and $y \in Y$. If $D$ contains an arc $y' \to x$, $y' \in Y$, then the arcs $y \to s$, $y' \to x$ and $s \to x$ imply that $y$ and $y'$ are adjacent, contradicting the fact that $Y$ is independent. This shows that $x \to Y$. Similarly we can show $X \to y$. This implies that $s$ is adjacent to all of $V(D')$. Thus, we must have $Y \Rightarrow s$ since if $sy'$ is an arc, then the arcs $s \to x$, $x \to y$ and $s \to y'$ would imply and arc between $y$ and $y'$. Similarly, we see that $s \Rightarrow X$. Now it is easy to show that $X \Rightarrow Y$ and thus $D = C_3(E_{n_1}, E_{n_2}, E_1)$. Hence it suffices to find a 3-cycle as above.

As $D$ is strong and not bipartite there is some pair $x \in X$, $y \in Y$ such that both are adjacent to $s$ and either $y \to s$, $s \to x$ or $x \to s$, $s \to y$. Suppose w.l.o.g. that $y \to s$ and $s \to x$. If $x \to y$ we are done so assume $y \to x$. As above we conclude that $x \Rightarrow (y - y)$ and $(x - x) \Rightarrow y$ (that argument did not use the orientation of the arc between $x$ and $y$). As $n \geq 4$ be may assume w.l.o.g. that $|X| \geq 2$. Let $x' \in X - x$ be arbitrary. The arcs $x' \to y$, $y \to x$ and $s \to x$ imply that there is an arc between $x'$ and $s$. If $s \to x'$ we are done so assume $x' \to s$. Since $D$ is strong there is some $y' \in Y - y$ such that $y' \to x'$. However now considering the arcs $y \to s$, $y' \to x'$ and $x' \to s$ we get that $y$ and $y'$ are adjacent contradicting the fact that $Y$ is independent. Hence we must have $x \to y$ and the proof is complete. \qed

Theorem 3.3. Let $D$ be a strong arc-locally semicomplete digraph, then $D$ is either semicomplete, semicomplete bipartite or an extension of a cycle.

Proof. We prove this by induction on $n$. It clearly holds for $n \leq 4$ so we go to the induction step and assume $n \geq 5$. If $D$ is just a directed cycle, there is nothing to prove so assume that this is not the case. We first prove that $D$ contains a strong subdigraph $D'$ containing either $n - 1$ or $n - 2$ vertices. Let $D'$ be chosen such that $D'$ is a strong induced subdigraph of $D$ such that $|V(D')| \leq n - 1$ and $|V(D')|$ is maximum among all such subdigraphs. If $|V(D')| = n - 1$ we do so assume this is the not the case. By Lemma 2.2, every vertex of $V - V(D')$ is adjacent to $V(D')$ and by the maximality of $V(D')$ no $x \in V - V(D')$ can have arcs in both directions to $V(D')$. Let $X$ denote those vertices of $V - V(D')$ that have only arcs from $V(D')$ and let $Y$ denote the remaining vertices. Since $D$ is strong there must exist an arc from $X$ to $Y$ and now it follows easily from the maximality of $V(D')$ that $|V(D')| = n - 2$, $X = \{x\}$, $Y = \{y\}$ and $x \to y$.

By induction $D'$ is either semicomplete, semicomplete bipartite or an extended cycle. We now show that the same is true for $D$.

Suppose first that $|V(D')| = n - 1$ and let $s$ be the unique vertex of $D$ not in $V(D')$. If $D'$ is an extension of a cycle of length at least $5$, then the claim follows from Lemma 3.1. If $D'$ is semicomplete bipartite, then by Lemma 3.2 we may assume that $D$ is also bipartite. Let $X$ and $Y$ be the bipartition of $V(D')$. As $D$ is bipartite and every vertex in $X$ is adjacent to every vertex in $Y$, we may assume w.l.o.g. that $s$ is not adjacent to any vertex in $Y$. Suppose some vertex $x$ is not adjacent to $s$. Choose $x' \in X$ and $y \in Y$ such that $s \to x'$ and $x \to y$ ($D$ is strong so $s$ must dominate some vertex in $X$). Since $x'$ and $y$ are also adjacent the arcs $x' \to y$, $s \to x'$ imply that there is an arc between $x$ and $s$, a contradiction. Thus $D$ is semicomplete bipartite. It remains to consider the case when $D'$ is an extended 3-cycle or $D'$ is semicomplete (any extended 4-cycle is a semicomplete bipartite digraph). Suppose first that $D' = C_3(x, y, z)$ where each of $x, y, z$ are independent sets. W.l.o.g. $s$ dominates some $s \in x$. Then $s$ is adjacent to every vertex in $Y$. Suppose $s \to y$ for some $y \in Y$, then $s$ is adjacent to all vertices of $Z$ (because $Z \Rightarrow X$ and $X \Rightarrow Y$) and $x \to y$ implies that $|Y| = 1$ (as $Y$ is independent). The presence of an arc between $s$ and $Z$ together with $s \to x$ and $Z \Rightarrow X$ implies that $|X| = 1$. If $z \Rightarrow s$ for some $z \in Z$, then $z$ is adjacent to all other vertices in $Z$ (as all vertices of $Z$ dominate $x$ and $s$ dominates $x$), implying that $|Z| = 1$, which is a contradiction as $|V(D')| \geq 4$. If $s \to z$ for some $z \in Z$ we also reach the contradiction that $|Z| = 1$ using that $y \Rightarrow Z$. Hence there are no arcs from $s$ to $Y$ and we have $Y \Rightarrow s$.

