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Long Range Scattering and Modified Wave Operators for some Hartree Type Equations, III. Gevrey Spaces and Low Dimensions[1]

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We study the theory of scattering for a class of Hartree type equations with long range interactions in arbitrary space dimension $n \geqslant 1$, including the case of Hartree equations with time dependent potential $V(t, x) = \kappa t^{\mu-\gamma} |x|^{-\mu}$ with $0 < \gamma \leqslant 1$ and $0 < \mu < n$. This includes the case of potential $V(x) = \kappa |x|^{-\gamma}$ and can be extended to the limiting case of nonlinear Schrödinger equations with cubic nonlinearity $\kappa t^{n-\gamma} |u|^2 u$. Using Gevrey spaces of asymptotic states and solutions, we prove the existence of modified local wave operators at infinity with no size restriction on the data and we determine the asymptotic behaviour in time of solutions in the range of the wave operators, thereby extending the results of previous papers which covered the range $0 < \gamma \leqslant 1$ but only $0 < \mu \leqslant n-2$ and were therefore restricted to space dimension $n \geqslant 3$. © 2001 Academic Press

Key Words: long range scattering; modified wave operators; Hartree equation; Gevrey spaces.

1. INTRODUCTION

This is the third paper where we study the theory of scattering and more precisely the existence of modified wave operators for a class of long range Hartree type equations

$$i\partial_t u + \tfrac{1}{2} \Delta u = \tilde{g}(|u|^2)\, u, \tag{1.1}$$

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where u is a complex function defined in space time $\mathbb{R}^{n+1}$, Δ is the Laplacian in $\mathbb{R}^n$, and

$$\tilde{g}(|u|^2) = \kappa t^{\mu-\gamma} |\nabla|^{\mu-n} |u|^2 \tag{1.2}$$

with $|\nabla| = (-\Delta)^{1/2}$, $\kappa \in \mathbb{R}$, $0 < \gamma \leqslant 1$ and $0 < \mu \leqslant n$. For $\mu < n$, the operator $|\nabla|^{\mu-n}$ can be represented by the convolution in $\mathbb{R}^n$

$$|\nabla|^{\mu-n} f = C_{n,\mu} |x|^{-\mu} * f \tag{1.3}$$

so that (1.2) is a Hartree type interaction with potential $V(x) = C |x|^{-\mu}$. The more standard Hartree equation corresponds to the case $\gamma = \mu$. In that case, the nonlinearity $\tilde{g}(|u|^2)$ becomes

$$\tilde{g}(|u|^2) = V * |u|^2 = \kappa |x|^{-\gamma} * |u|^2 \tag{1.4}$$

with a suitable redefinition of κ.

A large amount of work has been devoted to the theory of scattering for the Hartree equation (1.1) with nonlinearity (1.4) as well as with similar nonlinearities with more general potentials. As in the case of the linear Schrödinger equation, one must distinguish the short range case, corresponding to $\gamma > 1$, from the long range case corresponding to $\gamma \leqslant 1$. In the short range case, it is known that the (ordinary) wave operators exist in suitable function spaces for $\gamma > 1$ [14]. Furthermore for repulsive interactions, namely for $\kappa \geqslant 0$, it is known that all solutions in suitable spaces admit asymptotic states in L^2 for $\gamma > 1$, and that asymptotic completeness holds for $\gamma > 4/3$ [12]. In the long range case $\gamma \leqslant 1$, the ordinary wave operators are known not to exist in any reasonable sense [12], and should be replaced by modified wave operators including a suitable phase in their definition, as is the case for the linear Schrödinger equation. A well developed theory of long range scattering exists for the latter. See for instance [1] for a recent treatment and for an extensive bibliography. In contrast with that situation, only partial results are available for the Hartree equation. For small solutions (or equivalently small asymptotic states) the existence of modified wave operators has been proved in the critical case $\gamma = 1$ [2]. On the other hand, it has been shown, first in the critical case $\gamma = 1$ and then in the whole range $0 < \gamma \leqslant 1$ [6, 7, 9, 10] that the global solutions of the Hartree equation (1.1) with nonlinearity (1.4) and with small initial data exhibit an asymptotic behaviour as $t \to \pm\infty$ of the expected scattering type characterized by scattering states $u_\pm$ and including suitable phase factors that are typical of long range scattering.

In the previous two papers of this series [4, 5] (hereafter referred to as I and II) we proved the existence of modified wave operators in suitable spaces for the equation (1.1) with nonlinearity (1.2), and we gave a description of the asymptotic behaviour in time of solutions in the ranges of those operators, with no size restriction on the data, first for $1/2 < \gamma < 1$ in I and then in the whole range $0 < \gamma \leqslant 1$ in II. The method is an extension of the energy method used in [6, 9, 10], and uses in particular an auxiliary system of equations introduced in [9] to study the asymptotic behaviour of small solutions. The spaces of initial data, namely in the present case of asymptotic states, are Sobolev spaces of finite order. However there occurs a loss of derivatives in the auxiliary system, which has to be compensated for by the smoothing effect of the operator $|\nabla|^{\mu-n}$ in (1.2). This is done in the framework of Sobolev spaces at the expense of assuming $\mu \leqslant n-2$, which in particular restricts the space dimension to $n \geqslant 3$. In the present paper, we overcome that difficulty by treating the problem in Gevrey spaces [13], following and extending the method used in [7] to treat the case of small solutions. This makes it possible to cover the whole range $0 < \mu \leqslant n$, and in particular the case of dimensions 1 and 2 and the case of cubic nonlinear Schrödinger (NLS) equations (with time dependent nonlinearity). More precisely we use Gevrey classes $G^{1/\nu}$ of order $1/\nu$ with $0 < \nu \leqslant 1$, and the method applies under the condition $\mu \leqslant n-2+2\nu$. In particular for cubic NLS equations we need $\nu = 1$, namely spaces of analytic functions. The previous restriction on μ and ν can still be weakened and has been weakened in [7] at the expense of introducing parabolic terms in the auxiliary system of equations. However those terms introduce a priviledged orientation of time, which is inconvenient for the study of scattering theory, where we like to go back and forth from finite to infinite time, and we shall not make use of that extension here. The origin of the derivative loss and the mechanism by which that loss is overcome in Gevrey spaces, which is the same as in [7], will be described in Section 3 after sufficient technical material has been introduced, namely after Lemma 3.4.

The construction of the modified wave operators is in its principle the same as in II and will be recalled in Section 2 below, which is mostly a summary of Section 2 of II. It involves the study of the same auxiliary system of equations as in II for an amplitude w and a phase φ which replace the original function u, and the definition of the same modified asymptotic dynamics for that system as in II.

We now give a brief outline of the contents of this paper. A more technical description will be given at the end of Section 2. In Section 3 we define the relevant Gevrey spaces and derive the basic estimates in those spaces that are needed to study the auxiliary system. In Section 4 we prove the existence of the large time dynamics associated with that system and some preliminary asymptotic properties of that dynamics. In Section 5 we study the asymptotic dynamics and we prove the existence of asymptotic states for the previously constructed solutions of the auxiliary system. In Section 6 we construct the local wave operators at infinity for the auxiliary system by solving the Cauchy problem for that system with infinite initial time. We then come back from the auxiliary system to the original equation (1.1) for u and construct the (local) wave operators (at infinity) for u in Section 7, where the main result is stated as Proposition 7.5.

We have tried to make this paper self-contained and at the same time to keep duplication with I and II to a minimum. Duplication occurs in Section 2, as already mentioned, and in part of Section 7. The more technical sections 3 to 6 follow the same pattern as in II, but there is almost no duplication because the functional framework is significantly different.

We conclude this section by giving some general notation which will be used freely throughout this paper. We shall work mostly in Fourier space. We denote by $*$ the convolution in $\mathbb{R}^n$, by F the Fourier transform, and by $\hat{u} = Fu$ the Fourier transform of u. We denote by $\| \cdot \|_r$ the norm in $L^r \equiv L^r(\mathbb{R}^n)$ and by $\langle \cdot , \cdot \rangle$ the scalar product in L^2. For any interval I and any Banach space X, we denote by $\mathscr{C}(I, X)$ the space of strongly continuous functions from I to X, by $L^\infty(I, X)$ (resp. $L^\infty_{loc}(I, X)$) the space of measurable essentially bounded (resp. locally essentially bounded) functions from I to X, and by $L^2(I, X)$ (resp. $L^2_{loc}(I, X)$, resp. $L^2_\rho(I, X)$) the space of measurable functions u from I to X such that $\|u(\cdot); X\|$ belongs to $L^2(I)$ (resp. $L^2_{loc}(I)$, resp. $L^2_\rho(I)$), where $L^2_\rho(I)$ is the weighted space $L^2(I, \rho(t)\, dt)$ for some positive function ρ. For real numbers a and b, we use the notation $a \vee b = \text{Max}(a, b)$ and $a \wedge b = \text{Min}(a, b)$. In the estimates of solutions of the relevant equations, we shall use the letter C to denote constants, possibly different from an estimate to the next, depending on various parameters such as γ, but not on the solutions themselves or on their initial data. Those constants will be bounded in γ for γ away from zero. We shall use the notation $A(a_1, a_2, \ldots)$ for estimating functions, also possibly different from an estimate to the next, depending in addition on suitable norms $a_1, a_2, \ldots$ of the solutions or of their initial data. If (p.q) is a double inequality, we denote by (p.qa) and (p.qb) the first and second inequality in (p.q). Finally, Item (p.q) of I or II will be referred to as Item (I.p.q.) or (II.p.q). Additional notation will be given when needed.

In all this paper, we assume that $0 < \mu \leqslant n$ and $0 < \gamma \leqslant 1$.

2. HEURISTICS

In this section, we describe in heuristic terms the construction of the modified wave operators for the equation (1.1). That construction is the same as that performed in II, and this section is mostly a summary of Section II.2, which we include in order to make this paper self-contained.

The problem that we address is that of classifying the possible asymptotic behaviours of the solutions of (1.1) by relating them to a set of model functions $\mathscr{V} = \{v = v(u_+)\}$ parametrized by some data u_+ and with suitably chosen and preferably simple asymptotic behaviour in time. For each $v \in \mathscr{V}$, one tries to construct a solution u of (1.1) such that $u(t)$ behaves as $v(t)$ when $t \to \infty$ in a suitable sense. The map $\Omega\colon u_+ \to u$ thereby obtained classifies the asymptotic behaviours of solutions of (1.1) and is a preliminary version of the wave operator for positive time. A similar question can be asked for $t \to -\infty$. From now on we restrict our attention to positive time.

In the short range case corresponding to $\gamma > 1$ in (1.2), the previous scheme can be implemented by taking for $\mathscr{V}$ the set $\mathscr{V} = \{v = U(t)\, u_+\}$ of solutions of the equation

$$i\partial_t v + \tfrac{1}{2}\, \Delta v = 0, \tag{2.1}$$

with $U(t)$ being the unitary group

$$U(t) = \exp(i(t/2)\, \Delta). \tag{2.2}$$

The initial data u_+ for v is called the asymptotic state for u.

In the long range case corresponding to $\gamma \leqslant 1$ in (1.2), the previous set is known to be inadequate and has to be replaced by a better set of model functions obtained by modifying the previous ones by a suitable phase. The modification that we use requires additional structure of $U(t)$. In fact $U(t)$ can be written as

$$U(t) = M(t)\, D(t)\, FM(t) \tag{2.3}$$

where $M(t)$ is the operator of multiplication by the function

$$M(t) = \exp(ix^2/2t), \tag{2.4}$$

and $D(t)$ is the dilation operator defined by

$$(D(t)\, f)(x) = (it)^{-n/2}\, f(x/t). \tag{2.5}$$

Let now $\varphi^{(0)} = \varphi^{(0)}(x, t)$ be a real function of space time and let $z^{(0)}(x, t) = \exp(-i\varphi^{(0)}(x, t))$. We replace $v(t) = U(t)\, u_+$ by the modified free evolution [16, 17]

$$v(t) = M(t)\, D(t)\, z^{(0)}(t)\, w_+, \tag{2.6}$$

where $w_+ = Fu_+$. In order to allow for easy comparison of u with v, it is then convenient to represent u in terms of a phase factor $z(t) = \exp(-i\varphi(t))$ and of an amplitude $w(t)$ in such a way that asymptotically $\varphi(t)$ behaves as $\varphi^{(0)}(t)$ and $w(t)$ tends to w_+. This is done by writing u in the form [8, 9]

$$u(t) = M(t)\, D(t)\, z(t)\, w(t) \equiv (\Lambda(w, \varphi))(t). \tag{2.7}$$

The construction of the wave operators for u proceeds by first constructing the wave operators for the pair (w, φ) and then recovering the wave operators for u therefrom by the use of (2.7). The evolution equation for (w, φ) is obtained by substituting (2.7) into the equation (1.1). One obtains the equation

$$(i\partial_t + (2t^2)^{-1} \Delta - D^* \tilde{g} D)\, zw = 0 \tag{2.8}$$

for zw, with

$$\tilde{g} \equiv \tilde{g}(|u|^2) = \tilde{g}(|Dw|^2), \tag{2.9}$$

or equivalently, by expanding the derivatives in (2.8),

$$\begin{aligned}\{i\partial_t + (2t^2)^{-1} \Delta - i(2t^2)^{-1} (2\nabla\varphi \cdot \nabla + (\Delta\varphi))\}\, w \\ + \{\partial_t \varphi - (2t^2)^{-1} |\nabla\varphi|^2 - D^* \tilde{g} D\}\, w = 0.\end{aligned} \tag{2.10}$$

We are now in the situation of a gauge theory. The equation (2.8) or (2.10) is invariant under the gauge transformation $(w, \varphi) \to (w \exp(i\omega), \varphi + \omega)$, where ω is an arbitrary function of space time, and the original gauge invariant equation is not sufficient to provide evolution equations for the two gauge dependent quantities w and φ. At this point we arbitrarily add the Hamilton–Jacobi equation as a gauge condition. This yields a system of evolution equations for (w, φ), namely

$$\partial_t w = i(2t^2)^{-1} \Delta w + (2t^2)^{-1} (2\nabla\varphi \cdot \nabla + (\Delta\varphi))\, w \tag{2.11}$$

$$\partial_t \varphi = (2t^2)^{-1} |\nabla\varphi|^2 + t^{-\gamma} g_0(w, w), \tag{2.12}$$

where we have defined

$$g_0(w_1, w_2) = \kappa \operatorname{Re} |\nabla|^{\mu - n} w_1 \bar{w}_2 \tag{2.13}$$

and rewritten the nonlinear interaction term in (2.10) as

$$D^* \tilde{g}(|Dw|^2)\, D = t^{-\gamma} g_0(w, w).$$

The gauge freedom in (2.11)–(2.12) is now reduced to that given by an arbitrary function of space only. It will be shown in Section 4 that the Cauchy problem for the system (2.11)–(2.12) is locally well-posed in a neighborhood of infinity in time. The solutions thereby obtained behave asymptotically as $w(t) = O(1)$ and $\varphi(t) \cong O(t^{1-\gamma})$ as $t \to \infty$, a behaviour that is immediately seen to be compatible with (2.11)–(2.12).

We next study the asymptotic behaviour of the solutions of the auxiliary system (2.11)–(2.12) in more detail and try to construct wave operators for that system. For that purpose, we need to choose a set of model functions playing the role of v, in the spirit of (2.6). We proceed as follows. Let $p \geqslant 0$ be an integer. We write

$$w = \sum_{0 \leqslant m \leqslant p} w_m + q_{p+1} \equiv W_p + q_{p+1} \tag{2.14}$$

$$\varphi = \sum_{0 \leqslant m \leqslant p} \varphi_m + \psi_{p+1} \equiv \phi_p + \psi_{p+1} \tag{2.15}$$

with the understanding that asymptotically in t

$$w_m(t) = O(t^{-m\gamma}), \qquad q_{p+1}(t) = o(t^{-p\gamma}), \tag{2.16}$$

$$\varphi_m(t) = O(t^{1-(m+1)\gamma}), \qquad \psi_{p+1}(t) = o(t^{1-(p+1)\gamma}). \tag{2.17}$$

Substituting (2.14)–(2.15) into (2.11)–(2.12) and identifying the various powers of $t^{-\gamma}$ yields the following system of equations for (w_m, φ_m):

$$\partial_t w_{m+1} = (2t^2)^{-1} \sum_{0 \leqslant j \leqslant m} (2\nabla\varphi_j \cdot \nabla + (\Delta\varphi_j))\, w_{m-j} \tag{2.18}$$

$$\partial_t \varphi_{m+1} = (2t^2)^{-1} \sum_{0 \leqslant j \leqslant m} \nabla\varphi_j \cdot \nabla\varphi_{m-j} + t^{-\gamma} \sum_{0 \leqslant j \leqslant m+1} g_0(w_j, w_{m+1-j}) \tag{2.19}$$

for $m+1 \geqslant 0$. Here it is understood that $w_j = 0$ and $\varphi_j = 0$ for $j < 0$. We supplement that system with the initial conditions

$$\begin{cases} w_0(\infty) = w_+, \quad w_m(\infty) = 0 & \text{for} \quad m \geqslant 1 \qquad (2.20) \\ \varphi_m(1) = 0 & \text{for} \quad 0 \leqslant m \leqslant p. \qquad (2.21) \end{cases}$$

The system (2.18)–(2.19) with the initial conditions (2.20)–(2.21) can be solved by successive integrations: knowing (w_j, φ_j) for $0 \leqslant j \leqslant m$, one constructs successively w_{m+1} by integrating (2.18) between t and ∞, and then φ_{m+1} by integrating (2.19) between 1 and t.

If $(p+1)\gamma < 1$, that method of resolution reproduces the asymptotic behaviour in time (2.16) (2.17) which was used in the first place to provide a heuristic derivation of the system (2.18)–(2.19). For sufficiently large p, ϕ_p is a sufficiently good approximation for φ to ensure that ψ_{p+1} has a limit as $t \to \infty$. In fact by comparing the system (2.18)–(2.19) with (2.11)–(2.12), one finds that $\partial_t \psi_{p+1}$ is essentially of the same order in t as $\partial_t \varphi_{p+1}$, namely $\partial_t \psi_{p+1} \cong O(t^{-(p+2)\gamma})$, which is integrable at infinity for $(p+2)\gamma > 1$. In this way every solution (w, φ) of the system (2.11)–(2.12) as obtained previously has asymptotic states consisting of $w_+ = \lim_{t \to \infty} w(t)$ and $\psi_+ = \lim_{t \to \infty} \psi_{p+1}(t)$.

Conversely, under the condition $(p+2)\gamma > 1$, we shall be able to solve the system (2.11)–(2.12) by looking for solutions in the form (2.14)–(2.15) with the additional initial condition $\psi_{p+1}(\infty) = \psi_+$, thereby getting a solution which is asymptotic to $(W_p, \phi_p + \psi_+)$ with

$$w - W_p \cong O(t^{-(p+1)\gamma}), \qquad \varphi - \phi_p - \psi_+ \cong O(t^{1-(p+2)\gamma}). \tag{2.22}$$

This allows to define a map Ω_0: $(w_+, \psi_+) \to (w, \varphi)$ which is essentially the wave operator for (w, φ).

We next discuss the gauge covariance properties of Ω_0. Two solutions (w, φ) and (w', φ') of the system (2.11)–(2.12) will be said to be gauge equivalent if they give rise to the same u through (2.7), namely if $w \exp(-i\varphi) = w' \exp(-i\varphi')$. If (w, φ) and (w', φ') are two gauge equivalent solutions, one can show easily that the difference $\varphi_- = \varphi' - \varphi$ has a limit ω when $t \to \infty$ and that $w'_+ = w_+ \exp(i\omega)$. Under that condition, it turns out that the phases $\{\varphi_j\}$ and ϕ_p (but not the amplitudes) obtained by solving (2.18)–(2.19) are gauge invariant, namely $\varphi_m = \varphi'_m$ for $0 \leqslant m \leqslant p$ and therefore $\phi_p = \phi'_p$, so that $\psi'_+ = \psi_+ + \omega$. It is then natural to define gauge equivalence of asymptotic states (w_+, ψ_+) and (w'_+, ψ'_+) by the condition $w_+ \exp(-i\psi_+) = w'_+ \exp(-i\psi'_+)$ and the previous result can be rephrased as the statement that gauge equivalent solutions of (2.11)–(2.12) in $\mathscr{R}(\Omega_0)$ have gauge equivalent asymptotic states. Conversely, we shall show that gauge equivalent asymptotic states have gauge equivalent images under Ω_0. Here however we meet with a technical problem coming from the construction of Ω_0 itself. For given (w_+, ψ_+) we construct (w, φ) in practice as follows. We take a (large) finite time t_0 and we define a solution (w_{t_0}, φ_{t_0}) of the system (2.11)–(2.12) by imposing a suitable initial condition at t_0, depending on (w_+, ψ_+), and solving the Cauchy problem with finite initial time. We then let t_0 tend to infinity and obtain (w, φ) as the limit of (w_{t_0}, φ_{t_0}). The simplest way to prove the gauge equivalence of two solutions (w, φ) and (w', φ') obtained in this way from gauge equivalent (w_+, ψ_+) and (w'_+, ψ'_+) consists in using an initial condition at t_0 which already ensures that (w_{t_0}, φ_{t_0}) and $(w'_{t_0}, \varphi'_{t_0})$ are gauge equivalent. However the natural choice $(w_{t_0}(t_0), \varphi_{t_0}(t_0)) = (W_p(t_0), \phi_p(t_0) + \psi_+)$ does not satisfy that requirement as soon as $p \geqslant 1$ because $\phi_p(t_0)$ is gauge invariant while $W_p(t_0) \exp(-\psi_+)$ is not. In order to overcome that difficulty, we introduce a new amplitude V and a new phase χ defined by solving the transport equations

$$\partial_t V = (2t^2)^{-1} (2\nabla\phi_{p-1} \cdot \nabla + (\Delta\phi_{p-1}))\, V \tag{2.23}$$

$$\partial_t \chi = t^{-2} \nabla\phi_{p-1} \cdot \nabla\chi \tag{2.24}$$

with initial condition

$$V(\infty) = w_+, \qquad \chi(\infty) = \psi_+. \tag{2.25}$$

It follows from (2.23) (2.24) that $V \exp(-i\chi)$ satisfies the same transport equation as V, now with gauge invariant initial condition $(V \exp(-i\chi))(\infty) = w_+ \exp(-i\psi_+)$, and is therefore gauge invariant. Furthermore, (V, χ) is a sufficiently good approximation of (W_p, ψ_+) in the sense that

$$V(t) - W_p(t) \cong O(t^{-(p+1)\gamma}), \qquad \chi(t) - \psi_+ \cong O(t^{-\gamma}). \tag{2.26}$$

One then takes $(w_{t_0}(t_0), \varphi_{t_0}(t_0)) = (V(t_0), \phi_p(t_0) + \chi(t_0))$ as an initial condition at time t_0, thereby ensuring that (w_{t_0}, φ_{t_0}) and $(w'_{t_0}, \varphi'_{t_0})$ are gauge equivalent. That equivalence is easily seen to be preserved in the limit $t_0 \to \infty$. Furthermore, the estimates (2.26) ensure that the asymptotic properties (2.22) are preserved by the modified construction. As a consequence of the previous discussion, the map Ω_0 is gauge covariant, namely induces an injective map of gauge equivalence classes of asymptotic states (w_+, ψ_+) to gauge equivalence classes of solutions (w, φ) of the system (2.11)–(2.12).

The wave operator for u is obtained from Ω_0 just defined and from Λ defined by (2.7). From the previous discussion it follows that the map $\Lambda \circ \Omega_0 \colon (w_+, \psi_+) \to u$ is injective from gauge equivalence classes of asymptotic states (w_+, ψ_+) to solutions of (1.1). In order to define a wave operator for u involving only the asymptotic state u_+ but not an arbitrary phase ψ_+, we choose a representative in each equivalence class (w_+, ψ_+), namely we define the wave operator for u as the map $\Omega \colon u_+ \to u = (\Lambda \circ \Omega_0)(Fu_+, 0)$. Since each equivalence class of asymptotic states contains at most one element with $\psi_+ = 0$, the map Ω is again injective.

The previous heuristic discussion was based in part on a number of time decay estimates in terms of negative powers of t. In practice however two complications occur, namely (i) for integer γ^{-1}, some of the estimates involve logarithmic factors in time, and (ii) the use of Gevrey spaces requires that of norms defined by integrals over time involving a convergence factor which eventually produces a small loss in the time decays. Both difficulties are handled by introducing suitable estimating functions of time, some of which are defined by integral representations and generalize in a natural way a similar family of functions defined in II.

In the same way as in I, the system (2.11)–(2.12) can be rewritten as a system of equations for w and for $s = \nabla\varphi$, from which φ can then be recovered by (2.12), thereby leading to a slightly more general theory since the system for (w, s) can be studied without even assuming that s is a gradient. For simplicity, and in the same way as in II, we shall not follow that track. However, we shall use systematically the notation $s = \nabla\varphi$, and for the purposes of estimation, we shall supplement the system (2.11)–(2.12) with the equation satisfied by s, which is simply the gradient of (2.12), namely

$$\partial_t s = t^{-2} s \cdot \nabla s + t^{-\gamma} \nabla g_0(w, w). \tag{2.27}$$

We are now in a position to describe in more detail the contents of the technical sections 3–7 of this paper. In Section 3, we introduce the relevant Gevrey spaces and derive the basic estimates in those spaces that are needed to study the system (2.11)–(2.12) (Lemmas 3.4–3.7), we explain in passing the mechanism by which those spaces make it possible to overcome the derivative loss in (2.11)–(2.12) for $\mu > n-2$ (after Lemma 3.4) and finally we introduce the estimating functions of time mentioned above and obtain some estimates for them. In Section 4, we prove that the Cauchy problem for the system (2.11)–(2.12) is well-posed for large time, with large but finite initial time (Proposition 4.1), we prove the existence of a limit for $w(t)$ as $t \to \infty$ for the solutions thereby obtained (Proposition 4.2) and we derive a uniqueness result of solutions with prescribed asymptotic behaviour (Proposition 4.3). In Section 5, we study the asymptotic behaviour in time of the solutions obtained in Section 4. We derive a number of properties and estimates for the solutions of the asymptotic system (2.18)–(2.19), defined inductively (Proposition 5.2). We then obtain asymptotic estimates on the approximation of the solutions of the system (2.11)–(2.12) by the asymptotic functions (W_m, ϕ_m) defined by (2.14)–(2.15), and in particular we complete the proof of the existence of asymptotic states for those solutions (Proposition 5.3). In Section 6, we study the Cauchy problem with infinite initial time, first for the transport equations (2.23) (Proposition 6.1) and (2.24) (Proposition 6.2), and then for the system (2.11)–(2.12). For a given solution (V, χ) of the system (2.23)–(2.24) and a given (large) t_0, we construct a solution (w_{t_0}, φ_{t_0}) of the system (2.11)–(2.12) which coincides with $(V, \phi_p + \chi)$ at t_0 and we estimate it uniformly in t_0 (Proposition 6.4). We then prove that when $t_0 \to \infty$, (w_{t_0}, φ_{t_0}) has a limit (w, φ) which is asymptotic both to $(V, \phi_p + \chi)$ and to $(W_p, \phi_p + \psi_+)$ (Proposition 6.5). In Section 7, we exploit the results of Section 6 to construct the wave operators for the equation (1.1) and to describe the asymptotic behaviour of solutions in their range. We first prove that the local wave operator at infinity for the system (2.11)–(2.12) defined through Proposition 6.5 in Definition 7.1 is gauge covariant in the sense of Definitions 7.2 and 7.3 in the best form that can be expected with the available regularity (Propositions 7.2 and 7.3). With the help of some information on the Cauchy problem for (1.1) at finite time (Proposition 7.1), we then define the wave operator $\Omega: u_+ \to u$ (Definition 7.4), and we prove that it is injective. We then collect all the available information on Ω and on solutions of (1.1) in its range in Proposition 7.5, which contains the main results of this paper. Finally some side results relevant for the definition and properties of the Gevrey spaces used here are collected in two Appendices.

3. GEVREY SPACES AND PRELIMINARY ESTIMATES

In this section, we define the Gevrey spaces where we shall study the auxiliary system (2.11)–(2.12) and we derive a number of energy type estimates which hold in those spaces and play an essential role in that study. We then introduce some estimating functions of time generalizing those of II and we derive a number of estimates for them.

The relevant spaces will be defined with the help of the functions

$$f_0(\xi) = \exp(\rho\, |\xi|^\nu), \qquad f(\xi) = \exp(\rho(|\xi|^\nu \vee 1)), \tag{3.1}$$

where $0 < \nu \leqslant 1$, ρ is a positive parameter to be specified later, and $\xi \in \mathbb{R}^n$. The dependence of f on ρ will always be omitted in the notation.

In all this paper, one could use instead of f_0 the function

$$\tilde{f}(\xi) = \sum_{j \geqslant 0} (j!)^{-1/\nu}\, \rho^{j/\nu}\, |\xi|^j \tag{3.2}$$

which satisfies the same basic estimates and would yield essentially the same results. The function $\tilde{f}$ is also convenient in order to relate the definition of the Gevrey spaces K_ρ^k and Y_ρ^ℓ (see (3.8) (3.9) below) to more standard definitions. Those points are discussed in Appendix A.

The functions f_0 and f satisfy the following estimates.

LEMMA 3.1. *Let $\xi, \eta \in \mathbb{R}^n$. Then f satisfies the estimates:*

$$f(\xi) \leqslant f(\xi-\eta)\, f(\eta) \qquad \text{for all} \quad \xi, \eta, \tag{3.3}$$

$$f(\xi) \leqslant f(\xi-\eta)\, f_0(\eta)^\nu \qquad \text{for} \quad |\xi| \wedge |\eta| \leqslant |\xi-\eta|, \tag{3.4}$$

$$|f(\xi)-f(\eta)|\, |\eta|^{1-\nu} \leqslant |\xi-\eta|^{1-\nu}\, f(\xi-\eta)\, f(\eta) \qquad \text{for all} \quad \xi, \eta, \tag{3.5}$$

$$|f(\xi)-f(\eta)|\, |\eta|^{1-\nu} \leqslant C\, |\xi-\eta|^{1-\nu}\, f_0(\xi-\eta)^\nu\, f(\eta) \qquad \text{for} \quad |\xi| \wedge |\xi-\eta| \leqslant |\eta|, \tag{3.6}$$

$$|f(\xi)-f(\eta)|\, |\eta|^{1-\nu} \leqslant C\, |\xi-\eta|^{1-\nu}\, f(\xi-\eta)\, f_0(\eta)^\nu \qquad \text{for} \quad |\xi| \wedge |\eta| \leqslant |\xi-\eta|. \tag{3.7}$$

In (3.6) and (3.7), one can take $C = 1$, except in the region $|\xi| \leqslant |\xi-\eta| \leqslant |\eta|$ where $C = 2^{1-\nu}$.

The function f_0 satisfies the same estimates as f.

Proof. We first prove the estimates for f_0.
(3.3) follows from the fact that $|\xi|^\nu \leqslant |\xi-\eta|^\nu + |\eta|^\nu$.
(3.4) is obvious for $|\xi| \leqslant |\xi-\eta|$. For $|\eta| \leqslant |\xi-\eta| \leqslant |\xi|$, we estimate

$$|\xi|^\nu \leqslant |\xi-\eta|^\nu + \nu\, |\xi-\eta|^{\nu-1}\, |\eta| \leqslant |\xi-\eta|^\nu + \nu\, |\eta|^\nu.$$

(3.5) follows from (3.3) for $|\eta| \leqslant |\xi-\eta|$ and from (3.6) with $C=1$ for $|\xi-\eta| \leqslant |\xi| \wedge |\eta|$. For $|\xi| \leqslant |\xi-\eta| \leqslant |\eta|$, we estimate

$$(f_0(\eta)-f_0(\xi))\,|\eta|^{1-\nu} \leqslant f_0(\eta)\,|\eta|^{1-\nu}\,(1-\exp(-\rho\,|\eta|^\nu)) \leqslant f_0(\eta)\,\rho\,|\eta|$$
$$\leqslant f_0(\eta)(|\eta|/|\xi-\eta|)\,|\xi-\eta|^{1-\nu}\,\rho\,|\xi-\eta|^\nu$$

and the result follows from the fact that $|\eta| \leqslant 2\,|\xi-\eta|$ and $\rho\,|\xi-\eta|^\nu \leqslant e^{-1} f_0(\xi-\eta)$.

