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Lyapunov inequalities for partial differential equations ☆

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Abstract

This paper is devoted to the study of L_p Lyapunov-type inequalities ($1 \leqslant p \leqslant +\infty$) for linear partial differential equations. More precisely, we treat the case of Neumann boundary conditions on bounded and regular domains in $\mathbb{R}^N$. It is proved that the relation between the quantities p and $N/2$ plays a crucial role. This fact shows a deep difference with respect to the ordinary case. The linear study is combined with Schauder fixed point theorem to provide new conditions about the existence and uniqueness of solutions for resonant nonlinear problems.

Keywords: Neumann boundary value problems; Resonance; Lyapunov inequalities; Partial differential equations

1. Introduction

The well-known Lyapunov inequality states that if $a \in L^1(b,c)$, then a necessary condition for the boundary value problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (b,c), \qquad u(b) = u(c) = 0 \tag{1.1}$$

to have nontrivial solutions is that

$$\int_b^c a^+(x)\,dx > 4/(c-b), \quad \text{where } a^+(x) = \max\{a(x), 0\}$$

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(see [8]). An analogous result is true for Neumann boundary conditions. In fact, if we consider the linear problem

$$u''(x) + a(x)u(x) = 0, \quad x \in (b, c), \qquad u'(b) = u'(c) = 0, \tag{1.2}$$

where $a \in \Lambda_0$ and Λ_0 is defined by

$$\Lambda_0 = \left\{ a \in L^1(b, c) \setminus \{0\}\colon \int_b^c a(x)\,dx \geqslant 0 \text{ and (1.2) has nontrivial solutions} \right\} \tag{1.3}$$

then

$$\int_b^c a^+(x)\,dx > 4/(c-b)$$

for any function $a \in \Lambda_0$ (see [3–5,9]). In the case of Neumann boundary conditions, the positivity of $\int_b^c a(x)\,dx$ is necessary in order to obtain this result (see [4, Remark 4]).

In [4] the authors generalize this result by considering, for each p with $1 \leqslant p \leqslant \infty$, the quantity

$$\beta_p \equiv \inf_{a \in \Lambda_0 \cap L^p(\Omega)} I_p(a), \tag{1.4}$$

where $\Omega = (b, c)$ and

$$I_p(a) = \left\| a^+ \right\|_p = \left(\int_b^c \left| a^+(x) \right|^p dx \right)^{1/p}, \quad \forall a \in \Lambda_0 \cap L^p(\Omega),\ 1 \leqslant p < \infty,$$
$$I_\infty(a) = \operatorname{sup\,ess} a, \quad \forall a \in \Lambda_0 \cap L^\infty(\Omega), \tag{1.5}$$

and obtaining an explicit expression for β_p as a function of p, b and c. One of the main applications of Lyapunov inequalities is to give optimal nonresonance conditions for the existence (and uniqueness) of solutions of nonlinear boundary value problems at resonance [4,9,10].

To the best of our knowledge, similar results for partial differential equations has not been yet proved. In this paper we carry out a complete qualitative study of this question pointing out the important role played by the dimension of the problem. More precisely, we consider the linear problem

$$\begin{cases} -\Delta u(x) = a(x)u(x), & x \in \Omega, \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded and regular domain and the function $a : \Omega \to \mathbb{R}$ belongs to the set Λ defined as

$$\Lambda = \left\{ a \in L^{N/2}(\Omega) \setminus \{0\}\colon \int_\Omega a(x)\,dx \geqslant 0 \text{ and (1.6) has nontrivial solutions} \right\} \tag{1.7}$$

if $N \geqslant 3$ and

$$\Lambda = \left\{ a : \Omega \to \mathbb{R} \text{ s.t. } \exists q \in (1, \infty] \text{ with } a \in L^q(\Omega) \setminus \{0\}, \int_\Omega a(x)\,dx \geqslant 0 \text{ and } \right. \\ \left. \text{(1.6) has nontrivial solutions} \right\}$$

if $N = 2$.

Since the positive eigenvalues of the eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda u(x), & x \in \Omega, \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega \end{cases} \tag{1.8}$$

belong to Λ, the quantity

$$\beta_p \equiv \inf_{a \in \Lambda \cap L^p(\Omega)} \|a^+\|_p, \quad 1 \leqslant p \leqslant \infty, \tag{1.9}$$

is well defined and it is a nonnegative real number. The first novelty of this paper is that $\beta_1 = 0$ for each $N \geqslant 2$. Moreover, we prove that if $N = 2$, then $\beta_p > 0$, $\forall p \in (1, \infty]$ and that if $N \geqslant 3$, then $\beta_p > 0$ if and only if $p \geqslant N/2$. Also, for each $N \geqslant 2$, β_p is attained if $p > N/2$. These results show a great difference with respect to the ordinary case, where $\beta_p > 0$ for each $1 \leqslant p \leqslant \infty$. Moreover, we prove some qualitative properties of β_p such as the continuity and monotonicity with respect to p. It seems difficult to obtain explicit expressions for β_p, as a function of p, Ω and N, at least for general domains (see [4,6,7] for the case $N = 1$). As in the ordinary case, we have imposed $\int_\Omega a \geqslant 0$ in the definition of the set Λ. This is not a technical but a natural assumption for Neumann boundary conditions. In fact, under this positivity condition on $\int_\Omega a$, there is no positive solution of (1.6) (see Remark 4).

The paper finishes with an application of our main linear result to nonlinear boundary value problems of the form

$$\begin{cases} -\Delta u(x) = f(x, u(x)), & x \in \Omega, \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.10}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is a bounded and regular domain and the function $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $(x, u) \mapsto f(x, u)$, satisfies the condition

(H) f, f_u are Caratheodory functions and $0 \leqslant f_u(x, u)$ in $\overline{\Omega} \times \mathbb{R}$.