Suppose that there is an arc between $s$ and $Z$. Then using the fact that $s \to x$ and $Z \Rightarrow X$ we get that $|X| = 1$ and using that $Y \Rightarrow Z \cup \{s\}$ we see that $|Y| = 1$. Now, considering the two possible directions of arcs between $s$ and a fixed vertex $z \in Z$ we can conclude that $|Z| = 1$, contradicting the fact that $|V(D')| \geq 4$. Thus $Z \cup \{s\}$ is independent. Since $Y \to s$ and $Z \to X$, $s$ is adjacent to every vertex of $X$. If there is some arc $x' \to s$ then the arcs $x' \to s$, $y \to z$, $x' \to y$ (where $y \in Y$ is arbitrary) would imply an arc between $s$ and $z$, contradicting the derivation above. Hence $D = C_3(x, y, z \cup \{s\})$.

Suppose next that $D'$ is semicomplete. Divide the vertices of $D'$ into 4 sets $X, Y, Z, W$ such that there is no arc between $s$ and $Y$, $s$ forms a 2-cycle with each vertex in $W$ and all arcs between $s$ and $X(Z)$ start in $s(Z)$. If $Y = \emptyset$ we are done so suppose $Y \neq \emptyset$. If $|W| > 1$ then consider some $y \in Y$ and $w_1, w_2 \in W$ such that $w_1 \neq w_2$. Because $w_1$ forms a 2-cycle with $s$ and $y$ is adjacent to $w_2$ and $w_1$ is adjacent to $w_1$ we get that $y$ and $s$ are adjacent, a contradiction. So $|W| \leq 1$. Suppose next that $W = \{w\}$. If $|Y| > 1$ then consider distinct vertices $y_1, y_2 \in Y$ such that $y_1 \to y_2$. Now the adjacency of $y_1$ and $w$ and the arcs $y_1 \to y_2, w \to w$ imply that $s$ is adjacent to $y_2$, a contradiction. Hence $|Y| = 1$. If $X \neq \emptyset$ then
consider an arbitrary \(x\) and the arcs between this and \(w, y\). Together with the two arcs between \(w\) and \(s\) this implies an arc between \(s\) and \(y\), a contradiction. Similarly, \(Z = \emptyset\), contradicting the fact that \(|V(D')| \geq 4\).

So \(W = \emptyset\) and we have \(X \neq \emptyset\) and \(Z \neq \emptyset\) as \(D\) is strong. Let \(x \in X\) and \(z \in Z\) be arbitrary. If \(x \to y\) for some \(y \in Y\), then using that \(z \to s\) and \(x\) and \(z\) are adjacent we see that \(s\) and \(y\) must be adjacent, a contradiction. Hence, we conclude that \(Y \not\to X\) and similarly that \(Z \not\to Y\). Now it is easy to see that we must have \(|X| = |Z| = 1\) (otherwise there would be an arc between \(s\) and \(y\) and since \(D'\) is strong the unique vertex \(x\) in \(X\) dominates the unique vertex \(z\) in \(Z\) implying that \(D = C_3([z], Y \cup \{s\}, \{x\})\).

Now, we consider the case when \(D'\) has \(n - 2\) vertices. Let \(x, y\) be the remaining vertices of \(V\) such that \(x \to y\) and there is no arc from \(x\) to \(V(D')\) and no arc from \(V(D')\) to \(y\). If \(D'\) is an extension of a cycle of length at least \(5\) we easily reach a contradiction using Lemma 3.1. Suppose \(D'\) is semicomplete bipartite with bipartition \(W, Z\). Suppose \(W\) o.g. that \(y\) dominates some vertex \(z \in Z\). Let \(w \in W\) be a vertex which dominates \(z\). Then \(w \to x\) (as \(x\) has no arc to \(V(D')\)). Since every \(w' \in W\) can be reached by a directed path of even length from \(w\), it follows easily that \(W \not\to X\).