(3.6) is obvious for $|\xi| \leqslant |\eta| \leqslant |\xi-\eta|$, with $C=1$.

For $|\xi| \leqslant |\xi-\eta| \leqslant |\eta|$, we estimate

$$(f_0(\eta)-f_0(\xi))\,|\eta|^{1-\nu} \leqslant 2^{1-\nu}\,|\xi-\eta|^{1-\nu}\,f_0(\eta)$$

which yields (3.6) with $C=2^{1-\nu}$ since $|\eta| \leqslant 2\,|\xi-\eta|$.

The really crucial case is the case $|\xi-\eta| \leqslant |\xi| \wedge |\eta|$.

For $|\xi-\eta| \leqslant |\xi| \leqslant |\eta|$, we estimate

$$(f_0(\eta)-f_0(\xi))\,|\eta|^{1-\nu} \leqslant f_0(\eta)\,|\eta|^{1-\nu}\,(1-\exp(-\rho\nu\,|\xi-\eta|\,|\xi|^{\nu-1})),$$

where we have used

$$|\eta|^\nu-|\xi|^\nu \leqslant \nu\,|\xi|^{\nu-1}\,|\xi-\eta|,$$
$$\cdots \leqslant f_0(\eta)(|\eta|/|\xi|)^{1-\nu}\,\rho\nu\,|\xi-\eta|$$
$$\leqslant f_0(\eta)\,|\xi-\eta|^{1-\nu}\,2^{1-\nu}e^{-1}f_0(\xi-\eta)^\nu$$

since $|\eta| \leqslant 2\,|\xi|$ and $\rho\nu\,|\xi-\eta|^\nu \leqslant e^{-1}f_0(\xi-\eta)^\nu$. This proves (3.6) with $C=1$ in that case.

For $|\xi-\eta| \leqslant |\eta| \leqslant |\xi|$, we estimate similarly

$$(f_0(\xi)-f_0(\eta))\,|\eta|^{1-\nu} \leqslant f_0(\eta)\{|\eta|^{1-\nu}\,(\exp(\rho\nu\,|\xi-\eta|\,|\eta|^{\nu-1})-1)\}.$$

Now for fixed $|\xi-\eta|$, the last bracket is a decreasing function of $|\eta|$, and is therefore bounded by its value for $|\eta|=|\xi-\eta|$, so that

$$\cdots \leqslant f_0(\eta)\,|\xi-\eta|^{1-\nu}\,(f_0(\xi-\eta)^\nu-1)$$

which proves (3.6) with $C=1$ in that case.

(3.7) is obvious for $|\xi| \leqslant |\eta| \leqslant |\xi-\eta|$ and follows from (3.4) with $C=1$ for $|\eta| \leqslant |\xi| \wedge |\xi-\eta|$. For $|\xi| \leqslant |\xi-\eta| \leqslant |\eta|$, we estimate

$$(f_0(\eta)-f_0(\xi))\,|\eta|^{1-\nu} \leqslant 2^{1-\nu}\,|\xi-\eta|^{1-\nu}\,f_0(\eta)$$

and (3.7) with $C=2^{1-\nu}$ follows from

$$|\eta|^\nu \leqslant \nu\,|\eta|^\nu+(1-\nu)\,2^\nu\,|\xi-\eta|^\nu \leqslant \nu\,|\eta|^\nu+|\xi-\eta|^\nu.$$

The estimates for f follow from those for f_0. This is obvious for (3.3) (3.4). For (3.5) (3.6) (3.7), it follows from the fact that for all ξ, η and all $a > 0$

$$|f_0(\xi) \vee a - f_0(\eta) \vee a| \leqslant |f_0(\xi) - f_0(\eta)|. \quad \blacksquare$$

We now turn to the definition of the spaces where we shall solve the system (2.11)–(2.12). For any tempered distribution u in $\mathbb{R}^n$ with $\hat{u} \in L^1_{loc}(\mathbb{R}^n)$, we define $u_{\gtrless}$ by $\hat{u}_>(\xi) = \hat{u}(\xi)$ for $|\xi| > 1$, $\hat{u}_>(\xi) = 0$ for $|\xi| \leqslant 1$, $\hat{u}_<(\xi) = 0$ for $|\xi| > 1$, $\hat{u}_<(\xi) = \hat{u}(\xi)$ for $|\xi| \leqslant 1$. Similarly, for $\xi \in \mathbb{R}^n$ and $m \in \mathbb{R}$, we define $|\xi|^m_>$ and $|\xi|^m_<$ to be equal to $|\xi|^m$ for $|\xi| > 1$ and $|\xi| \leqslant 1$ respectively, and zero otherwise. Occasionally we shall make the separation between low and high $|\xi|$ at some value $a \neq 1$. In that case we shall denote by $u_{<a}$ and $u_{>a}$ the corresponding components of u.

Let now $\rho > 0$, $k \in \mathbb{R}$, $\ell \in \mathbb{R}$ and $0 \leqslant \ell_< < n/2$. Starting from Lemma 3.4 below (but not until then), we shall assume in addition that $\ell_< > n/2 - \mu$. We define

$$K^k_\rho = \{w : |w|^2_k \equiv \|w; K^k_\rho\|^2 \equiv \| |\xi|^k f(\xi)\, \hat{w}_>(\xi)\|^2_2 + \|f(\xi)\, \hat{w}_<(\xi)\|^2_2 < \infty\}, \quad (3.8)$$

$$Y^\ell_\rho = \{\varphi : \hat{\varphi} \in L^1_{loc}(\mathbb{R}^n) \text{ and } |\varphi|^2_\ell \equiv \|\varphi; Y^\ell_\rho\|^2 \equiv \| |\xi|^{\ell+2} f(\xi)\, \hat{\varphi}_>(\xi)\|^2_2$$
$$+ \| |\xi|^{\ell_<} f(\xi)\, \hat{\varphi}_<(\xi)\|^2_2 < \infty\}. \quad (3.9)$$

The apparent ambiguity in the notation $|\cdot|_b$ will be lifted by the fact that the symbol b will always contain the letter k when referring to K^k_ρ spaces and the letter ℓ when referring to Y^ℓ_ρ spaces. The spaces K^k_ρ and Y^ℓ_ρ are Hilbert spaces and satisfy the embeddings $K^k_\rho \subset K^{k'}_\rho$ for $k \geqslant k'$ and $Y^\ell_\rho \subset Y^{\ell'}_\rho$ for $\ell \geqslant \ell'$, $K^k_\rho \subset K^k_{\rho'}$ and $Y^\ell_\rho \subset Y^\ell_{\rho'}$, for $\rho \geqslant \rho'$.

Remark 3.1. The norms in the spaces K^k_ρ and Y^ℓ_ρ are both of the form $\|f_1 f \hat{u}\|_2$ with

$$f_1 = |\xi|^{k_>}_> + |\xi|^{k_<}_< \quad (3.10)$$

where $(k_>, k_<) = (k, 0)$ for K^k_ρ and $(k_>, k_<) = (\ell+2, \ell_<)$ for Y^ℓ_ρ. In particular this implies that

$$\begin{aligned} \|w; K^k_\rho\| &= \|F^{-1}(f\hat{w}); K^k_0\| \\ \|\varphi; Y^\ell_\rho\| &= \|F^{-1}(f\hat{\varphi}); Y^\ell_0\|. \end{aligned} \quad (3.11)$$

If $k_> \geqslant k_<$, namely if $k \geqslant 0$ for K^k_ρ and if $\ell + 2 \geqslant \ell_<$ for Y^ℓ_ρ (which will always be the case in the applications), one can omit either or both of the upper and lower restrictions in the definition of K^k_ρ or Y^ℓ_ρ, thereby obtaining equivalent norms uniformly in ρ. In fact, in that case

$$|\xi|^{2k_>}_> + |\xi|^{2k_<}_< \leqslant |\xi|^{2k_>} + |\xi|^{2k_<} \leqslant 2(|\xi|^{2k_>}_> + |\xi|^{2k_<}_<).$$

Furthermore, the relations (3.11) are preserved under thoses changes.

We shall use the spaces K_ρ^k and Y_ρ^ℓ with a time dependent parameter $\rho \in \mathscr{C}^1(\mathbb{R}^+, \mathbb{R}^+)$. The form (3.8)–(3.9) of the norms has been chosen so as to ensure that for fixed (time independent) w and φ, the following relations hold

$$\begin{aligned} \frac{d}{dt}|w|_k^2 &= 2\rho'\, |w|_{k+\nu/2}^2\,, \\ \frac{d}{dt}|\varphi|_\ell^2 &= 2\rho'\, |\varphi|_{\ell+\nu/2}^2\,, \end{aligned} \tag{3.12}$$

where $\rho' = d\rho/dt$.

We shall look for w as complex K_ρ^k valued functions of time and for φ as real Y_ρ^ℓ valued functions of time. More precisely, we shall look for (w, φ) such that for some interval $I \subset [1, \infty)$

$$(w, \varphi) \in \mathscr{X}_{\rho,\, loc}^{k,\, \ell}(I) \equiv \mathscr{C}(I, K_\rho^k \oplus Y_\rho^\ell) \cap L_{loc}^2(I, K_\rho^{k+\nu/2} \oplus Y_\rho^{\ell+\nu/2}) \tag{3.13}$$

by which is meant, especially as regards continuity, that

$$(F^{-1} f \hat{w}, F^{-1} f \hat{\varphi}) \in \mathscr{X}_{0,\, loc}^{k,\, \ell}(I) \equiv \mathscr{C}(I, K_0^k \oplus Y_0^\ell) \cap L_{loc}^2(I, K_0^{k+\nu/2} \oplus Y_0^{\ell+\nu/2})$$

in the usual sense. In particular, when taking norms such as $|w(t)|_k$ or $|\varphi(t)|_\ell$ with time dependent ρ, it will always be understood that ρ in the definition of the relevant space is taken at the same value of the time as w or φ.

We shall also need global versions of the previous spaces, especially when the interval I is unbounded. The definition of those global versions will require assumptions on ρ that are irrelevant for the considerations of this section and will be postponed until the beginning of Section 4.

We shall need the following elementary estimates.

Lemma 3.2. *Let $m \in \mathbb{R}$. The following estimates hold:*

$$\| |\xi|^m \,\widehat{(u_1 u_2)}_> \|_2 \leqslant C\, \| \langle \xi \rangle^{k_1} \hat{u}_1 \|_2 \, \| \langle \xi \rangle^{k_2} \hat{u}_2 \|_2 \tag{3.14}$$

for $k_1, k_2 \geqslant m \vee 0$ and $k_1 + k_2 > m + n/2$, where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$,

$$\| |\xi|^m \,\widehat{(u_1 u_2)}_< \|_2 \leqslant C\, \| u_1 \|_2 \, \| u_2 \|_2 \tag{3.15}$$

for $m > -n/2$.

Proof. (3.14) for $m \geqslant 0$ follows from (3.14) (3.15) for $m = 0$ and from

$$\| |\xi|^m \,\widehat{(u_1 u_2)} \|_2 \leqslant C(\| (|\xi|^m |\hat{u}_1|) * |\hat{u}_2| \|_2 + \| |\hat{u}_1| * (|\xi|^m |\hat{u}_2|) \|_2).$$

For $m<0$, we estimate by the Hölder and Young inequalities

$$\| |\xi|^m \widehat{(u_1 u_2)}_> \|_2 \leqslant C \, \| |\xi|^m_> \|_s \, \|\hat{u}_1\|_{\bar{r}_1} \, \|\hat{u}_2\|_{\bar{r}_2}$$

with $1/s+1/\bar{r}_1+1/\bar{r}_2=3/2$, $n/s<|m|$ and $1 \leqslant \bar{r}_1, \; \bar{r}_2 \leqslant 2$. The last two norms are estimated by $\|\langle\xi\rangle^{k_i} \hat{u}_i\|_2$ provided $k_i>n/\bar{r}_i-n/2$. One can find $\bar{r}_1$, $\bar{r}_2$ satisfying all previous conditions under the assumptions made on k_i.

For $m=0$, we apply the same argument with $s=\infty$.

For the proof of (3.15) we estimate by the Schwarz and Young inequalities

$$\| |\xi|^m \widehat{(u_1 u_2)}_< \|_2 \leqslant \| |\xi|^m_< \|_2 \, \|u_1\|_2 \, \|u_2\|_2$$

for $m>-n/2$. ∎

For future reference, we also state the following elementary inequalities

$$\| |\xi|^m \hat{\varphi}_< \|_1 \leqslant C \, \| |\xi|^{\ell_<} \hat{\varphi}_< \|_2 \qquad \text{for all} \quad m \geqslant 0, \tag{3.16}$$

$$\| |\xi|^m \hat{\varphi}_> \|_1 \leqslant C \, \| |\xi|^{\ell+2} \hat{\varphi}_> \|_2 \qquad \text{for} \quad \ell+2>m+n/2. \tag{3.17}$$

In what follows, we shall repeatedly estimate norms such as $\| |\xi|^m f \widehat{u_1 u_2} \|_2$ with $m \geqslant 0$. For that purpose, using (3.3), we shall write

$$\begin{aligned} |\xi|^m f \, |\widehat{u_1 u_2}| &\leqslant |\xi|^m \int d\eta f(\xi) \, |\hat{u}_1(\xi-\eta)| \, |\hat{u}_2(\eta)| \\ &\leqslant 2^m \int d\eta (|\xi-\eta|^m + |\eta|^m) \, f(\xi-\eta) \, f(\eta) \, |\hat{u}_1(\xi-\eta)| \, |\hat{u}_2(\eta)| \\ &= 2^m \{ (|\cdot|^m \, f \, |\hat{u}_1|) * (f \, |\hat{u}_2|) + (1 \leftrightarrow 2) \}. \end{aligned} \tag{3.18}$$

That inequality will be often combined with restrictions to low or high values of $|\xi|$, either in the product $(u_1 u_2)$ or in u_1 or u_2 separately.

The next lemma states that under suitable assumptions on k and ℓ, Y^ℓ_ρ is an algebra under ordinary multiplication and acts boundedly on K^k_ρ by multiplication.

LEMMA 3.3. *Let $\ell+2>n/2$ and $0 \leqslant k \leqslant \ell+2$. Then there exist constants C_1 and C_2, independent of ρ, such that*

$$|\varphi\psi|_\ell \leqslant C_1 \, |\varphi|_\ell \, |\psi|_\ell \qquad \textit{for all} \quad \varphi, \psi \in Y^\ell_\rho, \tag{3.19}$$

$$|\varphi w|_k \leqslant C_2 \, |\varphi|_\ell \, |w|_k \qquad \textit{for all} \quad \varphi \in Y^\ell_\rho, \, w \in K^k_\rho. \tag{3.20}$$

In particular

$$|(\exp(-i\varphi)-1) \, w|_k \leqslant C_2 C_1^{-1} (\exp(C_1 |\varphi|_\ell)-1) \, |w|_k \tag{3.21}$$

for all $\varphi \in Y^\ell_\rho$, $w \in K^k_\rho$.

Proof. From the definitions (3.8) (3.9) of the norms and in particular from (3.11), and from (3.3), it follows that (3.19) (3.20) for $\rho = 0$ imply the same estimates for arbitrary $\rho > 0$ with the same constants. We therefore restrict our attention to the case $\rho = 0$. In that case (3.19) (3.20) are almost standard properties of Sobolev spaces, except for the presence of $|\xi|_<^{\ell_<}$ with possibly $\ell_< > 0$ in (3.9). We give a proof for completeness.

(3.19) From the definition (3.9) with $\rho = 0$, it follows that it is sufficient to estimate $\| |\xi|^{\ell+2} (\widehat{\varphi\psi})_{>2} \|_2$ and $\| |\xi|^{\ell_<} \widehat{\varphi\psi} \|_2$. We estimate

$$\begin{aligned} \| |\xi|^{\ell+2} (\widehat{\varphi\psi})_{>2} \|_2 &\leqslant C\{ \| (|\xi|^{\ell+2} |\hat{\varphi}_>|) * |\hat{\psi}| \|_2 \\ &\quad + \| |\hat{\varphi}_<| * |\hat{\psi}_>| \|_2 + (\varphi \leftrightarrow \psi)\} \\ &\leqslant C\{ \| |\xi|^{\ell+2} \hat{\varphi}_> \|_2 \, \|\hat{\psi}\|_1 + \|\hat{\varphi}_<\|_1 \, \|\hat{\psi}_>\|_2 + (\varphi \leftrightarrow \psi)\} \\ &\leqslant C \, |\varphi|_\ell \, |\psi|_\ell \end{aligned}$$

by (3.3), by the Young inequality and by (3.16) (3.17). The lower restriction $|\xi| > 2$ in $\varphi\psi$ implies that there is no $\varphi_< \psi_<$ contribution.

On the other hand

$$\| |\xi|^{\ell_<} \widehat{\varphi\psi} \|_2 \leqslant C\{ \| |\xi|^{\ell_<} \hat{\varphi} \|_2 \, \|\hat{\psi}\|_1 + (\varphi \leftrightarrow \psi)\} \leqslant C \, |\varphi|_\ell \, |\psi|_\ell$$

by the Young inequality and (3.16) (3.17).

(3.20) We estimate similarly by (3.3) and the Young inequality

$$\begin{aligned} \| \langle \xi \rangle^k \widehat{\varphi w} \|_2 &\leqslant C\{ \|\hat{\varphi}\|_1 \, \| \langle \xi \rangle^k \hat{w} \|_2 + \|\hat{\varphi}_<\|_1 \, \|\hat{w}\|_2 \\ &\quad + \| (|\xi|^k |\hat{\varphi}_>|) * |\hat{w}| \|_2 \} \end{aligned}$$

and the result follows from (3.16) (3.17) and from Lemma 3.2 with $m = 0$, $k_1 = \ell + 2 - k$ and $k_2 = k$.

(3.21) follows immediately from a repeated application of (3.19) (3.20). ∎

Remark 3.2. In Lemma 3.3 we have used only (3.3) from Lemma 3.1. For $\nu < 1$, by using (3.4), one can obtain more general results. In particular K_ρ^k and Y_ρ^ℓ are algebras under multiplication for any k and ℓ in $\mathbb{R}^+$ (see Appendix B). Similarly in what follows, we shall use only (3.3) and (3.5). For $\nu < 1$, by using in addition (3.4) and (3.6), one could generalize some of the results by weakening the assumptions made on k and ℓ. That extension however would not hold uniformly in ν for $\nu \to 1$, and we shall therefore refrain from following that track.

We now turn to the derivation of the basic estimates needed to study the auxiliary system (2.11)–(2.12). For that purpose, we shall use a regularization. Let $j \in \mathscr{C}^1(\mathbb{R}^n, \mathbb{R})$ with $0 \leqslant j \leqslant 1$ and $j(0) = 1$. We denote by j_ε both the function $j_\varepsilon(\xi) = j(\varepsilon\xi)$ and the operator of multiplication by that function in Fourier space variables, and by J_ε the operator $F^* j_\varepsilon F$.

We recall that g_0 is defined by (2.13). We shall use systematically the notation $s = \nabla\varphi$ and whenever convenient, start from the equation (2.27) satisfied by s instead of the equation (2.12) satisfied by φ.

We first state the basic estimates.

LEMMA 3.4. *Let $m \geqslant 0$. The following estimates hold:*

$$|\mathrm{Re}\langle j_\varepsilon\,|\xi|^k\, f\hat{w},\, j_\varepsilon\,|\xi|^k\, f(\widehat{s\cdot\nabla w})\rangle| + |\mathrm{Re}\langle j_\varepsilon f\hat{w},\, j_\varepsilon f(\widehat{s\cdot\nabla w})\rangle| \leqslant C\,|w|^2_{k+\nu/2}\,|\varphi|_\ell \tag{3.22}$$

uniformly in ε, for $\ell > n/2 - \nu$, $k \geqslant \nu/2$, $\ell + 1 \geqslant k - \nu/2$.

$$\|\,|\xi|^m\, f(\widehat{s\cdot\nabla w})\|_2 \leqslant C\,|\varphi|_\ell\,|w|_k \tag{3.23}$$

for $k + \ell > m + n/2$, $k \geqslant m + 1$, $\ell + 1 \geqslant m$.

$$\|\,|\xi|^m\, f(\widehat{(\nabla\cdot s)\,w})\|_2 \leqslant C\,|\varphi|_\ell\,|w|_k \tag{3.24}$$

for $k + \ell > m + n/2$, $k \geqslant m$, $\ell \geqslant m$.

$$|\mathrm{Re}\langle j_\varepsilon\,|\xi|^{\ell'+1}\, f\hat{s}'_>,\, j_\varepsilon\,|\xi|^{\ell'+1}\, f(\widehat{s\cdot\nabla s'})_>\rangle| \leqslant C\,|\varphi'|^2_{\ell'+\nu/2}\,|\varphi|_\ell \tag{3.25}$$

uniformly in ε, for $\ell > n/2 - \nu$, $\ell' + 1 \geqslant \nu/2$, $\ell \geqslant \ell' - \nu/2$.

$$\|\,|\xi|^m\, f(\widehat{s\cdot\nabla s'})_>\|_2 \leqslant C\,|\varphi|_\ell\,|\varphi'|_{\ell'} \tag{3.26}$$

for $\ell + \ell' > m - 1 + n/2$, $\ell + 1 \geqslant m$, $\ell' \geqslant m$.

$$\|\,|\xi|^{\ell_<}\, f(\widehat{s\cdot s'})\|_2 \leqslant C\,|\varphi|_\ell\,|\varphi'|_{\ell'} \tag{3.27}$$

for $\ell, \ell' > n/2 - 2$.

$$\|\,|\xi|^m\, f\widehat{g_0(w_1 w_2)}_>\|_2 \leqslant C\{|w_1|_{k_1}\,|w_2|_{k_2} + |w_1|_{k'_1}\,|w_2|_{k'_2}\} \tag{3.28}$$

for $(k_1 + k_2) \wedge (k'_1 + k'_2) > \beta + n/2$, $k_1 \wedge k'_2 \geqslant \beta \vee 0$, $k_2 \wedge k'_1 \geqslant 0$, where $\beta = m + \mu - n$.

$$\|\,|\xi|^{\ell_<}\, \widehat{g_0(w_1 w_2)}_<\|_2 \leqslant C\,\|w_1\|_2\,\|w_2\|_2 \tag{3.29}$$

for $\ell_< > n/2 - \mu$.

The constants C in (3.22)–(3.29) *can be taken independent of ρ.*

Proof. (3.22) We have to estimate

$$\mathrm{Im}\int d\xi\, d\eta\; j_\varepsilon(\xi)\,|\xi|^k\, f(\xi)\,\bar{\hat{w}}(\xi)\; j_\varepsilon(\xi)\,|\xi|^k\, f(\xi)\,\hat{s}(\xi-\eta)\cdot\eta\hat{w}(\eta) \tag{3.30}$$

and a similar expression with $k=0$. We consider only the former one. The proof for the latter is similar and simpler. We split the domain of integration into three regions, namely

$$|\xi-\eta| \leqslant |\xi| \wedge |\eta|, \qquad |\xi| \leqslant |\xi-\eta| \wedge |\eta| \qquad \text{and} \qquad |\eta| \leqslant |\xi| \wedge |\xi-\eta|$$

and correspondingly the integral (3.30) is written as the sum $I_1+I_2+I_3$ of three terms which we estimate successively.

Region $|\xi-\eta| \leqslant |\xi| \wedge |\eta|$, *estimate of* I_1. In this region we decompose the integrand according to the identity

$$j_\varepsilon(\xi)\, |\xi|^k\, f(\xi)\, \eta = j_\varepsilon(\xi)\, |\xi|^k\, (f(\xi)-f(\eta))\, \eta$$
$$+ (j_\varepsilon(\xi)\, |\xi|^k - j_\varepsilon(\eta)\, |\eta|^k)\, f(\eta)\, \eta + j_\varepsilon(\eta)\, |\eta|^k\, f(\eta)\, \eta \tag{3.31}$$

and correspondingly I_1 is written as the sum $I_{1,1}+I_{1,2}+I_{1,3}$ of three terms which we estimate successively.

Estimate of $I_{1,1}$. From (3.5) of Lemma 3.1, we obtain

$$|j_\varepsilon(\xi)\, |\xi|^{2k}\, (f(\xi)-f(\eta))\, \eta| \leqslant C\, |\xi|^{k+\nu/2}\, |\xi-\eta|^{1-\nu}\, |\eta|^{k+\nu/2}\, f(\xi-\eta)\, f(\eta)$$

and therefore by the Schwarz and Young inequalities

$$|I_{1,1}| \leqslant C\, |w|^2_{k+\nu/2}\, \| |\xi|^{1-\nu}\, f\hat{s} \|_1$$

so that by (3.16) (3.17)

$$|I_{1,1}| \leqslant C\, |w|^2_{k+\nu/2}\, |\varphi|_\ell. \tag{3.32}$$

Estimate of $I_{1,2}$. We rewrite

$$(j_\varepsilon(\xi)\, |\xi|^k - j_\varepsilon(\eta)\, |\eta|^k)\, \eta = j_\varepsilon(\xi)(|\xi|^k - |\eta|^k)\, \eta + |\eta|^k\, (j_\varepsilon(\xi)-j_\varepsilon(\eta))\, \eta. \tag{3.33}$$

We estimate

$$||\xi|^k - |\eta|^k|\, |\eta| \leqslant k 2^{|k-1|}\, |\xi-\eta|\, |\eta|^k \tag{3.34}$$

and we rewrite

$$(j_\varepsilon(\xi)-j_\varepsilon(\eta))\, \eta = j_\varepsilon(\xi)\, \xi - j_\varepsilon(\eta)\, \eta - j_\varepsilon(\xi)(\xi-\eta)$$
$$= \int_0^1 d\theta \{(\xi-\eta)\cdot \nabla j_\varepsilon(\xi_\theta)\, \xi_\theta + (j_\varepsilon(\xi_\theta)-j_\varepsilon(\xi))(\xi-\eta)\} \tag{3.35}$$

with $\xi_\theta = \theta\xi + (1-\theta)\, \eta$, so that

$$|j_\varepsilon(\xi)-j_\varepsilon(\eta)|\, |\eta| \leqslant (\| |\cdot|\, \nabla j\|_\infty + 2)\, |\xi-\eta|. \tag{3.36}$$

Comparing (3.33) (3.34) (3.36), we obtain

$$||j_\varepsilon(\xi)|\xi|^k - j_\varepsilon(\eta)|\eta|^k|\,|\eta| \leqslant C\,|\xi-\eta|\,|\eta|^k \leqslant C\,|\xi|^{\nu/2}\,|\xi-\eta|^{1-\nu}\,|\eta|^{k+\nu/2} \tag{3.37}$$

from which we obtain as previously

$$|I_{1,2}| \leqslant C\,|w|^2_{k+\nu/2}\,|\varphi|_\ell. \tag{3.38}$$

Estimate of $I_{1,3}$. By reality and symmetry, which is the Fourier space version of integration by parts, $I_{1,3}$ can be rewritten as

$$I_{1,3} = (i/2) \int_{|\xi-\eta| \leqslant |\xi| \wedge |\eta|} d\xi\, d\eta\, j_\varepsilon(\xi)\,|\xi|^k\, f(\xi)\,\bar{\hat{w}}(\xi)(\xi-\eta)$$
$$\cdot\,\hat{s}(\xi-\eta)\, j_\varepsilon(\eta)\,|\eta|^k\, f(\eta)\,\hat{w}(\eta).$$

Using the second inequality in (3.37) we obtain as previously

$$|I_{1,3}| \leqslant C\,|w|^2_{k+\nu/2}\,|s|_\ell. \tag{3.39}$$

We now turn to the contribution of the region $|\xi| \leqslant |\xi-\eta| \wedge |\eta|$.

Estimate of I_2. Using (3.3) and

$$|\xi|^{2k}\,|\eta| \leqslant C\,|\xi|^{k+\nu/2}\,|\xi-\eta|^{1-\nu}\,|\eta|^{k+\nu/2} \tag{3.40}$$

we obtain as previously

$$|I_2| \leqslant C\,|w|^2_{k+\nu/2}\,|\varphi|_\ell. \tag{3.41}$$

We finally consider the region $|\eta| \leqslant |\xi| \wedge |\xi-\eta|$.

Estimate of I_3. Using (3.3) and

$$|\xi|^{2k}\,|\eta| \leqslant C\,|\xi|^{k+\nu/2}\,|\xi-\eta|^{k-\nu/2+1-\theta}\,|\eta|^\theta$$

with $0 \leqslant \theta \leqslant 1$ and decomposing $s = s_> + s_<$, we estimate

$$|I_3| \leqslant C\,|w|_{k+\nu/2}\{\|(|\xi|^{k-\nu/2+1}\, f\,|\hat{s}_<|) * (f\,|\hat{w}|)\|_2$$
$$+ \|(|\xi|^{k-\nu/2+1-\theta}\, f\,|\hat{s}_>|) * (|\xi|^\theta\, f\,|\hat{w}|)\|_2\}. \tag{3.42}$$

Using the Young inequality for the term in $s_<$ and Lemma 3.2 for the term in $s_>$, we obtain

$$|I_3| \leqslant C\,|w|_{k+\nu/2}\{\||\xi|^{k-\nu/2+1}\, f\hat{s}_<\|_1\,\|f\hat{w}\|_2 + \||\xi|^{\ell+1}\, f\hat{s}_>\|_2\,\||\xi|^{k+\nu/2}\, f\hat{w}\|_2\}$$

with $\ell > n/2 - \nu$, $k + \nu/2 \geqslant \theta$ and $\ell \geqslant k - \nu/2 - \theta$. We choose $\theta = 1 \wedge (k + \nu/2)$ and the last two conditions reduce to $\ell + 1 \geqslant k - \nu/2$. Using (3.16), we obtain

$$|I_3| \leqslant C\, |w|_{k+\nu/2}^2\, |\varphi|_\ell. \tag{3.43}$$

Collecting (3.32) (3.38) (3.39) (3.41) (3.43) yields (3.22).

The constants C appearing in the proof of (3.22) are independent of ρ. This can be checked explicitly on each case. More generally, it is a consequence of the following facts. We treat f only through the inequalities (3.3) and (3.5) of Lemma 3.1, until we end up with integrals of the type (3.30) with however f appearing only through the product $f(\xi)\, f(\xi - \eta)\, f(\eta)$ (possibly after adding one missing f and using the fact that $f \geqslant 1$). It then follows from (3.11) that an estimate of the type (3.22) of such a quantity for $\rho = 0$ implies the same estimate for any $\rho > 0$ with the same constant. The same argument applies to all subsequent estimates of Lemma 3.4 that contain f. Therefore from now on and in the same way as in the proof of Lemma 3.3, we omit f in the proofs.

(3.23) We estimate

$$\| |\xi|^m\, (\widehat{s \cdot \nabla w}\,) \|_2 \leqslant C\{ \| |\hat{s}| * (|\xi|^{m+1}\, |\hat{w}|) \|_2 + \| (|\xi|^m\, |\hat{s}|) * (|\xi|\, |\hat{w}|) \|_2 \}.$$

We decompose $s = s_< + s_>$, we estimate the contribution of $s_<$ by the Young inequality and the contribution of $s_>$ by Lemma 3.2, thereby obtaining

$$\cdots \leqslant C\{ \|\hat{s}_<\|_1\, (\| |\xi|^{m+1}\, \hat{w}\|_2 + \| |\xi|\, \hat{w}\|_2) + \| |\xi|^{\ell+1}\, \hat{s}_>\|_2\, \| |\xi|^k\, \hat{w}\|_2 \}$$

under the condition stated on k, ℓ, m. (3.23) then follows by (3.16).

