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# Discrete Mathematics

journal homepage: [www.elsevier.com/locate/disc](http://www.elsevier.com/locate/disc)

## On dynamic monopolies of graphs with general thresholds

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### ARTICLE INFO

#### Article history:

Received 23 January 2011

Received in revised form 21 November 2011

Accepted 29 November 2011

Available online 24 December 2011

#### Keywords:

Spread of influence

Dynamic monopolies

### ABSTRACT

Let  $G$  be a graph and  $\tau : V(G) \rightarrow \mathbb{N}$  be an assignment of thresholds to the vertices of  $G$ . A subset of vertices  $D$  is said to be dynamic monopoly (or simply dynamo) if the vertices of  $G$  can be partitioned into subsets  $D_0, D_1, \dots, D_k$  such that  $D_0 = D$  and for any  $i = 1, \dots, k-1$  each vertex  $v$  in  $D_{i+1}$  has at least  $t(v)$  neighbors in  $D_0 \cup \dots \cup D_i$ . Dynamic monopolies are in fact modeling the irreversible spread of influence such as disease or belief in social networks. We denote the smallest size of any dynamic monopoly of  $G$ , with a given threshold assignment, by  $\text{dyn}(G)$ . In this paper, we first define the concept of a resistant subgraph and show its relationship with dynamic monopolies. Then we obtain some lower and upper bounds for the smallest size of dynamic monopolies in graphs with different types of thresholds. Next we introduce dynamo-unbounded families of graphs and prove some related results. We also define the concept of a homogeneous society that is a graph with probabilistic thresholds satisfying some conditions and obtain a bound for the smallest size of its dynamos. Finally, we consider dynamic monopoly of line graphs and obtain some bounds for their sizes and determine the exact values in some special cases.

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### 1. Motivation and related works

In recent years, great attention has been paid to the modeling and analysis of the spread of belief or influence in complex networks. Various problems in social and virtual networks such as world wide web or models of distributed computing can be formalized in terms of the spread of influence. Elections in societies where individuals decide whether to vote for a certain candidate, spread of disease among people or virus in world wide web or any web of computers are some examples of these problems. A network in all of these examples which is simply consisted of a set of elements (e.g. agents in social networks or computing units in distributed computing systems) and some relationships or interactions between these elements can be conveniently modeled by a graph whose nodes represent the elements and edges represent the links of the network. For the graph theoretical notions, not defined in this paper, we refer the reader to [3].

The model to be studied in this paper is as follows. A graph  $G$  on the vertex set  $V(G)$  and the edge set  $E(G)$  together with an assignment of thresholds  $\tau : V(G) \rightarrow \mathbb{N}$  to the vertices of  $G$  is given. The discrete time dynamic process corresponding to the threshold assignment  $\tau$  is defined as follows.

The process starts with a subset  $D$  of vertices which consists of the vertices having the state  $+$  at time 0. We denote the set of vertices of state  $+$  in time  $i$  by  $D_i$ . So at the beginning i.e. at time 0 we have  $D_0 = D$ . Then at any time  $i + 1 \geq 1$ , the state of any vertex  $v$  changes to the state  $+$  provided that at least  $t(v)$  neighbors of  $v$  belong to  $D_i$ . Also if the state of  $v$  is already  $+$  in time  $i$  (i.e. when  $v \in D_i$ ) then its state remains  $+$  as before. If at a certain time  $i$  of the process a vertex  $v$  has state  $+$  then  $v$  is said to be an active vertex. Note that the process defined above is progressive or irreversible i.e. when the state of a vertex becomes  $+$  at some step of the process then its state remains unchanged until the end of the process.

By a  $\tau$ -dynamic monopoly we mean any subset  $D$  of the vertices of  $G$  such that by starting from  $D$ , all the vertices of  $G$  get state  $+$  at the end of the process. Throughout the paper by  $(G, \tau)$  we mean a graph  $G$  together with a threshold

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assignment  $\tau$  to the vertices of  $G$ . We simply write dynamic monopoly or (following some authors) dynamo instead of  $\tau$ -dynamic monopoly. By the size of a dynamo  $D$  we mean the cardinality of  $D$ . It is easy to see that a subset of vertices  $D$  in a graph  $(G, \tau)$  is dynamo if and only if there exists a partition of  $V(G)$  into subsets  $D_0, D_1, \dots, D_k$  such that  $D_0 = D$  and for any  $i = 1, \dots, k-1$  each vertex  $v$  in  $D_{i+1}$  has at least  $t(v)$  neighbors in  $D_0 \cup \dots \cup D_i$ . Dynamic monopolies have been widely studied by various authors. Some related graph theoretical and algorithmic results concerning dynamic monopolies have been obtained in [7,10]. More polynomial time or approximation algorithms were obtained in [5]. Dynamic monopolies of special families of graphs were studied in [9,11,14]. Dynamic monopolies in random graphs were studied in [4]. Also [12] studies optimization formulations of dynamic monopolies and obtains some approximation algorithms. The reversible (or non-progressive) version of dynamic monopolies which we defined in this paper was introduced in [16] in terms of repetitive polling games and under the name of dynamic monopolies. In [17], controlling monopolies, a concept similar to dynamic monopolies have been introduced. The survey paper [17] also surveys some of the results concerning various kinds of monopolies including dynamic monopolies. It also presents various applications of these concepts in distributed computing and probabilistic polling models. *The outline of the paper is as follows.* In Section 2, we first introduce resistant subgraphs and show their relationships with dynamic monopolies. Then we obtain some lower and upper bounds for the dynamo size of graphs with various types of thresholds. Finally, in Section 2, we determine the smallest dynamo of Generalized Petersen graph  $GP(n, k)$ . In Section 3, we introduce dynamo-unbounded families of graphs and obtain some results in this regard. We show the importance of the concept of dynamo-unbounded families by its applications in some famous social network problems. Homogeneous societies are also defined in Section 3 and a result concerning their dynamo sizes is given in Section 3. In Section 4, we consider line graphs and obtain some lower bounds for their dynamo numbers and determine the exact value for the line graph of the complete bipartite graphs. The final section is devoted to some complementary concepts and results, where some open questions are also proposed.

