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Cohomogeneity one Riemannian manifolds and Killing fields

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Abstract: This paper is devoted to the study of isometries of cohomogeneity one Riemannian spaces, namely Riemannian manifolds acted on by a Lie group of isometries G with principal orbits of codimension one. We show that, for a class of such manifolds, every one parameter group of isometries preserves the foliation induced by the action of the Lie group G.

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Introduction

Cohomogeneity one Riemannian manifolds, namely Riemannian spaces acted on by a closed Lie group of isometries with one orbit of codimension one, have been used quite intensively in recent literature. In particular in 1982 Bérard Bergery ([4]) used cohomogeneity one Riemannian spaces to produce new examples of Einstein spaces. Bryant and Salamon ([7]) succeeded in constructing explicit examples of metrics with holonomy G_2 and Spin(7), by considering cones on homogeneous spaces endowed with a warped product metric. Such examples are cohomogeneity one Riemannian spaces and have maximal degree of symmetry: indeed, if the holonomy group of a Riemannian space is contained in G_2 or in Spin(7), the Ricci curvature is forced to vanish and, by a result of Alekseevsky–Kimelfeld (see e.g. [5]), it is clear that there are no homogeneous non-flat examples of such spaces.

More recently, Alekseevsky ([1,2,3]) began a more systematic study of cohomogeneity one Riemannian manifolds by using a Lie group theoretic approach; he developed some basic ideas explained in the book of Bredon ([6]) about general actions of compact Lie groups on manifolds and used the general notion of slice, as given in [12], in the Riemannian setting. In particular, in [2], a general description of cohomogeneity one manifolds is provided from the viewpoint of group theory, by reducing the problem of their classification to a problem in representation theory.

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Nevertheless, it seems that cohomogeneity one Riemannian manifolds have never been investigated from the differential geometric point of view. Actually, several fields of differential geometry, like foliation theory or curvature analysis, could be successfully applied to provide a better understanding of the geometric properties of such manifolds.

The present paper is aimed at starting such research, by examining at first a particular class of cohomogeneity one Riemannian manifolds in which every orbit is an umbilical hypersurface of the ambient manifold. We shall interpret this condition in terms of the metric structure and will give a theorem on the isometry group of such spaces.

Section 1 will be devoted to some basic facts on cohomogeneity one Riemannian manifolds and to explanation of our main result together with a corollary (Corollary A), which we think could be of some general interest.

Section 2 will be devoted to the proof of the stated results.

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1. Preliminaries

Throughout the following (M, g) will denote a C^∞ Riemannian manifold. In the present paper we shall deal with cohomogeneity one Riemannian manifolds and the behaviour of the flow of Killing vector fields with respect to the induced foliation. For sake of completeness we shall now go through some basic facts about cohomogeneity one Riemannian manifolds, which have been largely used in recent literature (see [2, 3, 5, 6]).

Definition 1.1. Let (M, g) be a complete Riemannian manifold and let $G \subset I(M, g)$ be a closed Lie group of isometries of M. We say that M is of *cohomogeneity one under the action of* G if G has an orbit of codimension one.

From the general theory of G-manifolds (see [10, 6, 3]), the condition given in Definition 1.1 is equivalent to the fact that the orbit space $\Omega = M/G$ is a topological space of dimension one. Indeed, we may affirm that the orbit space Ω is always a 1-dimensional Hausdorff space homeomorphic to one of the following topological spaces: $\mathbb{R}$, S^1, $\mathbb{R}^+ = [0, +\infty)$, $[0, \pi] \subset \mathbb{R}$. In the following we will denote by $\kappa : M \to \Omega$ the projection onto the orbit space Ω.

Let us now introduce some basic terminology for G-manifolds:

Definition 1.2. Given a point $x \in M$, the orbit $P_x = Gx$ is called *principal* (*singular*) if the corresponding image in the orbit space Ω is an internal (resp. boundary) point of Ω. The point x will be called *regular* (resp. *singular*) and the set of all regular points will be denoted by M_{reg}.

From the structure theorems, a point x is regular if and only if there exists a neighborhood $V_{\kappa(x)} \subset \Omega$ of $\kappa(x)$ so that $U = \kappa^{-1}(V_{\kappa(x)})$ admits a G-invariant diffeomorphism with $G/K \times V_{\kappa(x)}$, where K is the stability subgroup of G at x. Indeed, for any *any* open and connected interval $I \subset \mathring{\Omega}$ (= the interior of Ω), $\kappa^{-1}(I)$ is a *non-compact, connected* G-manifold with only regular orbits: in this case, the whole $\kappa^{-1}(I)$ is G-invariantly diffeomorphic to $G/K \times I$. This implies also that *all regular orbits are diffeomorphic to* G/K.