(3.24) We estimate

$$\| |\xi|^m\, (\widehat{(\nabla \cdot s)\, w}\,) \|_2 \leqslant C\{ \| (|\xi|^{m+1}\, |\hat{s}|) * |\hat{w}| \|_2 + \| (|\xi|\, |\hat{s}|) * (|\xi|^m\, |\hat{w}|) \|_2 \}.$$

Proceeding as above, we obtain

$$\cdots \leqslant C\{ \|\hat{s}_<\|_1\, (\| |\xi|^m\, \hat{w}\|_2 + \|\hat{w}\|_2) + \| |\xi|^{\ell+1}\, \hat{s}_>\|_2\, \| |\xi|^k\, \hat{w}\|_2 \}$$

from which (3.24) follows by the same argument as above.

(3.25) and (3.26) We decompose $s' = s'_> + s'_<$. The contribution of $s'_>$ to (3.25) and (3.26) is estimated by (3.22) and (3.24) respectively, by replacing w by $s'_>$ and k by $\ell' + 1$. In order to complete the proof it remains to estimate $\| |\xi|_>^m\, (\widehat{s \cdot \nabla s'_<}\,) \|_2$ with $m = \ell' + 1 - \nu/2$ for (3.25) and with general m for (3.26), namely to estimate the L^2 norm in ξ of the integral

$$J = |\xi|_>^m\, f(\xi) \int d\eta\, \hat{s}(\xi - \eta) \cdot \eta \hat{s}'_<(\eta).$$

For that purpose, we decompose $s = s_{>1/2} + s_{<1/2}$ and correspondingly $J = J_> + J_<$. In $J_>$ we have $|\eta| \leqslant 1 \leqslant 2\,|\xi-\eta|$ and therefore $|\xi| \leqslant 3\,|\xi-\eta|$, so that

$$\begin{aligned} \|J_>\|_2 &\leqslant C\,\|(|\xi|^m\,|\hat{s}_{>1/2}|) * (|\eta|\,|\hat{s}'_<|)\|_2 \\ &\leqslant C\,|s|_\ell\,\|\hat{s}'_<\|_1 \end{aligned} \tag{3.44}$$

by the Young inequality, provided $m \leqslant \ell+1$, a condition which appears explicitly in (3.25) and which reduces to $\ell \geqslant \ell' - \nu/2$ in (3.26) for $m = \ell' + 1 - \nu/2$.

In $J_<$, we have $|\xi-\eta| \leqslant 1/2$, $|\xi| \geqslant 1$ and $|\eta| \leqslant 1$, and therefore $1/2 \leqslant |\eta| \leqslant 1$ and $|\xi| \leqslant 3/2$, so that

$$\|J_<\|_2 \leqslant C\,\|\hat{s}_{<1/2}\|_1\,\|s'_{>1/2}\|_2. \tag{3.45}$$

(3.25) and (3.26) now follow from (3.22) (3.24), (3.44) (3.16) and (3.45).

(3.27) follows immediately from

$$\begin{aligned} \|\,|\xi|^{\ell_<}\,(\widehat{s\cdot s'}\,)\|_2 &\leqslant C\{\|(|\xi|^{\ell_<}\,|\hat{s}|) * |\hat{s}'|\,\|_2 + (s \leftrightarrow s')\} \\ &\leqslant C\{\|\,|\xi|^{\ell_<}\,\hat{s}\|_2\,\|\hat{s}'\|_1 + (s \leftrightarrow s')\} \end{aligned}$$

and from (3.16) (3.17).

(3.28) follows from (3.14) either directly for $\beta \leqslant 0$, or through the inequality

$$\|\,|\xi|^m\,\widehat{g_0(w_1 w_2)_>}\,\|_2 \leqslant C\{\|(|\xi|^\beta\,|\hat{w}_1|) * (|\hat{w}_2|)\|_2 + (1 \leftrightarrow 2)\}$$

for $\beta \geqslant 0$.

(3.29) follows from (3.15) with $m = \beta$. ∎

We now explain the origin of the derivative loss in the system (2.11)–(2.12) for $\lambda \equiv \mu - n + 2 > 0$ and the mechanism by which that loss is overcome through the use of the spaces $\mathscr{X}^{k,\ell}_{\rho,\,loc}$ defined by (3.13). If we try to solve the system (2.11)–(2.12) by the energy method in a space like $\mathscr{C}(I, H^k \oplus \dot{H}^{\ell+2})$, we have to estimate in particular

$$\begin{aligned} \partial_t\,\|\partial^k w\|_2^2 &= 2\,\mathrm{Re}\langle \partial^k w, \partial^k \partial_t w\rangle \\ \partial_t\,\|\partial^{\ell+2}\varphi\|_2^2 &= 2\langle \partial^{\ell+2}\varphi, \partial^{\ell+2}\partial_t\varphi\rangle. \end{aligned} \tag{3.46}$$

The term with $\Delta\phi$ from $\partial_t w$ forces us to apply $k+2$ derivatives to φ and requires therefore $\ell \geqslant k$, while the term with g_0 from $\partial_t\varphi$ forces us to apply $\ell+2$ derivatives to g_0 or equivalently $\ell+\lambda$ derivatives to $|w|^2$ and requires therefore $k \geqslant \ell+\lambda$. The terms with $\nabla\varphi\cdot\nabla w$ from $\partial_t w$ and with $|\nabla\varphi|^2$ from

$\partial_t \varphi$ can be handled essentially under the same assumptions, possibly after an integration by parts. The method therefore applies only if $\lambda \leqslant 0$, namely $\mu \leqslant n-2$, which is the case treated in I and II.

If we try instead to solve the same problem in the space $\mathscr{X}^{k,\ell}_{\rho, loc}$ with time dependent ρ, we have by (3.12)

$$\begin{aligned} \partial_t\, |w|_k^2 &= 2\rho'\, |w|^2_{k+\nu/2} + 2\,\mathrm{Re}\langle w, \partial_t w\rangle_k \\ \partial_t\, |\varphi|_\ell^2 &= 2\rho'\, |\varphi|^2_{\ell+\nu/2} + 2\langle \varphi, \partial_t \varphi\rangle_\ell, \end{aligned} \tag{3.47}$$

where $\langle\cdot, \cdot\rangle_k$ and $\langle\cdot, \cdot\rangle_\ell$ denote the scalar products in K^k_ρ and Y^ℓ_ρ. If ρ' has a favourable sign, namely if ρ decreases away from the initial time, the terms containing ρ' provide a control of the norm in $L^2(K^{k+\nu/2}_\rho \oplus Y^{\ell+\nu/2}_\rho)$, and it suffices to control the scalar products at most quadratically in terms of the norms in $K^{k+\nu/2}_\rho$ and $Y^{\ell+\nu/2}_\rho$. In the term with $\Delta\phi$ from $\partial_t w$, after distributing the function f with the help of (3.3) and shifting $\nu/2$ derivatives on the first vector of the scalar product, it suffices to apply $k-\nu/2$ derivatives on $\Delta\phi$ while one is allowed to use the norm $|\varphi|_{\ell+\nu/2}$. This requires only $\ell \geqslant k-\nu$. Similarly in the term with g_0 from $\partial_t\varphi$, it suffices to apply $\ell+2-\nu/2$ derivatives to g_0 or equivalently $\ell+\lambda-\nu/2$ derivatives to $|w|^2$, while one is allowed to use $|w|_{k+\nu/2}$. This requires only $k \geqslant \ell+\lambda-\nu$. The two conditions on (k, ℓ) are compatible provided $\lambda \leqslant 2\nu$, which allows for $\mu \leqslant n$ under that condition. It remains to estimate the terms $\nabla\varphi\cdot\nabla w$ from $\partial_t w$ and $|\nabla\varphi|^2$ from $\partial_t \varphi$, which in the Sobolev case requires the integration by part of one derivative. Here however, by the same argument as above, it suffices to integrate by parts $1-\nu$ derivative. Now it turns out that integration by parts of $1-\nu$ derivative is exactly what is allowed by the inequality (3.5), which is exploited through (3.31) to derive the estimates (3.22) (3.25) where that integration by parts occurs. Actually the inequality (3.5) is optimal in the dangerous part $|\xi-\eta| \ll |\xi| \sim |\eta|$ of the region where it is used, and more precisely when $|\eta| \to \infty$ for fixed $|\xi-\eta|$. The conditions $\ell \geqslant k-\nu$ and $k \geqslant \ell+\lambda-\nu$ and therefore their consequence $\lambda \leqslant 2\nu$ will appear from the next lemma onward as the most important part of the condition (3.48) and will propagate throughout this paper (except in Section 5) up to the main and final results of Propositions 6.5 and 7.5.

We now exploit Lemma 3.4 to derive energy like estimates for the solutions of the auxiliary system (2.11)–(2.12). In the following three lemmas, I is an interval contained in $[1, \infty)$, ρ is a nonnegative continuous and piecewise $\mathscr{C}^1$ function defined in I. We shall be interested in solutions (w, φ) in spaces of the type $\mathscr{X}^{k,\ell}_{\rho, loc}(I)$ for suitable values of k and ℓ. The estimates will hold in integrated form in any compact subinterval of I under the available regularity, but will be stated in differential form for brevity.

LEMMA 3.5. *Let k, ℓ satisfy*

$$\ell > n/2 - \nu, \qquad k \geqslant \nu/2, \qquad \ell \geqslant k - \nu,$$
$$k \geqslant \ell + \lambda - \nu, \qquad 2k > \ell + \lambda - \nu + n/2, \tag{3.48}$$

where $\lambda = \mu - n + 2$.

Let $(w, \varphi) \in \mathscr{X}^{k,\ell}_{\rho, loc}(I)$ be a solution of the system (2.11)–(2.12). *Then the following estimates hold:*

$$|\partial_t\, |w|_k^2 - 2\rho'\, |w|^2_{k+\nu/2}| \leqslant Ct^{-2}\{|w|^2_{k+\nu/2}\, |\varphi|_\ell + |w|_{k+\nu/2}\, |\varphi|_{\ell+\nu/2}\, |w|_k\}, \tag{3.49}$$

$$|\partial_t\, |\varphi|_\ell^2 - 2\rho'\, |\varphi|^2_{\ell+\nu/2}| \leqslant Ct^{-2}\, |\varphi|^2_{\ell+\nu/2}\, |\varphi|_\ell + Ct^{-\gamma}\, |\varphi|_{\ell+\nu/2}\, |w|_{k+\nu/2}\, |w|_k. \tag{3.50}$$

Proof. We use the same regularization as in the proof of Lemma 3.4. From (3.12) we obtain

$$\partial_t\, |J_\varepsilon w|_k^2 - 2\rho'\, |J_\varepsilon w|^2_{k+\nu/2} = 2\,\mathrm{Re}(\langle j_\varepsilon\, |\xi|^k\, f\hat{w}_>\,,\, j_\varepsilon\, |\xi|^k\, f\partial_t \hat{w}_>\rangle$$
$$+ \langle j_\varepsilon f\hat{w}_<,\, j_\varepsilon f\partial_t \hat{w}_<\rangle). \tag{3.51}$$

We substitute $\partial_t w$ from (2.11) and we estimate the various terms successively. The term with Δw does not contribute. The term $s\cdot\nabla w$ is estimated by (3.22) under the conditions

$$\ell > n/2 - \nu, \qquad k \geqslant \nu/2, \qquad \ell + 1 \geqslant k - \nu/2.$$

The term $(\nabla\cdot s)\, w$ is estimated by (3.24) with $m = k - \nu/2$ or $m = 0$, and ℓ replaced by $\ell + \nu/2$, under the conditions

$$\ell > n/2 - \nu, \qquad \ell \geqslant k - \nu.$$

Substituting those two estimates into (3.51), integrating over time and taking the limit $\varepsilon \to 0$ yields the integrated form of (3.49). The required conditions on k, ℓ are implied by (3.48).

Similarly, we obtain from (3.12)

$$\partial_t\, |J_\varepsilon \varphi|_\ell^2 - 2\rho'\, |J_\varepsilon \varphi|^2_{\ell+\nu/2} = 2\langle j_\varepsilon\, |\xi|^{\ell+1}\, f\hat{s}_>\,,\, j_\varepsilon\, |\xi|^{\ell+1}\, f\partial_t \hat{s}_>\rangle$$
$$+ 2\langle j_\varepsilon\, |\xi|^{\ell_<}\, f\hat{\varphi}_<,\, j_\varepsilon\, |\xi|^{\ell_<}\, f\partial_t \hat{\varphi}_<\rangle. \tag{3.52}$$

We substitute $\partial_t s$ and $\partial_t \varphi$ from (2.27) (2.12) into (3.52) and we estimate the various terms successively. The term $s\cdot\nabla s$ from $\partial_t s$ is estimated by (3.25) with $s' = s$, $\ell' = \ell$ under the conditions

$$\ell > n/2 - \nu, \qquad \ell + 1 \geqslant \nu/2.$$

The term ∇g_0 from $\partial_t s$ is estimated by (3.28) with $m = \ell + 2 - v/2$, $w_1 = w_2 = w$, $k_1 (= k'_2) = k + v/2$, $k_2 (= k'_1) = k$, under the conditions

$$2k > \ell + \lambda - v + n/2, \qquad k \geqslant \ell + \lambda - v.$$

The terms $|s|^2$ and g_0 from $\partial_t \varphi_<$ are estimated by (3.27) and (3.29) respectively. Using those estimates yields (3.50) in the same way as above. The required conditions on k, ℓ are implied by (3.48). In particular the condition $\ell + 1 \geqslant v/2$ follows from $\ell > n/2 - v$. ∎

The next lemma is a regularity result.

LEMMA 3.6. *Let k, ℓ satisfy* (3.48) *and let $\bar{k}$, $\bar{\ell}$ satisfy*

$$\bar{k} - k = \bar{\ell} - \ell \geqslant 0. \tag{3.53}$$

Let $(w, \varphi) \in \mathscr{X}^{\bar{k}, \bar{\ell}}_{\rho, loc}(I)$ be a solution of the system (2.11)–(2.12). *Then the following estimates hold:*

$$|\partial_t\, |w|^2_{\bar{k}} - 2\rho'\, |w|^2_{\bar{k}+v/2}| \leqslant Ct^{-2}\{|w|^2_{\bar{k}+v/2}\, |\varphi|_\ell + |w|_{\bar{k}+v/2}\, |\varphi|_{\bar{\ell}+v/2}\, |w|_k\}, \tag{3.54}$$

$$|\partial_t\, |\varphi|^2_{\bar{\ell}} - 2\rho'\, |\varphi|^2_{\bar{\ell}+v/2}| \leqslant Ct^{-2}\, |\varphi|^2_{\bar{\ell}+v/2}\, |\varphi|_\ell + Ct^{-\gamma}\, |\varphi|_{\bar{\ell}+v/2}\, |w|_{\bar{k}+v/2}\, |w|_k. \tag{3.55}$$

Proof. The proof follows the same pattern as that of Lemma 3.5 and we concentrate on the differences, which bear on the estimates connected with Lemma 3.4, omitting the regularization for brevity. We estimate only the contribution of the high $|\xi|$ region, since that of the low $|\xi|$ region is already estimated by Lemma 3.5.

We first estimate $\partial_t\, |w|^2_{\bar{k}}$, starting from (3.51) with k replaced by $\bar{k}$ and with J_ε omitted and we estimate successively the contribution of the various terms from (2.11).

The contribution of $s \cdot \nabla w$, namely

$$2\, \mathrm{Re}\langle |\xi|^{\bar{k}}\, f\hat{w}, |\xi|^{\bar{k}}\, f(\widehat{s \cdot \nabla w})\rangle$$

is estimated in the same way as in the proof of (3.22). The contributions I_1 and I_2 of the regions $|\xi - \eta| \leqslant |\xi| \wedge |\eta|$ and $|\xi| \leqslant |\xi - \eta| \wedge |\eta|$ are estimated as in the latter with k replaced by $\bar{k}$ under the conditions $\bar{k} \geqslant v/2$, $\ell > n/2 - v$.

The contribution of the region $|\eta| \leqslant |\xi| \wedge |\xi - \eta|$ is estimated by (3.42) with k replaced by $\bar{k}$ and with $\theta = 0$, or equivalently

$$|I_3| \leqslant C\, |w|_{\bar{k}+v/2}\{\|f\hat{s}_<\|_1\, \|f\hat{w}\|_2 + \|(|\xi|^{\bar{k}-v/2+1}\, f\, |\hat{s}_>|) * (f\, |\hat{w}|)\|_2\}.$$

The last norm is then estimated by Lemma 3.2, thereby yielding

$$|I_3| \leqslant C\, |w|_{\bar{k}+v/2}\, |\varphi|_{\bar{\ell}+v/2}\, |w|_k$$

under the conditions

$$k+\bar{\ell} > \bar{k}+n/2-\nu, \qquad \bar{\ell} \geqslant \bar{k}-\nu$$

which for $\bar{k}-k=\bar{\ell}-\ell$ reduce to

$$\ell > n/2-\nu, \qquad \ell \geqslant k-\nu.$$

The contribution of $(\nabla \cdot s)\, w$ is estimated by

$$|w|_{\bar{k}+\nu/2}\, \| |\xi|^{\bar{k}-\nu/2} f(\widehat{(\nabla \cdot s)\, w})\|_2$$

and subsequently in a way similar to the proof of (3.24) with $m=\bar{k}-\nu/2$. Using the elementary inequality

$$|\xi|^m\, |\xi-\eta| \leqslant 2^m(|\xi-\eta|^{m+1}+|\eta|^{m+\theta}\, |\xi-\eta|^{1-\theta}) \tag{3.56}$$

valid for $m \geqslant 0$ and $0 \leqslant \theta \leqslant 1$, we estimate the last norm by

$$\cdots C\{\|(|\xi|^{m+1} f\, |\hat{s}|) * (f\, |\hat{w}|)\|_2 + \|(|\xi|^{1-\theta} f\, |\hat{s}|) * (|\xi|^{m+\theta} f\, |\hat{w}|)\|_2\}.$$

We then estimate the first term by Lemma 3.2 and the second term with $\theta=\nu$ by the Young inequality and (3.16) (3.17), thereby obtaining

$$\cdots \leqslant C(|\varphi|_{\bar{\ell}+\nu/2}\, |w|_k + |\varphi|_\ell\, |w|_{\bar{k}+\nu/2})$$

under the same conditions as before, namely

$$k+\bar{\ell} > \bar{k}+n/2-\nu, \qquad \bar{\ell} \geqslant \bar{k}-\nu.$$

This completes the proof of (3.54).

We next estimate $\partial_t\, |\varphi|_{\bar{\ell}}^2$ by substituting similarly the various terms from (2.12) (2.27) into the right-hand side of (3.52) with ℓ replaced by $\bar{\ell}$ and J_ε omitted. The contribution of $s \cdot \nabla s$ to the high $|\xi|$ part of the norm, namely

$$\langle |\xi|^{\bar{\ell}+1} f \hat{s}_>,\, |\xi|^{\bar{\ell}+1} f(\widehat{s \cdot \nabla s})_> \rangle$$

is estimated by modifying the proof of (3.25) along the same lines as that of (3.22) above.

The contribution of $g_0(w, w)_>$ is estimated by

$$|\varphi|_{\bar{\ell}+\nu/2}\, \| |\xi|^{\bar{\ell}+1-\nu/2} f \widehat{\nabla g_0(w, w)}_> \|_2$$

followed by (3.28) with $m=\bar{\ell}+2-\nu/2$, $k_1(=k_2')=\bar{k}+\nu/2$, $k_2(=k_1')=k$ under the conditions

$$k+\bar{k} > \bar{\ell}+\lambda-\nu+n/2, \qquad \bar{k} \geqslant \bar{\ell}+\lambda-\nu$$

which reduce to the same conditions with $(\bar{k}, \bar{\ell})$ replaced by (k, ℓ) under the condition $\bar{k}-k=\bar{\ell}-\ell$.

This completes the proof of (3.55). ∎

We next estimate the difference between two solutions of the system (2.11)–(2.12).

LEMMA 3.7. *Let k, ℓ satisfy* (3.48) *and let k', ℓ' satisfy*

$$k' \geqslant \nu/2, \qquad k-k'=\ell-\ell' \geqslant 1-\nu. \tag{3.57}$$

Let (w_1, φ_1) and $(w_2, \varphi_2) \in \mathscr{X}^{k,\ell}_{\rho, loc}(I)$ be two solutions of the system (2.11)–(2.12) *and let $w_\pm = w_1 \pm w_2$, $\varphi_\pm = \varphi_1 \pm \varphi_2$. Then the following estimates hold:*

$$\begin{aligned} |\partial_t \, |w_-|_{k'}^2 - 2\rho' \, |w_-|^2_{k'+\nu/2}| &\leqslant Ct^{-2}\{|w_-|^2_{k'+\nu/2} \, |\varphi_+|_\ell \\ &\quad + |w_-|_{k'+\nu/2} \, (|w_+|_{k+\nu/2} \, |\varphi_-|_{\ell'} + |\varphi_+|_{\ell+\nu/2} \, |w_-|_{k'} + |\varphi_-|_{\ell'+\nu/2} \, |w_+|_k)\}, \end{aligned} \tag{3.58}$$

$$\begin{aligned} |\partial_t \, |\varphi_-|_{\ell'}^2 - 2\rho' \, |\varphi_-|^2_{\ell'+\nu/2}| &\leqslant Ct^{-2}\{|\varphi_-|^2_{\ell'+\nu/2} \, |\varphi_+|_\ell + |\varphi_-|_{\ell'+\nu/2} \, |\varphi_-|_{\ell'} \, |\varphi_+|_{\ell+\nu/2}\} \\ &\quad + Ct^{-\gamma} \, |\varphi_-|_{\ell'+\nu/2} \, \{|w_+|_{k+\nu/2} \, |w_-|_{k'} + |w_-|_{k'+\nu/2} \, |w_+|_k\}. \end{aligned} \tag{3.59}$$

Proof. The proof follows the same pattern as that of Lemma 3.5, using the estimates of Lemma 3.4. We omit again the regularization for brevity. The equations satisfied by (w_-, φ_-) are

$$\begin{aligned} \partial_t w_- &= i(2t^2)^{-1} \, \Delta w_- + (2t)^{-2} \, \{2s_+ \cdot \nabla w_- + 2s_- \cdot \nabla w_+ + (\nabla \cdot s_+) \, w_- \\ &\qquad + (\nabla \cdot s_-) \, w_+\}, \end{aligned} \tag{3.60}$$

$$\partial_t \varphi_- = (2t^2)^{-1} \, (s_+ \cdot s_-) + t^{-\gamma} g_0(w_+, w_-), \tag{3.61}$$

and in the same way as in the proof of Lemma 3.5, we shall also use the equation for s_- obtained by taking the gradient of (3.61), namely

$$\partial_t s_- = (2t^2)^{-1} \, (s_+ \cdot \nabla s_- + s_- \cdot \nabla s_+) + t^{-\gamma} \nabla g_0(w_+, w_-). \tag{3.62}$$

From (3.12), we obtain

$$\begin{aligned} |\partial_t \, |w_-|^2_{k'} - 2\rho' \, |w_-|^2_{k'+\nu/2}| &= 2 \, \mathrm{Re}(\langle |\xi|^{k'} f \hat{w}_{->}, |\xi|^{k'} f \partial_t \hat{w}_{->} \rangle \\ &\qquad + \langle f \hat{w}_{-<}, f \partial_t \hat{w}_{-<} \rangle). \end{aligned} \tag{3.63}$$

We substitute $\partial_t w_-$ from (3.60) into (3.63) and we estimate the various terms successively. The term Δw_- does not contribute. We consider only the contribution of the high $|\xi|$ region. The low $|\xi|$ region is treated in a similar and simpler way.

Term with $s_+ \cdot \nabla w_-$. We apply (3.22) with k replaced by k', thereby obtaining a contribution

$$|w_-|^2_{k'+\nu/2} \, |\varphi_+|_\ell$$

under conditions which follow from (3.48) and from $\nu/2 \leqslant k' \leqslant k$.

Term with $s_- \cdot \nabla w_+$. We apply (3.23) with $m = k' - \nu/2$, ℓ replaced by ℓ' and k replaced by $k+\nu/2$, thereby obtaining a contribution

$$|w_-|_{k'+\nu/2} \, |w_+|_{k+\nu/2} \, |\varphi_-|_{\ell'}$$

under the conditions

$$k+\ell' > k' + n/2 - \nu, \qquad k+\nu \geqslant k'+1, \qquad \ell+1 \geqslant k' - \nu/2,$$

which follow from (3.48) (3.57).

Term with $(\nabla \cdot s_+)\, w_-$. We apply (3.24) with $m = k' - \nu/2$, k replaced by k' and ℓ replaced by $\ell + \nu/2$, thereby obtaining a contribution

$$|w_-|_{k'+\nu/2} \, |\varphi_+|_{\ell+\nu/2} \, |w_-|_{k'}$$

under the conditions

$$\ell > n/2 - \nu, \qquad \ell \geqslant k' - \nu,$$

which follow from (3.48) and from $k' \leqslant k$.

Term with $(\nabla \cdot s_-)\, w_+$. We apply (3.24) with $m = k' - \nu/2$, and ℓ replaced by $\ell' + \nu/2$, thereby obtaining a contribution

$$|w_-|_{k'+\nu/2} \, |\varphi_-|_{\ell'+\nu/2} \, |w_+|_k$$

under the conditions

$$k+\ell' > k' + n/2 - \nu, \qquad \ell' \geqslant k' - \nu, \qquad k \geqslant k' - \nu/2,$$

which follow from (3.48) (3.57).

Collecting the previous four estimates together with the contribution of the low $|\xi|$ region, yields (3.58).

We now turn to the estimate of φ_-. From (3.12) we obtain

$$\partial_t \, |\varphi_-|^2_{\ell'} - 2\rho' \, |\varphi_-|^2_{\ell'+\nu/2} = 2\langle |\xi|^{\ell'+1} f \hat{s}_{->}, |\xi|^{\ell'+1} f \partial_t \hat{s}_{->} \rangle$$
$$+ 2\langle |\xi|^{\ell_<} f \hat{\varphi}_{-<}, |\xi|^{\ell_<} f \hat{\varphi}_{-<} \rangle. \qquad (3.64)$$

We substitute $\partial_t s_-$ and $\partial_t \varphi_-$ from (3.62) and (3.61) into (3.64) and we estimate the various terms successively.

Term with $s_+ \cdot \nabla s_-$. We apply (3.25) with $s' = s_-$ and obtain a contribution

$$|\varphi_-|^2_{\ell'+\nu/2}\, |\varphi_+|_\ell$$

under the conditions

$$\ell > n/2 - \nu, \qquad \ell' + 1 \geqslant \nu/2, \qquad \ell \geqslant \ell' - \nu/2,$$

which follow from (3.48), from $\ell' + 1 \geqslant k' + 1 - \nu \geqslant 1 - \nu/2 \geqslant \nu/2$ and from $\ell \geqslant \ell'$.

Term with $s_- \cdot \nabla s_+$. We apply (3.26) with $m = \ell' + 1 - \nu/2$, ℓ replaced by ℓ' and ℓ' replaced by $\ell + \nu/2$, thereby obtaining a contribution

$$|\varphi_-|_{\ell'+\nu/2}\, |\varphi_-|_{\ell'}\, |\varphi_+|_{\ell+\nu/2}$$

under the conditions

$$\ell > n/2 - \nu, \qquad \ell \geqslant \ell' + 1 - \nu,$$

which follow from (3.48) (3.57).

Term with $\nabla g_0(w_+, w_-)_>$. We apply (3.28) with $m = \ell' + 1 - \nu/2$, $w_1 = w_+$, $w_2 = w_-$, $k_1 = k + \nu/2$, $k_2 = k'$, $k_1' = k$, $k_2' = k' + \nu/2$, thereby obtaining a contribution

$$|\varphi_-|_{\ell'+\nu/2}\, \{|w_+|_{k+\nu/2}\, |w_-|_{k'} + |w_-|_{k'+\nu/2}\, |w_+|_k\}$$

under the conditions

$$k + k' + \nu > \ell' + \mu - n + 2 + n/2, \qquad k' + \nu \geqslant \ell' + \mu - n + 2$$

which follow from (3.48) (3.57).

The terms with $(s_+, s_-)_<$ and $g_0(w_+ w_-)_<$ are treated by the use of (3.27) and (3.29) as in the proof of Lemma 3.5.

Collecting the previous estimates yields (3.59). ∎

We conclude this section by introducing a number of estimating functions of time generalizing those introduced in Section II.3 and by deriving a number of estimates for them. Those functions will be defined in terms of the derivative h_0' of a given function h_0 on which we make the following assumptions

$$h_0 \in \mathscr{C}^1([1, \infty), \mathbb{R}^+), \qquad h_0' \geqslant 0, \qquad t^{-2} h_0(t) \in L^1([1, \infty)),$$
$$t^{-1} h_0'(t) \in L^1([1, \infty)). \tag{3.65}$$

From the relation

$$t_2^{-1}h_0(t_2)-t_1^{-1}h_0(t_1)=\int_{t_1}^{t_2} dt\ t^{-1}h_0'(t)-\int_{t_1}^{t_2} dt\ t^{-2}h_0(t) \tag{3.66}$$

it follows that the last condition on h_0 in (3.65) can be replaced by the condition that $t^{-1}h_0(t)$ tends to zero when $t\to\infty$. A typical example for h_0' is that considered in Section II.3, namely $h_0'(t)=t^{-\gamma}$.

The first and basic estimating function h is defined by

$$h(t)=\int_1^\infty dt_1 (t\vee t_1)^{-1}\, h_0'(t_1) \tag{3.67}$$

from which it follows that $h(t)$ is decreasing in t and tends to zero when $t\to\infty$, while $th(t)$ is increasing in t. We next define for any $m\geqslant 0$

$$N_m(t)=\int_1^t dt_1\ h_0'(t_1)\, h^m(t_1), \tag{3.68}$$

$$Q_m(t)=\int_1^\infty dt_1 (t\vee t_1)^{-1}\, h_0'(t_1)\, h^m(t_1), \tag{3.69}$$

where the integral in (3.69) is convergent since $t^{-1}h_0'(t)\in L^1([1,\infty))$ and since $h(t)$ is decreasing in t. It follows from (3.68) that $N_m(t)$ is increasing in t and from (3.69) that $Q_m(t)$ is decreasing in t and tends to zero when $t\to\infty$, while $tQ_m(t)$ is increasing in t, so that $Q_m(t)\geqslant t^{-1}Q_m(1)$. Moreover, for any nonnegative integers i and j

$$N_{i+j}(t)\leqslant h(1)^i\, N_j(t)\leqslant h(1)^{i+j}\, N_0(t), \tag{3.70}$$

$$Q_{i+j}(t)\leqslant h(1)^i\, Q_j(t)\leqslant h(1)^{i+j}\, h(t)\leqslant h(1)^{i+j+1}. \tag{3.71}$$

Clearly $N_0(t)=h_0(t)-h_0(1)$ and $Q_0=h$. It will be convenient to introduce the notation $Q_{-1}=1$.

Finally we set

$$P_m(t)=\int_1^\infty dt_1\ h(t\vee t_1)\, h_0'(t_1)\, h^m(t_1) \tag{3.72}$$

which is well defined provided

$$P_m(1)=\int_1^\infty dt_1\ h_0'(t_1)\, h^{m+1}(t_1)<\infty. \tag{3.73}$$

It follows from (3.72) that $P_m(t)$ is decreasing in t and tends to zero when $t \to \infty$ while $h(t)^{-1} P_m(t)$ is increasing in t, so that

$$P_m(1)\, h(t) \leqslant h(1)\, P_m(t) \leqslant h(1)\, P_m(1). \tag{3.74}$$

We now collect a number of identities and inequalities satisfied by the previous estimating functions.