## 2. Some bounds for dynamo size of graphs

We begin with the following concept. Given  $(G, \tau)$ , by a resistant subgraph of  $G$  we mean any subgraph  $K$  such that for any vertex  $v \in K$  one has  $d_K(v) \geq d_G(v) - t(v) + 1$ , where  $d_G(v)$  is the degree of  $v$  in  $G$ . The following proposition provides a sufficient and necessary condition for graphs containing no resistant subgraphs.

**Proposition 1.** *A subgraph  $H$  of  $(G, \tau)$  does not contain any resistant subgraph of  $G$  if and only if the vertices of  $H$  can be labeled as  $v_1, v_2, \dots, v_n$  such that  $v_i$  has at most  $d_G(v_i) - t(v_i)$  neighbors among  $\{v_i, v_{i+1}, \dots, v_n\}$ .*

**Proof.** Assume first that  $H$  contains no resistant subgraph. Then  $H$  itself is not a resistant graph and so there exists a vertex  $v_1 \in H$  such that  $d_H(v_1) \leq d_G(v_1) - t(v_1)$ . Set  $H_1 = H \setminus \{v_1\}$ . Since  $H_1$  too is not resistant then there exists  $v_2$  such that  $d_{H_1}(v_2) \leq d_G(v_2) - t(v_2)$ . This means that  $v_2$  has at most  $d_G(v_2) - t(v_2)$  neighbors in  $H[v_2, \dots, v_n]$ . We repeat this argument and obtain  $v_1, v_2, \dots, v_n$  such that  $d_{H_{i-1}}(v_i) \leq d_G(v_i) - t(v_i)$  where  $H_{i-1} = H[v_i, \dots, v_n]$ .

Assume now that the vertices of  $H$  are labeled as specified in the proposition. Let by contrary that  $H$  has a resistant subgraph  $K$ . Let  $i$  be the smallest index with  $v_i \in K$ . Then  $K \subseteq H_{i-1} = H[v_i, \dots, v_n]$ . This implies  $d_K(v_i) \leq d_{H_{i-1}}(v_i)$ . From one side we have  $d_G(v_i) - t(v_i) + 1 \leq d_K(v_i)$  and from other side  $d_{H_{i-1}}(v_i) \leq d_G(v_i) - t(v_i)$ . This contradiction completes the proof.  $\square$

**Proposition 2.** *A subset  $D$  in  $(G, \tau)$  is dynamic monopoly if and only if  $G \setminus D$  does not contain any resistant subgraph.*

**Proof.** Assume first that there exists a subgraph  $K$  of  $G \setminus D$  which is resistant. Let  $D_i$  be the set of vertices of  $G$  which become active at time  $i$ , by starting from  $D_0 = D$ . Let  $v$  be any arbitrary vertex of  $K$ . Then  $v$  has at most  $t(v) - 1$  neighbors in  $G \setminus K$ . Since  $D \subseteq G \setminus K$  then this shows that  $v$  has at most  $t(v) - 1$  neighbors in  $D$ . Therefore  $K \cap D_1 = \emptyset$  and in particular  $v$  has at most  $t(v) - 1$  neighbors in  $D_1$ . We repeat the argument we made above, for  $v$  and  $D_1$  and also for other  $D_i, i = 2, \dots$  to conclude that  $K$  remains outside  $D_i$  for any  $i$ . This shows that  $D$  cannot be a dynamic monopoly.

Now assume that  $H = G \setminus D$  does not contain any resistant subgraph. This shows that  $H$  itself is not resistant. Then there exists a vertex  $v_1$  of  $H$  with  $d_H(v_1) \leq d_G(v_1) - t(v_1)$ . Namely  $v_1$  has at least  $t(v_1)$  neighbors in  $K$ . Then  $v_1 \in D_1$ . Now we repeat this argument for  $H \setminus v_1$  which is not a resistant subgraph of  $H$  by the hypothesis on  $H$ . We obtain another vertex  $v_2$  of  $H$  with  $v_2 \in D_2$ . Naturally this technique can be applied until all the vertices of  $H$  get state + in at most  $|H|$  time steps. This completes the proof.  $\square$

The following useful remark comes immediately.

**Remark 1.** If a vertex  $v \in G$  is such that  $t(v) \geq d_G(v) + 1$  then the subgraph of  $G$  consisting of the single vertex  $v$  is a resistant subgraph.

Throughout the paper for any subset  $S \subseteq V(G)$  we denote the subgraph of  $G$  induced on  $S$  by  $G[S]$ . Also  $\delta(G)$  denotes the minimum degree in  $G$ .

**Theorem 1.** *Let  $D$  be a dynamic monopoly of size  $k$  in  $G$ . Set  $H = G \setminus D$  and let  $t_{\max}$  be the maximum threshold among the vertices of  $H$ . Then*

- (i)  $\sum_{v \in H} t(v) \leq |E(G)| - |E(G[D])| - \delta(G) + t_{\max}$
- (ii)  $\sum_{v \in H} t(v) \leq |E(G)|$  provided that  $t(v) \leq d_G(v)$  for any vertex  $v \in H$ .