The existence of a solution of (1.10) implies

$$\int_\Omega f(x, s_0)\,dx = 0 \tag{1.11}$$

for some $s_0 \in \mathbb{R}$. Trivially, conditions (H) and (1.11) are not sufficient for the existence of solutions of (1.10). Indeed, consider the problem

$$\begin{cases} -\Delta u(x) = \lambda_1 u(x) + \varphi_1(x), & x \in \Omega, \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.12}$$

where φ_1 is a nontrivial eigenfunction associated to λ_1. Here λ_1 is the first positive eigenvalue of the eigenvalue problem (1.8). The function $f(x,u) = \lambda_1 u + \varphi_1(x)$ satisfies (H) and (1.11), but the Fredholm alternative theorem shows that there is no solution of (1.12).

If, in addition to (H) and (1.11), f satisfies a non-uniform non-resonance condition of the type

(h1) $f_u(x,u) \leqslant \beta(x)$ in $\overline{\Omega} \times \mathbb{R}$ with $\beta(x) \leqslant \lambda_1$ in Ω and $\beta(x) < \lambda_1$ in a subset of Ω of positive measure,

then it has been proved in [10] that (1.10) has solution. Let us observe that supplementary condition (h1) is given in terms of $\|\beta\|_\infty$. In this paper we provide new supplementary conditions in terms of $\|\beta\|_p$, where $N/2 < p \leqslant \infty$, obtaining a generalization of [10, Theorem 2].

Moreover, we consider other situations, where the condition $0 \leqslant f_u(x,u)$ in $\overline{\Omega} \times \mathbb{R}$ is not necessary (see Theorem 7.1).

2. Lyapunov-type inequalities for the linear problem

This section will be concerned with the existence of nontrivial solutions of a homogeneous linear problem of the form

$$\begin{cases} -\Delta u(x) = a(x)u(x), & x \in \Omega, \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.1}$$

Here $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded and regular domain and the function $a : \Omega \to \mathbb{R}$ belongs to Λ, where

(1) If $N \geqslant 3$, then

$$\Lambda = \left\{ a \in L^{N/2}(\Omega) \setminus \{0\}\text{: } \int_\Omega a(x)\,dx \geqslant 0 \text{ and (2.1) has nontrivial solutions} \right\}.$$

(2) If $N = 2$, then

$$\Lambda = \left\{ a : \Omega \to \mathbb{R} \text{ s.t. } \exists q \in (1,\infty] \text{ with } a \in L^q(\Omega) \setminus \{0\},\ \int_\Omega a(x)\,dx \geqslant 0 \text{ and (2.1)} \right.$$
$$\left. \text{has nontrivial solutions} \right\}.$$

Obviously, the positive eigenvalues of the eigenvalue problem (1.8) belong to Λ. Therefore Λ is not empty and

$$\beta_p \equiv \inf_{a\in\Lambda\cap L^p(\Omega)} \|a^+\|_p, \quad 1 \leqslant p \leqslant \infty, \tag{2.2}$$

is a well-defined real number ($\|\cdot\|_p$ denotes the usual L_p-norm).

Now we state the main result of this paper.

Theorem 2.1. *The following statements hold*:

(1) *If* $N = 2$ *then* $\beta_p > 0 \Leftrightarrow 1 < p \leqslant \infty$. *If* $N \geqslant 3$ *then* $\beta_p > 0 \Leftrightarrow N/2 \leqslant p \leqslant \infty$.
(2) *If* $N/2 < p \leqslant \infty$ *then* β_p *is attained. In this case, any function* $a \in \Lambda \cap L^p(\Omega)$ *in which* β_p *is attained is of the form*:
 (i) $a(x) \equiv \lambda_1$, *if* $p = \infty$; *where* λ_1 *is the first strictly positive eigenvalue of* (1.8).
 (ii) $a(x) \equiv |u(x)|^{2/(p-1)}$, *if* $N/2 < p < \infty$; *where* u *is a solution of the problem*

$$\begin{cases} -\Delta u(x) = |u(x)|^{\frac{2}{p-1}} u(x), & x \in \Omega, \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega. \end{cases} \tag{2.3}$$

(3) *The mapping* $(N/2, \infty) \to \mathbb{R}$, $p \mapsto \beta_p$, *is continuous and the mapping* $[N/2, \infty) \to \mathbb{R}$, $p \mapsto |\Omega|^{-1/p}\beta_p$, *is strictly increasing.*
(4) *There exists always the limits* $\lim_{p\to\infty} \beta_p$ *and* $\lim_{p\to(N/2)^+} \beta_p$ *and take the values*
 (i) $\lim_{p\to\infty} \beta_p = \beta_\infty$, *if* $N \geqslant 2$;
 (ii) $\lim_{p\to(N/2)^+} \beta_p \geqslant \beta_{N/2} > 0$, *if* $N \geqslant 3$,
 $\lim_{p\to 1^+} \beta_p = 0$, *if* $N = 2$.

Remark 1. Since any nontrivial solution of (2.3) is a $C^1(\overline{\Omega})$ function which changes sign, we deduce that, for $N/2 < p < \infty$, any function $a \in \Lambda \cap L^p(\Omega)$ in which β_p is attained is a continuous nonnegative function which vanishes at some point of Ω.

For the proof of Theorem 2.1, we will distinguish three cases: the subcritical case ($1 \leqslant p < N/2$ if $N \geqslant 3$, and $p = 1$ if $N = 2$), the supercritical case ($p > N/2$ if $N \geqslant 2$), and the critical case ($p = N/2$ if $N \geqslant 3$).