LEMMA 3.8. *Let i, j and m be nonnegative integers, let $1 \leqslant a \leqslant b$ and $t \geqslant 1$. Then the following identities and inequalities hold:*

$$\int_t^\infty dt_1\, t_1^{-2} N_m(t_1) = Q_m(t_1) \tag{3.75}$$

$$\int_1^t dt_1\, t_1^{-2} N_0(t_1)\, N_m(t_1) = N_{m+1}(t) - h(t)\, N_m(t) \leqslant N_{m+1}(t) \tag{3.76}$$

$$\int_t^\infty dt_1\, t_1^{-2} N_0(t_1)\, N_m(t_1) = P_m(t) \qquad \textit{if (3.73) holds} \tag{3.77}$$

$$N_i(t)\, N_j(t) \leqslant N_0(t)\, N_{i+j}(t) \tag{3.78}$$

$$N_i(t)\, Q_j(t) \leqslant h(t)\, N_{i+j}(t) \leqslant N_{i+j+1}(t) \tag{3.79}$$

$$Q_i(t)\, Q_j(t) \leqslant h(t)\, Q_{i+j}(t) \leqslant 2Q_{i+j+1}(t) \tag{3.80}$$

$$\int_t^\infty dt_1\, h_0'(t_1)\, h(t_1)\, Q_{m-1}(t_1) \leqslant \int_t^\infty dt_1\, h_0'(t_1)\, Q_m(t_1) \tag{3.81}$$

$$\int_t^\infty dt_1\, h_0'(t_1)\, Q_m(t_1) \leqslant P_m(t_1) \qquad \textit{if (3.73) holds} \tag{3.82}$$

$$\int_1^t dt_1\, h_0'(t_1)\, h(t_1)\, Q_{m-1}(t_1) \leqslant N_{m+1}(t_1) \tag{3.83}$$

$$\int_1^t dt_1\, h_0'(t_1)\, Q_m(t_1) \leqslant N_{m+1}(t) \tag{3.84}$$

$$\int_a^b dt\, h_0'(t)\, Q_m(t) \leqslant Q_m(a)(h_0(b) - h_0(a)) \tag{3.85}$$

$$\int_a^b dt\, h_0'(t)\, h(t)\, Q_{m-1}(t) \leqslant 2Q_m(a)(h_0(b) - h_0(a)). \tag{3.86}$$

Proof. This lemma is a generalization of Lemma II.3.6 and most of the proofs are obtained by manipulations of integrals similar to those of the corresponding integrals in Lemma II.3.6, after the replacement of $t^{-\gamma}$ by $h_0'(t)$. This applies to (3.75) (3.76) (3.77) (3.81) (3.82) (3.83) (3.84) and

(3.80b). (The latter is the generalization of (II.3.49)). The estimates (3.78) and (3.80a) follow from the Hölder inequality. The estimate (3.79a) is the pointwise counterpart of (II.3.39) and is proved in the same way. The estimate (3.79b) follows from the decrease of h, (3.85) follows from the decrease of Q_m, and (3.86) follows from (3.80b) and from (3.85). ▮

4. CAUCHY PROBLEM AND PRELIMINARY ASYMPTOTICS FOR THE AUXILIARY SYSTEM

In this section, we study the existence of solutions in a neighborhood of infinity in time for the auxiliary system (2.11)–(2.12) and we derive some preliminary results on the asymptotic behaviour in time of the solutions thereby obtained. We first introduce some notation.

We choose once for all a strictly positive $\mathscr{C}^1$ function defined in $[1, \infty)$, which we call $|\rho'|$ for reasons that will soon become obvious. We assume that $|\rho'|$ satisfies the following properties

(i) $|\rho'| \in L^1([1, \infty))$,

(ii) the function $t^{-\gamma}|\rho'|^{-1}$ is nondecreasing (and therefore $|\rho'|$ is nonincreasing), and the function $t^{-2}|\rho'|^{-1}$ is nonincreasing and tends to zero at infinity.

Typical examples of suitable functions $|\rho'|$ are $t^{-1-\varepsilon}$ for ε sufficiently small, depending on γ, or $t^{-1}(\alpha+\ell n t)^{-\alpha}$ for $\alpha > 1$. It will be useful to keep those examples in mind in order to understand the time decay implied by the subsequent estimates.

Let now $I \subset [1, \infty)$ be an interval, possibly unbounded, and let $t_0 \in I$ (or $t_0 = \infty$ if I is unbounded). We define a function ρ in I by

$$\rho(t) = \rho(t_0) - \left| \int_{t_0}^{t} dt_1 \; |\rho'(t_1)| \right| \tag{4.1}$$

so that ρ is increasing (resp. decreasing) for $t \leqslant t_0$ (resp. $t \geqslant t_0$) and has $|\rho'|$ as the absolute value of its derivative, which justifies the notation. We take $\rho(t_0)$ sufficiently large so that ρ is nonnegative in I. All subsequent estimates will be independent of $\rho(t_0)$ (they will however depend on $|\rho'|$). The previous choice of ρ will be used in this section without further comment unless otherwise stated. We now define the global version of the fundamental spaces, corresponding to the local version (3.13). We define

$$\mathscr{X}_\rho^{k,\ell}(I) \equiv (\mathscr{C} \cap L^\infty)(I, K_\rho^k \oplus Y_\rho^\ell) \cap L^2_{|\rho'|}(I, K_\rho^{k+\nu/2} \oplus Y_\rho^{\ell+\nu/2}), \tag{4.2}$$

where $L^2_{|\rho'|}$ denotes weighted L^2 in time with weight $|\rho'|$. More precisely, especially as regards continuity, $(w, \varphi) \in \mathscr{X}^{k,\ell}_\rho(I)$ is understood to mean that

$$(F^{-1} f \hat{w}, F^{-1} f \hat{\varphi}) \in \mathscr{X}^{k,\ell}_0(I) \equiv (\mathscr{C} \cap L^\infty)(I, K^k_0 \oplus Y^\ell_0) \cap L^2_{|\rho'|}(I, K^{k+\nu/2}_0 \oplus Y^{\ell+\nu/2}_0). \tag{4.3}$$

Note that in (4.3) we keep the weight $|\rho'|$ in the L^2 part.

The norm of (w, φ) in $\mathscr{X}^{k,\ell}_\rho$ is made up of several pieces for which we introduce additional notation which will be most helpful in the derivation of the estimates. Let H be a continuous strictly positive function defined in I. We define

$$y(w; I, H, k) = \operatorname*{Sup}_{t \in I} H^{-1}(t) \, |w(t)|_k, \tag{4.4}$$

$$y_1(w; I, H, k) = \left\{ \int_I dt \, |\rho'(t)| \, H^{-2}(t) \, |w(t)|^2_{k+\nu/2} \right\}^{1/2}, \tag{4.5}$$

$$Y(w; I, H, k) = (y \vee y_1)(w; I, H, k), \tag{4.6}$$

$$z(\varphi; I, H, \ell) = \operatorname*{Sup}_{t \in I} H^{-1}(t) \, |\varphi(t)|_\ell, \tag{4.7}$$

$$z_1(w; I, H, \ell) = \left\{ \int_I dt \, |\rho'(t)| \, H^{-2}(t) \, |\varphi(t)|^2_{\ell+\nu/2} \right\}^{1/2}, \tag{4.8}$$

$$Z(\varphi; I, H, \ell) = (z \vee z_1)(\varphi; I, H, \ell). \tag{4.9}$$

We then take

$$\|(w, \varphi); \mathscr{X}^{k,\ell}_\rho(I)\| = Y(w; I, 1, k) + Z(\varphi; I, 1, \ell). \tag{4.10}$$

In order to study the Cauchy problem for the auxiliary system (2.11)–(2.12) we need additional a priori estimates of solutions of that system, which are a continuation of those of Lemmas 3.5, 3.6 and 3.7 in which we now take into account the dependence on time in the framework of the spaces $\mathscr{X}^{k,\ell}_\rho$ just defined.

LEMMA 4.1. *Let k, ℓ satisfy* (3.48). *Let $1 \leqslant T \leqslant t_0 < \infty$ and let $(w, \varphi) \in \mathscr{X}^{k,\ell}_{\rho, loc}([T, \infty))$ be a solution of the system* (2.11)–(2.12). *Let h_0 and h_1 be $\mathscr{C}^1$ positive functions defined in $[T, \infty)$, with h_0 nondecreasing, h_1 nonincreasing, $h_0 \geqslant t^{-\gamma} |\rho'|^{-1}$ and $h_1 \geqslant t^{-2} |\rho'|^{-1} h_0$. Let $a_0 = |w(t_0)|_k$ and $b_0 = h_0(t_0)^{-1} |\varphi(t_0)|_\ell$.*

(1) *There exist constants c and C such that if*

$$(b_0 + a_0^2) \, h_1(t_0) \leqslant c, \tag{4.11}$$

then $(w, h_0^{-1}\varphi) \in \mathscr{X}_\rho^{k,\ell}([t_0, \infty))$ *and* (w, φ) *satisfies the estimates*

$$\begin{aligned} Y(w; [t_0, \infty), 1, k) &\leqslant C a_0, \\ Z(\varphi; [t_0, \infty], h_0, \ell) &\leqslant C(b_0 + a_0^2). \end{aligned} \tag{4.12}$$

(2) *There exist constants c and C such that if*

$$(b_0 + a_0^2)\, T^{-2}\, |\rho'(T)|^{-1}\, h_0(t_0) \leqslant c, \tag{4.13}$$

then (w, φ) *satisfies the estimates*

$$\begin{aligned} Y(w; [T, t_0], 1, k) &\leqslant C a_0, \\ Z(\varphi; [T, t_0], 1, \ell) &\leqslant C(b_0 + a_0^2)\, h_0(t_0). \end{aligned} \tag{4.14}$$

Proof. The proof requires the same regularization procedure as that of (the integrated form of) the estimates of Lemma 3.5, but we omit it for brevity. We begin the proof by treating simultaneously the cases $t \geqslant t_0$ and $t \leqslant t_0$. Let H be a $\mathscr{C}^1$ positive function of time, increasing for $t \geqslant t_0$ and decreasing for $t \leqslant t_0$ and let $\tilde{\varphi} = H^{-1}\varphi$. From Lemma 3.5 we obtain

$$\partial_t\, |w|_k^2 \lesseqgtr 2\rho'\, |w|_{k+\nu/2}^2 \pm C t^{-2} \{ |w|_{k+\nu/2}^2\, |\varphi|_\ell + |w|_{k+\nu/2}\, |\varphi|_{\ell+\nu/2}\, |w|_k \} \tag{4.15}$$

$$\partial_t\, |\tilde{\varphi}|_\ell^2 \lesseqgtr 2\rho'\, |\tilde{\varphi}|_{\ell+\nu/2}^2 \pm C t^{-2}\, |\tilde{\varphi}|_{\ell+\nu/2}^2\, |\varphi|_\ell \pm C t^{-\gamma} H^{-1}\, |\tilde{\varphi}|_{\ell+\nu/2}\, |w|_{k+\nu/2}\, |w|_k \tag{4.16}$$

for $t \gtrless t_0$. In (4.16), we have dropped the term $-2H'H^{-1}\, |\tilde{\varphi}|_\ell^2$ coming from the derivative of H. Let now $y_{(1)} = y_{(1)}(w; I, 1, k)$ and $z_{(1)} = z_{(1)}(\varphi; I, H, \ell)$ where $I = [t_0, t]$ for $t \geqslant t_0$ and $I = [t, t_0]$ for $t \leqslant t_0$. Integrating (4.15) (4.16) over time and using the Schwarz inequality, we obtain

$$|w(t)|_k^2 + y_1^2 \leqslant y_0^2 + C \{ \operatorname*{Sup}_{t \in I}\, t^{-2}\, |\rho'|^{-1}\, H \} (y_1^2 z + y_1 z_1 y) \tag{4.17}$$

$$|\tilde{\varphi}(t)|_\ell^2 + z_1^2 \leqslant z_0^2 + C \{ \operatorname*{Sup}_{t \in I}\, (t^{-2}\, |\rho'|^{-1}\, H) \}\, z_1^2 z + C \{ \operatorname*{Sup}_{t \in I}\, t^{-\gamma}\, |\rho'|^{-1}\, H^{-1} \}\, z_1 y_1 y \tag{4.18}$$

where $y_0 = |w(t_0)|_k = a_0$ and $z_0 = |\tilde{\varphi}(t_0)|_\ell$, and since the RHS of (4.17) (4.18) is increasing in $|t - t_0|$,

$$\begin{aligned} y^2 \vee y_1^2 &\leqslant \text{RHS of (4.17)} \\ z^2 \vee z_1^2 &\leqslant \text{RHS of (4.18)} \end{aligned}$$

so that $Y = y \vee y_1$ and $Z = z \vee z_1$ satisfy

$$\begin{aligned} Y^2 &\leqslant y_0^2 + m_1 Y^2 Z \\ Z^2 &\leqslant z_0^2 + m_1 Z^3 + m_2 Z Y^2 \end{aligned}$$

with

$$m_1 = C \operatorname*{Sup}_{t \in I} t^{-2} |\rho'|^{-1} H, \qquad m_2 = C \operatorname*{Sup}_{t \in I} (t^{-\gamma} |\rho'|^{-1} H^{-1}).$$

If we can arrange that $m_1 Z \leqslant 1/2$, then we obtain

$$Y^2 \leqslant 2y_0^2$$
$$Z^2 \leqslant 2z_0^2 + 2m_2 Z Y^2 \leqslant 2z_0^2 + 4m_2 y_0^2 Z$$

so that

$$\begin{aligned} Y &\leqslant 2a_0 \\ Z &\leqslant 2z_0 + 4m_2 a_0^2 \end{aligned} \tag{4.19}$$

which will yield the final estimates, while the condition $m_1 Z \leqslant 1/2$ is implied by

$$4m_1(z_0 + 2m_2 a_0^2) \leqslant 1. \tag{4.20}$$

We now consider separately the cases $t \geqslant t_0$ and $t \leqslant t_0$.

For $t \geqslant t_0$, we choose $H = h_0$, so that $m_2 \leqslant 1$ and $m_1 \leqslant h_1(t_0)$. Furthermore $z_0 = b_0$. Then (4.19) reduces to (4.12) and (4.20) reduces to (4.11).

For $t \leqslant t_0$, we choose $H = 1$, so that $m_2 = t_0^{-\gamma} |\rho'(t_0)|^{-1} \leqslant h_0(t_0)$, $z_0 = b_0 h_0(t_0)$ and $m_1 = t^{-2} |\rho'(t)|^{-1}$. Then (4.19) reduces to (4.14) and (4.20) reduces to (4.13) with t replaced by T. ∎

Remark 4.1. The optimal time decay results in Lemma 4.1 are obtained by saturating the conditions on h_0 and h_1, namely by taking $h_0 = t^{-\gamma} |\rho'|^{-1}$, which we have already assumed to be nondecreasing, and $h_1 = t^{-2} |\rho'|^{-1} h_0 = t^{-2-\gamma} |\rho'|^{-2}$, in so far as that function is nonincreasing, a property that we could have (but have not) assumed. We have stated Lemma 4.1 with inequalities instead of the previous special choices, in order not to hide the flexibility allowed by the proof, and we shall proceed in the same way in all subsequent similar estimates. In the special case $|\rho'| = t^{-1-\varepsilon}$, the optimal decays obtained with the previous special choices are $h_0 = t^{1-\gamma+\varepsilon}$ and $h_1 = t^{-\gamma+2\varepsilon}$.

We now turn to the extension of the regularity result of Lemma 3.6.

LEMMA 4.2. *Let k, ℓ satisfy* (3.48) *and let $\bar{k}$, $\bar{\ell}$ satisfy $\bar{k} - k = \bar{\ell} - \ell \geqslant 0$. Let $1 \leqslant T \leqslant t_0 < \infty$ and let $(w, \varphi) \in \mathscr{X}^{\bar{k}, \bar{\ell}}_{\rho, loc}([T, \infty))$ be a solution of the system* (2.11)–(2.12). *Let h_0 and h_1 be as in Lemma* 4.1 *and assume that (w, φ) satisfies*

$$|w(t)|_k \leqslant a, \qquad |\varphi(t)|_\ell \leqslant b h_0(t) \qquad \textit{or} \qquad |\varphi(t)|_\ell \leqslant b h_0(t \vee t_0). \tag{4.21}$$

(1) *There exist constants c and C such that if*

$$(b+a^2)\, h_1(t_0) \leqslant c, \tag{4.22}$$

then $(w, h_0^{-1}\varphi) \in \mathscr{X}_\rho^{\bar{k},\bar{\ell}}([t_0, \infty))$ *and* (w, φ) *satisfies the estimates*

$$\begin{aligned} Y(w; [t_0, \infty), 1, \bar{k}) &\leqslant C\{|w(t_0)|_{\bar{k}} + ah_1(t_0)\, h_0(t_0)^{-1}\, |\varphi(t_0)|_{\bar{\ell}}\}, \\ Z(\varphi; [t_0, \infty), h_0, \bar{\ell}) &\leqslant C\{h_0(t_0)^{-1}\, |\varphi(t_0)|_{\bar{\ell}} + a\, |w(t_0)|_{\bar{k}}\}. \end{aligned} \tag{4.23}$$

(2) *In the case where* $|\varphi(t)|_\ell \leqslant bh_0(t \vee t_0)$, *there exist constants c and C such that if*

$$(b+a^2)\, T^{-2}\, |\rho'(T)|^{-1}\, h_0(t_0) \leqslant c, \tag{4.24}$$

then (w, φ) *satisfies the estimates*

$$\begin{aligned} Y(w; [t, t_0], 1, \bar{k}) &\leqslant C\{|w(t_0)|_{\bar{k}} + at^{-2}\, |\rho'|^{-1}\, |\varphi(t_0)|_{\bar{\ell}}\} \\ &\qquad \textit{for all} \quad t \in [T, t_0], \\ Z(\varphi; [T, t_0], 1, \bar{\ell}) &\leqslant C\{|\varphi(t_0)|_{\bar{\ell}} + ah_0(t_0)\, |w(t_0)|_{\bar{k}}\}. \end{aligned} \tag{4.25}$$

(3) *In the case where* $|\varphi(t)|_\ell \leqslant bh_0(t)$, *there exist constants c and C such that if*

$$(b+a^2)\, h_1(T) \leqslant c, \tag{4.26}$$

then (w, φ) *satisfies the estimates*

$$\begin{aligned} Y(w; [t, t_0], h_0^{-1}, \bar{k}) &\leqslant C\{h_0(t_0)\, |w(t_0)|_{\bar{k}} + ah_1(t)\, |\varphi(t_0)|_{\bar{\ell}}\} \\ &\qquad \textit{for all} \quad t \in [T, t_0], \\ Z(\varphi; [T, t_0], 1, \bar{\ell}) &\leqslant C\{|\varphi(t_0)|_{\bar{\ell}} + ah_0(t_0)\, |w(t_0)|_{\bar{k}}\}. \end{aligned} \tag{4.27}$$

Proof. The proof follows closely that of Lemma 4.1. Let h_2 and h_3 be $\mathscr{C}^1$ positive functions of time, increasing for $t \geqslant t_0$ and decreasing for $t \leqslant t_0$, and let $\tilde{w} = h_2^{-1} w$, $\tilde{\varphi} = h_3^{-1}\varphi$. From Lemma 3.6 we obtain for $t \gtrless t_0$

$$\begin{aligned} \partial_t\, |\tilde{w}|_{\bar{k}}^2 &\lesseqgtr 2\rho'\, |\tilde{w}|_{\bar{k}+\nu/2}^2 \\ &\quad \pm Ct^{-2}\{|\tilde{w}|_{\bar{k}+\nu/2}^2\, |\varphi|_\ell + h_3 h_2^{-1}\, |\tilde{w}|_{\bar{k}+\nu/2}\, |\tilde{\varphi}|_{\bar{\ell}+\nu/2}\, |w|_k\}, \end{aligned} \tag{4.28}$$

$$\begin{aligned} \partial_t\, |\tilde{\varphi}|_{\bar{\ell}}^2 &\lesseqgtr 2\rho'\, |\tilde{\varphi}|_{\bar{\ell}+\nu/2}^2 \\ &\quad \pm Ct^{-2}\, |\tilde{\varphi}|_{\bar{\ell}+\nu/2}^2\, |\varphi|_\ell \pm Ct^{-\gamma} h_3^{-1} h_2\, |\tilde{\varphi}|_{\bar{\ell}+\nu/2}\, |\tilde{w}|_{\bar{k}+\nu/2}\, |w|_k \end{aligned} \tag{4.29}$$

where we have omitted the terms containing h_2' and h_3'. We define $y_{(1)} = y_{(1)}(w; I, h_2, \bar{k})$ and $z_{(1)} = z_{(1)}(\varphi; I, h_3, \bar{\ell})$ where $I = [t_0, t]$ for $t \geqslant t_0$

and $I = [t, t_0]$ for $t \leqslant t_0$. Proceeding as in the proof of Lemma 4.1, we obtain from (4.28) (4.29)

$$\begin{aligned} y^2 \vee y_1^2 &\leqslant y_0^2 + m_0 y_1^2 + m_1 a y_1 z_1 \\ z^2 \vee z_1^2 &\leqslant z_0^2 + m_0 z_1^2 + m_2 a y_1 z_1, \end{aligned} \tag{4.30}$$

where $y_0 = |\tilde{w}(t_0)|_{\bar{k}}$, $z_0 = |\tilde{\varphi}(t_0)|_{\bar{\ell}}$,

$$m_0 = C \operatorname*{Sup}_{t \in I} t^{-2} |\rho'|^{-1} |\varphi(t)|_{\ell},$$

$$m_1 = C \operatorname*{Sup}_{t \in I} t^{-2} |\rho'|^{-1} h_3 h_2^{-1}, \qquad m_2 = C \operatorname*{Sup}_{t \in I} t^{-\gamma} |\rho'|^{-1} h_2 h_3^{-1}. \tag{4.31}$$

If we can arrange that $m_0 \leqslant 1/2$, then $Y = y \vee y_1$ and $Z = z \vee z_1$ satisfy

$$\begin{aligned} Y^2 &\leqslant 2y_0^2 + 2m_1 a YZ \\ Z^2 &\leqslant 2z_0^2 + 2m_2 a YZ. \end{aligned} \tag{4.32}$$

By an elementary computation, one obtains from (4.32) the estimates

$$\begin{aligned} Y &\leqslant 4(y_0 + 2am_1 z_0) \\ Z &\leqslant 4(z_0 + 2am_2 y_0) \end{aligned} \tag{4.33}$$

under the condition

$$8a^2 m_1 m_2 \leqslant 1. \tag{4.34}$$

We now consider separately the various cases at hand.

For $t \geqslant t_0$, we take $h_2 = 1$ and $h_3 = h_0$, so that

$$y_0 = |w(t_0)|_{\bar{k}}, \qquad z_0 = h_0(t_0)^{-1} |\varphi(t_0)|_{\bar{\ell}},$$

$$m_0 \leqslant Cbh_1(t_0), \qquad m_1 \leqslant Ch_1(t_0), \qquad m_2 \leqslant C.$$

The estimate (4.33) then reduces to (4.23), while the conditions $m_0 \leqslant 1/2$ and (4.34) recombine to yield (4.22).

For $t \leqslant t_0$, in the case where $|\varphi(t)|_{\ell} \leqslant bh_0(t_0)$, we take $h_2 = h_3 = 1$, so that

$$y_0 = |w(t_0)|_{\bar{k}}, \qquad z_0 = |\varphi(t_0)|_{\bar{\ell}},$$

$$m_0 \leqslant Cbt^{-2}\rho'^{-1}h_0(t_0), \qquad m_1 \leqslant Ct^{-2}\rho'^{-1} \qquad \text{and} \qquad m_2 \leqslant Ch_0(t_0),$$

thereby obtaining (4.25) under the condition (4.24).

For $t \leqslant t_0$, in the case where $|\varphi(t)|_\ell \leqslant bh_0(t)$, we take $h_2 = h_0^{-1}$, $h_3 = 1$, so that

$$y_0 = h_0(t_0)\,|w(t_0)|_{\bar{k}}, \qquad z_0 = |\varphi(t_0)|_{\bar{\ell}},$$
$$m_0 \leqslant Cbh_1(t), \qquad m_1 \leqslant Ch_1(t) \qquad \text{and} \qquad m_2 \leqslant C,$$

thereby obtaining (4.27) under the condition (4.26). ∎

We now turn to the estimate of the difference of two solutions of the system (2.11)–(2.12).

LEMMA 4.3. *Let k, ℓ satisfy* (3.48) *and let k', ℓ' satisfy* (3.57). *Let $1 \leqslant T \leqslant t_0 < \infty$. Let h_0 and h_1 be as in Lemma* 4.1 *and let (w_i, φ_i), $i = 1, 2$, be two solutions of the system* (2.11)–(2.12) *such that $(w_i, h_0^{-1}\varphi_i) \in \mathscr{X}_\rho^{k,\ell}([T, \infty))$ and such that (w_i, φ_i) satisfy the estimates*

$$Y(w_i; [T, \infty), 1, k) \leqslant a, \tag{4.35}$$

$$Z(\varphi_i; [t_0, \infty), h_0, \ell) \leqslant b, \tag{4.36}$$

and either

$$Z(\varphi_i; [T, t_0), 1, \ell) \leqslant bh_0(t_0) \tag{4.37}$$

or

$$Z(\varphi_i; [T, t_0), h_0, \ell) \leqslant b. \tag{4.38}$$

Let $w_\pm = w_1 \pm w_2$ and $\varphi_\pm = \varphi_1 \pm \varphi_2$.

(1) *There exist constants c and C such that under the condition* (4.22), *(w_-, φ_-) satisfies the estimates*

$$\begin{aligned} Y(w_-; [t_0, \infty), 1, k') &\leqslant C\{|w_-(t_0)|_{k'} + ah_1(t_0)\,h_0(t_0)^{-1}\,|\varphi_-(t_0)|_{\ell'}\}, \\ Z(\varphi_-; [t_0, \infty), h_0, \ell') &\leqslant C\{h_0(t_0)^{-1}\,|\varphi_-(t_0)|_{\ell'} + a\,|w_-(t_0)|_{k'}\}. \end{aligned} \tag{4.39}$$

(2) *In the case where φ_i satisfy* (4.37), *there exist constants c and C such that under the condition* (4.24), *(w_-, φ_-) satisfies the estimates*

$$\begin{aligned} Y(w_-; [t, t_0,], 1, k') &\leqslant C\{|w_-(t_0)|_{k'} + at^{-2}\,|\rho'|^{-1}\,|\varphi_-(t_0)|_{\ell'}\} \\ &\qquad \textit{for all} \quad t \in [T, t_0], \\ Z(\varphi_-; [T, t_0], 1, \ell') &\leqslant C\{|\varphi_-(t_0)|_{\ell'} + ah_0(t_0)\,|w_-(t_0)|_{k'}\}. \end{aligned} \tag{4.40}$$

(3) *In the case where* φ_i *satisfy* (4.38), *there exist constants* c *and* C *such that under the condition* (4.26) (w_-, φ_-) *satisfies the estimates*

$$Y(w_-; [t, t_0,], h_0^{-1}, k') \leqslant C\{h_0(t_0)\, |w_-(t_0)|_{k'} + a h_1(t)\, |\varphi_-(t_0)|_{\ell'}\}$$
$$\textit{for all} \quad t \in [T, t_0], \tag{4.41}$$
$$Z(\varphi_-; [T, t_0], 1, \ell') \leqslant C\{|\varphi_-(t_0)|_{\ell'} + a h_0(t_0)\, |w_-(t_0)|_{k'}\}.$$

(The estimates (4.39) (4.40) *and* (4.41) *are obtained from* (4.23) (4.25) *and* (4.27) *by replacing* (w, φ) *by* (w_-, φ_-) *and* $(\bar{k}, \bar{\ell})$ *by* (k', ℓ').

Proof. The proof follows closely that of Lemma 4.2. Let h_2 and h_3 be $\mathscr{C}^1$ positive functions of time, increasing for $t \geqslant t_0$ and decreasing for $t \leqslant t_0$ and let $\tilde{w}_- = h_2^{-1} w_-$, $\tilde{\varphi}_- = h_3^{-1} \varphi_-$. Let $H(t) = h_0(t)$ in cases where (4.36) (4.38) are relevant, $H(t) = h_0(t_0)$ in the case where (4.37) is relevant. From Lemma 3.7 we obtain for $t \gtrless t_0$

$$\partial_t\, |\tilde{w}_-|_{k'}^2 \lesseqgtr 2\rho'\, |\tilde{w}_-|_{k'+\nu/2}^2 \pm C t^{-2}\{|\tilde{w}_-|_{k'+\nu/2}^2\, |\varphi_+|_{\ell} + |\tilde{w}_-|_{k'+\nu/2}\, |\tilde{w}_-|_{k'}\, |\varphi_+|_{\ell+\nu/2}\}$$
$$\pm C t^{-2} h_3 h_2^{-1}\, |\tilde{w}_-|_{k'+\nu/2}\, \{|\tilde{\varphi}_-|_{\ell'}\, |w_+|_{k+\nu/2} + |\tilde{\varphi}_-|_{\ell'+\nu/2}\, |w_+|_k\}, \tag{4.42}$$
$$\partial_t\, |\tilde{\varphi}_-|_{\ell'}^2 \lesseqgtr 2\rho'\, |\tilde{\varphi}_-|_{\ell'+\nu/2}^2 \pm C t^{-2}\{|\tilde{\varphi}_-|_{\ell'+\nu/2}^2\, |\varphi_+|_{\ell} + |\tilde{\varphi}_-|_{\ell'+\nu/2}\, |\tilde{\varphi}_-|_{\ell'}\, |\varphi_+|_{\ell+\nu/2}\}$$
$$+ C t^{-\gamma} h_2 h_3^{-1}\, |\tilde{\varphi}_-|_{\ell'+\nu/2}\, \{|\tilde{w}_-|_{k'}\, |w_+|_{k+\nu/2} + |\tilde{w}_-|_{k'+\nu/2}\, |w_+|_k\}, \tag{4.43}$$

where we have omitted the terms containing h_2' and h_3'. We define $y_{(1)} = y_{(1)}(w_-; I, h_2, k')$ and $z_{(1)} = z_{(1)}(\varphi_-; I, h_3, \ell')$ where $I = [t_0, t]$ for $t \geqslant t_0$ and $I = [t, t_0]$ for $t \leqslant t_0$. Proceeding as in the proof of Lemmas 4.1 and 4.2, we obtain from (4.42) (4.43) supplemented by (4.35)–(4.38) applied to (w_+, φ_+)

$$\begin{aligned} y^2 \vee y_1^2 &\leqslant y_0^2 + m_0 b y_1 (y + y_1) + m_1 a y_1 (z + z_1) \\ z^2 \vee z_1^2 &\leqslant z_0^2 + m_0 b z_1 (z + z_1) + m_2 a z_1 (y + y_1), \end{aligned} \tag{4.44}$$

where

$$y_0 = |\tilde{w}_-(t_0)|_{k'}, \qquad z_0 = |\tilde{\varphi}_-(t_0)|_{\ell'},$$
$$m_0 = C \operatorname*{Sup}_{t \in I} \, (t^{-2}\, |\rho'|^{-1}\, H(t))$$

and m_1, m_2 are defined by (4.31). From there on, the proof is identical with that of Lemma 4.2, with (4.30) replaced by (4.44). ▮

Remark 4.2. It is an unfortunate feature of Lemmas 4.2 and 4.3 that the derivation of regularity and of difference estimates requires a large time restriction (see (4.22) (4.24) (4.26)) whereas one would expect those estimates to hold for all times where the solution is a priori defined, since those estimates are linear in the higher or difference norm and are expected to follow from some kind of Gronwall's inequality. The reason for that fact is the occurrence of integral norms in the definition of the spaces, for which we obtain algebraic inequalities which require some kind of smallness condition in order to enable us to conclude.

In practice, the conditions (4.22) (4.24) (4.26) required for those estimates to hold have the same form and the same dependence on basic parameters such as a_0, b_0 as the conditions that will be needed anyway in order to derive the a priori estimates on one single solution that are needed to solve the Cauchy problem. We shall impose all such conditions together, without any significant limitation on the range of validity of the results (see the proof of Proposition 4.1 below).

We now turn to the Cauchy problem for large time for the system (2.11)–(2.12).