**Proof.** By Proposition 2,  $H$  does not contain any resistant subgraph and so by Proposition 1 the vertices of  $H$  can be labeled as  $v_1, \dots, v_n$  in such a way that for any  $i$ ,  $d_{H_i}(v_i) \leq d_G(v_i) - t(v_i)$  where  $H_i = H[v_i, \dots, v_n]$ . In particular  $H = H_1$  and  $H_2 = H \setminus v_1$ . For the number of edges of  $H_i$  we have

$$|E(H_i)| = |E(H_{i-1})| - d_{H_{i-1}}(v_{i-1}) \geq |E(H_{i-1})| - d_G(v_{i-1}) + t(v_{i-1}).$$

We use recursively the above inequality and obtain the following

$$|E(H_i)| \geq |E(H)| - \sum_{j=1}^{i-1} d_G(v_j) + \sum_{j=1}^{i-1} t(v_j).$$

For  $i = |H|$  we have  $|E(H_i)| = 0$  and obtain the following

$$|E(H)| \leq \sum_{v \in H} d_G(v) - d_G(v_n) - \sum_{v \in H} t(v) + t(v_n). \tag{1}$$

Now we estimate the number of edges of  $G$ . Let  $e$  be the number of edges between  $D$  and  $H$ . We have  $|E(G)| = |E(G[D])| + e + |E(H)|$ . Using (1) we obtain

$$|E(G)| \leq |E(G[D])| + e + \sum_{v \in H} d_G(v) - d_G(v_n) - \sum_{v \in H} t(v) + t(v_n).$$

We note that  $\sum_{v \in H} d_G(v) = e + 2|E(H)|$ . Let  $t_{\max}$  be the maximum threshold among the vertices of  $H$ . We have now

$$|E(G)| \leq |E(G[D])| + e + e + 2|E(H)| - \delta(G) + t_{\max} - \sum_{v \in H} t(v).$$

Therefore

$$\sum_{v \in H} t(v) \leq |E(G)| - |E(G[D])| - \delta(G) + t_{\max}.$$

To prove part (ii), note that by (1) we have  $|E(H)| \leq \sum_{v \in H} d_G(v) - \sum_{v \in H} t(v)$ . Now by  $|E(G)| = |E(G[D])| + e + |E(H)|$  and  $\sum_{v \in H} d_G(v) = e + 2|E(H)|$  the desired inequality is obtained.  $\square$

As a corollary of Theorem 1, part (ii) we have the following result for regular graphs.

**Corollary 1.** Let  $G$  be a regular graph of degree  $2r + 1$ . Let also  $t(v) = r + 1$  for any vertex of  $G$ . Then any dynamo for  $G$  has at least  $\frac{n+2r}{2(r+1)}$  vertices.

In Theorem 3, we show that the bound obtained in Corollary 1 is a tight bound when  $r = 1$  i.e. for cubic graphs. Of course we believe that it is also tight for regular graphs of higher degrees. Before stating Theorem 3, we present an upper bound for the dynamic monopoly of graphs in which the threshold of any vertex is 2.

**Theorem 2.** Let  $G$  be a graph on  $n$  vertices such that no connected component of  $G$  is isomorphic to an odd cycle, with  $\delta(G) \geq 2$  and  $t(v) = 2$  for any  $v \in G$ . Let also  $S$  be any domination set of  $G$  and let  $c$  be the number of connected components of  $G \setminus S$ . Then the following bounds hold.

- (i)  $\text{dyn}(G) \leq n/2$
- (ii)  $\text{dyn}(G) \leq \text{dyn}(G[S]) + c$
- (iii)  $\text{dyn}(G) \leq \frac{|S|}{2} + c$ .

**Proof.** We may assume that  $G$  is connected. To prove (i), let  $\{C_1, \dots, C_k\}$  be a set of vertex disjoint cycles in  $G$  with the maximum cardinality. This set is non-empty because  $\delta(G) \geq 2$ . In the following we obtain a dynamo denoted by  $D$  of cardinality at most  $n/2$ . The subgraph  $G \setminus (C_1 \cup \dots \cup C_k)$  is a forest, denote it by  $F$ . Let  $T$  be any connected component of  $F$ . The tree  $T$  contains a domination set  $S$  of the cardinality at most  $|T|/2$ . From any connected component  $T$  of  $F$  we consider a minimum domination, say  $S$  for  $T$  and add the vertices of  $S$  to  $D$ . Note that  $|S| \leq |T|/2$ .

Now we consider the odd cycles of  $\{C_1, \dots, C_k\}$  and let  $C_i$  be any one of such cycles. There are three possibilities.

Case 1. There is an edge between  $C_i$  and another odd cycle say  $C_j$ . Let  $u \in C_i$ ,  $v \in C_j$  and  $uv \in E(G)$ . We put one of  $u$  or  $v$  in  $D$ . It is easy now to find  $(|C_i| + |C_j|)/2$  suitable vertices from  $C_i \cup C_j$  in order to activate whole vertices of  $C_i \cup C_j$ . We add these vertices to  $D$ .