3. The subcritical case

In this section, we study the subcritical case, i.e. $1 \leqslant p < N/2$, if $N \geqslant 3$, and $p = 1$ if $N = 2$. In all those cases we will prove that $\beta_p = 0$. Roughly speaking if, for instance $N \geqslant 3$, the main idea of the proof is to take first a function u and to calculate the corresponding function a for which u is a solution of (2.1). Obviously, if u is smooth enough, then we must impose two conditions: (i) $\partial u/\partial n = 0$ on $\partial\Omega$, (ii) the zeros of u are also zeros of Δu. For instance, if $\Omega = B(0, 1)$ (the open ball in $\mathbb{R}^N$ of center zero and radius one) we can take radial functions $u(x) = f(|x|)$ of the form $f(r) = \alpha r^{-a} - \beta r^{-b}$ ($r \in (\varepsilon, 1]$) for certain a, b, α, β such that the two mentioned conditions are satisfied.

Lemma 3.1. *Let* $N \geqslant 3$ *and* $1 \leqslant p < N/2$. *Then* $\beta_p = 0$.

Proof. First of all, note that if we define $\Omega + x_0 = \{x + x_0\colon x \in \Omega\}$ (for arbitrary $x_0 \in \mathbb{R}^N$), then $\beta_p(\Omega + x_0) = \beta_p(\Omega)$. On the other hand, if we define $r\Omega = \{rx\colon x \in \Omega\}$ (for arbitrary $r \in \mathbb{R}^+$), then $\beta_p(r\Omega) = r^{N/p-2}\beta_p(\Omega)$. Hence

$$\beta_p(\Omega) = 0 \quad \Leftrightarrow \quad \beta_p(r\Omega + x_0) = 0.$$

Then, we can suppose without loss of generality that $\overline{B}(0,1) \subset \Omega$.

Take now arbitrary real numbers $a > b > 0$ satisfying $a + b = N - 2$ and choose $0 < \varepsilon < (a/b)^{1/(b-a)} < 1$. Define $u : \Omega \to \mathbb{R}$ as the radial function

$$u(x) = \begin{cases} -\alpha|x|^2 + \beta, & \text{if } |x| \leqslant \varepsilon, \\ b|x|^{-a} - a|x|^{-b}, & \text{if } \varepsilon < |x| < 1, \\ b - a, & \text{if } |x| \geqslant 1,\ x \in \Omega, \end{cases} \tag{3.1}$$

where α and β are defined such that $u \in C^1(\overline{\Omega})$; i.e.

$$\alpha = \frac{ab}{2}\left(\varepsilon^{-a-2} - \varepsilon^{-b-2}\right) > 0 \quad \text{and} \quad \beta = \varepsilon^{-a} b\left(\frac{a}{2} + 1\right) - \varepsilon^{-b} a\left(\frac{b}{2} + 1\right).$$

Then, it is easy to check that u is a solution of (2.1), being $a : \Omega \to \mathbb{R}$ the radial function

$$a(x) = \begin{cases} \frac{2N\alpha}{-\alpha|x|^2 + \beta}, & \text{if } |x| \leqslant \varepsilon, \\ \frac{ab}{|x|^2}, & \text{if } \varepsilon < |x| < 1, \\ 0, & \text{if } |x| \geqslant 1,\ x \in \Omega. \end{cases} \tag{3.2}$$

It is easily seen that $a(x) \geqslant 0$ and $a(x) \in L^\infty(\Omega)$. Hence $a(x) \in \Lambda$. Let us estimate the L_p-norm of $a(x)$. To this aim, taking into account that the maximum of $a(x)$ in $\overline{B}(0,\varepsilon)$ is attained in $|x| = \varepsilon$, we have

$$\begin{aligned} \|a\|_p &\leqslant \left(\int_{B(0,\varepsilon)} \left(\frac{2N\alpha}{-\alpha\varepsilon^2 + \beta}\right)^p + \int_{B(0,1)\setminus B(0,\varepsilon)} \left(\frac{ab}{|x|^2}\right)^p \right)^{1/p} \\ &= \left(\left(\frac{Nab(\varepsilon^{-a-2} - \varepsilon^{-b-2})}{b\varepsilon^{-a} - a\varepsilon^{-b}}\right)^p \frac{w_N}{N}\varepsilon^N + \frac{(ab)^p w_N (1 - \varepsilon^{N-2p})}{N - 2p} \right)^{1/p}. \end{aligned} \tag{3.3}$$

Then β_p is smaller than this expression. But (for fixed real numbers $a > b > 0$ with $a + b = N - 2$) we can take limit when ε tends to zero in (3.3). This gives (taking into account that $p < N/2$):

$$\beta_p \leqslant \frac{ab w_N^{1/p}}{(N - 2p)^{1/p}}.$$

Finally, taking limit when b tends to zero in the last formula, we conclude $\beta_p = 0$. $\square$

Lemma 3.2. *Let $N = 2$ and $p = 1$. Then $\beta_1 = 0$.*

Proof. As we have argued in Lemma 3.1 it is easy to check that $\beta_1(r\Omega+x_0)=\beta_1(\Omega)$, for every $x_0\in\mathbb{R}^2$, $r\in\mathbb{R}^+$. Then, we can suppose again without loss of generality that $\overline{B}(0,1)\subset\Omega$.