We note moreover, that the action of the Lie group G induces a codimension one foliation $\mathfrak{F}$ of the open dense subset M_{reg} consisting of regular points: the maximal leaf of $\mathfrak{F}$ through a regular point $x \in M_{\text{reg}}$ is given by the orbit of G through x. The normal bundle L of such foliation is a rank one subbundle of TM defined on M_{reg} and it is always trivial: indeed the mapping κ is a regular submersion of M_{reg} onto $\mathring{\Omega}$ which is naturally oriented.

On the other hand, if a point x is singular, there exists a neighborhood $U \subset M$ of x, so that there exists a G-invariant diffeomorphism between U and $G \times_H V$, where H is the stability subgroup of G at x, V is an euclidean space and H acts orthogonally on V and transitively on the unit sphere in V.

A singular point x is called *exceptional* if the codimension of the orbit $G(x)$ is still one; in this case the foliation $\mathfrak{F}$ is still defined on the open subset $A = M_{\text{reg}} \cup G(x)$, but the normal bundle $L|_A$ is not trivial any more.

Definition 1.3. A (complete) geodesic γ on a Riemannian manifold of cohomogeneity one is called *normal geodesic* if it crosses each orbit orthogonally.

In the next proposition, we list some of most relevant properties of Riemannian manifolds of cohomogeneity one.

Proposition 1.4. a) *A geodesic γ is normal if and only if it is orthogonal to the orbit Gx at one point $x \in \gamma$;*

b) *each regular point belongs to a unique normal geodesic;*

c) *the map $\kappa|_\gamma : \gamma \to \Omega$ is surjective and it defines a covering over the set $\mathring{\Omega}$ of internal points of Ω;*

d) *the group G transforms normal geodesics into normal geodesics and acts on the set of normal geodesics transitively;*

e) *let x be a regular point of M; then there exists a neighborhood U of x such that (U, g) is locally isometric to $((G/K \times I), g_t + dt^2)$. Here, K is the isotropy group of G at x, I is an open interval of $\mathbb{R}$, g_t is a family of left invariant metrics on G/K, depending smoothly on $t \in I$, dt^2 is the standard metric of $\mathbb{R}$.*

For a), b), c) and d), see [1,2,3]; for e) consider the following facts. Let γ be the normal geodesic such that $\gamma(0) = x$ and let I be an open interval which contains 0 and such that $\gamma(I)$ crosses only principal orbits. Then, by means of κ (by point c)), we may identify I with an open set of $\mathring{\Omega}$ and $\gamma|_I$ with a section of the fibering $\kappa^{-1}(I)(\simeq G/K \times I) \to I$. Being γ a geodesic, it is then clear that the G-invariant diffeomorphism between $\kappa^{-1}(I)$ and $G/K \times I$ induces an isometry between the metric g and $g_t + dt^2$.

A particular case is given when the one parameter family of metrics g_t is expressed by

$$g_t = \phi(t)^2 g_o,$$

where $\phi(t)$ is a nowhere vanishing C^∞ function and g_o is a fixed G-invariant Riemannian metric on G/K; in this case we shall say that the given cohomogeneity one Riemannian space inherits an *adapted warped product structure* (for a general treatment of warped product structures, we refer to [11] and [5]). Many examples of such spaces are given by revolution hypersurfaces in the euclidean space: actually in a joint work ([13]), it has been shown that a compact cohomogeneity one Riemannian manifold, which is isometrically immersed as a hypersurface of the euclidean space, is a revolution hypersurface if and only if it has an adapted warped product structure. In [13], we also gave a necessary and sufficient condition for a cohomogeneity one Riemannian manifold to have a warped product structure, namely we proved that

Proposition 1.5. *A cohomogeneity one Riemannian manifold (M, g) has an adapted warped product structure if and only if the induced foliation $\mathfrak{F}$ is umbilical in M.*

This proposition motivates the definition

Definition. A cohomogeneity one Riemannian manifold (M, g) will be said to be *umbilical* if the induced foliation $\mathfrak{F}$ is umbilical. Our main theorems, we shall prove in the next section, deals with isometries on umbilical cohomogeneity one Riemannian manifolds.