Case 2. The cycle  $C_i$  is only adjacent to even cycle(s). Let  $C_i$  be adjacent to  $C_l$  where  $C_l$  is an even cycle and it is not already activated. Let  $u \in C_l$  be a vertex adjacent to  $C_i$ . We add  $u$  to  $D$ . Now we can activate the whole  $C_i \cup C_l$  using  $(|C_i| + |C_l| - 1)/2$  vertices including the vertex  $u$ .

Case 3. The cycle  $C_i$  is only adjacent to a vertex say  $v$  of  $F$ . In this case we put  $v$  in  $D$ . Note that by  $(|C_i| + 1)/2$  vertices we can activate whole  $C_i \cup \{v\}$ .

So far we have activated all the odd cycles of  $\{C_1, C_2, \dots, C_k\}$  and whole vertices of the forest  $F$  except possibly some of its leaf vertices. Also some of the even cycles of the collection are activated during the above steps. The remaining even cycles can easily be activated using half of their vertices. We add these new vertices by which we activate even cycles in  $D$  too. The construction process of the dynamo  $D$  is finished. We only have to check the leaf vertices of  $F$ . Let  $v$  be any vertex of degree one in a connected component  $T$  of  $F$ . We already know that  $v$  has an active neighbor in the tree  $T$ . Since the degree of  $v$  in the whole graph is at least two then it should have another neighbor in  $C_1 \cup \dots \cup C_k$ . But the latter set is activated by  $D$ . Hence we obtain two active neighbors for the vertex  $v$ . Note that the set  $D$  has at most  $n/2$  vertices by the way of its construction. This completes the proof of part (i).

To prove part (ii), let  $D$  be any dynamo of size  $\text{dyn}(G[S])$ . Using  $D$  we can activate all vertices of  $G[S]$ . Let  $T$  be any connected component of  $G \setminus S$ . Any vertex of  $T$  has an active neighbor since  $S$  is a domination set of  $G$ . Now since  $T$  is connected one extra vertex from  $T$  is enough to activate all vertices of  $T$ . We conclude that there exists a dynamo of size  $\text{dyn}(G[S]) + c$ , where  $c$  is the number of components of  $G \setminus S$ .

Part (iii) can be derived from part (ii) and applying the proof of part (i) for  $G[S]$ .  $\square$

In the following theorem by determining the minimum dynamo of Generalized Petersen graphs we show that the lower bound of Corollary 1 is tight. We first recall the definition of Generalized Petersen graphs  $GP(n, k)$ . For any positive integers  $n$  and  $k$  with  $k \leq n - 2$  and  $\text{gcd}(n, k) = 1$  we define  $GP(n, k)$  as follows. The vertex set of  $GP(n, k)$  is  $\{a_1, a_2, \dots, a_n\} \cup \{b_1, b_2, \dots, b_n\}$ . The edges of  $GP(n, k)$  are  $a_i b_i$  for  $i = 1, \dots, n$ ,  $a_1 a_n, a_j a_{j+1}$  for  $j = 1, \dots, n - 1$  and also  $a_i b_j$  for any  $i$  and  $j$  such that  $|i - j| = k$ . We note that  $GP(n, k)$  is a cubic graph and its order is  $2n$ . The following theorem determines the exact value of the minimum dynamic monopoly in  $GP(n, k)$ , where the threshold of any vertex is two.

**Theorem 3.** *The size of smallest dynamic monopoly in  $GP(n, k)$  with  $t(v) = 2$  for any vertex  $v$ , is  $\lceil \frac{n+1}{2} \rceil$ .*

**Proof.** Since  $|GP(n, k)| = 2n$ , then by Corollary 1 it is enough to show that  $GP(n, k)$  contains a dynamo of size  $\lceil \frac{2n+2}{4} \rceil = \lceil \frac{n+1}{2} \rceil$ . Assume first that  $n$  is an even number. In this case we use directly Theorem 2. The subset  $S = \{a_1, a_2, \dots, a_n\}$  forms a dominating set in  $GP(n, k)$  and since  $\text{gcd}(n, k) = 1$  then  $GP(n, k) \setminus S = G[b_1, \dots, b_n]$  is a connected subgraph of the graph. From the other side,  $\text{dyn}(G[S]) = n/2$ . Hence by Theorem 2 there exists a dynamo of size  $\lceil \frac{n+1}{2} \rceil$  for  $GP(n, k)$  when  $n$  is even. In fact  $\{a_1, a_3, \dots, a_{n-1}, b_{n-1}\}$  is a dynamo of this size.

When  $n$  is odd, it can be shown that  $S' = \{a_1, a_3, a_5, \dots, a_{n-2}, b_{n-1}\}$  is a dynamic monopoly in  $GP(n, k)$ . The argument uses Theorem 2 but the point is that using  $S'$  we first activate all vertices of  $\{a_1, a_2, \dots, a_n\}$  and then use the connectivity of  $GP(n, k) \setminus \{a_1, a_2, \dots, a_n\}$ .  $\square$

### 3. Dynamo-unbounded families of graphs

Consider an election where people vote YES or NO for a certain candidate. Any individual (represented by a vertex  $v$  in the underlying network) votes YES if at least  $t(v)$  number of her friends have decided to vote YES. A dynamic monopoly  $D$  for the underlying network of this election has the property that if the vote of the members of  $D$  is YES then the whole community will eventually vote for that candidate. The following strategic question arises. If the population of the community increases then does it imply that the size of the smallest dynamo too increases (as a function of the size of community)? Another example where the same question becomes important is the adoption of a new product in viral marketing (for a formulation of viral marketing in terms of dynamic monopolies, see [6]). In the following by introducing the concept of dynamo-unbounded families, we present a method to analyze the question we mentioned above.