Take now an arbitrary real number $K>\log(4)$ and $\varepsilon>0$ satisfying $\log(\varepsilon^2)+K<0$. Define $u:\Omega\to\mathbb{R}$ as the radial function

$$u(x)=\begin{cases}\left(\frac{|x|}{\varepsilon}\right)^2+\log(\varepsilon^2)+K-1, & \text{if } |x|\leqslant\varepsilon,\\ \log(|x|^2)+K, & \text{if } \varepsilon<|x|\leqslant\frac{1}{2},\\ -4(1-|x|)^2+1+K-\log(4), & \text{if } \frac{1}{2}<|x|<1,\\ 1+K-\log(4), & \text{if } |x|\geqslant 1,\ x\in\Omega.\end{cases}\tag{3.4}$$

Then, it is easy to check that u is a solution of (2.1), being $a:\Omega\to\mathbb{R}$ the radial function

$$a(x)=\begin{cases}\frac{-4}{|x|^2+\varepsilon^2(\log(\varepsilon^2)+K-1)}, & \text{if } |x|\leqslant\varepsilon,\\ 0, & \text{if } \varepsilon<|x|\leqslant\frac{1}{2},\\ \frac{16-8/|x|}{-4(1-|x|)^2+1+K-\log(4)}, & \text{if } \frac{1}{2}<|x|<1,\\ 0, & \text{if } |x|\geqslant 1,\ x\in\Omega.\end{cases}\tag{3.5}$$

It is easily seen that $a(x)\geqslant 0$ and $a(x)\in L^\infty(\Omega)$. Hence $a(x)\in\Lambda$. Let us estimate the L_1-norm of $a(x)$:

$$\begin{aligned}\|a\|_1&=\int_{B(0,\varepsilon)}a(x)\,dx+\int_{B(0,1)\setminus B(0,1/2)}a(x)\,dx\\&=2\pi\int_0^\varepsilon\frac{-4r\,dr}{r^2+\varepsilon^2(\log(\varepsilon^2)+K-1)}+2\pi\int_{1/2}^1\frac{(16r-8)\,dr}{-4(1-r)^2+1+K-\log(4)}.\end{aligned}$$

It is possible to evaluate the first integral and to estimate the second one:

$$\|a\|_1\leqslant 4\pi\log\left(\frac{\varepsilon^2(\log(\varepsilon^2)+K-1)}{\varepsilon^2+\varepsilon^2(\log(\varepsilon^2)+K-1)}\right)+2\pi\int_{1/2}^1\frac{(16\cdot 1-8)\,dr}{-4(1-1/2)^2+1+K-\log(4)}.$$

Then β_1 is smaller than this expression. But (for fixed real number $K>\log(4)$) we can take limit when ε tends to zero in this formula. This gives

$$\beta_1\leqslant\frac{8\pi}{K-\log(4)}.$$

Finally, taking limit when K tends to $+\infty$ we conclude $\beta_1=0$. □

4. The supercritical case

In this section, we study the supercritical case, i.e. $p > N/2$, if $N \geqslant 2$. In all those cases we will prove that the positive quantity β_p is attained. We begin by studying the case $p = \infty$.

Lemma 4.1. *β_∞ is attained in a unique element $a_\infty \in \Lambda$. Moreover $a_\infty(x) \equiv \lambda_1$, where λ_1 is the first strictly positive eigenvalue of the Neumann eigenvalue problem.*

Proof. If $a \in \Lambda \cap L^\infty(\Omega)$ and $u \in H^1(\Omega)$ is a nontrivial solution of (2.1) then

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega auv, \quad \forall v \in H^1(\Omega).$$

In particular, we have

$$\int_\Omega |\nabla u|^2 = \int_\Omega au^2, \qquad \int_\Omega au = 0. \tag{4.1}$$

Therefore, for each $k \in \mathbb{R}$, we have

$$\begin{aligned}\int_\Omega \left|\nabla(u+k)\right|^2 &= \int_\Omega |\nabla u|^2 = \int_\Omega au^2 \leqslant \int_\Omega au^2 + k^2 \int_\Omega a \\ &= \int_\Omega au^2 + \int_\Omega k^2 a + 2k \int_\Omega au = \int_\Omega a(u+k)^2 \leqslant \int_\Omega a^+(u+k)^2.\end{aligned}$$

This implies

$$\int_\Omega \left|\nabla(u+k)\right|^2 \leqslant \|a^+\|_\infty \int_\Omega (u+k)^2.$$

Also, since u is a nonconstant solution of (2.1), $u+k$ is a nontrivial function. Consequently

$$\|a^+\|_\infty \geqslant \frac{\int_\Omega |\nabla(u+k)|^2}{\int_\Omega (u+k)^2}.$$

Now, choose $k_0 \in \mathbb{R}$ satisfying $\int_\Omega (u+k_0) = 0$. Then,

$$\|a^+\|_\infty \geqslant \frac{\int_\Omega |\nabla(u+k_0)|^2}{\int_\Omega (u+k_0)^2} \geqslant \inf_{v \in X_\infty \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2}{\int_\Omega v^2} = \lambda_1, \quad \forall a \in \Lambda, \tag{4.2}$$

where

$$X_\infty = \left\{ v \in H^1(\Omega): \int_\Omega v = 0 \right\}.$$

Hence $\beta_\infty \geqslant \lambda_1$. Since the constant function λ_1 is an element of Λ, we deduce $\beta_\infty = \lambda_1$. Furthermore, if $a \in \Lambda$ is such that $\|a^+\|_\infty = \lambda_1$, then all the inequalities of the previous proof become equalities. In particular, it follows from (4.2) that

$$\frac{\int_\Omega |\nabla(u+k_0)|^2}{\int_\Omega (u+k_0)^2} = \lambda_1.$$

The variational characterization of λ_1 (this constant is the second eigenvalue of the eigenvalue problem (1.8)) implies that $u(x)+k_0$ is an eigenfunction associated to λ_1. Therefore

$$-\Delta(u+k_0) = \lambda_1(u+k_0) = a(x)u.$$

Multiplying by $u+k_0$ we obtain

$$\int_\Omega (\lambda_1 - a)(u+k_0)^2 \leqslant 0.$$

Since $\|a^+\|_\infty = \lambda_1$, we deduce

$$\int_\Omega (\lambda_1 - a)(u+k_0)^2 = 0.$$

The unique continuation property of the eigenfunctions implies that $u(x)+k_0$ vanishes in a set of measure zero and therefore $a(x) \equiv \lambda_1$. This completes the proof of the lemma. □

Next we concentrate on the case $N/2 < p < \infty$. We will need some auxiliary lemmas.