Theorem I. *Let (M, g) be a compact umbilical cohomogeneity one Riemannian manifold. If the orbit space Ω is diffeomorphic to S^1, then every Killing vector field preserves the induced foliation.*

Theorem II. *Let (M, g) be a compact, irreducible homogeneous space. Suppose moreover (M, g) is a cohomogeneity one Riemannian manifold under the action of a connected Lie group G, such that the principal orbit G/K is an isotropy irreducible space. Then (M, g) is isometric to a sphere or to a real projective space.*

As a concluding remark, we quote here that in Remark 1 of the next section, we explain how one can deal with cohomogeneity one Riemannian manifolds having exactly two exceptional singular orbits; in particular we show that there is a two fold covering of the manifold on which the same group G acts with codimension one principal orbits and no singular ones.

2. Proof of the Theorems

To fix notations, we denote by (M, g) the compact cohomogeneity one Riemannian manifold and by G the group of isometries acting on it with principal orbits of codimension one. Moreover we are supposing that the orbit space $\Omega = M/G$ is diffeomorphic to S^1 and that the induced foliation $\mathfrak{F}$, given by the orbits under the action of G, is umbilical.

We shall indicate by ξ a unit vector field everywhere normal to the foliation $\mathfrak{F}$: we recall that the field ξ is geodesic, that is $\nabla_\xi \xi = 0$. By our assumption of umbilicity, there is a C^∞ function $f : M \to \mathbb{R}$ such that for all $X \in T\mathfrak{F}$

$$\nabla_X \xi = -fX. \tag{2.1}$$

We take any Killing vector field Z on M and we want to show that for any $X \in T\mathfrak{F}$

$$[Z, X] \in T\mathfrak{F}. \tag{2.2}$$

This is actually equivalent to showing that, for any normal geodesic $\gamma : \mathbb{R} \to M$ and for any Killing vector field X induced by the action of G, the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(s) = g([Z, X], \dot\gamma(s)) \qquad s \in \mathbb{R}, \tag{2.3}$$

vanishes identically. Now we note immediately that, since $\dot\gamma(s) = \xi_{\gamma(s)}$ and since $[Z, X]$ is a Killing vector field,

$$\frac{d}{ds} g(s) = g(\nabla_\xi [Z, X], \xi) + g([Z, X], \nabla_\xi \xi) = 0$$

that is the function g is constant. We have to prove that this constant is zero.

First of all we prove the following

Lemma 2.1. *For every normal geodesic γ parametrized by an arc parameter, the mapping $\kappa \circ \gamma : \mathbb{R} \to S^1$ is periodic.*

Proof. We fix a point $x_o \in M$ and consider the normal geodesic $\gamma : \to M$ issuing from x_o, that is $\gamma(0) = x_o$. We know that the mapping $\kappa \circ \gamma : \mathbb{R} \to S^1$ defines a covering, hence we may define the real number

$$T = \inf\{t \in \mathbb{R}^+ \mid \kappa(\gamma(t)) = \kappa(x_o)\}. \tag{2.4}$$

Since $\gamma(T) \in G(x_o)$, there exists $g \in G$ with $\gamma(T) = g(x_o)$. Since every $g \in G$ preserves the foliation $\mathfrak{F}$, we have only two possibilities, namely

$$dg_{x_o}(\dot\gamma(0)) = \pm\ \dot\gamma(T).$$

We shall examine these two cases separately. If $dg_{x_o}(\dot\gamma(0)) = \dot\gamma(T)$, then by the uniqueness of geodesics, we have that $g \circ \gamma(s) = \gamma(s + T)$ for all $s \in \mathbb{R}$ and the map $\kappa \circ \gamma$ is obviously periodic.

Let us now assume that $dg_{x_o}(\dot\gamma(0)) = -\dot\gamma(T)$: again by uniqueness we have

$$g(\gamma(s)) = \gamma(T - s), \qquad \forall s \in \mathbb{R}.$$

But then $g(\gamma(\frac{1}{2}T)) = \gamma(\frac{1}{2}T)$, that is g fixes the point $\gamma(\frac{1}{2}T)$; moreover, since g preserves globally but not pointwise the geodesic γ, we have

$$dg_{\gamma(\frac{1}{2}T)}(\dot\gamma(\tfrac{1}{2}T)) = -\dot\gamma(\tfrac{1}{2}T). \tag{2.5}$$