By a threshold pattern we mean any threshold assignment  $\tau$  such that for any graph  $G$  and any  $v \in V(G)$ ,  $\tau$  assigns a non-negative value  $t(v)$  such that if  $\sigma$  is any automorphism of  $G$  with  $\sigma(u) = v$  for some vertex  $u \in G$ , then  $t(v) = t(u)$ . Without loss of generality we may restrict a threshold pattern  $\tau$  so that  $t(v) \leq d_G(v)$ . The common examples are when  $t(v)$  is a function of  $d_G(v)$  for any vertex  $v$ . In this section by a family  $\mathcal{F}$  we mean any set of graphs equipped with a threshold pattern. Such a family is called dynamo-unbounded if there exists a function  $f(x) \rightarrow \infty$  as  $x \rightarrow \infty$  such that for any graph  $G$  from  $\mathcal{F}$  one has  $f(n) \leq \text{dyn}(G)$ , where  $n = |G|$ . Corollary 1 implies that the family of  $2r + 1$ -regular graphs with threshold  $t(v) = r + 1$  for any vertex, is dynamo-unbounded family. In this section we obtain more results concerning dynamo-unbounded graphs.

In the following corollary, we denote the edge density of a graph  $G$  by  $\epsilon(G)$  which is defined as  $\epsilon(G) = |E(G)|/|V(G)|$ .

**Corollary 2.** *Let  $(G, \tau)$  be a graph of order  $n$ . Set  $t = \min\{t(v) : v \in V(G)\}$ . Then*

$$n \left( 1 - \frac{\epsilon(G)}{t} \right) \leq \text{dyn}(G).$$

**Proof.** Let  $D$  be any dynamo of size  $k = \text{dyn}(G)$  and  $H = G \setminus K$ . Then by part (ii) of Theorem 1,  $(n - k)t \leq |E(G)|$ . This easily implies the desired inequality.  $\square$

The following corollary follows immediately from Corollary 2.

**Corollary 3.** *Let  $\mathcal{F}$  be any family of graphs such that for some positive constant  $\delta$ ,  $\min\{t(v) : v \in G\} \geq \epsilon(G) + \delta$  for any graph  $G \in \mathcal{F}$ . Then  $\mathcal{F}$  is dynamo-unbounded.*

The following theorem concerns graphs with probabilistic thresholds. In proving the following theorem we shall make use of the following concentration result of McDiarmid [15]. Let  $X_1, \dots, X_n$  be a sequence of nonnegative independent random variables and set  $X = \sum X_i$ . Then for any  $\lambda \geq 0$

$$\mathbb{P}(X \leq \mathbb{E}(X) - \lambda) \leq e^{-\frac{\lambda^2}{2 \sum_{i=1}^n \mathbb{E}(X_i^2)}}.$$

**Theorem 4.** Assume that any vertex  $v$  of  $G$  chooses a random threshold  $a \leq i \leq b$  with probability  $p_i$  where  $p_i$  is independent of  $v$ . Set  $\alpha = \sum_a^b ip_i$ . If  $\alpha > \epsilon(G)$  then for any positive constant  $\delta$ , with high probability no dynamo of  $G$  contains less than  $n^{1-\delta}$  elements.

**Proof.** Let  $n$  be a sufficiently large integer so that  $\alpha/(\alpha - \epsilon) < n^\delta$ . Let  $D$  be any subset of vertices of cardinality  $k$  where  $k < n^{1-\delta}$  and set  $H = G \setminus D$ . For any vertex  $v \in H$  define a random variable  $X_v$  as the threshold  $t(v)$  chosen by the vertex  $v$ . Set  $X = \sum_{v \in H} X_v$ . We have the following by Theorem 1, part (ii).

$$\mathbb{P}(D \text{ is a dynamo}) \leq \mathbb{P}(X \leq |E(G)|).$$

For  $X$  we have the following information

$$\mathbb{E}(X) = (n - k) \sum_{i \in [a,b]} ip_i = \alpha(n - k) \quad \text{and} \quad \sum_{v \in H} \mathbb{E}(X_v^2) = (n - k) \sum_{i \in [a,b]} i^2 p_i.$$

Also write  $\beta = \sum_{i \in [a,b]} i^2 p_i$  for simplicity. By our hypothesis  $k < n^{1-\delta}$  which by  $\alpha/(\alpha - \epsilon) < n^\delta$  imply  $k < n(1 - \epsilon/\alpha)$  or  $\mathbb{E}(X) - |E(G)| > 0$ . Now by considering  $\lambda = \mathbb{E}(X) - |E(G)|$  we may use the above-mentioned result of McDiarmid, since  $\lambda > 0$ . We obtain

$$\begin{aligned} \mathbb{P}(D \text{ is a dynamo}) &\leq \mathbb{P}(X \leq |E(G)|) \\ &\leq \mathbb{P}(X \leq \mathbb{E}(X) - \lambda) \\ &\leq e^{-\frac{\lambda^2}{2 \sum_{v \in H} \mathbb{E}(X_v^2)}} \\ &= e^{-\frac{(\alpha(n-k) - n\epsilon(G))^2}{2(n-k)\beta}} \\ &= e^{-\frac{(n(\alpha-\epsilon) - k\alpha)^2}{2(n-k)\beta}} \\ &\leq e^{-\frac{(n(\alpha-\epsilon) - \alpha n^{1-\delta})^2}{2n\beta}}. \end{aligned}$$