Lemma 4.2. *Assume $N/2 < p < \infty$ and let*

$$X_p = \left\{ u \in H^1(\Omega)\colon \int_\Omega |u|^{\frac{2}{p-1}} u = 0 \right\}.$$

If $J_p : X_p \setminus \{0\} \to \mathbb{R}$ is defined by

$$J_p(u) = \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}} \tag{4.3}$$

and $m_p \equiv \inf_{X_p \setminus \{0\}} J_p$, m_p is attained. Moreover, if $u_p \in X_p \setminus \{0\}$ is a minimizer, then u_p satisfies the problem

$$\begin{cases} -\Delta u_p(x) = A_p(u_p)|u_p(x)|^{\frac{2}{p-1}} u_p(x), & x \in \Omega, \\ \frac{\partial u_p}{\partial n}(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.4}$$

where

$$A_p(u_p) = m_p \left(\int_\Omega |u_p|^{\frac{2p}{p-1}} \right)^{-1/p}. \tag{4.5}$$

Proof. It is clear that for any $u \in H^1(\Omega)$, there exists some constant $k \in \mathbb{R}$ such that $u + k \in X_p$. Hence m_p is well defined. Now, let $\{u_n\} \subset X_p \setminus \{0\}$ be a minimizing sequence. Since the sequence $\{k_n u_n\}$, $k_n \neq 0$, is also a minimizing sequence, we can assume without loss of generality that

$$\int_\Omega |u_n|^{\frac{2p}{p-1}} = 1.$$

Then $\{\int_\Omega |\nabla u_n|^2\}$ is also bounded. On the other hand, since $p > N/2$, then

$$2 < \frac{2p}{p-1} < \frac{2N}{N-2},$$

which is the critical Sobolev exponent. Hence, $\{u_n\}$ is bounded in $H^1(\Omega)$. So, we can suppose, up to a subsequence, that $u_n \rightharpoonup u_0$ in $H^1(\Omega)$ and $u_n \to u_0$ in $L^{2p/(p-1)}$. The strong convergence in $L^{2p/(p-1)}$ gives us

$$\int_\Omega |u_0|^{\frac{2p}{p-1}} = 1 \quad \text{and} \quad u_0 \in X_p \setminus \{0\}.$$

The weak convergence in $H^1(\Omega)$ implies $J_p(u_0) \leqslant \liminf J_p(u_n) = m_p$. Then u_0 is a minimizer.

Since

$$X_p = \{u \in H^1(\Omega)\colon \varphi(u) = 0\}, \quad \varphi(u) = \int_\Omega |u|^{\frac{2}{p-1}} u,$$

if $u_0 \in X_p \setminus \{0\}$ is any minimizer of J_p, Lagrange multiplier theorem implies that there is $\lambda \in \mathbb{R}$ such that

$$H'(u_0) + \lambda \varphi'(u_0) \equiv 0,$$

where $H : H^1(\Omega) \to \mathbb{R}$ is defined by

$$H(u) = \int_\Omega |\nabla u|^2 - m_p \left(\int_\Omega |u|^{\frac{2p}{p-1}} \right)^{(p-1)/p}.$$

Also, since $u_0 \in X_p$ we have $H'(u_0)(1) = 0$. Moreover, $H'(u_0)(v) = 0$, $\forall v \in H^1(\Omega)$: $\varphi'(u_0)(v) = 0$. Finally, as any $v \in H^1(\Omega)$ may be written in the form $v = \alpha + w$, $\alpha \in \mathbb{R}$, and w satisfying $\varphi'(u_0)(w) = 0$, we conclude $H'(u_0)(v) = 0$, $\forall v \in H^1(\Omega)$, i.e., $H'(u_0) \equiv 0$ which is (4.4). □

Remark 2. Similar minimization problems, for the ordinary case, has been considered in [4,6,7].

Lemma 4.3. *If $N/2 < p < \infty$, then β_p is attained and $\beta_p = m_p$. Moreover, any function $a \in \Lambda \cap L^p(\Omega)$ in which β_p is attained is of the form*

$$a(x) \equiv \left|u(x)\right|^{\frac{2}{p-1}},$$

where $u(x)$ is a solution of (2.3).

Proof. As in Lemma 4.1, if $a \in \Lambda \cap L^p(\Omega)$ and $u \in H^1(\Omega)$ is a nontrivial solution of (2.1), then for each $k \in \mathbb{R}$ we have

$$\int_\Omega \left|\nabla(u+k)\right|^2 \leqslant \int_\Omega a(u+k)^2 \leqslant \int_\Omega a^+(u+k)^2. \tag{4.6}$$

It follows from Hölder inequality

$$\int_\Omega \left|\nabla(u+k)\right|^2 \leqslant \left\|a^+\right\|_p \left\|(u+k)^2\right\|_{p/(p-1)}.$$

Also, since u is a nonconstant solution of (2.1), $u+k$ is a nontrivial function. Consequently

$$\left\|a^+\right\|_p \geqslant \frac{\int_\Omega |\nabla(u+k)|^2}{\|(u+k)^2\|_{p/(p-1)}}.$$

Now, choose $k_0 \in \mathbb{R}$ satisfying $u + k_0 \in X_p$. Then, $\|a^+\|_p \geqslant m_p$, $\forall a \in \Lambda \cap L^p(\Omega)$ and consequently $\beta_p \geqslant m_p$. Conversely, if $u_p \in X_p \setminus \{0\}$ is any minimizer of J_p, then u_p satisfies (4.4). Therefore, $A_p(u_p)|u_p|^{2/(p-1)} \in \Lambda$. Also,

$$\left\|A_p(u_p)|u_p|^{\frac{2}{p-1}}\right\|_p^p = A_p(u_p)^p \int_\Omega |u_p|^{\frac{2p}{p-1}} = m_p^p.$$

Then $\beta_p = m_p$ and β_p is attained.