PROPOSITION 4.1. *Let (k, ℓ) satisfy (3.48) and in addition $k \geqslant 1 - \nu/2$. Let h_0 and h_1 be $\mathscr{C}^1$ positive functions defined in $[1, \infty)$ with h_0 nondecreasing, h_1 nonincreasing and tending to zero at infinity, $h_0 \geqslant t^{-\gamma} |\rho'|^{-1}$ and $h_1 \geqslant t^{-2} |\rho'|^{-1} h_0$. Let $a_0 > 0$, $b_0 > 0$.*

(1) *There exists $T_0 < \infty$, depending on a_0, b_0, such that for all $t_0 \geqslant T_0$, there exists $T \leqslant t_0$, depending on a_0, b_0 and t_0, such that for any $(w_0, \varphi_0) \in K^k_{\rho_0} \oplus Y^\ell_{\rho_0}$, where $\rho_0 = \rho(t_0)$, satisfying $|w_0|_k \leqslant a_0$, $|\varphi_0|_\ell \leqslant h_0(t_0)\, b_0$, the system (2.11)–(2.12) has a unique solution in the interval $[T, \infty)$ with $w(t_0) = w_0$, $\varphi(t_0) = \varphi_0$, such that $(w, h_0^{-1}\varphi) \in \mathscr{X}^{k,\ell}_\rho([T, \infty))$. One can define T_0 and T by*

$$(b_0 + a_0^2)\, h_1(T_0) = c \tag{4.45}$$

$$T^{-2} |\rho'(T)|^{-1} h_0(t_0)\, h_1(T_0)^{-1} = 1 \tag{4.46}$$

and the solution (w, φ) satisfies the estimates

$$Y(w; [T, \infty), 1, k) \leqslant C a_0, \tag{4.47}$$

$$Z(\varphi; [t_0, \infty), h_0, \ell) \leqslant C(b_0 + a_0^2), \tag{4.48}$$

$$Z(\varphi; [T, t_0], 1, \ell) \leqslant C(b_0 + a_0^2)\, h_0(t_0). \tag{4.49}$$

(2) *If $(w_0, \varphi_0) \in K^{\bar{k}}_{\rho_0} \oplus Y^{\bar{\ell}}_{\rho_0}$ for some $\bar{k}$, $\bar{\ell}$ with $\bar{k} - k = \bar{\ell} - \ell > 0$, then $(w, h_0^{-1}\varphi) \in \mathscr{X}^{\bar{k},\bar{\ell}}_\rho([T, \infty))$, possibly after changing the constant c in (4.45) (see Remark 4.2).*

(3) *The map $(w_0, \varphi_0) \to (w, \varphi)$ is norm continuous on the bounded sets of $K^k_{\rho_0} \oplus Y^\ell_{\rho_0}$ from the norm of (w_0, φ_0) in $K^{k'}_{\rho_0} \oplus Y^{\ell'}_{\rho_0}$ to the norm of $(w, h_0^{-1}\varphi)$ in $\mathscr{X}^{k'',\ell''}_\rho[T, \infty))$ for $k' \geqslant \nu/2$, $k - k' = \ell - \ell' \geqslant 1 - \nu$, $k - k'' = \ell - \ell'' > 0$. Furthermore the same map is continuous from the same topology on (w_0, φ_0) to the weak-$*$ topology of $(w, h_0^{-1}\varphi)$ in $\mathscr{X}^{k,\ell}_\rho([T, \infty))$.*

Proof. *Part* (1). The proof proceeds in several steps using a parabolic regularization and a limiting procedure. We consider first the case $t \geqslant t_0$. We shall then indicate briefly the modifications needed in the case $t \leqslant t_0$. We shall need the function $\tilde{w}$ defined by $\tilde{w}(t) = U(1/t)\, w(t)$. Since the operator $U(1/t)$ is unitary in K^k_ρ for all k and ρ, all subsequent norm estimates for $\tilde{w}$ in K^k_ρ are identical with the same estimates for w.

Step 1: Parabolic regularization and local resolution. We rewrite the system (2.11)–(2.12) in the equivalent form

$$\begin{aligned} \partial_t \tilde{w} &= (2t^2)^{-1}\, U(1/t)(2s\cdot\nabla + (\nabla\cdot s))\, U(1/t)^*\, \tilde{w} \equiv G_1(\tilde{w}, s), \\ \partial_t \varphi &= (2t^2)^{-1}\, |s|^2 + t^{-\gamma} g_0(U(1/t)^*\, \tilde{w}) \equiv G_2(\tilde{w}, s). \end{aligned} \tag{4.50}$$

We introduce a parabolic regularization and consider the regularized system

$$\begin{aligned} \partial_t \tilde{w} &= \theta \Delta \tilde{w} + G_1(\tilde{w}, s) \\ \partial_t \varphi &= \theta \Delta \varphi + G_2(\tilde{w}, s) \end{aligned} \tag{4.51}$$

with $\theta > 0$. We also regularize the initial data $(\tilde{w}_0, \varphi_0)$ at time t_0 to $(\bar{\tilde{w}}_0, \bar{\varphi}_0) \in X^{\bar{k}, \bar{\ell}}_{\rho_0} \equiv K^{\bar{k}}_{\rho_0} \oplus Y^{\bar{\ell}}_{\rho_0}$ with $\rho_0 = \rho(t_0)$, $\bar{k} \geqslant k \vee (1 + v/2)$, $\bar{\ell} - \ell = \bar{k} - k$. For the purpose of proving Part (1), one can take equality in the previous condition, namely $\bar{k} = k \vee (1 + v/2)$ and in particular that second regularization is unnecessary if $k \geqslant 1 + v/2$. We continue the argument with general $(\bar{k}, \bar{\ell})$ because it will be useful for the proof of Part (2).

We rewrite the Cauchy problem for the system (4.51) in the integral form

$$\begin{pmatrix} \tilde{w} \\ \varphi \end{pmatrix}(t) = V_\theta(t - t_0) \begin{pmatrix} \bar{\tilde{w}}_0 \\ \bar{\varphi}_0 \end{pmatrix} + \int_{t_0}^t dt' V_\theta(t - t') \begin{pmatrix} G_1(\tilde{w}, s) \\ G_2(\tilde{w}, s) \end{pmatrix}(t'), \tag{4.52}$$

where $V_\theta(t) \equiv \exp(\theta t \Delta)$ is a contraction in $X^{\bar{k}, \bar{\ell}}_{\rho_0}$ and satisfies the bound

$$\| \nabla V_\theta(t); \mathscr{L}(X^{\bar{k}, \bar{\ell}}_{\rho_0}) \| \leqslant C(\theta t)^{-1/2}. \tag{4.53}$$

By (3.23) (3.24) (3.26) (3.27) (3.28) (3.29) of Lemma 3.4, we estimate

$$\begin{aligned} |G_1(\tilde{w}, s)|_{\bar{k}-1} &\leqslant C t^{-2} (|\tilde{w}|_{\bar{k}}\, |\varphi|_{\bar{\ell}-1} + |\tilde{w}|_{\bar{k}-1}\, |\varphi|_{\bar{\ell}}) \\ |G_2(\tilde{w}, s)|_{\bar{\ell}-1} &\leqslant C t^{-2}\, |\varphi|_{\bar{\ell}}\, |\varphi|_{\bar{\ell}-1} + C t^{-\gamma}\, |\tilde{w}|_{\bar{k}}\, |\tilde{w}|_{\bar{k}-1} \end{aligned} \tag{4.54}$$

under the conditions (which follow from (3.48))

$$\bar{\ell} > n/2, \qquad \bar{\ell} \geqslant \bar{k} - 1 \geqslant 0, \qquad \bar{k} \geqslant \bar{\ell} + \lambda - 1, \qquad 2\bar{k} > \bar{\ell} + \lambda + n/2,$$

where $\lambda = \mu - n + 2$. In (4.54) the various norms are taken with constant $\rho(t) \equiv \rho_0$. By a standard contraction argument, the system (4.52) has a unique solution

$$(\tilde{w}, \varphi) \in \mathscr{C}([t_0, t_0 + \tau)], X_{\rho_0}^{\bar{k}, \bar{\ell}}), \tag{4.55}$$

where one can take

$$\tau = C\theta(t_0^{-2} h_0(t_0) + t_0^{-\gamma} h_0(t_0)^{-1})^{-2} (\bar{a}_0 + \bar{b}_0)^{-2} \tag{4.56}$$

with

$$\bar{a}_0 = |\bar{\tilde{w}}_0|_{\bar{k}}, \qquad \bar{b}_0 = h_0(t_0)^{-1} |\bar{\varphi}_0|_{\bar{\ell}}.$$

Furthermore, from the estimates

$$\begin{aligned}
\partial_t |\tilde{w}|_{\bar{k}}^2 + 2\theta |\nabla \tilde{w}|_{\bar{k}}^2 &= 2 \operatorname{Re}\langle |\xi|^{\bar{k}+1} f\hat{\tilde{w}}, |\xi|^{\bar{k}-1} f\hat{G}_1(\tilde{w}, \varphi)\rangle + \text{lower order terms} \\
&\leqslant |\tilde{w}|_{\bar{k}+1} A_1(t, |\tilde{w}|_{\bar{k}}, |\varphi|_{\bar{\ell}}), && (4.57) \\
\partial_t |\varphi|_{\bar{\ell}}^2 + 2\theta |\nabla \varphi|_{\bar{\ell}}^2 &= 2\langle |\xi|^{\bar{\ell}+3} f\hat{\varphi}, |\xi|^{\bar{\ell}+1} f\hat{G}_2(\tilde{w}, \varphi)\rangle + \text{lower order terms} \\
&\leqslant |\varphi|_{\bar{\ell}+1} A_2(t, |\tilde{w}|_{\bar{k}}, |\varphi|_{\bar{\ell}}) && (4.58)
\end{aligned}$$

for some estimating functions A_1 and A_2, it follows that

$$(\tilde{w}, \varphi) \in L^2([t_0, t_0 + \tau], X_{\rho_0}^{\bar{k}+1, \bar{\ell}+1}). \tag{4.59}$$

The estimates (4.57) (4.58) are derived with the help of a regularization j_ε, which we have omitted for brevity, and of the estimate (4.54).

Step 2: Uniform estimates and globalisation. From the regularity conditions (4.55) (4.59), from the fact that ρ is decreasing and from Lemmas 4.1 and 4.2, especially (4.12) and (4.23), it follows that $(\tilde{w}, \varphi) \in \mathscr{X}_\rho^{\bar{k}, \bar{\ell}}([t_0, t_0 + \tau])$ and that $(\tilde{w}, \varphi)$ satisfies the estimates

$$\begin{aligned}
&Y(\tilde{w}; I, 1, k) \leqslant Ca_0 \\
&Z(\varphi; I, h_0, \ell) \leqslant C(b_0 + a_0^2),
\end{aligned} \tag{4.60}$$

$$\begin{aligned}
&Y(\tilde{w}; I, 1, \bar{k}) \leqslant C(\bar{a}_0 + h_1(t_0)\, a_0 \bar{b}_0) \\
&Z(\varphi; I, h_0, \bar{\ell}) \leqslant C(\bar{b}_0 + a_0 \bar{a}_0),
\end{aligned} \tag{4.61}$$

for $I = [t_0, t_0 + \tau]$ and $t_0 \geqslant T_0$, under the condition (4.45) which ensures (4.11) (4.22). The estimates (4.60) (4.61) are uniform in θ. From (4.56), from (4.60) (4.61) for general I and from the fact that ρ is decreasing, it follows by a minor modification of a standard globalisation argument that $(\tilde{w}, \varphi)$ can be continued to a solution of the system (4.51) such that

$(\tilde{w}, h_0^{-1}\varphi)$ belongs to $\mathscr{X}_\rho^{\bar{k},\bar{\ell}}([t_0, \infty))$ and that $(\tilde{w}, \varphi)$ satisfies (4.60) (4.61) with $I = [t_0, \infty)$.

Step 3: Limiting procedure. We now take the limits $\theta \to 0$ and $(\bar{\tilde{w}}_0, \bar{\varphi}_0) \to (\tilde{w}_0, \varphi_0)$ in that order. We first keep $(\bar{\tilde{w}}_0, \bar{\varphi}_0)$ fixed and consider two solutions $(\tilde{w}_1, \varphi_1)$ and $(\tilde{w}_2, \varphi_2)$ with $(\tilde{w}_i, h_0^{-1}\varphi_i) \in \mathscr{X}_{\rho_0}^{\bar{k},\bar{\ell}}([t_0, \infty))$ as obtained in Step 2, corresponding to two values θ_1 and θ_2. We estimate the difference $(\tilde{w}_-, \varphi_-) = (\tilde{w}_1 - \tilde{w}_2, \varphi_1 - \varphi_2)$ by a minor variation of Lemma 4.3 with (k, ℓ) replaced by $(\bar{k}, \bar{\ell})$ and $(k', \ell') = (\bar{k}-1, \bar{\ell}-1)$, under the condition (4.22) which follows from $t_0 \geqslant T_0$ and from (4.45) (4.60), possibly after changing the constant c. More precisely in the proof of (4.39), we take the initial condition $\tilde{w}_-(t_0) = 0$, $\varphi_-(t_0) = 0$, but we have an additional term coming from the parabolic regularization in the analogue of (4.42) (4.43), namely

$$\begin{aligned} \partial_t \, |\tilde{w}_-|_{\bar{k}-1}^2 &\leqslant 2\, |\tilde{w}_-|_{\bar{k}} \, \{\theta_1 \, |\tilde{w}_1|_{\bar{k}} + \theta_2 \, |\tilde{w}_2|_{\bar{k}}) + \text{previous terms}, \\ \partial_t \, |\varphi_-|_{\bar{\ell}-1}^2 &\leqslant 2 |\varphi_-|_{\bar{\ell}} \, (\theta_1 \, |\varphi_1|_{\bar{\ell}} + \theta_2 \, |\varphi_2|_{\bar{\ell}}) + \text{previous terms}. \end{aligned} \tag{4.62}$$

(The tildas in (4.62) and in (4.42) (4.43) have different meanings, but this has no implication on the argument). From (4.61) and (4.62), for any t_1 with $t_0 < t_1 < \infty$, we obtain estimates of the type (4.44) for the quantities $y_{(1)} = y_{(1)}(w_-; [t_0, t_1], 1, \bar{k}-1)$ and $z_{(1)} = z_{(1)}(\varphi_-; [t_0, t_1], h_0, \bar{\ell}-1)$ where now

$$y_0^2 \vee z_0^2 \leqslant |t_1 - t_0| \, (\theta_1 + \theta_2) \, A(t_0, \bar{a}_0, \bar{b}_0). \tag{4.63}$$

Therefore, by the same argument as in the proof of (4.39)

$$\begin{aligned} Y(\tilde{w}_-; [t_0, t_1], 1, \bar{k}-1) &\leqslant C(y_0 + a_0 h_1(t_0)\, z_0) \\ Z(\varphi_-; [t_0, t_1], h_0, \bar{\ell}-1) &\leqslant C(z_0 + a_0 y_0), \end{aligned} \tag{4.64}$$

which implies that the solution $(\tilde{w}_\theta, \varphi_\theta)$ associated with θ converges in norm in $\mathscr{X}_\rho^{\bar{k}-1,\bar{\ell}-1}([t_0, t_1])$ when $\theta \to 0$ for all $t_1 \geqslant t_0$. Furthermore, the limit $(\tilde{w}, \varphi)$ is such that $(\tilde{w}, h_0^{-1}\varphi) \in \mathscr{X}_\rho^{\bar{k},\bar{\ell}}([t_0, \infty))$ and $(\tilde{w}, \varphi)$ satisfies (4.47) (4.48). This follows from the bound (4.61) with $I = [t_0, \infty)$, which is uniform in θ, and from the previous convergence by standard compactness arguments, except for the strong continuity in time. The latter follows from the weak continuity, which also follows from a compactness argument, and from the fact that the $K_\rho^{\bar{k}} \oplus Y_\rho^{\bar{\ell}}$ norm of $(\tilde{w}, \varphi)$ is (absolutely) continuous in t by Lemma 3.5 with k, ℓ replaced by $\bar{k}$, $\bar{\ell}$. The limit obviously satisfies the system (4.50).

We let now $(\bar{\tilde{w}}_0, \bar{\varphi}_0)$ tend to $(\tilde{w}_0, \varphi_0)$ in $K_{\rho_0}^k \oplus Y_{\rho_0}^\ell$ (that step is not needed if $\bar{k} = k \geqslant 1 + \nu/2$). Let $(\bar{\tilde{w}}_{0i}, \bar{\varphi}_{0i}) \in K_{\rho_0}^{\bar{k}} \oplus Y_{\rho_0}^{\bar{\ell}}$, $i = 1, 2$, be two sets of regularized initial conditions and let $(\tilde{w}_i, \varphi_i)$ be the solutions of the system (4.50) obtained previously. The difference $(\tilde{w}_-, \varphi_-) = (\tilde{w}_1 - \tilde{w}_2, \varphi_1 - \varphi_2)$ is then estimated by (4.39) with a replaced by a_0 as follows from (4.60), under the condition $t_0 \geqslant T_0$ and (4.45) as previously. One can (but need not) take $k' = k - 1 + \nu$, $\ell' = \ell - 1 + \nu$. This implies that the solution $(\bar{\tilde{w}}, \bar{\varphi})$ associated with $(\bar{\tilde{w}}_0, \bar{\varphi}_0)$ converges in the norm of $(\bar{\tilde{w}}, h_0^{-1}\bar{\varphi})$ in $\mathscr{X}_\rho^{k', \ell'}([t_0, \infty))$ when $(\bar{\tilde{w}}_0, \bar{\varphi}_0)$ converges to $(\tilde{w}_0, \varphi_0)$ in the norm of $K_{\rho_0}^{k'} \oplus Y_{\rho_0}^{\ell'}$ on the bounded sets of $K_{\rho_0}^k \oplus Y_{\rho_0}^\ell$ (and a fortiori in the norm of $K_{\rho_0}^k \oplus Y_{\rho_0}^\ell$). Let $(\tilde{w}, \varphi)$ be the limit. By the same arguments as above, $(\tilde{w}, h_0^{-1}\varphi) \in \mathscr{X}_\rho^{k, \ell}([t_0, \infty))$ and $(\tilde{w}, \varphi)$ satisfies the system (4.50) and the estimates (4.47) (4.48).

Step 4: Uniqueness follows immediately from Lemma 4.3, part (1).

We now turn to the case $t \leqslant t_0$. The proof proceeds exactly in the same way, with Parts 1 of Lemmas 4.1, 4.2 and 4.3 replaced by Parts 2 of the same Lemmas. In the same way as before, (4.14) implies that (4.24) follows from (4.13), possibly after a change of constant c. With T_0 defined by (4.45) and possibly with another change of constant c, the condition (4.13) follows from (4.46).

Part (2). If $\bar{k} \geqslant 1 + \nu/2$, the result follows from the proof of Part (1) with the second limiting procedure omitted. If $\bar{k} < 1 + \nu/2$, the result follows from the proof of Part (1) with $\bar{k}$ replaced by $1 + \nu/2$ and with the second limiting procedure going down from $1 + \nu/2$ to $\bar{k}$ instead of going down from $1 + \nu/2$ to k.

Note that in the previous proof, the constant c in (4.45) comes from a successive application of Lemma 4.1, especially from (4.11) (4.13), and of Lemmas 4.2 and 4.3, especially from (4.22) (4.24). In particular that constant depends on (k, ℓ) and on $(\bar{k}, \bar{\ell})$ (the pair (k', ℓ') in the applications of Lemma 4.3 is chosen as a function of (k, ℓ) or $(\bar{k}, \bar{\ell})$). In the proof of Part (1), one can in addition choose $\bar{k} = k \vee (1 + \nu/2)$, $\bar{k} - k = \bar{\ell} - \ell$, and the constant c therefore depends only on (k, ℓ). In contrast with that, in the proof of Part (2), the pairs (k, ℓ) and $(\bar{k}, \bar{\ell})$ are independent, and the constant c in (4.45) needed for Part (2) to hold may (and is expected to) depend on both (k, ℓ) and $(\bar{k}, \bar{\ell})$. The crucial information contained in Part (2) is the fact that (4.45) does not involve $|w_0|_{\bar{k}}$, $|\varphi_0|_{\bar{\ell}}$, but only $|w_0|_k$ and $|\varphi_0|_\ell$.

Part (3). Continuity with respect to initial data follows from Lemma 4.3 parts 1 and 2, and from the a priori estimates (4.47) (4.48) (4.49) by interpolation and compactness arguments. ∎

We conclude this section with two properties of the behaviour at infinity of solutions of the system (2.11)–(2.12) in spaces $\mathscr{X}_\rho^{k, \ell}$. There is no initial time involved in those properties and ρ is only required to satisfy a suitable monotony condition at infinity. The first property is the existence of a limit for $w(t)$ as $t \to \infty$. It applies in particular to the solutions constructed in Proposition 4.1. There is a large flexibility on the assumptions under which such a limit exist. We shall give another example in the next section in a different context.

PROPOSITION 4.2. *Let $T \geqslant 1$, $\rho_\infty \geqslant 0$ and*

$$\rho(t) = \rho_\infty + \int_t^\infty dt_1 \, |\rho'(t_1)|$$

for $t \geqslant T$. Let k, ℓ satisfy

$$\ell > n/2 - 1, \qquad 1 - v/2 \leqslant k \leqslant \ell + 1. \tag{4.65}$$

Let h_0 and h_1 be as in Proposition 4.1. *Let (w, φ) be a solution of the system* (2.11)–(2.12) *such that $(w, h_0^{-1}\varphi) \in \mathscr{X}_\rho^{k,\ell}([T, \infty))$. Let*

$$a = Y(w; [T, \infty), 1, k), \qquad b = Z(\varphi; [T, \infty), h_0, \ell).$$

Then there exists $w_+ \in K^k_{\rho_\infty}$ such that $w(t)$ tends to w_+ strongly in $K^{k'}_{\rho_\infty}$ for $k' < k$ and weakly in $K^k_{\rho_\infty}$ when $t \to \infty$. Furthermore, the following estimates hold:

$$|w_+|_k \equiv \| w_+; K^k_{\rho_\infty} \| \leqslant a, \tag{4.66}$$

$$Y(\tilde{w} - \tilde{w}(t_1); [t_1, \infty), 1, k') \leqslant C \, a \, b \, h_1(t_1), \tag{4.67}$$

$$\| \tilde{w}(t) - w_+; K^{k'}_{\rho_\infty} \| \leqslant C \, a \, b \, h_1(t), \tag{4.68}$$

for $k' = k - 1 + v$, for all t, $t_1 \geqslant T$, and with $\tilde{w}(t) = U(1/t) \, w(t)$.

Proof. We first prove (4.67). Let $t \geqslant t_1$ and $w_1(t) = \tilde{w}(t) - \tilde{w}(t_1)$ so that by (4.50)

$$\partial_t w_1 = (2t^2)^{-1} \, U(1/t)(2s \cdot \nabla + (\nabla \cdot s)) \, U(1/t)^* \, \tilde{w}.$$

Using (3.12) and (3.23) (3.24) from Lemma 3.4 with $k' = k - 1 + v$, $m = k' - v/2$, we obtain

$$\partial_t \, |w_1|_{k'}^2 - 2\rho' \, |w_1|_{k'+v/2}^2 \leqslant C t^{-2} \, |w_1|_{k'+v/2} \, \{ |w|_{k+v/2} \, |\varphi|_\ell + |\varphi|_{\ell+v/2} \, |w|_k \} \tag{4.69}$$

under the conditions $k' \geqslant v/2$, $\ell > n/2 - 1$, $\ell \geqslant k - 1$, which reduce to (4.65). Let $y_{(1)} = y_{(1)}(w_1; [t_1, t], 1, k')$. Integrating (4.69) over time in the same way as in Lemma 4.1, we obtain

$$y^2 \vee y_1^2 \leqslant C \, ab \, y_1 \{ \operatorname*{Sup}_{t' \in [t_1, t]} \, t'^{-2} \, |\rho'(t')|^{-1} \, h_0(t') \} \leqslant C \, ab \, y_1 h_1(t_1)$$

from which (4.67) follows. From (4.67) and from the fact that ρ is decreasing, it follows that $w(t)$ has a strong limit $w_+ \in K^{k'}_{\rho_\infty}$ and that (4.68) holds. By standard compactness and interpolation arguments, $w_+ \in K^{k}_{\rho_\infty}$, w_+ satisfies (4.66) and $w(t)$ converges to w_+ weakly in $K^{k}_{\rho_\infty}$ and strongly in $K^{k'}_{\rho_\infty}$ for $k' < k$. ▮

The second property is a uniqueness property for solutions with suitable restrictions on the behaviour at infinity. It will be used in Section 6. Since it corresponds to a situation with infinite initial time, it requires ρ to be increasing.

PROPOSITION 4.3. *Let $T \geqslant 1$, $\rho_\infty > 0$ and*

$$\rho(t) = \rho_\infty - \int_t^\infty dt_1 \, |\rho'(t_1)|$$

with $\rho(t) \geqslant 0$ for $t \geqslant T$. Let (k, ℓ) satisfy (3.48) *and $k \geqslant 1 - \nu/2$. Let h_0 and h_1 be as in Proposition* 4.1. *Let (w_i, φ_i), $i = 1, 2$ be two solutions of the system* (2.11)–(2.12) *such that $(w_i, h_0^{-1}\varphi_i) \in \mathscr{X}^{k,\ell}_\rho([T, \infty))$ and such that*

$$|w_1(t) - w_2(t)|_{k'} \, h_0(t) \to 0, \qquad |\varphi_1(t) - \varphi_2(t)|_{\ell'} \to 0$$

when $t \to \infty$, for some (k', ℓ') such that $\nu/2 \leqslant k' \leqslant k - 1 + \nu$, $\ell - \ell' = k - k'$. Let

$$a = \operatorname*{Max}_i \; Y(w_i; [T, \infty), 1, k), \qquad b = \operatorname*{Max}_i \; Z(\varphi_i; [T, \infty), h_0, \ell).$$

Then there exists a constant c such that if (4.26) *holds, then $(w_1, \varphi_1) = (w_2, \varphi_2)$.*

Proof. The result follows immediately from Lemma 4.3, part (3) by taking the limit $t_0 \to \infty$ in (4.41). ▮

5. ASYMPTOTICS OF SOLUTIONS OF THE AUXILIARY SYSTEM

In this section, we continue the study of the asymptotic properties of the solutions of the auxiliary system (2.11)–(2.12) obtained in Proposition 4.1. We have already proved the existence of a limit w_+ for $w(t)$ as $t \to \infty$ for such solutions. Under suitable additional regularity assumptions in the form of stronger lower bounds on (k, ℓ), we shall obtain estimates on the asymptotic behaviour in time of the asymptotic functions (w_m, φ_m) and (W_m, ϕ_m) defined by (2.14)–(2.15) and estimates on the remainders $(q_{p+1}, \psi_{p+1}) = (w - W_p, \varphi - \phi_p)$, also defined by (2.14)–(2.15), eventually leading to the existence of an asymptotic state for the phase φ in the form of a limit ψ_+ for ψ_{p+1} as $t \to \infty$ for sufficiently large t.

At a technical level however, the situation here differs significantly from that in Section 4. We are no longer trying to solve the system (2.11)–(2.12), but instead we assume a solution of that system to be given, and we estimate successively the asymptotic functions (w_m, φ_m) and the remainders (q_m, ψ_m), which are defined by triangular systems of equations. As a consequence, there is no need to control the loss of derivatives as in (2.11)–(2.12), and we can simply let that loss accumulate in the solution of the triangular system. Therefore we no longer need a time dependent ρ (see (3.12)), integral norms in $\mathscr{X}_\rho^{k,\ell}$, and integration by parts (see (3.22) (3.25)). We do not even need Gevrey spaces, and we could use instead ordinary Sobolev spaces, in the same way as in II. We shall of course nevertheless keep Gevrey spaces in order to make contact with Section 4, but in all this section we shall take ρ to be constant. Instead of the spaces $\mathscr{X}_\rho^{k,\ell}$ defined by (4.2), we shall use the simpler spaces

$$\mathscr{Y}_\rho^{k,\ell}(I) = (\mathscr{C} \cap L^\infty)(I, K_\rho^k \oplus Y_\rho^\ell). \tag{5.1}$$

The contact with Section 4, in particular the applicability of the results of this section to the solutions obtained in Proposition 4.1, will be achieved through the fact that if ρ is defined in $[t_0, \infty)$ by (4.1) or equivalently by

$$\rho(t) = \rho_\infty + \int_t^\infty dt_1\, |\rho'(t_1)|, \tag{5.2}$$

then

$$\mathscr{X}_\rho^{k,\ell}([t_0, \infty)) \hookrightarrow \mathscr{Y}_{\rho\infty}^{k,\ell}([t_0, \infty)). \tag{5.3}$$

In all the estimates of this section, the function f occurring in the definition (3.8) (3.9) of the spaces plays no role whatsoever, and is consistently eliminated from the proofs by using the submultiplicativity property (3.3). As a consequence all the estimates are uniform in (actually independent of) ρ and v, and no assumption is made connecting v to other parameters such as μ or (k, ℓ): we only assume $0 \leqslant v \leqslant 1$ and $\rho \geqslant 0$.

As a preliminary result, we give an existence result of the limit w_+ of $w(t)$ as $t \to \infty$ for a solution of the system (2.11)–(2.12). That result is a simplified version of Proposition 4.2 appropriate to the present context.

PROPOSITION 5.1. *Let (k, ℓ) satisfy*

$$k \geqslant 1, \qquad \ell \geqslant 0, \qquad k + \ell > n/2. \tag{5.4}$$

Let $1 \leqslant T < \infty$ and let h_0 be a $\mathscr{C}^1$ positive nondecreasing function defined in $[T, \infty)$ with $t^{-2}h_0 \in L^1([T, \infty))$. Let (w, φ) be a solution of the system

(2.11)–(2.12) *such that* $(w, h_0^{-1}\varphi) \in \mathscr{Y}_\rho^{k,\ell}([T,\infty))$. *Then there exists* $w_+ \in K_\rho^k$ *such that* $w(t)$ *tends to* w_+ *strongly in* $K_\rho^{k'}$ *for* $k' < k$ *and weakly in* K_ρ^k. *Furthermore, the following estimate holds:*

$$|\tilde{w}(t) - w_+|_{k'} \leqslant C\, a\, b\, h(t) \tag{5.5}$$

for $0 \leqslant k' \leqslant k-1$, $k' \leqslant \ell$, $k' < k+\ell-n/2$, *where*

$$a = \operatorname*{Sup}_t |w(t)|_k, \qquad b = \operatorname*{Sup}_t h_0^{-1}(t)\, |\varphi(t)|_\ell, \tag{5.6}$$

$h(t) = \int_t^\infty dt_1\, t_1^{-2} h_0(t_1)$ *and* $\tilde{w}(t) = U(1/t)\, w(t)$.

Proof. By (3.23) (3.24), we estimate

$$|\partial_t(\tilde{w}(t) - \tilde{w}(t_1))|_{k'} \leqslant Ct^{-2}\, |w(t)|_k\, |\varphi(t)|_\ell \leqslant Ct^{-2} h_0(t)\, a\, b$$

under the conditions stated on k', and therefore by integration

$$|\tilde{w}(t) - \tilde{w}(t_1)|_{k'} \leqslant C\, a\, b\, h(t \wedge t_1)$$

which implies the existence of $w_+ \in K_\rho^{k'}$ satisfying the estimate (5.5). By standard compactness arguments, $w_+ \in K_\rho^k$, $|w_+|_k \leqslant a$ and $w(t)$ tends weakly to w_+ in K_ρ^k when $t \to \infty$. The other convergences follow by interpolation. ∎

We now estimate the asymptotic functions (w_m, φ_m) defined by successive integrations from the system (2.18)–(2.19) with initial condition (2.20)–(2.21). For that purpose, we need the function

$$\bar{h}_0(t) = \int_1^t dt_1\, t_1^{-\gamma} \tag{5.7}$$

and the associated estimating functions defined by (3.68) (3.69) (3.72), which we denote by $\bar{N}_m$, $\bar{Q}_m$ and $\bar{P}_m$ for that special choice of h_0. (Those functions appeared already in II where they were called h_0, N_m, Q_m and P_m, see (II.3.19) (II.3.25) (II.3.26) (II.3.31)). We recall that $\lambda = \mu - n + 2$ and we define $\bar{\lambda} = \lambda \vee 1$. The following proposition is the extension to the present context of Proposition II.5.1.

