PERGAMON

Applied Mathematics Letters 12 (1999) 27–30

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Pointwise Convergence of Approximation Schemes for Parameter Estimation in Parabolic Equations

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(Received December 1998; accepted January 1999)

Abstract—A finite element-based approximation scheme is presented for parameter estimation problems for parabolic PDEs on a two-dimensional domain. Pointwise convergence results relating the approximating subspaces to the full infinite-dimensional state space are discussed. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Finite elements, Bicubic splines, 2-D domain, Transport models, Parameter estimation.

1. INTRODUCTION

In this note, we discuss approximation schemes for solving parameter estimation problems for parabolic partial differential equations posed on a two-dimensional domain. The technique is based on the standard Galerkin method, with approximating subspaces generated by bicubic spline basis functions. When pointwise state space data are given, it is necessary to obtain pointwise convergence results relating the approximating subspaces to the full infinite-dimensional state space. In [1], such results were derived for parabolic problems posed on a one-dimensional domain. Here we establish convergence for the analogous two-dimensional problem. (For another approach to this problem see [2].) We note that we are motivated by the parameter estimation problem for transport models used in population dispersal experiments, in which pointwise data (representing population counts) are collected over a two-dimensional domain, see [3].

PII:S0893-9659(99)00097-X

2. MATHEMATICAL FORMULATION

We consider the Initial Value-Boundary Value Problem (IV-BVP) on $\Omega = [0,1] \times [0,1]$

$$\begin{aligned} u_t + \nabla \cdot (\mathbf{V}u) &= \nabla \cdot (\mathbf{D} \otimes \nabla u) + \alpha u + f, && t \in (0,T], \\ u(0,x,y) &= u_0(\gamma(x,y)), && \\ u(t,x,y) &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $f = f(\beta,t,x,y)$ and $\mathbf{D}$, $\mathbf{V}$, α, and β are assumed to be functions of t, x, and y, with $\mathbf{D} = (D_1, D_2)$ and $\mathbf{V} = (V_1, V_2)$. With respect to our motivating applications, the dependent variable, $u = u(t,x,y)$, represents the population density of a species, whose dispersal over a two-dimensional domain is assumed to result from an innate diffusive mechanism, $\mathbf{D}$, and a convective or "directed transport" mechanism, $\mathbf{V}$. The parameter α represents a general "source/sink" or "growth/decay" term. Homogeneous Dirichlet boundary conditions are assumed, since any nonhomogeneous boundary conditions can be incorporated into the parameters β and γ.

To demonstrated the well-posedness of this IV-BVP and to facilitate the discussion of the approximation of the associated parameter estimation problem, we recast equation (1) as a weak variational system on the Hilbert space $H = L^2(\Omega)$:

$$\begin{aligned} \langle u_t, \phi \rangle + L(q)(u,\phi) &= \langle f, \phi \rangle, \qquad \text{for all } \phi \text{ in } H_0^1(\Omega), \\ u(0) &= u_0(\gamma), \end{aligned} \tag{2}$$

where $L(q)$ is the bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ defined by:

$$L(q)(\phi,\psi) = \langle \mathbf{D}\nabla\phi, \nabla\psi \rangle - \langle \mathbf{V}\phi, \nabla\psi \rangle - \langle \alpha\phi, \psi \rangle. \tag{3}$$

Here q is a "vector" of unknown parameters $(\mathbf{D}, \mathbf{V}, \alpha, \beta, \gamma)$ belonging to some set Q of admissible parameters belonging to the space $\mathbf{X} = \{H^1([0,T]; L^2(\Omega))\}^2 \times \{H^0((0,T) \times \Omega)\}^3$. For a general account of this approximation framework for parameter estimation problems for distributed parameter systems, see [4,5]. Under appropriate compactness and boundedness assumptions on Q (essentially guaranteeing the coercivity of the bilinear form L) as well as continuity assumptions on f and u_0 (see [6]), the well-posedness of this problem follows from the abstract theory of Lions [7].