On the other hand, let $a \in \Lambda \cap L^p(\Omega)$ be such that $\|a^+\|_p = \beta_p$. Then all the inequalities we have used become equalities. In particular, since the above Hölder inequality become equality, taking into account (4.6) we have that there exists $M > 0$ such that $a(x) \equiv M|u(x) + k_0|^{2/(p-1)}$. Hence $a(x) \geqslant 0$ and consequently $\int_\Omega a > 0$. Therefore, since $\int_\Omega au^2 = \int_\Omega a(u+k_0)^2$ we deduce $k_0 = 0$. Finally, if we define $w(x) = M^{(p-1)/2}u(x)$ we have that

$$\left|w(x)\right|^{\frac{2}{p-1}} = M\left|u(x)\right|^{\frac{2}{p-1}} = a(x).$$

Moreover, since $u(x)$ is a solution of (2.1) and $w(x)$ is a multiple of $u(x)$, then also $w(x)$ is a solution of (2.1) and consequently a solution of (2.3), and the lemma follows. □

5. The critical case

In this section, we study the critical case, i.e. $p = N/2$, if $N \geqslant 3$. We will prove that $\beta_p > 0$.

Lemma 5.1. *If $N \geqslant 3$ and $p = N/2$ we have $\beta_p > 0$.*

Proof. As in Lemma 4.1, if $a \in \Lambda$ and $u \in H^1(\Omega)$ is a nontrivial solution of (2.1), then for each $k \in \mathbb{R}$ we have

$$\int_\Omega \left|\nabla(u+k)\right|^2 \leqslant \int_\Omega a(u+k)^2 \leqslant \int_\Omega a^+(u+k)^2.$$

It follows from Hölder inequality

$$\int_\Omega \left|\nabla(u+k)\right|^2 \leqslant \left\|a^+\right\|_{N/2}\left\|(u+k)^2\right\|_{N/(N-2)}.$$

Also, since u is a nonconstant solution of (2.1), $u+k$ is a nontrivial function. Consequently

$$\left\|a^+\right\|_{N/2} \geqslant \frac{\int_\Omega |\nabla(u+k)|^2}{\|u+k\|^2_{2N/(N-2)}}.$$

Now, choose $k_0 \in \mathbb{R}$ satisfying $\int_\Omega (u+k_0) = 0$. Then,

$$\left\|a^+\right\|_{N/2} \geqslant \frac{\int_\Omega |\nabla(u+k_0)|^2}{\|u+k_0\|^2_{2N/(N-2)}} \geqslant \inf_{v \in X_\infty \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2}{\|v\|^2_{2N/(N-2)}} = C, \quad \forall a \in \Lambda, \tag{5.1}$$

where

$$X_\infty = \left\{ v \in H^1(\Omega)\colon \int_\Omega v = 0 \right\}.$$

Finally, the continuous inclusions

$$X_\infty \subset H^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$$

gives us $C > 0$, which completes the proof. □

6. Qualitative properties of β_p

In this section we will study some qualitative aspects of the function $p \mapsto \beta_p$. Specifically, we will prove some results of continuity, monotonicity and behavior of β_p when p is near $N/2$ and $+\infty$.

Proof of (3) and (4) of Theorem 2.1. We first prove the continuity of β_p in $(N/2, \infty)$. To this aim, consider a sequence $\{p_n\} \to p \in (N/2, \infty)$. Take a nonnegative function $a_p \in \Lambda \cap L^p(\Omega)$

such that $\|a_p^+\|_p = \beta_p$. By (2) of Theorem 2.1, and using standard regularity arguments, we have $a_p \in L^\infty(\Omega)$. Hence $\|a_p\|_{p_n} \to \|a_p\|_p$ and it follows that

$$\limsup \beta_{p_n} \leqslant \limsup \|a_p\|_{p_n} = \|a_p\|_p = \beta_p.$$

In order to obtain the inverse inequality, and using that $\beta_p = m_p$, consider a nonzero sequence

$$\{u_{p_n}\} \subset X_{p_n} = \left\{u \in H^1(\Omega): \int_\Omega |u|^{\frac{2}{p_n-1}} u = 0\right\}$$

and $J_{p_n}(u_{p_n}) = \beta_{p_n}$. We can suppose without loss of generality that $\|u_{p_n}\|_{2p_n/(p_n-1)} = 1$ (and consequently $\|u_{p_n}\|_q$ is bounded for some $q < 2N/(N-2)$). Hence

$$\int_\Omega |\nabla u_{p_n}|^2 = \beta_{p_n}$$

and we have that $\{u_{p_n}\}$ is bounded in $H^1(\Omega)$. Therefore, there exists $u_0 \in H^1(\Omega)$ such that, up to a subsequence, $\{u_{p_n}\} \rightharpoonup u_0$ in $H^1(\Omega)$ and $\{u_{p_n}\} \to u_0$ in $L^q(\Omega)$ for every $q < 2N/(N-2)$. So $u_0 \in X_p$. Using this facts we have

$$\liminf \beta_{p_n} = \liminf \frac{\int_\Omega |\nabla u_{p_n}|^2}{\left(\int_\Omega |u_{p_n}|^{\frac{2p_n}{p_n-1}}\right)^{\frac{p_n-1}{p_n}}} \geqslant \frac{\int_\Omega |\nabla u_0|^2}{\left(\int_\Omega |u_0|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}} \geqslant \beta_p$$

and the continuity of β_p is proved.

We now prove that the mapping $[N/2, \infty) \to \mathbb{R}$, $p \mapsto |\Omega|^{-1/p}\beta_p$ is strictly increasing in $[N/2, \infty)$. To do this, take $N/2 \leqslant q < p < \infty$. Taking into account that $|\Omega|^{-1/q}\|f\|_q \leqslant |\Omega|^{-1/p}\|f\|_p$ for every $f \in L^p(\Omega)$ (strict inequality if $|f|$ is not constant) we have

$$|\Omega|^{-1/q}\beta_q \leqslant |\Omega|^{-1/q}\|a_p\|_q \leqslant |\Omega|^{-1/p}\|a_p\|_p = |\Omega|^{-1/p}\beta_p.$$

Since $|a_p|$ is not constant, we have that the above inequality is strict.