PROPOSITION 5.2. *Let* $p \geqslant 0$ *be an integer, let* k *satisfy*

$$k > n/2, \qquad k \geqslant (p+2)\,\bar{\lambda} - 1,$$

and let

$$k_m = k - m\bar{\lambda}, \qquad \ell_m = k - m\bar{\lambda} - \lambda, \qquad 0 \leqslant m \leqslant p+1. \tag{5.8}$$

Let $w_+ \in K^k_\rho$ *and let* $a = |w_+|_k$. *Let* $\{w_0 = w_+, w_{m+1}\}$ *and* $\{\varphi_m\}$, $0 \leqslant m \leqslant p$ *be the solution of the system* (2.18)–(2.19) *with initial conditions* (2.20)–(2.21). *Then*

(1) $w_{m+1} \in \mathscr{C}([1,\infty), K^{k_{m+1}}_\rho)$, $\varphi_m \in \mathscr{C}([1,\infty), Y^{\ell_m}_\rho)$ *and the following estimates hold for all* $t \geqslant 1$:

$$|w_{m+1}(t)|_{k_{m+1}} \leqslant A(a)\, \bar{Q}_m(t), \tag{5.9}$$

$$|\varphi_m(t)|_{\ell_m} \leqslant A(a)\, \bar{N}_m(t), \tag{5.10}$$

for some estimating function $A(a)$.

If in addition $(p+2)\,\gamma > 1$ *and if we define* φ_{p+1} *by* (2.19) *with initial condition* $\varphi_{p+1}(\infty) = 0$, *then* $\varphi_{p+1} \in \mathscr{C}([1,\infty), Y^{\ell_{p+1}})$ *and the following estimate holds:*

$$|\varphi_{p+1}(t)|_{\ell_{p+1}} \leqslant A(a)\, \bar{P}_p(t). \tag{5.11}$$

(2) *The functions* $\{\varphi_m\}$ *are gauge invariant, namely if* $w'_+ = w_+ \exp(i\omega)$ *for some real valued function* ω *and if* w'_+ *gives rise to* $\{\varphi'_m\}$, *then* $\varphi'_m = \varphi_m$ *for* $0 \leqslant m \leqslant p+1$.

(3) *The map* $w_+ \to \{w_{m+1}, \varphi_m\}$ *is uniformly Lipschitz continuous on the bounded sets from the norm topology of* w_+ *in* K^k_ρ *to the norms* $\|\bar{Q}_m^{-1} w_{m+1}; L^\infty([1,\infty), K^{k_{m+1}}_\rho)\|$ *and* $\|\bar{N}_m^{-1} \varphi_m; L^\infty([1,\infty), Y^{\ell_m}_\rho)\|$, $0 \leqslant m \leqslant p$. *A similar continuity holds for* φ_{p+1}.

Proof. The proof is essentially the same as that of Proposition II.5.1.

Part (1). We proceed by induction on m, starting from the assumption on w_+. We assume the results to hold for (w_j, φ_j) for $j \leqslant m$ and we prove them for w_{m+1} and φ_{m+1}. We first consider w_{m+1}. Letting the exponents (k_m, ℓ_m) be undefined for the moment except for being nonincreasing in m, we obtain from (2.18) and from (3.23) (3.24)

$$|\partial_t w_{m+1}|_{k_{m+1}} \leqslant A(a)\, t^{-2} \left\{ \sum_{0 \leqslant j \leqslant m-1} \bar{N}_j(t)\, \bar{Q}_{m-j-1}(t) + \bar{N}_m(t) \right\} \tag{5.12}$$

under the conditions

$$\begin{aligned} &k_{m+1} \geqslant 0, \qquad k_{m+1} \leqslant (k_m - 1) \wedge \ell_m, \\ &k_{m+1} + n/2 < k_j + \ell_{m-j}, \qquad 0 \leqslant j \leqslant m. \end{aligned} \tag{5.13}$$

Integrating (5.12) between t and ∞ with $w_{m+1}(\infty) = 0$ and using (3.79) (3.75) yields the result for w_{m+1}.

We next consider φ_{m+1}. From (2.19) and from (3.26) (3.27) (3.28) (3.29) we obtain

$$|\partial_t \varphi_{m+1}|_{\ell_{m+1}} \leqslant A(a) \left\{ t^{-2} \sum_{0 \leqslant j \leqslant m} \bar{N}_j(t)\, \bar{N}_{m-j}(t) \right. \\ \left. + t^{-\gamma} \left(\sum_{0 \leqslant j \leqslant m-1} \bar{Q}_j(t)\, \bar{Q}_{m-1-j}(t) + \bar{Q}_m(t) \right) \right\} \tag{5.14}$$

under the conditions

$$\begin{aligned} &\ell_{m+1}+1 \geqslant 0, \qquad \ell_{m+1} \leqslant (\ell_m - 1) \wedge (k_{m+1} - \lambda), \\ &\ell_{m+1} + n/2 < \ell_j + \ell_{m-j}, \qquad \ell_{m+1} + \lambda + n/2 < k_j + k_{m+1-j}, \qquad 0 \leqslant j \leqslant m, \end{aligned} \tag{5.15}$$

and for φ_0

$$\ell_0 + \lambda \leqslant k, \qquad \ell_0 + \lambda + n/2 < 2k. \tag{5.16}$$

Integrating (5.14) between 1 and t with $\varphi_m(1) = 0$ and using (3.78) (3.76) and (3.80) (3.84) yields the result for φ_{m+1} (and similarly for φ_0).

We saturate the nonstrict part of (5.13) (5.15) (5.16) by the choice (5.8), where in addition we optimize (maximize) $\{\ell_m\}$ for given k. The strict conditions then reduce to $k > n/2$, while the condition $k \geqslant (p+2)\,\bar{\lambda} - 1$ is simply the condition $k_{p+1} \wedge (\ell_{p+1} + 1) \geqslant 0$.

Finally, if $(p+2)\,\gamma > 1$, we integrate (5.14) with $m = p$ between t and ∞ and use (3.77) (3.78) and (3.80) (3.82) with $m = p$.

Part (2). The proof is identical with that of Proposition II.5.1 and will be omitted.

Part (3). From the fact that the RHS of (2.18)–(2.19) are bilinear, it follows by induction of m that the difference between two solutions $\{w_m, \varphi_m\}$ and $\{w'_m, \varphi'_m\}$ associated with w_+ and w'_+ is estimated by

$$|w_{m+1} - w'_{m+1}|_{k_{m+1}} \leqslant A(a)\, |w_+ - w'_+|_k\, \bar{Q}_m(t), \tag{5.17}$$

$$|\varphi_m - \varphi'_m|_{\ell_m} \leqslant A(a)\, |w_+ - w'_+|_k\, \bar{N}_m(t) \tag{5.18}$$

for $0 \leqslant m \leqslant p$, and if $(p+2)\,\gamma > 1$,

$$|\varphi_{p+1} - \varphi'_{p+1}|_{\ell_{p+1}} \leqslant A(a)\, |w_+ - w'_+|_k\, \bar{P}_p(t), \tag{5.19}$$

where $a = |w_+|_k \vee |w'_+|_k$. The continuity stated in Part (3) follows from those estimates. ∎

We now turn to the main result of this section, namely to the proof of existence of asymptotic states (w_+, ψ_+) for solutions of the auxiliary system (2.11)–(2.12). That result relies heavily on suitable estimates of the remainders

$$q_{m+1}(t) = w(t) - W_m(t) \tag{5.20}$$

$$\psi_{m+1}(t) = \varphi(t) - \phi_m(t) \tag{5.21}$$

where W_m and ϕ_m are defined (see (2.14) (2.15)) by

$$W_m = \sum_{0 \leqslant j \leqslant m} w_j, \qquad \phi_m = \sum_{0 \leqslant j \leqslant m} \varphi_j. \tag{5.22}$$

In view of Proposition 4.1, we shall consider solutions (w, φ) of the system (2.11)–(2.12) such that $(w, h_0^{-1}\varphi) \in \mathscr{Y}_\rho^{k,\ell}([T, \infty))$, where h_0 is a suitable $\mathscr{C}^1$ positive increasing function of time. We shall assume in addition that $t^{-\gamma} \leqslant c h_0'$, a property which occurs naturally in the interesting examples relevant for Section 4. In addition to the estimating functions $\bar{N}_m$, $\bar{Q}_m$ and $\bar{P}_m$ associated with $\bar{h}_0$, we shall also need the estimating functions N_m, Q_m and P_m associated with h_0, defined by (3.68) (3.69) (3.72). From the relation $t^{-\gamma} \leqslant c h_0'$, it follows that $\bar{N}_m$, $\bar{Q}_m$ and $\bar{P}_m$ are estimated by N_m, Q_m, and P_m, and more precisely

$$\bar{N}_m \leqslant c^{m+1} N_m, \qquad \bar{Q}_m \leqslant c^{m+1} Q_m, \qquad \bar{P}_m \leqslant c^{m+2} P_m. \tag{5.23}$$

We can now state the main result of this section, which is the extension of Proposition II.6.1 to the present context.

PROPOSITION 5.3. *Let $p \geqslant 0$ be an integer. Let (k, ℓ, k_0) satisfy*

$$k > n/2 + (p-2)\,\bar{\lambda} + \lambda \vee 0, \qquad k \geqslant k_0 - \bar{\lambda} + 2, \qquad \ell \geqslant k_0 - \lambda, \tag{5.24}$$

$$k_0 > n/2, \qquad k_0 \geqslant (p+2)\,\bar{\lambda} - 1, \tag{5.25}$$

and let

$$k_m = k_0 - m\bar{\lambda}, \qquad \ell_m = k_0 - \lambda - m\bar{\lambda}, \qquad 0 \leqslant m \leqslant p+1. \tag{5.26}$$

Let h_0 be a $\mathscr{C}^1$ positive increasing function defined in $[1, \infty)$, such that $t^{-2}h_0$, $t^{-1}h_0' \in L^1([1, \infty))$ and $t^{-\gamma} \leqslant c h_0'$. Let $T \geqslant 2$, let (w, φ) be a solution of the system (2.11)–(2.12) such that $(w, h_0^{-1}\varphi) \in \mathscr{Y}_\rho^{k,\ell}([T, \infty))$ and define a, b by (5.6). Let $w_+ = \lim_{t \to \infty} w(t) \in K_\rho^k$ be defined by Proposition 5.1, so that in particular

$$|w(t) - w_+|_{k_1} \to 0 \qquad \text{when} \quad t \to \infty. \tag{5.27}$$

Let (w_{m+1}, φ_m), $0 \leqslant m \leqslant p$ *be defined by Proposition* 5.2 *and let* (W_m, ϕ_m), $0 \leqslant m \leqslant p$, *be defined by* (5.22). *Then the following estimates hold for all* $t \in [T, \infty)$:

$$|w(t) - W_m(t)|_{k_{m+1}} \leqslant A(a, b)\, Q_m(t), \tag{5.28}$$

$$|\varphi(t) - \phi_m(t)|_{\ell_{m+1}} \leqslant A(a, b)\, N_{m+1}(t) \tag{5.29}$$

for $0 \leqslant m \leqslant p$, *and for some estimating function* $A(a, b)$.

If in addition $(p+2)\,\gamma > 1$ *and if* $P_p(1) < \infty$, *then the following limit exists*

$$\lim_{t \to \infty} (\varphi(t) - \phi_p(t)) = \psi_+ \tag{5.30}$$

as a strong limit in $Y_\rho^{\ell_{p+1}}$, *and the following estimate holds*

$$|\varphi(t) - \phi_p(t) - \psi_+|_{\ell_{p+1}} \leqslant A(a, b)\, P_p(t). \tag{5.31}$$

Remark 5.1. When applied to solutions (w, φ) of the system (2.11)–(2.12) obtained in Proposition 4.1, the time decay estimates of Proposition 5.3 will take the following typical form. Take $|\rho'| = t^{-1-\varepsilon}$ and $h_0 = t^{1-\gamma+\varepsilon}$, which is adequate for Propostion 4.1. Then for $(p+1)\,\gamma < 1$ and sufficiently small ε

$$N_m(t) \sim t^{1-(m+1)(\gamma-\varepsilon)}, \qquad Q_m \sim t^{-(m+1)(\gamma-\varepsilon)}, \qquad P_p(t) \sim t^{1-(p+2)(\gamma-\varepsilon)}$$

and the condition $P_p(1) < \infty$ reduces to $(p+2)(\gamma-\varepsilon) > 1$. For $(p+1)(\gamma-\varepsilon) > 1$, the time decay saturates at $N_p(t) \sim 1$, $Q_p(t) \sim t^{-1}$ and $P_p(t) \sim t^{-\gamma}$, as explained in Section II.3.

Remark 5.2. We have kept the parameter k_0 in the statement of Proposition 5.3 because it plays a central role in the proof. For a given solution (w, φ), namely for given (k, ℓ), we can optimize the results by maximizing k_0 as allowed by (5.24), namely by taking $k_0 = (k + \bar{\lambda} - 2) \vee (\ell + \lambda)$. The condition (5.25) then reduces to lower bounds on (k, ℓ). Those bounds are stronger than (5.4) and therefore allow for the application of Proposition 5.1. Note also that the regularity obtained for the remainders (q_m, ψ_m) for $m \geqslant 1$ is weaker than that of the estimating functions of the same level (w_m, φ_m) since (k_m, ℓ_m) in (5.26) contain only $k_0 \leqslant k$ whereas (k_m, ℓ_m) in (5.8) contain k and $w_+ \in K_\rho^k$ in both cases.

Proof of Proposition 5.3. The proof is essentially the same as that of Proposition II.6.1 and proceeds by an induction on m, the starting point of which is the estimate for q_1. We assume the estimates (5.28) (5.29) to hold for (q_j, ψ_j), $0 \leqslant j \leqslant m$, and we derive them for (q_{m+1}, ψ_{m+1}), with (q_m, ψ_m) defined by (5.20) (5.21) and $(q_0, \psi_0) = (w, \varphi)$.

We substitute the decompositions $w = W_m + q_{m+1}$ and $\varphi = \phi_m + \psi_{m+1}$ in the LHS of (2.11)–(2.12) and the decompositions $w = W_{m-1} + q_m$ and $\varphi = \phi_{m-1} + \psi_m$ in the RHS of the same, thereby obtaining

$$\partial_t q_{m+1} = (2t^2)^{-1} \Big\{ i\Delta w + (2\nabla\varphi \cdot \nabla + (\Delta\varphi))\, q_m + (2\nabla\psi_m \cdot \nabla + (\Delta\psi_m))\, W_{m-1}$$
$$+ \sum_{\substack{0 \leqslant i, j \leqslant m-1 \\ i+j \geqslant m}} (2\nabla\varphi_i \cdot \nabla + (\Delta\varphi_i))\, w_j \Big\} \tag{5.32}$$

$$\partial_t \psi_{m+1} = (2t^2)^{-1} \Big\{ (\nabla\varphi + \nabla\phi_{m-1}) \cdot \nabla\psi_m + \sum_{\substack{0 \leqslant i, j \leqslant m-1 \\ i+j \geqslant m}} \nabla\varphi_i \cdot \nabla\varphi_j \Big\}$$
$$+ t^{-\gamma} \Big\{ g_0(q_m, q_1) + g_0(q_m, W_{m-1} - w_0) + 2g_0(q_{m+1}, w_0)$$
$$+ \sum_{\substack{0 \leqslant i, j \leqslant m-1 \\ i+j \geqslant m+1}} g_0(w_i, w_j) \Big\} \tag{5.33}$$

for $m \geqslant 1$ and

$$\begin{aligned} \partial_t q_1 &= (2t^2)^{-1} \{ i\Delta w + 2\nabla\varphi \cdot \nabla w + (\Delta\varphi)\, w \} \\ \partial_t \psi_1 &= (2t^2)^{-1} |\nabla\varphi|^2 + t^{-\gamma} g_0(q_1, w + w_+) \end{aligned} \tag{5.34}$$

for $m = 0$. We let the exponents (k_m, ℓ_m) be undefined for the moment, except for the property of being decreasing in m and of being not larger than the corresponding exponents of Proposition 5.2, which we denote momentarily by $(\bar{k}_m, \bar{\ell}_m)$ inside this proof, namely

$$k_m \leqslant \bar{k}_m = k - m\bar{\lambda}, \qquad \ell_m \leqslant \bar{\ell}_m = k - \lambda - m\bar{\lambda}.$$

We estimate (5.32) by (3.23) (3.24), by (5.9) (5.10) (5.23) and by the induction assumption, and we estimate similarly (5.33) by (3.26) (3.27) (3.28) (3.29) and the same other ingredients, thereby obtaining

$$|\partial_t q_{m+1}|_{k_{m+1}} \leqslant A(a, b)\, t^{-2} \Big\{ 1 + h_0 Q_{m-1} + N_m + \sum_{m \leqslant i+j \leqslant 2(m-1)} N_i Q_{j-1} \Big\} \tag{5.35}$$

$$|\partial_t \psi_{m+1}|_{\ell_{m+1}} \leqslant A(a, b) \Big\{ t^{-2} \Big(h_0 N_m + \sum_{m \leqslant i+j \leqslant 2(m-1)} N_i N_j \Big)$$
$$+ t^{-\gamma} \Big(Q_0 Q_{m-1} + Q_m + \sum_{m+1 \leqslant i+j \leqslant 2(m-1)} Q_{i-1} Q_{j-1} \Big) \Big\} \tag{5.36}$$

for $m \geqslant 1$, under the conditions

$$k_{m+1} \leqslant (k_m - 1) \wedge \ell_m, \quad \ell_{m+1} \leqslant (\ell_m - 1) \wedge (k_{m+1} - \lambda), \tag{5.37}$$

$$\begin{aligned} k_{m+1} + n/2 &< (\ell + k_m) \wedge (\ell_m + \bar{k}_{m-1}) \\ \ell_{m+1} + n/2 &< (\ell + \ell_m) \wedge (\ell_m + \bar{\ell}_{m-1}) \\ \ell_{m+1} + n/2 + \lambda &< (k_m + k_1) \wedge (k_m + \bar{k}_{m-1}) \wedge (k_{m+1} + k), \end{aligned} \tag{5.38}$$

and

$$|\partial_t q_1|_{k_1} \leqslant C t^{-2}(a + h_0\, ab) \tag{5.39}$$

$$|\partial_t \psi_1|_{\ell_1} \leqslant C t^{-2} h_0^2\, b^2 + C t^{-\gamma} A(a, b)\, Q_0 \tag{5.40}$$

under the conditions

$$k_1 \leqslant (k-2) \wedge \ell, \qquad \ell_1 \leqslant (\ell - 1) \wedge (k_1 - \lambda), \tag{5.41}$$

$$k_1 + n/2 < k + \ell, \qquad \ell_1 + n/2 < 2\ell \wedge (k_1 + k - \lambda) \tag{5.42}$$

for $m = 0$.

As in the proof of Proposition 5.2, we saturate (5.37) and maximize ℓ_m with respect to k_m by the choice (5.26). Then (5.41) reduces to the last two conditions of (5.24), while (5.38) is easily seen to reduce to $k_0 > n/2$ and to the first condition of (5.24), and (5.42) follows from (5.24) and (5.25). The conditions $k_m \leqslant \bar{k}_m$ and $\ell_m \leqslant \bar{\ell}_m$ follow from the fact that $k \geqslant k_0$, implied by (5.24).

Integrating (5.39) between t and ∞ with initial condition (5.27) yields

$$|q_1(t)|_{k_1} \leqslant C(a + h_0(1))\, t^{-1} + C\, ab\, Q_0(t), \tag{5.43}$$

namely the estimate (5.28) for $m = 0$ (since $Q_0(t) \geqslant t^{-1} Q_0(1)$) which is the starting point of the induction procedure. Integrating (5.35) between t and ∞ with the initial condition $q_{m+1}(\infty) = 0$ for $m \geqslant 1$ and using (3.79) (3.75) yields (5.28). Similarly, integrating (5.40) and (5.36) between T and t and using (3.78) (3.76) and (3.80) (3.84) yields (5.29). Finally, if $(p+2)\,\gamma > 1$ and $P_p(1) < \infty$, the RHS of (5.36) with $m = p$ is integrable at infinity in time, which proves the existence of the limit (5.30). Integrating (5.36) between t and ∞ and using (3.78) (3.77) and (3.80) (3.82) yields (5.31). ∎

6. CAUCHY PROBLEM AT INFINITY AND WAVE OPERATORS FOR THE AUXILIARY SYSTEM

In this section we derive the main technical result of this paper, which is in some sense the converse of Proposition 5.3, namely we prove that sufficiently regular asymptotic states (w_+, ψ_+) generate solutions (w, φ) of the system (2.11)–(2.12) in the sense described in Section 2, thereby allowing for the definition of the local wave operator at infinity Ω_0: $(w_+, \psi_+) \to (w, \varphi)$. As a preliminary, and in order to allow for an easy proof of the gauge invariance of the construction, we first solve the linear transport equations (2.23) (2.24) with initial condition (2.25) and derive some asymptotic properties of their solutions.

In all this section, as in Proposition 4.1 and in contrast with Section 5, we are again solving the system (2.11)–(2.12) and we have to solve the simpler equations (2.23) (2.24) in the same framework. As a consequence, we again need the full machinery of Gevrey spaces with time dependent ρ so as to be able to use (3.12), we need the integral norms in $\mathscr{X}_\rho^{k,\ell}$ and the integration by parts (3.22) and (3.25). We therefore begin by choosing $|\rho'|$ exactly as in Section 4. Now however in contrast with Proposition 4.1 where we kept t_0 finite, we want to take $t_0 = \infty$, and therefore we must take ρ to be increasing. Therefore, in all this section, we take

$$\rho(t) = \rho_\infty - \int_t^\infty dt_1\, |\rho'(t_1)|, \tag{6.1}$$

taking ρ_∞ sufficiently large for $\rho(t)$ to be nonnegative in the (asymptotic) region of interest, a sufficient condition for which being that $\rho_\infty \geqslant \| |\rho'|; L^1([1, \infty))\|$. Except for that condition, ρ_∞ will be arbitrary (but fixed), and all subsequent estimates will be independent of ρ_∞, for the same reasons as in the previous sections.

Solving the Cauchy problem either for the system (2.11)–(2.12) or for the transport equations (2.23) (2.24) with infinite initial time will be done as in II by first solving that system (or equation) with large but finite initial time t_0 and then letting t_0 tend to infinity. In order not to make this paper too cumbersome, we shall restrict our attention to solving that problem only for $t \leqslant t_0$ when t_0 is finite. The solutions could easily be extended to $[t_0, \infty)$ with a modified ρ of the type (4.1), but that extension would be useless in the limit $t_0 \to \infty$, and we shall refrain from performing it.

By analogy with the spaces $\mathscr{X}_\rho^{k,\ell}(I)$ defined by (4.2) (4.3), we extend the definition (5.1) of the spaces $\mathscr{Y}_\rho^{k,\ell}(I)$ from the case of constant ρ to the case of variable ρ considered in this section by

$$\mathscr{Y}_\rho^{k,\ell}(I) = \{(w, \varphi) \colon (F^{-1} f \hat{w}, F^{-1} f \hat{\varphi}) \in \mathscr{Y}_0^{k,\ell}(I)\} \tag{6.2}$$

with f defined by (3.1), ρ defined by (6.1) and $\mathscr{Y}_0^{k,\ell}(I)$ defined by (5.1). Occasionally, we shall have to state that a single function w or φ belongs to the w-subspace or to the φ-subspace of some space $\mathscr{X}_\rho^{k,\ell}$ or $\mathscr{Y}_\rho^{k,\ell}$. In order to avoid introducing additional notation, we shall then write $(w, 0) \in \mathscr{X}_\rho^{k,0}(I)$ or $(0, \varphi) \in \mathscr{X}_\rho^{0,\ell}(I)$, and similarly with $\mathscr{X}$ replaced by $\mathscr{Y}$.

We shall have to consider norms of the type $|w_-|_k$ or $|\varphi_-|_\ell$ for w_- or φ_- that are differences of functions taken at different times, possibly leaving in doubt the value of t appearing in $\rho(t)$ in the definition of the norm. In such cases it will be understood that the value of t appearing in $\rho(t)$ should be the smaller of the times appearing in w_- or φ_-, thereby yielding the smaller of the corresponding values of ρ.

We begin with the study of the transport equation (2.23). As compared with the treatment of that problem given in II, however, a new difficulty arises. In order to compare V with the asymptotic approximation W_p to the anticipated solution of (2.11)–(2.12), it is no longer sufficient to take for V the initial condition $V(t_0) = w_+$ when solving (2.23) with finite initial time t_0, and we have to use instead the better initial condition $V(t_0) = W_p(t_0)$. On the other hand, the results on the Cauchy problem for (2.23) (2.24) do not depend on detailed properties of ϕ_{p-1} and W_p. They are therefore stated in Propositions 6.1 and 6.2 in terms of general functions ϕ and W, to be taken as ϕ_{p-1} and W_p from Proposition 6.3 on.

In all this section, we use systematically the notation $y_{(1)}$, Y, $z_{(1)}$, Z defined by (4.4)–(4.9).

We begin with the study of the transport equation (2.23) which we rewrite with general ϕ as

$$\partial_t V = (2t^2)^{-1} (2\nabla\phi \cdot \nabla + (\Delta\phi))\, V. \tag{6.3}$$

Proposition 6.1. *Let $(\tilde{k}, \tilde{\ell}, \bar{k}, k)$ satisfy*

$$\tilde{\ell} > n/2 - \nu, \quad \tilde{k} \wedge (\tilde{\ell} + \nu/2) \geqslant \bar{k} \geqslant k + 1 - \nu, \qquad k \geqslant \nu/2. \tag{6.4}$$

Let $1 \leqslant T < \infty$. Let h_0 and h_1 be $\mathscr{C}^1$ positive functions defined in $[T, \infty)$ with h_0 nondecreasing, h_1 nonincreasing and tending to zero at infinity, and $h_1 \geqslant t^{-2}\rho'^{-1} h_0$. Let $w_+ \in K^{\tilde{k}}_{\rho\infty}$. Let (W, ϕ) be such that $(W, h_0^{-1}\phi) \in \mathscr{Y}^{\tilde{k}, \tilde{\ell}}_\rho([T, \infty))$ and that $W(t)$ tends to w_+ as $t \to \infty$, with an estimate

$$|W(t) - w_+|_k \leqslant c_1 h_1(t) \tag{6.5}$$

for some constant c_1. Let

$$a = \operatorname*{Sup}_t\, |W(t)|_{\tilde{k}}, \qquad b = \operatorname*{Sup}_t\, h_0^{-1}(t)\, |\phi(t)|_{\tilde{\ell}}. \tag{6.6}$$

Then

(1) *There exist constants c and C such that if*

$$b\, h_1(T) \leqslant c, \tag{6.7}$$

there exists a unique solution V of the equation (6.3) *such that* $(V, 0) \in \mathscr{X}_\rho^{\bar{k},0}([T, \infty))$ *and such that the following estimates hold:*

$$Y(V; [T, \infty), 1, \bar{k}) \leqslant C\, a, \tag{6.8}$$

$$Y(V - w_+; [T, \infty), h_1, k) \leqslant C\, a\, b. \tag{6.9}$$

(2) *V is the limit as $t_0 \to \infty$ of solutions V_{t_0} of* (6.3) *such that $V_{t_0}(t_0) = W(t_0)$ and $(V_{t_0}, 0) \in \mathscr{X}_\rho^{\bar{k},0}([T, t_0])$. The convergence is in the strong sense in $\mathscr{X}_\rho^{k',0}([T, T_1])$ for $k' < \bar{k}$ and in the weak-$*$ sense in $\mathscr{X}_\rho^{\bar{k},0}([T, T_1])$ for every T_1, $T < T_1 < \infty$ and the following estimate holds for all $t_0 > T$*

$$Y(V - V_{t_0}; [T, t_0], 1, k) \leqslant C(ab + c_1)\, h_1(t_0). \tag{6.10}$$

(3) *The solution V is unique in $L^\infty([T, \infty), L^2)$ under the condition that $\|V(t) - w_+\|_2$ tends to zero when $t \to \infty$.*

Remark 6.1. From the uniqueness statement of Proposition 6.1, Part (3), it follows that for given ϕ and w_+, V is independent of W. Actually Parts (1) and (3) make no reference to W and could be proved by taking $W(t) \equiv w_+$. W appears only in the limiting process of Part (2), which however will be esssential to derive the more accurate estimates of Proposition 6.3.

Proof. We prove Parts (1) and (2) together.

We first solve (6.3) with initial data $V_{t_0}(t_0) = W(t_0)$ at finite t_0. This is a linear transport equation with $\mathscr{C}^\infty$ vector field $\nabla\phi$ and $\mathscr{C}^\infty$ initial data and the existence and uniqueness of a solution, for instance with value in H^N, is a standard result. We concentrate on the Gevrey estimates and on the subsequent limit $t_0 \to \infty$.

In the same way as in the proof of Lemma 3.5, from (3.22) (3.24) we obtain for $t \leqslant t_0$

$$\partial_t\, |V_{t_0}|_{\bar{k}}^2 \geqslant 2\rho'\, |V_{t_0}|_{\bar{k}+\nu/2}^2 - Ct^{-2}\, |V_{t_0}|_{\bar{k}+\nu/2}^2\, |\phi|_{\tilde{\ell}} \tag{6.11}$$

under the conditions

$$\tilde{\ell} > n/2 - \nu, \qquad \bar{k} \geqslant \nu/2, \qquad \tilde{\ell} + 1 \geqslant \bar{k} - \nu/2$$

and

$$\bar{k} + \tilde{\ell} + \nu/2 > \bar{k} - \nu/2 + n/2, \qquad \tilde{\ell} \geqslant \bar{k} - \nu/2$$

which follow from (6.4). Integrating (6.11) over time and using (6.6), we obtain as in the proof of Lemma 4.1

$$y^2 \vee y_1^2 \leqslant y_0^2 + C\, b\, y_1^2\, h_1(t), \tag{6.12}$$

where $y_{(1)} = y_{(1)}(V_{t_0}; [t, t_0], 1, \bar{k})$ and $y_0 = |W(t_0)|_{\bar{k}}$, which under the condition (6.7) yields

$$Y(V_{t_0}; [T, t_0], 1, \bar{k}) \leqslant C\, a. \tag{6.13}$$

We next estimate the difference $v_1(t) \equiv V_{t_0}(t) - V_{t_0}(t_0) \equiv V_{t_0}(t) - W(t_0)$, which satisfies the equation

$$\partial_t v_1 = (2t^2)^{-1} (2\nabla\phi \cdot \nabla + (\Delta\phi))\, V_{t_0} \tag{6.14}$$

with initial condition $v_1(t_0) = 0$. Let $\tilde{v}_1 = h_1^{-1} v_1$. From (3.23) (3.24) with $m = k - \nu/2$, $k \to \bar{k} + \nu/2$ and $\ell = \tilde{\ell}$, we obtain

$$\partial_t\, |\tilde{v}_1|_k^2 \geqslant 2\rho' |\tilde{v}_1|_{k+\nu/2}^2 - Ct^{-2} h_1^{-1}\, |\tilde{v}_1|_{k+\nu/2}\, |V_{t_0}|_{\bar{k}+\nu/2}\, |\phi|_{\tilde{\ell}} \tag{6.15}$$

under the conditions

$$\bar{k} + \tilde{\ell} > k + n/2 - \nu, \qquad \tilde{\ell} \geqslant k - \nu/2 \geqslant 0, \qquad \bar{k} + \nu/2 \geqslant k + 1 - \nu/2,$$

which follow from (6.4).