The latter inequality implies that for some positive constants  $c'$  and  $c$ ,  $\mathbb{P}(D \text{ is a dynamo}) \leq e^{-\frac{c'n^2}{2n\beta}} = e^{-cn}$ . Therefore the probability that there exists a monopoly of size  $k$  is at most

$$n^k e^{-cn} = e^{k \ln n - cn}.$$

Now since  $k \leq n^{1-\delta}$  then  $e^{k \ln n - cn} = o(1)$ . This shows that no subset of cardinality less than  $n^{1-\delta}$  is a dynamo. This completes the proof.  $\square$

We call any graph satisfying the assumptions of Theorem 4, a homogeneous society. The proof of Theorem 4 shows that if  $G$  is a homogeneous society and  $D$  any subset of  $G$  with  $|D| \leq n(1 - \epsilon/\alpha)$  then with high probability  $D$  is not a dynamo. We pose the following question: is it true that with high probability  $\text{dyn}(G) \geq n(1 - \epsilon/\alpha)$  for any homogeneous society?

A result related to the concept of dynamo-unbounded graphs is that of [2]. In [2] the reversible version of the model we studied in this paper has been considered. At each time step of the process any vertex updates its state as follows. Any vertex  $v$  takes a new state which is the state of the majority of its neighbors. In case that the number of active neighbors is the same as the number of non-active neighbors of  $v$  then the state of  $v$  is remained unchanged. Berger proved that for any  $n$  there exists a graph of more than  $n$  vertices which contains a dynamic monopoly of at most 18 vertices.

The following proposition shows that the result of Corollary 3 is best possible.

**Proposition 3.** For any positive integers  $r$  and  $n$  with  $r|n$ , there exists a  $2r$ -regular graph on  $n$  vertices which contains a dynamo of size  $r$ , where the threshold of any vertex is taken  $r$ .

**Proof.** Write  $n = rq$  for some  $q > 0$ . Let also  $C_1, \dots, C_q$  be  $q$  vertex disjoint copies of  $\overline{K_r}$  where  $\overline{K_r}$  is the empty graph on  $r$  vertices. Denote the vertex set of  $C_i$  by  $V_i$ . We define a graph  $G$  as follows. The vertex set of  $G$  is  $V_1 \cup V_2 \cup \dots \cup V_q$ . In  $G$  the subgraph induced on  $V_i \cup V_{i+1}$  for any  $i = 1, 2, \dots, q$  (when  $i = q$  we take  $q + 1$  as 1) is a complete bipartite subgraph whose bipartition sets are  $V_i$  and  $V_{i+1}$ . Set the threshold of each vertex of  $G$  as  $r$ . It is easily seen that  $\text{dyn}(G) = \epsilon(G) = r$ .  $\square$

#### 4. Dynamic monopolies in line graphs

By the line graph of a graph  $G$  denoted by  $L(G)$  we mean a graph whose vertex set is the edge set of  $G$  where two vertices  $e$  and  $e'$  of  $L(G)$  (as two edges in  $G$ ) are adjacent if and only if  $e$  intersects  $e'$  in  $G$ . A dynamic monopoly of  $L(G)$  can be considered

as the dynamic monopoly of the edges of  $G$ . In this section we study dynamic monopolies in line graphs when  $G$  is a regular graph where our studies will be in terms of the edges of  $G$  instead of working with vertices of  $L(G)$ . Note that if  $e$  is any edge between two vertices  $u$  and  $v$  in a graph  $G$  then the degree of  $e$  as a vertex of  $L(G)$  is  $d_G(u) + d_G(v) - 2$ .

**Theorem 5.** *Let an  $r$ -regular graph  $G$  with an assignment of thresholds to the edges of  $G$  be given. Set  $t = \min\{t(e) : e \in E(G)\}$ . Let  $D \subseteq E(G)$  be a dynamic monopoly of size  $k$  in  $L(G)$ . Then*

$$k \geq \left\lfloor \frac{4(t - r + 1)n + (2r - t)^2}{8} \right\rfloor.$$

**Proof.** Since  $G$  is  $r$ -regular then  $L(G)$  is  $2r - 2$ -regular. Set  $H = G \setminus D$ . The graph  $H$  has  $n$  vertices and  $|E(G)| - k = rn/2 - k$  edges. Since  $D$  is a dynamo then there exists  $e_1 \in E(H)$  such that  $d_H(e_1) = d_G(e_1) - d_D(e_1) \leq d_G(e_1) - t$ . Note that if  $e = uv$  then  $d_H(e_1) = d_H(u) + d_H(v) - 2$ .

Set now  $H_1 = H \setminus \{u, v\}$  we have  $|H_1| = n - 2$  and when we remove  $u$  and  $v$  from  $H$ , we lose exactly  $d_H(u) + d_H(v) - 1$  edges from  $H$ .

$$\begin{aligned} |E(H_1)| &= |E(H)| - (d_H(u) + d_H(v) - 1) \\ &= |E(H)| - d_H(e_1) - 1 \\ &\geq |E(H)| - d_G(e_1) + t - 1 \\ &\geq |E(H)| - (2r - 2) + t - 1. \end{aligned}$$

We repeat the above technique and obtain  $H_i$  on  $n - 2i$  vertices such that

$$|E(H_i)| \geq |E(H)| - i(2r - 2) + i(t - 1).$$

Now we use the obvious upper bound  $|E(H_i)| \leq \binom{n-2i}{2}$  and obtain the following inequalities for any  $i$

$$\begin{aligned} |E(H)| &\leq i(2r - t - 1) + \frac{4i^2 - 4in + 2i + n^2 - n}{2} \\ |E(H)| &\leq 2i^2 + i(2r - t - 1 - 2n + 1) + (n^2 - n)/2. \end{aligned}$$

The value in the right hand of the above inequality minimizes at  $i = \frac{2n-2r+t}{4}$ . Its minimum value is  $\frac{8rn-4nt-4n+4rt-4r^2-t^2}{8}$ . It turns out that

$$k \geq \left\lfloor \frac{4(t - r + 1)n - 4rt + 4r^2 + t^2}{8} \right\rfloor. \quad \square$$

The following theorem is concerning the line graphs of bipartite graphs.