We now consider the point $x_1 = \gamma(\frac{1}{2}T)$ and repeat the same argument as above: we denote by $\eta : \mathbb{R} \to M$ the normal geodesic through x_1 (actually $\eta(s) = \gamma(s + \frac{1}{2}T)$ for all $s \in \mathbb{R}$) and define a real number S as in (2.4) for the geodesic η. Again we have that $\eta(S) = h(x_1)$ for some $h \in G$; if $dh_{x_1}(\dot\eta(0)) = \dot\eta(S)$, then $\kappa \circ \eta$ is periodic. If it happens that $dh_{x_1}(\dot\eta(0)) = -\dot\eta(S)$, then $h' = h \circ g \in G$ is such that $h'(x_1) = \eta(S)$, since h stabilizes x_1, and moreover $dh'_{x_1}(\dot\eta(0)) = \dot\eta(S)$, so that $\kappa \circ \eta$ is periodic.

Now since G acts transitively on the set of normal geodesics, we get that for every normal geodesic γ the mapping $\kappa \circ \gamma$ is periodic. □

Corollary 2.2. *For all normal geodesic $\gamma : \mathbb{R} \to M$, the function $f \circ \gamma$ is periodic.*

Proof. Indeed the function f is given by $(n-1)\mathrm{Tr}\, A$, where A is the shape operator of the leaves of the foliation $\mathfrak{F}$, so that f is invariant under the action of G. Now if $\gamma : \mathbb{R} \to M$ is a normal geodesic, we know by the previous lemma, that there exists $T > 0$ with $\gamma(s) \equiv \gamma(s+T) \bmod G$ for all $s \in \mathbb{R}$: then $f(\gamma(s+T)) = f(\gamma(s))$ for all $s \in \mathbb{R}$, by the invariance of f under the action of G. □

We now turn to the function g defined in (2.3) and note that

$$\begin{aligned} g(s) &= g(\nabla_Z X, \xi)_{\gamma(s)} - g(\nabla_X Z, \xi)_{\gamma(s)} \\ &= -g(\nabla_\xi X, Z)_{\gamma(s)} + g(\nabla_\xi Z, X)_{\gamma(s)} \\ &= -2g(\nabla_\xi X, Z)_{\gamma(s)} + \xi(g(Z,X)) \\ &= \xi(g(Z,X)) + 2fg(Z,X). \end{aligned}$$

So if we consider the function $\phi : \mathbb{R} \to \mathbb{R}$ given by

$$\phi(s) = g(Z,X)_{\gamma(s)}, \qquad s \in \mathbb{R},$$

we may write

$$g(s) = \phi'(s) + 2f(s)\phi(s) = C,$$

where we have indicated by f also the composition mapping $f \circ \gamma$ and by C a real constant. We have to prove that $C = 0$. First of all we show that we may suppose $\phi(0) = 0$: indeed if $g(Z,X)_{\gamma(0)} \neq 0$, then there exists a real number $a \in \mathbb{R}$ with $g(Z + aX, X)_{\gamma(0)} = 0$; on the other hand $Z + aX$ is still a Killing vector field and $[Z + aX, X] = [Z, X]$, so that we may consider $Z + aX$ instead of Z. The following lemma will conclude the proof.

Lemma 2.3. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function which belongs to $L^\infty(\mathbb{R})$ and which satisfies $\phi(0) = 0$. If $f : \mathbb{R} \to \mathbb{R}$ is a C^∞ periodic function such that*

$$\phi'(s) + 2f(s)\phi(s) = C \tag{2.5}$$

for some constant, then $C = 0$.

Proof. We suppose that $C \neq 0$ and for sake of simplicity we may suppose $C > 0$, otherwise we change ϕ into $-\phi$. We define the function $F : \mathbb{R} \to \mathbb{R}$ given by

$$F(s) = \int_0^s f(t)dt$$

and note that, since f is periodic, we may find two constants A, B such that

$$As - B \leqslant F(s) \leqslant As + B, \qquad s \in \mathbb{R}. \tag{2.6}$$

Now equation (2.5) can be integrated in the following way

$$\phi(s) = C\frac{\int_0^s \exp(2F(t))\,dt}{\exp(2F(s))}, \qquad s \in \mathbb{R}.$$