Defining $y_{(1)} = y_{(1)}(v_1; [T, t_0], h_1, k)$, integrating over time and using (6.6) (6.13) and the Schwarz inequality we obtain in the same way as before

$$y^2 \vee y_1^2 \leqslant C\, a\, b\, y_1 \{\operatorname*{Sup}_t\, t^{-2} \rho'^{-1} h_0 h_1^{-1}\} = C\, a\, b\, y_1$$

and therefore

$$Y(V_{t_0} - W(t_0); [T, t_0], h_1, k) \leqslant C\, a\, b. \tag{6.16}$$

We now take the limit $t_0 \to \infty$, and for that purpose we estimate the difference $V_{t_1} - V_{t_0}$ of two solutions corresponding to t_0 and t_1, with $T < t_0 \leqslant t_1$. Since the equation (6.3) is linear in V, the difference of two solutions is estimated in the same way as a single solution. In the same way as in the proof of (6.13), we obtain

$$\begin{aligned} Y(V_{t_1} - V_{t_0}; [T, t_0], 1, k) &\leqslant C\, |V_{t_1}(t_0) - V_{t_0}(t_0)|_k \\ &\leqslant C\{|V_{t_1}(t_0) - W(t_1)|_k + |W(t_1) - W(t_0)|_k\}. \end{aligned} \tag{6.17}$$

We estimate the first norm in the last member of (6.17) by (6.16) with t_1 replacing t_0, and where we use the pointwise estimate taken at $t = t_0$, and we estimate the second norm by (6.5) used both for $t = t_0$ and $t = t_1$, thereby obtaining

$$Y(V_{t_1} - V_{t_0}; [T, t_0], 1, k) \leqslant C(ab + c_1)\, h_1(t_0). \tag{6.18}$$

From (6.18) it follows that when $t_0 \to \infty$, V_{t_0} converges to some V such that $(V, 0) \in \mathscr{X}_\rho^{k,0}([T, \infty))$ strongly in $\mathscr{X}_\rho^{k,0}([T, T_1])$ for all T_1, $T < T_1 < \infty$. By standard compactness arguments and by (6.13) $(V, 0) \in \mathscr{X}_\rho^{\bar{k},0}([T, \infty))$, V satisfies (6.8) and the convergence holds in the sense of Part (2) of the proposition. Furthermore, taking the limit $t_1 \to \infty$ in (6.18) yields (6.10).

It remains only to prove (6.9). For that purpose, we take again $T < t_0 < t_1$ and we estimate

$$\begin{aligned} Y(V - w_+; [T, t_0], h_1, k) \leqslant\; & Y(V_{t_1} - W(t_1); [T, t_0], h_1, k) \\ & + h_1(t_0)^{-1} \{Y(V - V_{t_1}; [T, t_0], 1, k) \\ & + Y(W(t_1) - w_+; [T, t_0], 1, k)\} \\ \leqslant\; & C\, a\, b + h_1(t_0)^{-1} \{C(ab + c_1)\, h_1(t_1) \\ & + C c_1 h_1(t_1)(\rho(t_1) - \rho(t_0))^{-1/2}\} \end{aligned}$$

by (6.16) and (6.10) with t_1 replacing t_0, and by (6.5) and the fact that $\rho(t)$ is strictly increasing, so that

$$\int_T^{t_0} dt\, \rho'\, |W(t_1) - w_+|^2_{\rho(t), k + \nu/2} \leqslant C\, |W(t_1) - w_+|^2_{\rho(t_1), k}\, |\rho(t_1) - \rho(t_0)|^{-1}\, \|\rho'\|_1$$

where $|\cdot|_{\rho, k}$ denotes the norm in K_ρ^k. Taking the limits $t_1 \to \infty$ and $t_0 \to \infty$ in that order yields (6.9), with the same constant as in (6.16).

Part (3) follows from an elementary energy estimate for the L^2 norm of the difference of two solutions, namely

$$\|V_1(t) - V_2(t)\|_2 \leqslant \|V_1(t') - V_2(t')\|_2 \exp(Cb\, |h(t) - h(t')|)$$

where $h(t) = \int_t^\infty dt_1\, t_1^{-2} h_0(t_1)$. ∎

We now turn to the transport equation (2.24), which we rewrite with general ϕ as

$$\partial_t \chi = t^{-2} \nabla\phi \cdot \nabla\chi. \tag{6.19}$$

PROPOSITION 6.2. *Let $(\tilde{\ell}, \bar{\ell}, \ell)$ satisfy*

$$\tilde{\ell} > n/2 - \nu, \qquad \tilde{\ell} + \nu/2 \geqslant \bar{\ell} + 1, \qquad \bar{\ell} \geqslant \ell + 1 - \nu, \qquad \ell + 1 \geqslant \nu/2. \tag{6.20}$$

Let $1 \leqslant T < \infty$ and let h_0, h_1 be as in Proposition 6.1. Let $\psi_+ \in Y_{\rho\infty}^{\bar{\ell}}$ and let $\beta = |\psi_+|_{\bar{\ell}}$. Let ϕ be such that $(0, h_0^{-1}\phi) \in Y_\rho^{0, \tilde{\ell}}([T, \infty))$ with

$$b = \operatorname*{Sup}_t\, h_0(t)^{-1}\, |\phi(t)|_{\tilde{\ell}}. \tag{6.21}$$

Then

(1) *There exist constants c and C such that if* (6.7) *holds, there exists a unique solution* χ *of the equation* (6.19) *such that* $(0, \chi) \in \mathscr{X}_{\rho}^{0, \bar{\ell}}([T, \infty))$ *and such that the following estimates hold*

$$Z(\chi; [T, \infty), 1, \bar{\ell}) \leqslant C\, \beta, \tag{6.22}$$

$$Z(\chi - \psi_+; [T, \infty), h_1, \ell) \leqslant C\, b\, \beta. \tag{6.23}$$

(2) χ *is the limit as* $t_0 \to \infty$ *of solutions* χ_{t_0} *of* (6.19) *such that* $\chi_{t_0}(t_0) = \psi_+$ *and* $(0, \chi_{t_0}) \in \mathscr{X}_{\rho}^{0, \bar{\ell}}([T, t_0))$. *The convergence is in the strong sense in* $\mathscr{X}_{\rho}^{0, \ell'}([T, T_1])$ *for* $\ell' \leqslant \bar{\ell}$ *and in the weak-$*$ sense in* $\mathscr{X}_{\rho}^{0, \bar{\ell}}([T, T_1])$ *for every* T_1, $T < T_1 < \infty$, *and the following estimate holds*

$$Z(\chi - \chi_{t_0}; [T, t_0], 1, \ell) \leqslant C\, b\, \beta\, h_1(t_0). \tag{6.24}$$

(3) *The solution* χ *is unique in* $L^\infty(I, L^\infty)$ *under the condition that* $\|\chi(t) - \psi_+\|_\infty$ *tends to zero as* $t \to \infty$.

(4) *Let in addition* $(\tilde{k}, \bar{k}, k)$ *satisfy* (6.4), *let* $w_+ \in K_{\rho_\infty}^{\tilde{k}}$ *and let V be defined by Proposition* 6.1 *for some W (for instance* $W(t) \equiv w_+$*). Then for fixed* ϕ, $V \exp(-i\chi)$ *is gauge invariant in the following sense. If* (V, χ) *and* (V', χ') *are the solutions obtained from* (w_+, ψ_+) *and* (w'_+, ψ'_+) *with* $w_+ \exp(-i\psi_+) = w'_+ \exp(-i\psi'_+)$, *then* $V(t) \exp(-i\chi(t)) = V'(t) \exp(-i\chi'(t))$ *for all* $t \in I$.

Proof. Parts (1) *and* (2). The proof is very similar to that of Proposition 6.1 and we concentrate again on the Gevrey estimates and on the limit $t_0 \to \infty$. Let χ_{t_0} be the solution of (6.19) with initial data $\chi_{t_0}(t_0) = \psi_+$. In addition to (6.19), it is convenient to use also the equation

$$\partial_t \tau = t^{-2}(S \cdot \nabla \tau + \tau \cdot \nabla S) \tag{6.25}$$

satisfies by $\tau = \nabla \chi$, with $S = \nabla \phi$.

We estimate $\partial_t\, |\chi_{t_0}|_{\bar{\ell}}^2$ in the same way as in the proof of Lemma 3.5. We estimate the contribution of $S \cdot \nabla \tau_{t_0}$ from (6.25) by (3.25) with (s, s', ℓ, ℓ') replaced by $(S, \tau_{t_0}, \tilde{\ell}, \bar{\ell})$ and the contribution of $\tau_{t_0} \cdot \nabla S$ by (3.26) with (s, s', ℓ, ℓ', m) replaced by $(\tau_{t_0}, S, \bar{\ell} + \nu/2, \tilde{\ell}, \bar{\ell} - \nu/2)$, thereby obtaining

$$\partial_t\, |\chi_{t_0}|_{\bar{\ell}}^2 \geqslant 2\rho'\, |\chi_{t_0}|_{\bar{\ell}+\nu/2}^2 - Ct^{-2}\, |\chi_{t_0}|_{\bar{\ell}+\nu/2}^2\, |\phi|_{\tilde{\ell}} \tag{6.26}$$

under the conditions

$$\tilde{\ell} > n/2 - \nu, \qquad \bar{\ell} + 1 \geqslant \nu/2, \qquad \tilde{\ell} \geqslant \bar{\ell} + 1 - \nu/2$$

which follow from (6.20). Introducing $z_{(1)} = z_{(1)}(\chi_{t_0}; [T, t_0], 1, \bar{\ell})$ and integrating (6.26) over time, we obtain in the same way as before

$$z^2 \vee z_1^2 \leqslant z_0^2 + C\, b\, h_1(T)\, z_1^2$$

with $z_0 = \beta$, which under the conditions (6.7) implies

$$Z(\chi_{t_0}; [T, t_0], 1, \bar{\ell}) \leqslant C\, \beta. \tag{6.27}$$

We next estimate the difference $\chi_1(t) \equiv \chi_{t_0}(t) - \chi_{t_0}(t_0) \equiv \chi_{t_0}(t) - \psi_+$ which satisfies the equations

$$\begin{aligned} \partial_t \chi_1 &= t^{-2} \nabla\phi \cdot \nabla \chi_{t_0} \\ \partial_t \tau_1 &= t^{-2}(S \cdot \nabla \tau_{t_0} + \tau_{t_0} \cdot \nabla S). \end{aligned}$$

Let $\tilde{\chi}_1 = h_1^{-1} \chi_1$. Using again (3.26) with $(s, s') = (S, \tau_{t_0})$ or (τ_{t_0}, S), with $m = \ell - \nu/2$ and with $(\ell, \ell') = (\tilde{\ell}, \bar{\ell} + \nu/2)$ or $(\bar{\ell} + \nu/2, \tilde{\ell})$, we obtain

$$\partial_t\, |\tilde{\chi}_1|_\ell^2 \geqslant 2\rho'\, |\tilde{\chi}_1|_{\ell+\nu/2}^2 - C t^{-2} h_1^{-1}\, |\tilde{\chi}_1|_{\ell+\nu/2}\, |\chi_{t_0}|_{\bar{\ell}+\nu/2}\, |\phi|_{\tilde{\ell}} \tag{6.28}$$

under the conditions

$$\tilde{\ell} + \bar{\ell} > \ell + n/2 - \nu, \qquad \tilde{\ell} \geqslant \ell + 1 - \nu/2, \qquad \bar{\ell} \geqslant \ell + 1 - \nu, \qquad \ell + 1 \geqslant \nu/2$$

which also follow from (6.20). Defining now $z_{(1)} = z_{(1)}(\chi_1; [T, t_0], h_1, \ell)$, integrating over time and using (6.21) (6.27), we obtain in the same way as before

$$z^2 \vee z_1^2 \leqslant C\, b\, \beta\, z_1 \operatorname*{Sup}_t \{t^{-2} \rho'^{-1} h_0 h_1^{-1}\} = C\, b\, \beta\, z_1$$

and therefore

$$Z(\chi_{t_0} - \psi_+; [T, t_0], h_1, \ell) \leqslant C\, b\, \beta. \tag{6.29}$$

Starting from the basic estimates (6.27) and (6.29), the end of the proof is the same as that of Proposition 6.1 based on (6.13) (6.16), with the simplification that the initial condition at t_0 is given by a fixed ψ_+ instead of a time dependent W. The difference between two solutions χ_{t_1} and χ_{t_0} with $T < t_0 < t_1$ is estimated with the help of the extension of (6.27) to that difference and of the pointwise part of (6.29) with t_0 replaced by t_1 taken at time $t = t_0$ as

$$Z(\chi_{t_1} - \chi_{t_0}; [T, t_0], 1, \ell) \leqslant C\, |\chi_{t_1}(t_0) - \psi_+|_\ell \leqslant C\, b\, \beta\, h_1(t_0) \tag{6.30}$$

from which the existence of χ with the properties and convergences stated in Part (2) follow. Taking the limit $t_1 \to \infty$ in (6.30) yields (6.24), while taking the limit $t_0 \to \infty$ in (6.27) (6.29) yields (6.22) (6.23) in the same way as in the proof of Proposition 6.1.

Part (3) follows from elementary estimates together with the estimate on ϕ expressed by (6.21).

Part (4). It follows from (6.3) and (6.19) that $V \exp(-i\chi)$ also satisfies (6.3), with gauge invariant initial condition $V(\infty)\exp(-i\chi(\infty)) = w_+ \exp(-i\psi_+)$. The result then follows from the uniqueness statement of Proposition 6.1, part (3). ∎

Remark 6.2. Because of the linearity of the equations (6.3) (6.19), the solutions V and χ constructed in Propositions 6.1 and 6.2 have obvious continuity properties with respect to w_+ and ψ_+ respectively. The continuity with respect to ϕ is more delicate and will not be considered here.

We shall use the results of Propositions 6.1 and 6.2 in the special case where $\phi = \phi_{p-1}$ and $W = W_p$ as defined by (5.22). In that case, V satisfies an asymptotic estimate which is much more accurate than (6.9) and which shows that V is a good approximation to W_p. In order to state that result, we need the results of Proposition 5.2. We need in particular the special function $\bar{h}_0$ defined by (5.7) and the associated functions $\bar{N}_m$ and $\bar{Q}_m$ associated with it according to (3.68) (3.69). The result can be stated as follows.

PROPOSITION 6.3. *Let $p \geqslant 1$ be an integer. Let $(k_+, \bar{k}, k)$ satisfy*

$$k_+ > n/2, \qquad \tilde{\ell} > n/2 - \nu, \qquad \tilde{k} \geqslant k + 1 - \nu/2,$$
$$\tilde{k} \geqslant \bar{k} \geqslant k + 1 - \nu, \qquad k \geqslant \nu/2 \tag{6.31}$$

where $\tilde{\ell}(\equiv \ell_{p-1}) = k_+ - \lambda - (p-1)\bar{\lambda}$ and $\tilde{k}(\equiv k_p) = k_+ - p\bar{\lambda}$. Let $w_+ \in K^{k_+}_{\rho\infty}$ and let $a_+ = |w_+|_{k_+}$. Let $\bar{h}_1$ and h_2 be $\mathscr{C}^1$ positive nonincreasing functions defined in $[1, \infty)$ and tending to zero at infinity, with $\bar{h}_1 \geqslant t^{-2}\rho'^{-1}\bar{h}_0$ and $h_2 \geqslant t^{-2}\rho'^{-1}\bar{N}_p$. Let $\phi = \phi_{p-1}$ and $W = W_p$ be defined by (5.22), *and let*

$$b = \operatorname*{Sup}_t \bar{h}_0(t)^{-1} |\phi|_{\tilde{\ell}} \equiv \operatorname*{Sup}_t \bar{h}_0(t)^{-1} |\phi_{p-1}(t)|_{\ell_{p-1}}. \tag{6.32}$$

Let T, $1 < T < \infty$ satisfy

$$b\bar{h}_1(T) \leqslant c \tag{6.33}$$

for a suitable constant c (see (6.7)*) and let V be the solution of the equation* (6.3) *constructed in Proposition 6.1, and such that $(V, 0) \in \mathscr{X}^{\bar{k},0}_p([T, \infty))$. Then V satisfies the following estimate*

$$Y(V - W_p; [T, \infty), h_2, k) \leqslant A(a_+). \tag{6.34}$$

Proof. One checks easily that the assumptions of Proposition 6.3 imply the relevant assumptions of Propositions 5.2 and 6.1. In particular (6.31) implies (6.4) and the exponents $\tilde{k}$ and $\tilde{\ell}$ are those given by Proposition 5.2. Let now V_{t_0} be defined in Proposition 6.1, part (2), let $r = V_{t_0} - W_p$ so that in particular $r(t_0) = 0$, and r satisfies the equation

$$\partial_t r = (2t^2)^{-1} \left\{ (2\nabla\phi_{p-1} \cdot \nabla + (\Delta\phi_{p-1}))\, r + \sum_{\substack{i \leqslant p-1,\, j \leqslant p \\ i+j \geqslant p}} (2\nabla\varphi_i \cdot \nabla + (\Delta\varphi_i))\, w_j \right\} \tag{6.35}$$

obtained by taking the difference between (6.3) and the appropriate sum of (2.18). Let $\tilde{r} = h_2^{-1} r$. We now estimate $\partial_t\, |\tilde{r}|_k^2$. The contribution of the terms containing r in the RHS is estimated exactly as in the proof of Proposition 6.1. The remaining terms are estimated by (3.23) (3.24) with $m = k - \nu/2$, $k \to \tilde{k}$ and $\ell \to \tilde{\ell}$ under the conditions $\tilde{\ell} > n/2 - \nu$, $0 \leqslant k - \nu/2 \leqslant \tilde{k} - 1$, which follow from (6.31). We obtain

$$\begin{aligned} \partial_t\, |\tilde{r}|_k^2 &\geqslant 2\rho'\, |\tilde{r}|_{k+\nu/2}^2 - Ct^{-2}\, |\tilde{r}|_{k+\nu/2}^2\, |\phi|_{\tilde{\ell}} - Ct^{-2} h_2^{-1}\, |\tilde{r}|_{k+\nu/2} \sum |\varphi_i|_{\tilde{\ell}}\, |w_j|_{\tilde{k}} \\ &\geqslant 2\rho'\, |\tilde{r}|_{k+\nu/2}^2 - Cbt^{-2}\bar{h}_0(t)\, |\tilde{r}|_{k+\nu/2}^2 - A(a_+)\, t^{-2} h_2^{-1} \bar{N}_p\, |\tilde{r}|_{k+\nu/2} \end{aligned} \tag{6.36}$$

by (6.32) (5.9) (5.10) (3.70) and (3.79). Defining $y_{(1)} = y_{(1)}(r; [T, t_0], h_2, k)$ and integrating (6.36) over time with the initial condition $r(t_0) = 0$, we obtain in the same way as before

$$y^2 \vee y_1^2 \leqslant C\, b\, \bar{h}_1(t)\, y_1^2 + A(a_+)\, y_1$$

by using the properties of $\bar{h}_1$ and h_2, and therefore, under the condition (6.33)

$$Y(r; [T, t_0], h_2, k) \leqslant A(a_+). \tag{6.37}$$

We now take $T < t_0 < t_1$, and take the limit $t_1 \to \infty$, $t_0 \to \infty$ in that order in the estimate

$$Y(V_{t_1} - W; [T, t_0], h_2, k) \leqslant A(a_+)$$

which follows from (6.37). The estimate (6.34) then follows from the convergence (6.10). ∎

Remark 6.2. In order to appreciate the improvement of the asymptotic accuracy of Proposition 6.3, especially (6.34) over Proposition 6.1, especially (6.9), it is useful to consider again the special case $\rho' = t^{-1-\varepsilon}$. Proposition 6.1 with $\phi = \phi_{p-1}$ is now applied with $h_0 = \bar{h}_0 \sim t^{1-\gamma}$ and therefore $h_1 = \bar{h}_1 \sim t^{-\gamma+\varepsilon}$, so that the pointwise part of (6.9) states that

$$|V(t) - w_+|_k \lesssim C\, ab\, t^{-\gamma+\varepsilon}.$$

On the other hand $\bar{N}_p \sim t^{1-(p+1)\gamma}$ for $(p+1)\gamma < 1$ thereby allowing for $h_2 \sim t^{-2}\rho'^{-1}\bar{N}_p \sim t^{-(p+1)\gamma+\varepsilon}$ in Proposition 6.3, so that the pointwise part of (6.34) states that

$$|V(t) - W_p(t)|_k \lesssim A(a_+)\, t^{-(p+1)\gamma+\varepsilon}.$$

That improvement will play an essential role in the estimates of Proposition 6.4 below.

We now turn to the construction of solutions (w, φ) of the system (2.11)–(2.12) with given asymptotic states (w_+, ψ_+). For that purpose, we first take a large positive t_0 and we construct a solution (w_{t_0}, φ_{t_0}) of (2.11)–(2.12) with initial data $(V(t_0), \phi_p(t_0) + \chi(t_0))$ at t_0. The solution (w, φ) will be obtained therefrom by taking the limit $t_0 \to \infty$, as explained in Section 2.

PROPOSITION 6.4. *Let (k, ℓ) satisfy (3.48) and $k \geqslant 1 - \nu/2$, let p be an integer such that $(p+2)\gamma > 1$ and let k_+ and ℓ_+ satisfy*

$$k_+ > n/2, \qquad k_+ \geqslant (k+2-\nu) \vee (\ell + \lambda + 1) + p\bar{\lambda}, \qquad \ell_+ \geqslant \ell + 1, \tag{6.38}$$

where $\lambda = \mu - n + 2$ and $\bar{\lambda} = \lambda \vee 1$. Let $\bar{h}_0$ be defined by (5.7) and let $\bar{N}_m$, $\bar{Q}_m$ be the associated estimating functions defined by (3.68) (3.69). Let $\bar{h}_1$, h_1, h_2 and h_3 be positive nonincreasing $\mathscr{C}^1$ functions defined in $[1, \infty)$, tending to zero at infinity, and satisfying

$$\bar{h}_1 \geqslant t^{-2}\rho'^{-1}\bar{h}_0, \qquad h_2 \geqslant t^{-2}\rho'^{-1}\bar{N}_p, \qquad h_2 \geqslant \bar{Q}_p \quad \textit{if} \quad p \geqslant 1, \tag{6.39}$$

$$h_3 \geqslant t^{-\gamma}\rho'^{-1}h_2, \qquad h_1 \geqslant t^{-2}\rho'^{-1}h_3 h_2^{-1}, \qquad h_3 \geqslant C\bar{h}_1. \tag{6.40}$$

Let $w_+ \in K^{k_+}_{\rho\infty}$ and let (W_m, ϕ_m), $0 \leqslant m \leqslant p$, be defined by (5.22) so that $(W_m, \bar{h}_0^{-1}\phi_m) \in \mathscr{Y}^{k_m, \ell_m}_{\rho\infty}([1, \infty))$ by Proposition 5.2. Let V be defined by Proposition 6.1 with $(W, \phi) = (W_p, \phi_{p-1})$ and $(\tilde{k}, \tilde{\ell}) = (k_p, \ell_{p-1})$. Let $\psi_+ \in Y^{\ell_+}_{\rho\infty}$ and let χ be defined by Proposition 6.2 with the same (W, ϕ). Let

$$a_+ = |w_+|_{\rho\infty, k_+}, \qquad b_+ = |\psi_+|_{\rho\infty, \ell_+}.$$

Then there exists T, $1 \leqslant T < \infty$, depending only on (γ, p, a_+, b_+) such that for all $t_0 \geqslant T$, the system (2.11)–(2.12) with initial data $w(t_0) = V(t_0)$, $\varphi(t_0) = \phi_p(t_0) + \chi(t_0)$ has a unique solution $(w_{t_0}, \varphi_{t_0}) \in \mathscr{X}^{k,\ell}_{\rho}([T, t_0])$. One can define T by a condition of the type

$$A(a_+, b_+)(h_1(T) \vee \bar{h}_1(T) \vee h_2(T)) = 1 \tag{6.41}$$

and the solution satifies the estimates

$$Y(w_{t_0}-V;[T,t_0],h_2,k)\vee Y(w_{t_0}-W_p;[T,t_0],h_2,k)\leqslant A(a_+,b_+) \qquad (6.42)$$

$$Z(\varphi_{t_0}-\phi_p-\chi;[T,t_0],h_3,\ell)\vee Z(\varphi_{t_0}-\phi_p-\psi_+;[T,t_0],h_3,\ell)\leqslant A(a_+,b_+) \qquad (6.43)$$

$$Y(w_{t_0};[T,t_0],1,k)\leqslant A(a_+,b_+), \qquad (6.44)$$

$$Z(\varphi_{t_0};[T,t_0],\bar{h}_0,\ell)\leqslant A(a_+,b_+). \qquad (6.45)$$

Remark 6.3. As mentioned previously, in the same way as in Propositions 6.1 and 6.2, we could easily (but we shall not) extend the solution (w_{t_0},φ_{t_0}) to the interval $[T,\infty)$.

Remark 6.4. In order to understand the time decay estimates implied by (6.42) (6.43), it is useful to consider again the special case $\rho'=t^{-1-\varepsilon}$. Saturating as far as possible the inequalities in (6.39) (6.40), we obtain

$$\bar{h}_1\sim t^{-\gamma+\varepsilon},\qquad h_2\sim t^{-(p+1)\gamma+\varepsilon},\qquad h_3\sim t^{1-(p+2)\gamma+2\varepsilon},\qquad h_1=t^{-\gamma+2\varepsilon}$$

for $(p+1)\gamma<1$, and the assumptions are satisfied for ε sufficiently small, namely $2\varepsilon<(p+2)\gamma-1$. In particular the condition that h_3 be decreasing in t essentially imposes the condition $(p+2)\gamma>1$.

Note also that in the condition (6.41) h_2 yields only a marginal restriction as soon as $p\geqslant 1$, while for $p=0$ it is natural to take $h_2=\bar{h}_1$ (see (6.39) with $\bar{N}_0=\bar{h}_0$). Finally, the condition $h_3\geqslant C\bar{h}_1$ will in general be automatically satisfied for any reasonable choice of the estimating functions $\bar{h}_1$ and h_3.

Proof. The proof follows the same pattern as that of Proposition 4.1, involving a parabolic regularization, possibly a regularization of the initial data, the local resolution of the regularized system by a fixed point method, the derivation of a priori estimates uniform in the regularization, and a limiting procedure. The only difference lies in the a priori estimates of the solutions in $\mathscr{X}_p^{k,\ell}([T,t_0])$. Those estimates are much more elaborate than previously and will ensure in particular that T can be taken independent of t_0, contrary to what happened in Lemma 4.1 (see especially (4.13)). We concentrate on the proof of those estimates, omitting the parabolic regularization terms for brevity. Their contribution will be briefly discussed at the end of that proof.

Let $(w_{t_0},\varphi_{t_0})\in\mathscr{X}_p^{k,\ell}([T,t_0])$ be a solution of the system (2.11)–(2.12) with initial condition $(w_{t_0}(t_0),\varphi_{t_0}(t_0))=(V(t_0),\phi_p(t_0)+\chi(t_0))$ where V and χ are defined by Propositions 6.1 and 6.2. Instead of estimating (w_{t_0},φ_{t_0}) directly, we estimate the differences $q=w_{t_0}-V$ and $\psi=\varphi_{t_0}-\phi_p-\chi$. For convenience we also introduce the gradients $\sigma=\nabla\psi$, $s_{t_0}=\nabla\varphi_{t_0}$, $\tau=\nabla\chi$, as well as $s_m=\nabla\varphi_m$, $S_m=\nabla\phi_m$ for $0\leqslant m\leqslant p$. Comparing the equations (2.11)–(2.12) and (6.3) (6.19) with $\phi=\phi_{p-1}$, we obtain

$$\partial_t q = (2t^2)^{-1} \{i\Delta w_{t_0} + 2s_{t_0} \cdot \nabla q + 2(\sigma + s_p + \tau) \cdot \nabla V + (\nabla \cdot \sigma)\, w_{t_0} + (\nabla \cdot (S_p + \tau))\, q + (\nabla \cdot (s_p + \tau))\, V\} \tag{6.46}$$

$$\partial_t \psi = (2t^2)^{-1} \left\{ |\sigma|^2 + 2\sigma \cdot (S_p + \tau) + |\tau|^2 + 2\tau \cdot s_p + \sum_{\substack{i,j \leqslant p \\ i+j \geqslant p}} s_i \cdot s_j \right\} + t^{-\gamma} \left\{ g_0(q,q) + 2g_0(q,V) + q_0(V - W_p, V + W_p) + \sum_{\substack{i,j \leqslant p \\ i+j > p}} g_0(w_i, w_j) \right\}. \tag{6.47}$$

It is convenient to write also the equation for $\sigma = \nabla\psi$, namely

$$\partial_t \sigma = t^{-2} \left\{ s_{t_0} \cdot \nabla\sigma + \sigma \cdot \nabla(S_p + \tau) + (\tau + s_p) \cdot \nabla\tau + \tau \cdot \nabla s_p + \sum_{\substack{i,j \leqslant p \\ i+j \geqslant p}} s_i \cdot \nabla s_j \right\} + t^{-\gamma} \nabla \left\{ g_0(q,q) + 2g_0(q,V) + q_0(V - W_p, V + W_p) + \sum_{\substack{i,j \leqslant p \\ i+j > p}} g_0(w_i, w_j) \right\}. \tag{6.48}$$

We define $\tilde{q} = h_2^{-1} q$ and $\tilde{\psi} = h_3^{-1} \psi$ and we estimate $\partial_t\, |\tilde{q}|_k^2$ and $\partial_t\, |\tilde{\psi}|_\ell^2$ by exactly the same method as in Lemmas 3.5, 3.6, 3.7, based on Lemma 3.4, using in particular (3.22) (3.23) (3.24) for $\tilde{q}$ and (3.25) (3.26) (3.28) for $\tilde{\psi}$, and omitting the terms with h_2' and h_3'. We obtain for $t \leqslant t_0$

$$\partial_t\, |\tilde{q}|_k^2 \geqslant 2\rho'\, |\tilde{q}|_{k+\nu/2}^2 - Ct^{-2}\, |\tilde{q}|_{k+\nu/2} \{h_2^{-1}\, |V|_{k+2-\nu/2} + |\tilde{q}|_{k+\nu/2}\, |\psi + \phi_p + \chi|_\ell + h_2^{-1}\, |V|_{k+1-\nu/2}\, |\psi + \varphi_p + \chi|_\ell + (|\tilde{q}|_k + h_2^{-1}\, |V|_k\,)|\psi|_{\ell+\nu/2} + |\tilde{q}|_k\, |\phi_p + \chi|_{\ell+\nu/2} + h_2^{-1}\, |V|_k\, |\varphi_p + \chi|_{\ell+\nu/2}\} \tag{6.49}$$

$$\partial_t\, |\tilde{\psi}|_\ell^2 \geqslant 2\rho'\, |\tilde{\psi}|_{\ell+\nu/2}^2 - Ct^{-2}\, |\tilde{\psi}|_{\ell+\nu/2} \Big\{ |\tilde{\psi}|_{\ell+\nu/2}\, |\psi + \phi_p + \chi|_\ell + |\tilde{\psi}|_\ell\, |\phi_p + \chi|_{\ell+1-\nu/2} + h_3^{-1}\, |\varphi_p + \chi|_\ell\, |\chi|_{\ell+1-\nu/2} + h_3^{-1}\, |\chi|_\ell\, |\varphi_p|_{\ell+1-\nu/2} + h_3^{-1} \sum_{\substack{i,j \leqslant p \\ i+j \geqslant p}} |\varphi_i|_\ell\, |\varphi_j|_{\ell+1-\nu/2} \Big\} - Ct^{-\gamma} h_3^{-1}\, |\tilde{\psi}|_{\ell+\nu/2} \Big\{ h_2\, |\tilde{q}|_{k+\nu/2} (|q|_k + |V|_k) + h_2\, |\tilde{q}|_k\, |V|_{k+\nu/2} + |V - W_p|_k\, |V + W_p|_{k+\nu/2} + |V - W_p|_{k+\nu/2}\, |V + W_p|_k + \sum_{\substack{i,j \leqslant p \\ i+j > p}} |w_i|_k\, |w_j|_{k+\nu/2} \Big\}. \tag{6.50}$$