On the other hand, similar arguments of the continuity of β_p in $(N/2, \infty)$ gives us $\lim_{p\to\infty} \beta_p = \beta_\infty$.

To study the behavior of β_p, for p near $N/2$, let us observe that, since $|\Omega|^{-1/p}\beta_p$ is strictly increasing in $[N/2, \infty)$, then

$$\exists \lim_{p\to(N/2)^+} |\Omega|^{-1/p}\beta_p \geqslant |\Omega|^{-2/N}\beta_{N/2}.$$

Hence

$$\exists \lim_{p\to(N/2)^+} \beta_p \geqslant \beta_{N/2} > 0, \quad \text{if } N \geqslant 3.$$

Finally, let us consider the case $N = 2$. If we fixed a function $a \in \Lambda \cap L^\infty(\Omega)$ we have

$$\limsup_{p\to1^+} \beta_p \leqslant \limsup_{p\to1^+} \left\|a^+\right\|_p = \left\|a^+\right\|_1.$$

But, when we prove $\beta_1 = 0$, for $N = 2$, we have used nonnegative minimizing functions $a \in \Lambda \cap L^\infty(\Omega)$. Then we can conclude

$$\lim_{p \to 1^+} \beta_p = 0. \qquad \square$$

As an application of Theorem 2.1 to the linear problem

$$\begin{cases} -\Delta u(x) = a(x)u(x) + f(x), & x \in \Omega, \\ \frac{\partial u}{\partial n}(x) = 0, & x \in \partial\Omega \end{cases} \tag{6.1}$$

we have the following corollary (see [4, Corollary 2.11] and [9, Theorem 3] for the ordinary case).

Corollary 6.1. *Let $a \in L^p(\Omega) \setminus \{0\}$ (for some $p > N/2$), $0 \leqslant \int_\Omega a(x)$, satisfying $\|a^+\|_p < \beta_p$ (or $\|a^+\|_p = \beta_p$ and $a(x)$ is not a minimizer of the L_p-norm in Λ). Then for each $f \in L^p(\Omega)$ the boundary value problem* (6.1) *has a unique solution.*

Remark 3. Let us observe that to prove Theorem 2.1 we have chosen nonnegative minimizing sequences in all the cases $N = 2$ and $N \geqslant 3$, $p \neq N/2$. Therefore, if we define

$$\tilde{\beta}_p \equiv \inf_{a \in \Lambda \cap L^p(\Omega)} \|a\|_p, \quad N = 2 \text{ and } 1 \leqslant p \leqslant \infty; \quad N \geqslant 3 \text{ and } p \neq N/2, \tag{6.2}$$

it is easily seen that $\tilde{\beta}_p = \beta_p$.

Remark 4. In the definition of the set Λ we have imposed $\int_\Omega a \geqslant 0$. This is not a technical but a natural assumption for Neumann boundary conditions. Otherwise, the corresponding infimum will be always zero. To see this, note that if $u \in H^1(\Omega)$ is a positive nonconstant solution of (1.6) and we consider $v = 1/u$ as test function in the weak formulation, we obtain

$$\int_\Omega \nabla u \cdot \nabla\left(\frac{1}{u}\right) = \int_\Omega a u \frac{1}{u},$$

which implies

$$\int_\Omega a = -\int_\Omega \frac{|\nabla u|^2}{u^2} < 0.$$

With this in mind, if we take a nonconstant $u_0 \in C^2(\overline{\Omega})$ such that $\partial u_0/\partial n(x) = 0$, $\forall x \in \partial\Omega$ then, for large $n \in \mathbb{N}$, we have that $u_n = u_0 + n$ is a positive nonconstant solution of (1.6), with $a_n = -\Delta u_0/(u_0 + n)$. Clearly $\|a_n\|_p \to 0$ as $n \to \infty$ for every $1 \leqslant p \leqslant \infty$ and, as we have seen before, $\int_\Omega a_n < 0$.

Remark 5. In this paper we have considered Neumann boundary conditions. In the case of Dirichlet conditions it is possible to obtain analogous results in an easier way. To be more precise, consider the linear problem

$$\begin{cases} -\Delta u(x) = a(x)u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{6.3}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded and regular domain and the function $a : \Omega \to \mathbb{R}$ belongs to the set Λ_D defined as

$$\Lambda_D = \{a \in L^{N/2}(\Omega) \text{ s.t. (6.3) has nontrivial solutions}\} \quad \text{if } N \geqslant 3,$$
$$\Lambda_D = \{a : \Omega \to \mathbb{R} \text{ s.t. } \exists q \in (1, \infty] \text{ with } a \in L^q(\Omega), \text{ and (6.3) has nontrivial solutions}\}$$
$$\text{if } N = 2.$$

Then, we can define the value $\beta_p^D \equiv \inf_{a \in \Lambda_D \cap L^p(\Omega)} \|a^+\|_p$, $1 \leqslant p \leqslant \infty$ and it is possible to prove that all the assertions of Theorem 2.1 remain true if we replaced β_p by β_p^D and Neumann boundary conditions of (2.3) by Dirichlet conditions.

In fact, as the Neumann case, it is possible to obtain a variational characterization of β_p^D for $N/2 < p < \infty$:

$$\beta_p^D = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}}.$$

If Ω is, moreover, a radial domain, previous minimization problem is related to a more general one which involves Rayleigh quotient

$$\frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega \rho(x)|u|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}}},$$

where $\rho \in L^q(\Omega)$, $q = N(p-1)/(2p-N)$, is a positive function. This has been used in the study of the existence of nonsymmetric ground states of symmetric problems for nonlinear PDE's (see [1,2,11]).