Similarly, \(y \to Z\) must hold. Suppose there is an arc \(z' \to x\). As every vertex in \(Z\) dominates some vertex in \(W\) and \(W \not\Rightarrow X\) this implies that \(Z = \{z'\}\). Now considering an in-neighbour \(w\) of \(z'\) in \(D'\) and using that \(W \Rightarrow x\) we see that \(|W| = 1\). Thus \(|V(D')| = 2\), contradicting that \(n \geq 5\). Hence \(D\) is semicomplete bipartite.

If \(D'\) is an extended 3-cycle then using that \(y(x)\) has only arcs to (from) \(V(D')\) it is easy to see that \(x\) and \(y\) are adjacent to every vertex of \(D'\) and that \(D'\) is just a 3-cycle (compare with the arguments above) and now we get that \(D\) is semicomplete. It only remains to consider the case when \(D'\) is semicomplete. Again using that \(D'\) is strong and that \(x\) and \(y\) have only arcs in one direction to \(V(D')\) it is easy to show that \(D\) is semicomplete. \(\Box\)

4. Concluding remarks

**Lemma 4.1.** Let \(D\) be a strong digraph containing no induced subdigraph from the class \(\mathcal{H}_3\) in Fig. 2. Then \(D\) is either semicomplete bipartite or semicomplete.

**Proof.** Suppose first that \(D\) is bipartite with bipartition \(X, Y\). Suppose \(x \in X\), \(y \in Y\) and there is no arc from \(x\) to \(y\). Let \(P=x_1; x_2; \ldots ; x_{k+2}\) be a shortest \((x, y)\)-path. By minimality of \(P\), \(x_1 \to x_2, x_{2k+2} \to x_{2k+1}\) and it is easy to show by induction that \(x_{2k} \to x_1\). Now the arcs \(x_{2k+2} \to x_{2k+1}, x_{2k-1} \to x_{2k}\) and \(x_{2k} \to x_1\) imply that \(y \to x\). Since \(x\) and \(y\) were arbitrary, it follows that \(D\) is semicomplete bipartite. If \(D\) is non-bipartite then it contains an \((x, y)\)-walk of odd length for every pair of vertices \(x, y\). Applying the same argument as above to a shortest odd \((x, y)\)-walk it is easy to see that the fact that \(D\) is \(\mathcal{H}_3\)-free implies that \(D\) is semicomplete. \(\Box\)

Theorem 3.3 and Lemma 4.1 show that one cannot obtain an interesting new common generalization of semicomplete and semicomplete bipartite digraphs, simply by replacing the definitions of locally semicomplete digraphs and quasi-transitive digraphs, respectively by \(\{\mathcal{H}_1, \mathcal{H}_2\}\)-free, respectively, \(\mathcal{H}_3\)-free digraphs.

The following result of [1] is an easy corollary of Theorem 2.1.

**Corollary 4.2** (Bang-Jensen [1]). An arc locally semicomplete digraph is hamiltonian if and only if it is strong and has a cycle factor.

The digraph \(D = T_4[I_x, I_y, w, z]\) obtained from the unique strong tournament \(T_4\) on four vertices \(x, y, w, z\), where \(d^+(x) = d^+(y) = 1\) and \(I_x, I_y\) are independent sets shows that Theorem 3.3 does not hold for the class of all \(\mathcal{H}_1\)-free digraphs and hence (by a similar example) it does not hold for the class of all \(\mathcal{H}_2\)-free digraphs. It is easy to check that the digraph \(D\) above is hamiltonian if and only if \(|I_x| = |I_y| = 1\). Several of the steps in the proof of Theorem 3.3 are still valid for digraphs which are only assumed to be \(\mathcal{H}_1\)-free. Thus the structure of these digraphs seems to be similarly restrictive when we consider only strong digraphs. Therefore we are convinced that it is still true that a \(\mathcal{H}_1\)-free digraph is hamiltonian if and only if it is strong and has a cycle factor.

The two digraphs in Fig. 4 are \(\mathcal{H}_4\)-free. None of them are semicomplete, semicomplete bipartite or extended cycles. Both are easily seen to be hamiltonian.

**Conjecture 4.3.** A \(\mathcal{H}_4\)-free digraph is hamiltonian if and only if it is strong and has a cycle factor.

**Conjecture 4.4.** The hamiltonian cycle problem and the hamiltonian path problems are polynomially solvable for digraphs that are \(\mathcal{H}_4\)-free.
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