**Theorem 6.** *Let  $G$  be an  $r$ -regular bipartite graph on  $n$  vertices and  $t$  an assignment of thresholds to the edges of  $G$ . Set  $t = \min\{t(e) : e \in E(G)\}$ . Let  $D \subseteq E(G)$  be a dynamic monopoly of size  $k$  in  $L(G)$ . Then*

$$k \geq \frac{n(2t - 2r + 2) + (2r - t)^2 - 4r + 2t}{4} + \epsilon$$

where  $\epsilon = 1/4$  if  $n - 2r + t + 1$  is an even integer and  $\epsilon = 0$  otherwise.

**Proof.** The proof is similar to the one of Theorem 5. Since  $G$  is regular then each bipartition of  $G$  contains  $n/2$  vertices. There exists  $e_i \in E(H_{i-1})$  such that  $d_{H_{i-1}}(e_i) \leq d_G(e_i) - t$ . Set  $H_i = H_{i-1} \setminus \{u_i, v_i\}$  where  $e_i = u_i v_i$ . We obtain  $|H_i| = n - 2i$  and

$$|E(H_i)| \geq |E(H)| - i(2r - 2) + i(t - 1).$$

We have now  $|E(H)| \leq |E(H_i)| + i(2r - t - 1)$ . From the other side  $|E(H_i)| \leq \binom{n}{2} - i^2$ , since  $H_i$  is a bipartite graph.

We have now

$$|E(H)| \leq i^2 + i(2r - t - n - 1) + \frac{n^2}{4}.$$

The minimum value of the right hand term in the above inequality is  $\frac{n^2}{4} - \frac{(n-2r+t+1)^2}{4}$  and it is achieved when  $i = (n+t-2r+1)/2$  is an integer, i.e. when  $n+t-2r+1$  is even. Namely when  $n+t-2r+1$  is even then  $|E(H)| \leq \frac{n^2}{4} - \frac{(n-2r+t+1)^2}{4}$ . But when  $n + t - 2r + 1$  is odd then  $|E(H)| \leq \frac{n^2}{4} - \frac{(n-2r+t+1)^2}{4} + 1/4$ . Therefore  $|E(H)| \leq \frac{n^2}{4} - \frac{(n-2r+t+1)^2}{4} + \varphi$ , where

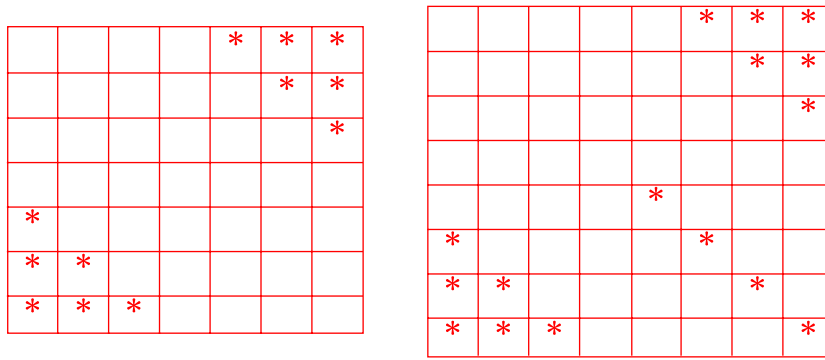


Fig. 1. Minimum dynamos for  $K_7 \square K_7$  and  $K_8 \square K_8$ .

$\varphi = 0$  when  $n + t - 2r + 1$  is even and  $\varphi = 1/4$  when  $n + t - 2r + 1$  is odd. Also  $|E(H)| = |E(G)| - k = mn/2 - k$ . We have the following

$$\begin{aligned}
 k &\geq mn/2 - |E(H)| \geq \frac{m}{2} - \frac{n^2}{4} + \frac{n^2 + 4r^2 + t^2 + 1 - 4nr + 2nt + 2n - 4rt - 4r + 2t}{4} - \varphi \\
 &\geq \frac{n(2t - 2r + 2) + (4r^2 + t^2 - 4rt - 4r + 2t + 1)}{4} - \varphi \\
 &= \frac{n(2t - 2r + 2) + (2r - t)^2 - 4r + 2t}{4} + \frac{1}{4} - \varphi.
 \end{aligned}$$

By taking  $\epsilon = 1/4 - \varphi$  the proof completes.  $\square$

The following result deals with the line graphs of complete graph and complete bipartite graph  $K_{n,n}$ , with constant edge thresholds  $n - 2$  (for  $K_n$ ) and  $n - 1$  (for  $K_{n,n}$ ). Note that  $L(K_{n,n}) = K_n \square K_n$ , where  $\square$  denotes the Cartesian product of graphs.

**Corollary 4.**

- (i) Any dynamic monopoly for the edges of  $K_n$  has at least  $\lfloor n^2/8 \rfloor$  vertices.
- (ii) The size of smallest dynamo in  $K_n \square K_n$  is  $\lfloor \frac{n^2}{4} \rfloor$ .