7. Nonlinear resonant problems

In this section we give some new results on the existence and uniqueness of solutions of nonlinear b.v.p. (1.10) in a domain $\Omega \subset \mathbb{R}^N$. As we will see, Theorem 7.1 is a generalization of [10, Theorem 2] in the sense that, the main hypothesis of $f(x,u)$ in [10] is given in terms of a L^∞-restriction, while we give here a more general L^p-restriction for $N/2 < p \leqslant \infty$. In the proof, the basic idea is to combine the results obtained in the previous section with the Schauder's fixed point theorem. In fact, once we have the results on the linear problem, the procedure is standard and may be seen, for example in [4,9].

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ be a bounded and regular domain and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $(x,u) \mapsto f(x,u)$, satisfying*:

(1) *f, f_u are Caratheodory functions and $f(\cdot, 0) \in L^p(\Omega)$ for some $N/2 < p \leqslant \infty$.*

(2) *There exist functions* $\alpha, \beta \in L^p(\Omega)$, *satisfying*

$$\alpha(x) \leqslant f_u(x,u) \leqslant \beta(x) \quad \text{in } \overline{\Omega} \times \mathbb{R} \tag{7.1}$$

with $\|\beta^+\|_p < \beta_p$ (*or* $\|\beta^+\|_p = \beta_p$ *and* $\beta(x)$ *is not a minimizer of the* L_p*-norm in* Λ), *where* β_p *is given by Theorem* 2.1.
Moreover, we assume one of the following conditions:
(a)

$$\int_\Omega \alpha \geqslant 0, \quad \alpha \not\equiv 0, \tag{7.2}$$

(b)

$$\alpha \equiv 0, \quad \exists s_0 \in \mathbb{R} \text{ s.t. } \int_\Omega f(x,s_0)\,dx = 0, \quad \text{and}$$

$$f_u(x,u(x)) \not\equiv 0, \quad \forall u \in C(\overline{\Omega}). \tag{7.3}$$

Then problem (1.10) *has a unique solution.*

Proof. We first prove uniqueness. Let u_1 and u_2 be two solutions of (1.10). Then, the function $u = u_1 - u_2$ is a solution of the problem

$$-\Delta u(x) = a(x)u(x), \quad x \in \Omega, \qquad \frac{\partial u}{\partial n} = 0, \quad x \in \partial\Omega, \tag{7.4}$$

where

$$a(x) = \int_0^1 f_u\big(x, u_2(x) + \theta u(x)\big)\,d\theta.$$

Hence $\alpha(x) \leqslant a(x) \leqslant \beta(x)$ and we deduce $a(x) \in \Lambda$ and $\|a^+\|_p \leqslant \|\beta\|_p$. Applying Theorem 2.1, we obtain $u \equiv 0$.

Next we prove existence. First, we write (1.10) in the equivalent form

$$\begin{cases} -\Delta u(x) = b(x,u(x))u(x) + f(x,0), & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.5}$$

where the function $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is defined by

$$b(x,z) = \int_0^1 f_u(x, \theta z)\,d\theta.$$

Hence $\alpha(x) \leqslant b(x,z) \leqslant \beta(x)$, $\forall(x,z) \in \Omega \times \mathbb{R}$ and our hypothesis permit us to apply Corollary 6.1 in order to have a well-defined operator $T : X \to X$, by $Ty = u_y$, being u_y the unique solution of the linear problem

$$\begin{cases} -\Delta u(x) = b(x, y(x))u(x) + f(x,0), & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega, \end{cases} \tag{7.6}$$

where $X = C(\overline{\Omega})$ with the uniform norm.

We will show that T is completely continuous and that $T(X)$ is bounded. The Schauder's fixed point theorem provides a fixed point for T which is a solution of (1.10).

The fact that T is completely continuous is a consequence of the compact embedding of the Sobolev space $W^{2,p}(\Omega) \subset C(\overline{\Omega})$. It remains to prove that $T(X)$ is bounded. Suppose, contrary to our claim, that $T(X)$ is not bounded. In this case, there would exist a sequence $\{y_n\} \subset X$ such that $\|u_{y_n}\|_X \to \infty$. Passing to a subsequence if necessary, we may assume that the sequence of functions $\{b(\cdot, y_n(\cdot))\}$ is weakly convergent in $L^p(\Omega)$ to a function a_0 satisfying $\alpha(x) \leqslant a_0(x) \leqslant \beta(x)$ a.e. in Ω. If $z_n \equiv u_{y_n}/\|u_{y_n}\|_X$, passing to a subsequence if necessary, we may assume that $z_n \to z_0$ strongly in X (we have used again the compact embedding $W^{2,p}(\Omega) \subset C(\overline{\Omega})$), where z_0 is a nonzero function satisfying

$$\begin{cases} -\Delta z_0(x) = a_0(x)z_0(x), & \text{in } \Omega, \\ \frac{\partial z_0}{\partial n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{7.7}$$

Now, if we are assuming (7.2), then $a_0 \not\equiv 0$, $a_0 \in \Lambda$ and we obtain a contradiction with Theorem 2.1. If we are assuming (7.3), there is no loss of generality if we suppose $s_0 = 0$. (Otherwise, we can do the change of variables $u(x) = v(x) + s_0$ and obtain a similar problem with the same original hypothesis.) Then for every $n \in \mathbb{N}$,

$$\int_\Omega b\big(x, y_n(x)\big)u_{y_n}(x)\,dx = -\int_\Omega f(x,0)\,dx = 0.$$

Therefore, for each $n \in \mathbb{N}$, the function u_{y_n} has a zero in $\overline{\Omega}$ and hence so does z_0. Thus, $a_0 \not\equiv 0$, $a_0 \in \Lambda$ and we obtain again a contradiction with Theorem 2.1. □

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