Now for $s \geqslant 0$, using (2.6), we get

$$C\frac{\int_0^s \exp(2F(t))\,dt}{\exp(2F(s))} \geqslant C\exp(-2B)\frac{(1-\exp(-As))}{A},$$

so that, if $A < 0$, we get

$$\lim_{s\to+\infty} C\exp(-2B)\frac{(1-\exp(-As))}{A} = +\infty,$$

which implies $||\phi||_\infty = +\infty$, contradicting our assumption $\phi \in L^\infty(\mathbb{R})$. So we get $A \geqslant 0$. If $A > 0$, then, for $s \leqslant 0$, using (2.6), we get

$$-C\frac{\int_0^s \exp(2F(t))\,dt}{\exp(2F(s))} \geqslant C\exp(-2B)\frac{\exp(-As)-1}{A}$$

and

$$\lim_{s\to-\infty} C\exp(-2B)\frac{\exp(-As)-1}{A} = +\infty\ ,$$

so that

$$\lim_{s\to-\infty} \phi(s) = -\infty\ ,$$

which is again contradictory. We are left with the case $A = 0$: in this case $F \in L^\infty(\mathbb{R})$ and if $C \neq 0$, we get that

$$\lim_{s\to+\infty} C\int_0^s \exp(2F(t))\,dt = +\infty,$$

showing again that ϕ were not bounded. □

Now we may apply the previous lemma to the function $\phi(s) = g(Z,X)_{\gamma(s)}$, noting that ϕ is bounded since the function $g(Z,X)$ is continuous on M, which is compact. This concludes the proof of the theorem. □

Corollary 2.4. *Suppose Z is a Killing vector field which is transverse to the foliation $\mathfrak{F}$ at some point p. Then the foliation $\mathfrak{F}$ is parallel and M is reducible.*

Proof. We note that our assumption implies that the orbit of the connected group $I^o(M,g)$ through the point p contains an open set, hence (M,g) is homogeneous. Now if Y is any Killing vector field, Y preserves the foliation $\mathfrak{F}$, hence the function f is constant along the flow of Y. It follows from homogeneity, that f is constant on the whole M. We now observe that

$$\text{div } \xi = -(n-1)f$$

and, by Stokes theorem,

$$\int_M (\text{div } \xi)\, \omega_g = -\int_M (n-1)f\, \omega_g = 0,$$

and since f is a constant, we get $f = 0$ everywhere on M. This means that the foliation $\mathfrak{F}$ is parallel. □

Remark 1. The same proof also applies in the case of a compact cohomogeneity one Riemannian manifold, whose singular points are all exceptional. Indeed in this case we may still define a rank one vector bundle L whose fibre over a point $x \in M$ is $L_x = T(G(x))^{\perp}$. Now this bundle is never trivial, since each exceptional orbit is not orientable. But we may consider the manifold $\tilde{M}$ given by

$$UL = \{(x,v) \in L \,|\, g_x(v,v) = 1\},$$

which turns out to be a compact, connected (since L is not trivial) two-fold covering manifold of M, with projection map $\pi(x,v) = x$. Moreover each element $h \in G$ acts on $\tilde{M}$ in a natural way by putting

$$h(x,v) = (h(x), dh_x(v)), \qquad (x,v) \in \tilde{M}, \quad h \in G$$

It is then clear that $\tilde{M}$, endowed with the lifted Riemannian metric g, is a cohomogeneity one Riemannian manifold under the action of the group G with orbit space $\tilde{M}/G = S^1$: indeed, any orbit of a point $\tilde{x} \in \tilde{M}$ is a covering of the orbit $G(\pi(\tilde{x}))$, hence, if there are singular orbits in $\tilde{M}$, they must be exceptional; but the pulled back vector bundle $\pi^*(L)$ is trivial on $\tilde{M}$, so that there are no exceptional orbits and the orbit space is S^1. So we may apply our main theorem or Corollary 2.4 to $\tilde{M}$.

Corollary 2.5. *Let (M,g) be a compact, homogeneous Riemannian manifold which is of cohomogeneity one under the action of a connected Lie group G. Suppose moreover that a principal orbit G/K is an isotropy irreducible homogeneous space. Then one of the following is true:*

1) *(M,g) is isometric to a sphere or to a real projective space;*

2) *the universal covering manifold $(\tilde{M},\tilde{g})$ splits isometrically as the Riemannian product $\mathbb{R} \times \tilde{L}$, where $\tilde{L}$ is the universal covering of a principal orbit G/K.*

Proof. First of all, we note that the existence of a Killing vector field which is not tangent to the foliation $\mathfrak{F}$ induced by G is equivalent to the homogeneity of (M, g). Moreover we observe that, if we denote by A the shape operator of the foliation $\mathfrak{F}$, then A is G-invariant, hence, by isotropy irreducibility, A is a multiple of the identity operator; by Proposition 1.5, this is equivalent to saying that the open subset of regular points of M has a warped product structure.