To formulate the parameter estimation problem, we assume that for each time $t_i \in (0,T]$, $i = 1,2,\ldots,P$ we have a matrix $\Lambda(t_i)$ of observations (taken at the $m \cdot n$ locations $(x_1,y_1),\ldots,(x_m,y_n)$). Associated with each $\Lambda(t_i)$ is a matrix $\Gamma(t_i;q) = (u(t_i,x_j,y_k;q))$ of model based "predictions". We then seek to solve the problem, $(\mathcal{P})$:

$$\text{minimize} \qquad J(q) = \sum_{i=1}^{P} \|\Lambda(t_i) - \Gamma(t_i;q)\|^2 \tag{4}$$

over all q belonging to Q. Here, the norm is the standard Euclidean norm on $\mathbf{R}^{mn}$. This problem is typically infinite-dimensional in both the state and parameter spaces, and we are led to develop appropriate approximation schemes. Here, we shall focus only on the state space approximations, as the convergence results which we wish to establish are independent of the approximation of the parameter space Q. (For details on these parameter space approximations see [8,9].)

Following the standard Galerkin technique, we define a sequence of finite-dimensional approximating state subspaces $H^N \subset L^2$, $N = 1,2,\ldots$, with $P^N : L^2 \to H^N$ the canonical orthogonal projection. Furthermore, we assume that $H^N \subset H_0^1(\Omega)$ and that for all $z \in C^2(\Omega) \cap H_0^1(\Omega)$:

$$\left\|P^N z - z\right\|_0 \quad \text{and} \quad \left\|\nabla\left(P^N z - z\right)\right\|_0 \le \epsilon(N)\{\|z_{xx}\|_0 + \|z_{yy}\|_0\},$$

where $\epsilon(N) \to 0$ as $N \to \infty$.

Restricting the bilinear form (3) to $H^N \times H^N$, we then define the Galerkin approximation, $u^N = u^N(t,\cdot,\cdot;q)$ to be the solution of:

$$\begin{aligned} \langle u_t^N, \psi\rangle + L(q)\left(u^N, \psi\right) &= \langle f, \psi\rangle, \qquad \text{for all } \psi \text{ in } H^N, \\ u^N(0) &= P^N u_0. \end{aligned} \tag{5}$$

These finite-dimensional initial-value problems then yield a sequence of approximate estimation problems (P^N):

$$\text{minimize} \qquad J^N(q) = \sum_{i=1}^{P} \left\|\Lambda(t_i) - u^N(t_i, x_j, y_k; q)\right\|^2 \tag{6}$$

over all q belonging to the set of admissible parameters, Q. (Again, the norm is the Euclidean norm on $\mathbf{R}^{mn}$.)

3. CONVERGENCE ARGUMENTS

The finite-dimensional approximation problems $(\mathcal{P}^N)$ produce solutions q^N that, under the compactness assumption on the set Q, converge at least subsequentially to some parameter q^*. To establish that q^* is a minimizer of the full cost functional J, we must prove the general statement that "convergence of q^N to q^* implies convergence of $J^N(q^N)$ to $J(q^*)$ as $N \to \infty$". Since we have chosen to minimize a *pointwise* least squares fit-to-data criterion, it is clear our approximation scheme must guarantee *pointwise* convergence of the state approximations. That is, under the topology on Q, we must show that convergence of q^N to q^* implies $u^N(t, x_j, y_k; q^N)$ converges to $u^*(t, x_j, y_k; q^*)$ as $N \to \infty$, for each of the data points (x_j, y_k), for $1 \le j \le m$ and $1 \le k \le n$.

Notice that we do not demand global pointwise convergence of the state approximations. Rather, it suffices to show pointwise convergence on a neighborhood D_{jk} of each of the points (x_j, y_k) and then piece together at most a finite number (in fact $m \cdot n$) of such results. The arguments for this rely on a general estimate that bounds the $L^\infty(D)$ norm of any member of a broad class of finite-element subspaces (including those generated from bicubic splines) by a global $H^1(\Omega)$ norm, where D is an open subset of Ω (see [10]). To establish the convergence in $H^1(\Omega)$ of $u^N(t; q^N)$ to $u^*(t; q^*)$ as $N \to \infty$, we observe that by the triangle inequality it suffices to argue that $\|u^N(t) - P^N u^*(t)\|_1 \to 0$, since $\|P^N - u^*(t)\|_1 \to 0$ by standard spline estimates (see [11]). We then use the weak variational formulation of our approximation scheme to derive a Gronwall inequality for each of the expressions $\|u^N(t) - P^N u^*(t)\|_0$ and $\|\nabla(u^N(t) - P^N u^*(t))\|_0$. For details, see [1,12]. Finally, since u^N and u^* are actually continuous (continuity of u^N follows from finite-dimensionality and continuity of u^* follows from standard results on parabolic equations, see [6]), we have that the convergence of u^N to u^* in $L_\infty(D)$ is actually convergence in $C(D)$.

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