**Proof.** By applying Theorem 4 for  $G = K_n$ ,  $r = n - 1$  and  $t = n - 2$ , we obtain  $\text{dyn}(K_n) \geq \lfloor n^2/8 \rfloor$ . To prove (ii) we note that a dynamic monopoly in  $K_n \square K_n$  is equivalent to an edge dynamic monopoly in  $K_{n,n}$ . In this case  $r = n$  and  $t = n - 1$  and the lower bound in Theorem 6 shows  $k \geq \lfloor n^2/4 \rfloor$ . In the following we obtain a dynamo of size  $\lfloor \frac{n^2}{4} \rfloor$  for  $K_n \square K_n$ . Consider the vertex set of  $K_n \square K_n$  as an  $n \times n$  square array, where each vertex is identified by a position say  $(i, j)$  in the array (i.e. the  $i$ -th row and  $j$ -th column). First, let  $n$  be an odd integer and write  $n = 2k + 1$ . In this case our dynamo  $D$  consists of two triangular subarrays (see Fig. 1 for  $n = 7$ ) in down-left and top-right parts of the whole array. The height and side of these two triangular arrays are  $k$ . It can be easily checked that the resulting subarray is a dynamo indeed and it has  $k(k + 1) = (n^2 - 1)/4$  entries. In fact the positions  $(1, 1)$  and  $(n, n)$  are the first vertices which become active. Then the first row and column and the last row and column of the array become active. We reach at an array of size  $(n - 2) \times (n - 2)$  where in addition to  $D$ , all the vertices in the first and last rows and columns are also activated. The rest of the array becomes active inductively in a similar manner. This proves the theorem for odd  $n$ .

When  $n$  is even of the form  $n = 2k$  we consider two similar triangular subarrays except that the height and side of these triangles are  $k - 1$  (see Fig. 1 for  $n = 8$ , where  $k = 4$ ). In addition to the vertices of these subarrays we also consider a subset of vertices on the principal diagonal of the array in our dynamo consisting of the positions  $(k + 1, k + 1), (k + 2, k + 2), \dots, (n, n)$ . The argument that the chosen vertices form a dynamo is similar to the previous one and we omit its proof.  $\square$

**5. Concluding remarks**

In this section we present some complementary concepts and results and also propose some open questions for further researches. We begin with another application of resistant subgraphs in the area of dynamic monopolies.

Let  $(G, \tau)$  be a graph on  $n$  vertices with a given  $\tau$  of thresholds for its vertices. Denote the smallest size of any resistant subgraph in  $(G, \tau)$  by  $\rho$ . Let also  $\ell$  be the smallest integer such that any subset of  $\ell$  vertices forms a  $\tau$ -dynamic monopoly in  $G$ . It can be easily shown that  $\rho + \ell = n + 1$ . For this purpose, let  $K$  be a resistant subgraph of  $G$  with  $|K| = \rho$ . By Proposition 2,  $G \setminus K$  is not a dynamo. It follows that  $n - \rho \leq \ell - 1$ . From the other side, there exists a subset of cardinality  $\ell - 1$  which is not a dynamo. Again by Proposition 2 we have  $\rho \leq n - \ell + 1$ . Therefore  $\rho + \ell = n + 1$ . Based on the latter

equality, to find the smallest resistant subgraph in  $(G, \tau)$  is a useful problem. A special case of this problem is when  $G$  is a regular graph of degree  $r$  where the threshold of any vertex  $v$  is  $t$ . In this case the size of smallest resistant subgraph is the same as the so-called  $(r - t + 1)$ -girth in  $G$ . In general the  $d$ -girth of a graph  $G$  is the smallest number of vertices in any subgraph  $H$  of  $G$  such that  $\delta(H) \geq d$ . The notion of  $d$ -girth was introduced in [8], where some extremal results concerning  $d$ -girth were obtained. It was shown in [1] that  $d$ -girth cannot be approximated within any constant factor unless  $P = NP$ . We propose the following complexity problem concerning  $d$ -girth in regular graphs.

**Question 1.** Let  $r$  and  $d$  be any two fixed positive integers with  $3 \leq d \leq r$ . Is it an  $NP$ -complete problem to determine the  $d$ -girth of  $r$ -regular graphs?

Another unexplored research area is obtaining suitable upper bounds for the  $d$ -girth of  $r$ -regular graphs in terms of the order of graph. Note that this is equivalent to bounding the size of smallest resistant subgraph in regular graphs with constant thresholds.

**Question 2.** Find suitable upper bounds for the  $d$ -girth of regular graphs.

**Theorem 4** shows the importance of the expectation of thresholds (denoted by  $\alpha$  in the theorem) in lower-bounding the size of dynamic monopolies in graphs with probabilistic thresholds. This leads us to the study of dynamic monopolies in terms of the deterministic version of the expectation, i.e. the average threshold  $\sum_{v \in G} t(v)/|G|$ . In [13], the authors have obtained some results for the size of dynamic monopolies in terms of the average threshold. As we mentioned after the proof of **Theorem 4**, when  $G$  is a homogeneous society and  $D$  any dynamo for  $G$ , then with high probability  $|D| \geq |G|(1 - \epsilon/\alpha)$ , where  $\epsilon$  is the edge density of  $G$ . We re-propose the following question.

**Question 3.** Is it true that with high probability  $\text{dyn}(G) \geq |G|(1 - \epsilon/\alpha)$  for homogeneous societies  $G$ ?

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