We now distinguish two cases, according to the nature of the orbit space $\Omega = M/G$, which can be diffeomorphic to S^1 or to $[0, \pi]$. If $\Omega \cong S^1$, then assertion 2) follows from Corollary 2.2 by standard arguments.

We may therefore suppose that $\Omega \cong [0, \pi]$. If all singular points in M are exceptional, then Corollary 2.2 applies to the two-fold covering $\tilde{M}$ of M and we get again assertion 2). So we may suppose that there is at least one non-exceptional singular point $p \in M$. We denote by H the stability subgroup of G at p, with $K \subset H \subset G$; but isotropy irreducibility implies that the Lie algebra $\mathfrak{H}$ of H coincides with the Lie algebra $\mathfrak{G}$ of G or with the Lie algebra $\mathfrak{K}$ of K. Now, the dimension of the singular orbit $G(p)$ is $\dim \mathfrak{G} - \dim \mathfrak{H}$ and this is strictly less than the dimension of a regular orbit G/K since p is not exceptional; we conclude that $\mathfrak{G} = \mathfrak{H}$, hence $G = H$, since G is supposed to be connected. This implies that the singular orbit $G(p)$ reduces to the point $\{p\}$ and that each regular orbit $G/K = H/K$ is isometric to a sphere. Moreover, we note that the linear isotropy representation of $G = H$ into $O(TM_p)$ acts transitively on the unit sphere in TM_p. From this it follows that the Ricci tensor at p is a multiple of the metric tensor, say $\mathrm{Ric}_p = \lambda g_p$. Now, from homogeneity, we deduce that (M, g) is an Einstein space: it is moreover clear that the scalar curvature τ of (M, g) cannot be negative, since otherwise there would exist no nonzero Killing vector fields by Bochner's theorem (see e.g. [8]). We first show that τ cannot be zero.

Indeed if the Ricci tensor vanishes identically, then (M, g), being homogeneous, is flat by Alekseevsky–Kimelfeld theorem (see e.g. [5, p.191]). But a flat homogeneous Riemannian manifold is isometric to a flat torus T^n: we shall show that this case can not occur, by examining the possible singular orbits.

We know that we have at least one singular orbit consisting of one point $\{p\}$: if also the other singular orbit reduces to a point $\{q\}$, then the fibration $\kappa : T^n - \{p, q\} \to (0, \pi)$ is trivial and we would have that $T^n - \{p, q\}$ is diffeomorphic to $(0, \pi) \times S^{n-1}$, since we know that the regular orbits are diffeomorphic to spheres. But a simple argument involving the first fundamental group excludes this possibility. Then the other singular orbit must be exceptional: in this case the G-manifold $N = T^n - \{p\}$ is of cohomogeneity one and G acts also on the two-fold covering $\tilde{N}$ with orbit space $\tilde{N}/G = \mathbb{R}$. Moreover a G-orbit in $\tilde{N}$ is a finite covering of a regular orbit in T^n, which is diffeomorphic to a sphere, hence it is a sphere itself. So we would have that $\tilde{N}$ is diffeomorphic to $\mathbb{R} \times S^{n-1}$. This would imply that the fundamental group $\pi_1(T^n - \{p\})$ contains a subgroup of index 2 isomorphic to $\pi_1(\mathbb{R} \times S^{n-1})$ and this is easily seen to be impossible.

So we have shown that the scalar curvature τ must be positive. Again we recall that the open dense subset of regular points M_{reg} is isometric to $(0, \pi) \times S^{n-1}$, endowed with a metric $dt^2 + \phi(t)^2 g_o$ for some C^∞ function ϕ on $(0, \pi)$. Our conclusion will follow from a result quoted in [5], which we reformulate here for the sake of completeness

Lemma 2.4. *Let (M, g) be a complete Einstein manifold with positive scalar curvature, containing an open dense subset U which is a warped product on an one dimensional basis with complete fibre. Then (M, g) is isometric to S^n or to $\mathbb{R}P^n$.*

For the proof of this lemma, we refer to [5], where it can be deduced from Theorem 113, p. 269, by a case by case checking of all spaces coming up in the classification list. □

Proof of Theorem II. It follows immediately from Corollary 2.5 and from irreducibility of (M, g). □

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