In order to continue the estimates, we need some information on V, χ and on the (φ_m, w_m). Applying Proposition 6.1 with $\phi = \phi_{p-1}$ and Proposition 6.3, both with $\bar{k} = k+2-v$, we rewrite (6.8) and (6.34) as

$$Y(V; [T, \infty), 1, k+2-v) \leqslant a \tag{6.51}$$

$$Y(V - W_p; [T, \infty), h_2, k) \leqslant a \tag{6.52}$$

for some a depending on a_+, under the conditions

$$k_+ > n/2, \qquad \ell_{p-1} > n/2 - v, \qquad k_p \geqslant k+2-v \tag{6.53}$$

where (cf (5.8))

$$k_m = k_+ - m\bar{\lambda}, \qquad \ell_m = k_+ - \lambda - m\bar{\lambda}.$$

Similarly, applying Proposition 6.2 with $\phi = \phi_{p-1}$, $\bar{\ell} = \ell + 1$, $h_0 = \bar{h}_0$ and therefore $h_1 = \bar{h}_1$, we rewrite (6.22) (6.23) as

$$Z(\chi; [T, \infty), 1, \ell+1) \leqslant b \tag{6.54}$$

$$Z(\chi - \psi_+; [T, \infty), \bar{h}_1, \ell) \leqslant b \tag{6.55}$$

for some b depending on a_+ and b_+, under the conditions

$$\ell_{p-1} > n/2 - v, \qquad \ell_{p-1} \geqslant \ell + 2 - v/2, \qquad \ell_+ \geqslant \ell + 1. \tag{6.56}$$

Finally, from Proposition 5.2, we obtain

$$|w_j(t)|_{k+v/2} \leqslant a\bar{Q}_{j-1}(t) \qquad \text{for} \quad 1 \leqslant j \leqslant p, \tag{6.57}$$

$$|W_p(t)|_{k+v/2} \leqslant a, \tag{6.58}$$

for some a depending on a_+, under the conditions

$$k_+ > n/2, \qquad k_p \geqslant k + v/2, \tag{6.59}$$

and

$$|\varphi_j(t)|_{\ell+1-v/2} \leqslant b\bar{N}_j(t) \qquad \text{for} \quad 0 \leqslant j \leqslant p, \tag{6.60}$$

$$|\phi_p(t)|_{\ell+1-v/2} \leqslant b\bar{h}_0(t) \tag{6.61}$$

for some b depending on a_+, under the conditions

$$k_+ > n/2, \qquad \ell_p \geqslant \ell + 1 - v/2. \tag{6.62}$$

In the estimates (6.51) (6.52) (6.54) (6.55) (6.57) (6.58) (6.60) (6.61), we use two common letters a and b to refer to estimates of amplitudes and phases respectively. The constant a depends only on a_+, while b depends on a_+ and on b_+. The conditions (6.53) (6.56) (6.59) (6.62) are implied by (6.38), which is the statement of

$$k_+ > n/2, \qquad k_p \geqslant k+2-\nu, \qquad \ell_p \geqslant \ell+1, \qquad \ell_+ \geqslant \ell+1.$$

With the estimates (6.51)–(6.61) available, we continue to estimate (q, ψ) by integrating (6.49) (6.50) over time in the interval $[t, t_0]$. We define $y_{(1)} = y_{(1)}(q; [t, t_0], h_2, k)$ and $z_{(1)} = z_{(1)}(\psi; [t, t_0], h_3, \ell)$. We proceed exactly as in the proofs of Lemmas 4.1, 4.2 and 4.3, using the Schwarz inequality for the time integrals whenever necessary. We use furthermore the fact that (6.49) (resp. (6.50)) contains $|\tilde{q}|_{k+\nu/2}$ (resp. $|\tilde{\psi}|_{\ell+\nu/2}$) as a factor in its RHS, thereby yielding a factor y_1 (resp. z_1) after integration, and the elementary fact that

$$y^2 \vee y_1^2 \leqslant A y_1 \Rightarrow y \vee y_1 \leqslant A$$

and its analogue for (z, z_1). We then obtain the following estimates, where Sup means that the Supremum of the function of time that follows is taken in the interval $[t, t_0]$, and where an overall constant C is omitted for brevity.

$$\begin{aligned} y \vee y_1 \leqslant\ & a \operatorname{Sup}(t^{-2}\rho'^{-1}h_2^{-1}) + ab \operatorname{Sup}(t^{-2}\rho'^{-1}\bar{N}_p h_2^{-1}) \\ & + b(y+y_1) \operatorname{Sup}(t^{-2}\rho'^{-1}\bar{h}_0) + a(z+z_1) \operatorname{Sup}(t^{-2}\rho'^{-1}h_2^{-1}h_3) \\ & + (yz_1 + y_1 z) \operatorname{Sup}(t^{-2}\rho'^{-1}h_3), \end{aligned} \tag{6.63}$$

$$\begin{aligned} z \vee z_1 \leqslant\ & b^2 \operatorname{Sup}(t^{-2}\rho'^{-1}\bar{h}_0 h_3^{-1}\bar{N}_p) + b(z+z_1) \operatorname{Sup}(t^{-2}\rho'^{-1}\bar{h}_0) \\ & + zz_1 \operatorname{Sup}(t^{-2}\rho'^{-1}h_3) + a(y+y_1) \operatorname{Sup}(t^{-\gamma}\rho'^{-1}h_3^{-1}h_2) \\ & + yy_1 \operatorname{Sup}(t^{-\gamma}\rho'^{-1}h_3^{-1}h_2^2) + pa^2 \operatorname{Sup}(t^{-\gamma}\rho'^{-1}h_3^{-1}(h_2+\bar{Q}_p)) \end{aligned} \tag{6.64}$$

where the factor p in the last term simply means that that term is absent for $p=0$. The various Sup in time are estimated in an obvious way with the help of the conditions (6.39) (6.40) which are taylored for that purpose. The only non-obvious term is the coefficient of b^2 in (6.64), which is estimated by (6.39) (6.40) as

$$(t^{-2}\rho'^{-1}\bar{N}_p h_2^{-1})(t^{-\gamma}\rho'^{-1}h_2 h_3^{-1})(t^{\gamma}\rho'\bar{h}_0) \leqslant \|\rho'\|_1$$

since

$$\bar{h}_0 = \int_1^t dt_1\ t_1^{-\gamma}\rho'(t_1)^{-1}\ \rho'(t_1) \leqslant t^{-\gamma}\rho'(t)^{-1} \int_1^t dt_1\ \rho'(t_1) \leqslant \|\rho'\|_1\ t^{-\gamma}\rho'^{-1} \tag{6.65}$$

by the monotony of $t^{-\gamma}\rho'^{-1}$. Absorbing the factor $\|\rho'\|_1$ in the (again omitted) overall constant and defining as previously $Y = y \vee y_1$ and $Z = z \vee z_1$, we end up with

$$Y \leqslant a + ab + bY\bar{h}_1 + aZh_1 + YZh_1h_2 \tag{6.66}$$

$$Z \leqslant b^2 + pa^2 + bZ\bar{h}_1 + Z^2h_1h_2 + aY + Y^2h_2 \tag{6.67}$$

where the functions h_1, $\bar{h}_1$ and h_2 are taken at time t where they take their Supremum in $[t, t_0]$, since they are assumed to be decreasing.

In order to conclude the estimates, we impose the conditions

$$4\,b\,\bar{h}_1(T) \leqslant 1, \qquad 4(1+b)\,h_2(T) \leqslant 1, \qquad Zh_1(t) \leqslant (1+b) \tag{6.68}$$

which together imply $4Z\,h_1(t)\,h_2(t) \leqslant 1$ for all $t \geqslant T$. We then obtain

$$Y \leqslant 2a(1+b) + 2a\,Z\,h_1 \leqslant 4a(1+b) \tag{6.69}$$

$$\begin{aligned} Z &\leqslant 2(b^2+pa^2) + 2a\,Y + 2Y(Y\,h_2) \leqslant 2(b^2+pa^2) + 4aY \\ &\leqslant 2(b^2+pa^2) + 16a^2(1+b). \end{aligned} \tag{6.70}$$

The condition $Zh_1 \leqslant 1+b$ then reduces to

$$(2(b^2+pa^2)(1+b)^{-1} + 16a^2)\,h_1 \leqslant 1$$

which is implied by

$$(2b + (16+2p)\,a^2)\,h_1(T) \leqslant 1. \tag{6.71}$$

The condition (6.71) together with the first two conditions of (6.68) then take the form (6.41), while the estimates (6.69) (6.70) yield the estimate of the first terms in the LHS of (6.42) (6.43). The estimates of the second terms follow from those of the first ones and from (6.52) and (6.55), together with the condition $h_3 \geqslant C\bar{h}_1$ from (6.40).

Finally the estimates (6.44) (6.45) follow immediately from (6.42) (6.43), from (6.51) or (6.58) and from (6.61) (6.54).

We now discuss briefly the contribution of the parabolic regularization terms in the previous proof. We regularize the system (2.11)–(2.12) in the same way as in the proof of Proposition 4.1 (see (4.51), where however the sign of the regularizing terms should be changed since we are now solving the equations for decreasing t starting from t_0). Instead of (6.49) (6.50), we then obtain

$$\partial_t\,|\tilde{q}|_k^2 \geqslant 2\theta(|\nabla\tilde{q}|_k^2 + h_2^{-1}\,\mathrm{Re}\langle\nabla\tilde{q}, \nabla V\rangle_k) + \text{previous terms},$$

$$\partial_t|\tilde{\psi}|_\ell^2 \geqslant 2\theta(|\nabla\tilde{\psi}|_\ell^2 + h_3^{-1}\langle\nabla\tilde{\psi}, \nabla(\phi_p+\chi)\rangle_\ell) + \text{previous terms},$$

where $\langle \cdot, \cdot \rangle_k$ and $\langle \cdot, \cdot \rangle_\ell$ denote the scalar products in K_ρ^k and Y_ρ^ℓ. The scalar products are controlled since we have assumed that $(V, \phi_p + \chi) \in \mathscr{X}_{\rho, loc}^{k+1, \ell+1}$ by imposing $\bar{k} = k + 2 - v$, $\bar{\ell} = \ell + 1$ and $\ell_p \geqslant \ell + 1$. They produce additional terms in the final estimates which are uniformly bounded in θ in a neighborhood of zero.

With the a priori estimates (6.44) (6.45) replacing Lemma 4.1, the proof of Proposition 6.4 proceeds in the same way as that of Proposition 4.1, as mentioned above. In particular, the required regularity estimates and difference estimates are provided by Parts 3 of Lemmas 4.2 and 4.3, which are taylored for that purpose. The assumption on (w, φ) made in those parts follow from (6.44) (6.45) and from the relation

$$\bar{h}_0 \leqslant \|\rho'\|_1 \, h_0$$

which follows from (6.65), while the regularity assumptions on the initial data required in Lemma 4.2 are ensured by the previous regularity of V and $\phi_p + \chi$. ∎

We can now take the limit $t_0 \to \infty$ of the solution (w_{t_0}, φ_{t_0}) constructed in Proposition 6.4, for fixed (w_+, ψ_+).

Proposition 6.5. *Let the assumptions of Proposition 6.4 be satisfied. Then*

(1) *There exists T, $1 \leqslant T < \infty$, depending only on (γ, p, a_+, b_+) and there exists a unique solution (w, φ) of the system* (2.11)–(2.12) *in the interval $[T, \infty)$ such that $(w, \bar{h}_0^{-1}\varphi) \in \mathscr{X}_\rho^{k,\ell}([T, \infty))$ and such that the following estimates hold*

$$Y(w - V; [T, \infty), h_2, k) \vee Y(w - W_p; [T, \infty), h_2, k) \leqslant A(a_+, b_+), \tag{6.72}$$

$$Z(\varphi - \phi_p - \chi; [T, \infty), h_3, \ell) \vee Z(\varphi - \phi_p - \psi_+; [T, \infty), h_3, \ell) \leqslant A(a_+, b_+), \tag{6.73}$$

$$Y(w; [T, \infty), 1, k) \leqslant A(a_+, b_+), \tag{6.74}$$

$$Z(\varphi; [T, \infty), \bar{h}_0, \ell) \leqslant A(a_+, b_+). \tag{6.75}$$

One can define T by a condition of the type (6.41).

(2) *Let $(w_{t_0}, \varphi_{t_0}) \in \mathscr{X}_\rho^{k,\ell}([T, t_0])$ be the solution of the system* (2.11)–(2.12) *constructed in Proposition* 6.4. *Then (w_{t_0}, φ_{t_0}) converges to (w, φ) in norm in $\mathscr{X}_\rho^{k',\ell'}([T, T_1])$ for $k' < k$, $\ell' < \ell$ and in the weak-∗ sense in $\mathscr{X}_\rho^{k,\ell}([T, T_1])$ for any T_1, $T < T_1 < \infty$, and in the weak-∗ sense in $K_\rho^k \oplus Y_\rho^\ell$ pointwise in t for all $t \geqslant T$.*

(3) *The map $(w_+, \psi_+) \to (w, \varphi)$ defined in Part* (1) *is continuous from the norm topology of (w_+, ψ_+) in $K_{\rho\infty}^{k_+} \oplus Y_{\rho\infty}^{\ell_+}$ to the norm topology of (w, φ)*

in $\mathscr{X}_\rho^{k',\ell'}([T, T_1])$ *for* $k' < k$, $\ell' < \ell$ *and to the weak-$*$ topology in* $\mathscr{X}_\rho^{k,\ell}([T, T_1])$ *for any* T_1, $T < T_1 < \infty$, *and to the weak-$*$ topology in* $K_\rho^k \oplus Y_\rho^\ell$ *pointwise in* t *for all* $t \geqslant T$.

Remark 6.5. For simplicity, we have not stated the strongest continuity properties that would follow by tracking more accurately the exponents in the proof of Part (3). Actually the required topology on (w_+, ψ_+) could be weakened to the norm topology of $K_{\rho\infty}^{k'} \oplus Y_{\rho\infty}^{\ell'}$ on the bounded sets of $K_{\rho\infty}^{k_+} \oplus Y_{\rho\infty}^{\ell_+}$ for suitable (k', ℓ') smaller than (k_+, ℓ_+).

Proof. Parts (1) and (2) will follow from the convergence of (w_{t_0}, φ_{t_0}) when $t_0 \to \infty$ in the topologies stated in Part (2). Let $T \leqslant t_0 \leqslant t_1$, let (w_{t_0}, φ_{t_0}) and (w_{t_1}, φ_{t_1}) be the corresponding solutions of the system (2.11)–(2.12) obtained in Proposition 6.4, and let $(w_-, \varphi_-) = (w_{t_0} - w_{t_1}, \varphi_{t_0} - \varphi_{t_1})$. From (6.42) (6.43) and their analogues for (w_{t_1}, φ_{t_1}), it follows that

$$
\begin{aligned}
Y(w_-; [T, t_0], h_2, k) &\leqslant A(a_+, b_+) \\
Z(\varphi_-; [T, t_0], h_3, \ell) &\leqslant A(a_+, b_+)
\end{aligned}
\tag{6.76}
$$

so that in particular

$$
\begin{aligned}
|w_-(t_0)|_k &\leqslant A(a_+, b_+)\, h_2(t_0) \\
|\varphi_-(t_0)|_\ell &\leqslant A(a_+, b_+)\, h_3(t_0).
\end{aligned}
\tag{6.77}
$$

We now apply Lemma 4.3, part (3) to (w_-, φ_-). For that purpose we take $h_0 = t^{-\gamma}\rho'^{-1}$, so that by (6.40)

$$
h_1 \geqslant t^{-2}\rho'^{-1} h_0, \qquad h_2 h_0 \leqslant h_3 \tag{6.78}
$$

and in particular (h_0, h_1) satisfy the assumptions of Lemma 4.3. The assumptions (4.35) (restricted to the relevant interval $[T, t_0]$) and (4.38) follow from (6.44) (6.45) and from (6.65), while the condition (4.26) can be included in (6.41). We now apply (4.41) with $k' = k - 1 + \nu$, $\ell' = \ell - 1 + \nu$, together with (6.77) (6.78), thereby obtaining

$$
\begin{aligned}
Y(w_-; [T, t_0], h_0^{-1}, k - 1 + \nu) &\leqslant A(a_+, b_+)\{h_0(t_0)\, |w_-(t_0)|_{k-1+\nu} \\
&\qquad + h_1(T)\, |\varphi_-(t_0)|_{\ell-1+\nu}\} \\
&\leqslant A(a_+, b_+)\, h_3(t_0)
\end{aligned}
\tag{6.79}
$$

$$
\begin{aligned}
Z(\varphi_-; [T, t_0], 1, \ell - 1 + \nu) &\leqslant A(a_+, b_+)\{|\varphi_-(t_0)|_{\ell-1+\nu} \\
&\qquad + h_0(t_0)\, |w_-(t_0)|_{k-1+\nu}\} \\
&\leqslant A(a_+, b_+)\, h_3(t_0).
\end{aligned}
\tag{6.80}
$$

From (6.79) (6.80) and from the fact that h_3 tends to zero at infinity, it follows that there exists $(w, \varphi) \in \mathscr{X}_{\rho, loc}^{k-1+\nu, \ell-1+\nu}([T, \infty))$ such that (w_{t_0}, φ_{t_0}) converges to (w, φ) in norm in $\mathscr{X}_{\rho}^{k-1+\nu, \ell-1+\nu}([T, T_1])$ for all T_1, $T < T_1 < \infty$. From that convergence, from (6.42)–(6.45) and from standard compactness, continuity and interpolation arguments, it follows that $(w, \bar{h}_0^{-1}\varphi) \in \mathscr{X}_{\rho}^{k,\ell}([T, \infty))$, that (w, φ) satisfies the estimates (6.72)–(6.75) and that (w_{t_0}, φ_{t_0}) converges to (w, φ) in the other topologies stated in Part (2). Furthermore (w, φ) satisfies the system (2.11)–(2.12), and uniqueness of (w, φ) under the conditions (6.72) (6.73) follows from Proposition 4.3 and from the fact that $h_3(t)$ tends to zero at infinity.

Part (3). Let (w_+, ψ_+) and (w'_+, ψ'_+) belong to a fixed bounded set of $K_{\rho\infty}^{k_+} \oplus Y_{\rho\infty}^{\ell_+}$, so that (5.17) (5.18) and (6.72)–(6.75) hold with fixed A. Let (W_p, ϕ_p) and (W'_p, ϕ'_p) be the associated functions defined by (5.22) and let (w, φ) and (w', φ') be the associated solutions of the system (2.11)–(2.12) defined in Part (1). We assume that (w'_+, ψ'_+) is close to (w_+, ψ_+) in the sense that

$$|w_+ - w'_+|_{k_+} \leqslant \varepsilon, \qquad |\psi_+ - \psi'_+|_\ell \leqslant \varepsilon_0. \tag{6.81}$$

Let $w_- = w - w'$, $\varphi_- = \varphi - \varphi'$ and let $t_0 > T$ be defined by $h_2(t_0) = \varepsilon$, which can be done for any (sufficiently small) $\varepsilon > 0$ and which implies that $t_0 \to \infty$ when $\varepsilon \to 0$. It follows from (6.72) and (5.17) that

$$|w_-(t_0)|_k \leqslant Ah_2(t_0) + |W_p(t_0) - W'_p(t_0)|_k \leqslant A(h_2(t_0) + \varepsilon) \leqslant Ah_2(t_0) \tag{6.82}$$

and from (6.73) and (5.18) that

$$\begin{aligned} |\varphi_-(t_0)|_\ell &\leqslant Ah_3(t_0) + |\phi_p(t_0) - \phi'_p(t_0)|_\ell + |\psi_+ - \psi'_+|_\ell \\ &\leqslant A(h_3(t_0) + \bar{h}_0(t_0)\,\varepsilon) + |\psi_+ - \psi'_+|_\ell \leqslant Ah_3(t_0) + \varepsilon_0. \end{aligned} \tag{6.83}$$

We now apply Lemma 4.3, part (3) with $k' = k - 1 + \nu$, $\ell' = \ell - 1 + \nu$, thereby obtaining

$$\begin{aligned} Y(w_-; [T, t_0], h_0^{-1}, k-1+\nu) &\leqslant A(h_0(t_0)\,h_2(t_0) + h_1(T)(h_3(t_0) + \varepsilon_0)) \\ &\leqslant A(h_3(t_0) + \varepsilon_0) \end{aligned} \tag{6.84}$$

$$\begin{aligned} Z(\varphi_-; [T, t_0], 1, \ell-1+\nu) &\leqslant A\{h_3(t_0) + \varepsilon_0 + h_0(t_0)\,h_2(t_0)\} \\ &\leqslant A(h_3(t_0) + \varepsilon_0). \end{aligned} \tag{6.85}$$

When ε tends to zero, $h_3(t_0)$ tends to zero, and therefore (6.84) (6.85) imply norm continuity in $\mathscr{X}_{\rho}^{k-1+\nu, \ell-1+\nu}([T, T_1])$ for all T_1, $T < T_1 < \infty$. The other continuities follow therefrom, from the estimates (6.74) (6.75) and from standard continuity, interpolation and compactness arguments. ∎

7. ASYMPTOTICS AND WAVE OPERATORS FOR u

In this section we complete the construction of the wave operators for the equation (1.1) and we derive asymptotic properties of solutions in their range. The construction relies in an essential way on those of Section 6, especially Proposition 6.5, and will require a discussion of the gauge invariance of those constructions. This section follows closely Section II.7.

We first define the wave operator for the auxiliary system (2.11)–(2.12).

DEFINITION 7.1. We define the wave operator Ω_0 as the map

$$\Omega_0 : (w_+, \psi_+) \to (w, \varphi) \tag{7.1}$$

from $K_{\rho\infty}^{k_+} \oplus Y_{\rho\infty}^{\ell_+}$ to the space of (w, φ) such that $(w, \bar{h}_0^{-1}\varphi) \in \mathscr{X}_\rho^{k,\ell}([T, \infty))$ for some T, $1 \leqslant T < \infty$, where ρ, k_+, ℓ_+, k, ℓ satisfy the assumptions of Proposition 6.5 and (w, φ) is the solution of the system (2.11)–(2.12) obtained in Part (1) of that proposition.

In order to study the gauge invariance of Ω_0, we need some information on the Cauchy problem at finite times for the equation (1.1). We define the operator $J \equiv J(t) = x + it\nabla$, which satisfies the commutation relation

$$i\, M\, D\, \nabla = J\, M\, D, \tag{7.2}$$

where M and D are defined by (2.4) (2.5). For any interval $I \subset [1, \infty)$, any nonnegative integer k and any nonnegative $\mathscr{C}^1$ function ρ defined in I, we define the space

$$\begin{aligned} X_\rho^k(I) &= \{u : D^* M^* u \in \mathscr{C}(I, K_\rho^k)\} \\ &= \{u : \langle J(t)\rangle^k f(J(t))\, u \in \mathscr{C}(I, L^2)\}. \end{aligned} \tag{7.3}$$

where $\langle \lambda \rangle = (1 + \lambda^2)^{1/2}$ for any real number or self-adjoint operator λ and where the second equality follows from (7.2) and from Remark 3.1. (The space $X_0^k(I)$ was denoted $\mathscr{X}^k(I)$ in II). We recall the following result (see Proposition I.7.1).

PROPOSITION 7.1. *Let k be a positive integer and let $0 < \mu < 2k$. Then the Cauchy problem for the equation* (1.1) *with initial data $u(t_0) = u_0$ such that $\langle J(t_0)\rangle^k u_0 \in L^2$ at some initial time $t_0 \geqslant 1$ is locally well posed in $\mathscr{X}_0^k(\cdot)$, namely*

(1) *There exists $T > 0$ such that* (1.1) *has a unique solution with initial data $u(t_0) = u_0$ in $\mathscr{X}_0^k([1 \vee (t_0 - T), t_0 + T])$.*

(2) *For any interval* I, $t_0 \in I \subset [1, \infty)$, (1.1) *with initial data* $u(t_0) = u_0$ *has at most one solution in* $\mathscr{X}_0^k(I)$.

(3) *The solution of Part* (1) *depends continuously on* u_0 *in the norms considered there.*

We come back from the system (2.11)–(2.12) to the equation (1.1) by reconstructing u from (w, φ) by (2.7) and accordingly we define the map

$$\Lambda: (w, \varphi) \to u = M\, D \exp(-i\varphi)\, w. \tag{7.4}$$

It follows immediately from Lemma 3.3 that the map Λ satisfies the following property.

LEMMA 7.1. *Let* $\ell + 2 > n/2$ *and* $0 \leqslant k \leqslant \ell + 2$. *Then for any interval* $I \subset [1, \infty)$ *and any nonnegative* $\mathscr{C}^1$ *function* ρ *defined in* I, *the map* Λ *is bounded and continuous from* $\mathscr{Y}^{k,\ell}_{\rho,\, loc}(I)$ *(defined by* (5.1) (6.2)*) to* $X^k_\rho(I)$, *with norm estimates in compact intervals independent of* ρ.

We now give the following definition

DEFINITION 7.2. Two solutions (w, φ) and (w', φ') of the system (2.11)–(2.12) in $\mathscr{Y}^{k,\ell}_{\rho,\, loc}(I)$ for some k, ℓ, ρ and some interval $I \subset [1, \infty)$ are said to be gauge equivalent if $\Lambda(w, \varphi) = \Lambda(w', \varphi')$, or equivalently if

$$\exp(-i\varphi(t))\, w(t) = \exp(-i\varphi'(t))\, w'(t) \tag{7.5}$$

for all $t \in I$.

The following sufficient condition for gauge equivalence follows immediately from Lemma 7.1 and from the uniqueness statement of Proposition 7.1, part (2).

LEMMA 7.2. *Let* $\ell + 2 > n/2$ *and* $0 \leqslant k \leqslant \ell + 2$. *Let* $I \subset [1, \infty)$ *be an interval, let* ρ *be a strictly positive* $\mathscr{C}^1$ *function defined in* I, *and let* (w, φ) *and* (w', φ') *be two solutions of the system* (2.11)–(2.12) *in* $\mathscr{Y}^{k,\ell}_{\rho,\, loc}(I)$. *In order that* (w, φ) *and* (w', φ') *be gauge equivalent, it is sufficient that* (7.5) *holds for one* $t \in I$.

We now turn to the study of the gauge equivalence of asymptotic states (if any) for solutions of the system (2.11)–(2.12) such as those obtained in Proposition 4.1. For that purpose, we need ρ to be decreasing, namely to be defined by (4.1) for some $t_0 \geqslant 1$.

PROPOSITION 7.2. *Let* $k \geqslant 0$ *and* $\ell > n/2 - \nu$. *Let* ρ *be defined by* (4.1) *for some* $t_0 \geqslant 1$ *and let* h_0 *and* h_1 *be as in Proposition* 4.1. *Let* (w, φ) *and*

(w', φ') be two gauge equivalent solutions of the system (2.11)–(2.12) *such that $(w, h_0^{-1}\varphi), (w', h_0^{-1}\varphi') \in \mathscr{X}_\rho^{k,\ell}([T, \infty))$ for some T, $t_0 \leqslant T < \infty$. Let*

$$b = Z(\varphi; [T, \infty), h_0, \ell) \vee Z(\varphi'; [T, \infty), h_0, \ell). \tag{7.6}$$

Then

(1) *There exists $\omega \in Y_{\rho(\infty)}^{\ell-1+\nu}$ such that $\varphi'(t) - \varphi(t)$ converges to ω strongly in $Y_{\rho(\infty)}^{\ell'}$ for $\ell' < \ell - 1 + \nu$ and weakly in $Y_{\rho(\infty)}^{\ell-1+\nu}$. The following estimate holds:*

$$|\varphi'(t) - \varphi(t) - \omega|_{\ell-2+\nu} \leqslant Cb\, |\varphi'(T_1) - \varphi(T_1)|_{\ell-1+\nu}\, h_1(t) \tag{7.7}$$

for $t \geqslant T_1$, for some T_1 sufficiently large, namely satisfying

$$b\, h_1(T_1) \leqslant c \tag{7.8}$$

for some constant c.

(2) *Assume in addition that $1 - \nu/2 \leqslant k \leqslant \ell + 1$ and let w_+ and w'_+ be the limits of $w(t)$ and $w'(t)$ as $t \to \infty$ obtained in Proposition* 4.2. *Then $w'_+ = w_+ \exp(-i\omega)$.*

(3) *Let $p \geqslant 0$ be an integer. Assume in addition that $w_+, w'_+ \in K_{\rho(\infty)}^{k_+}$ where k_+ satisfies*

$$k_+ > n/2, \qquad k_+ \geqslant (p+1)\, \bar{\lambda} - 1 \tag{7.9}$$

and let ϕ_p, ϕ'_p be associated with (w_+, w'_+) according to (5.22) *and Proposition* 5.2. *Assume that the following limits exist*

$$\lim_{t \to \infty} (\varphi(t) - \phi_p(t)) = \psi_+, \qquad \lim_{t \to \infty} (\varphi'(t) - \phi'_p(t)) = \psi'_+ \tag{7.10}$$

as strong limits in L^∞. Then $\psi'_+ = \psi_+ + \omega$.

Proof. *Part* (1). Let $\varphi_\pm(t) = \varphi'(t) \pm \varphi(t)$. By the same estimates as in the proof of Lemma 3.7, we obtain

$$\partial_t\, |\varphi_-|_{\ell'}^2 - 2\rho'\, |\varphi_-|_{\ell'+\nu/2}^2 \leqslant Ct^{-2}\, |\varphi_-|_{\ell'+\nu/2}\, \{|\varphi_-|_{\ell'+\nu/2}\, |\varphi_+|_\ell + |\varphi_-|_{\ell'}\, |\varphi_+|_{\ell+\nu/2}\} \tag{7.11}$$

for $\nu/2 \leqslant \ell' + 1 \leqslant \ell + \nu$. Defining $z_{(1)} = z_{(1)}(\varphi_-; [T_1, t], 1, \ell')$ with $T \leqslant T_1 < t$, integrating over time and using (7.6), we obtain

$$\begin{aligned} z^2 \vee z_1^2 &\leqslant z_0^2 + C(\operatorname*{Sup}_{[T_1, t]} t^{-2}\, |\rho'|^{-1}\, h_0)\, bz_1(z + z_1) \\ &\leqslant z_0^2 + Cbh_1(T_1)\, z_1(z + z_1), \end{aligned}$$

where $z_0 = |\varphi_-(T_1)|_{\ell'}$, which under the condition (7.8) with suitable c, yields

$$Z(\varphi_-; [T_1, \infty), 1, \ell') \leqslant C\, |\varphi_-(T_1)|_{\ell'}. \tag{7.12}$$

From

$$\partial_t \varphi_- = (2t^2)^{-1} \,(\nabla\varphi_- \cdot \nabla\varphi_+)$$

we next estimate directly

$$\begin{aligned} |\partial_t \varphi_-|_{\ell'-1} &\leqslant C\, t^{-2}\, |\varphi_-|_{\ell'}\, |\varphi_+|_{\ell} \\ &\leqslant C\, t^{-2}\, h_0\, b\, |\varphi_-(T_1)|_{\ell'} \end{aligned} \tag{7.13}$$

by (7.6) and (7.12). The last member of (7.13) is integrable in time since

$$\int_t^\infty dt_1\, t_1^{-2}\, h_0(t_1) \leqslant \|\rho'\|_1\, h_1(t), \tag{7.14}$$

which for $\ell' = \ell - 1 + v$ proves the existence of the limit $\omega \in Y^{\ell-2+v}_{\rho(\infty)}$ for φ_- together with the estimate (7.7). The fact that actually $\omega \in Y^{\ell-1+v}_{\rho(\infty)}$ and the other convergences stated in Part (1) follow therefrom and from (7.12) by standard compactness and interpolation arguments.

Parts (2) *and* (3). The proof is identical with that of the corresponding statements in Proposition II.7.2 and will be omitted. ∎

Remark 7.1. The assumptions made on (k, ℓ) in Parts (1) and (2) of Proposition 7.2 are implied by the assumptions (3.48) and $k \geqslant 1 - v/2$ of Proposition 4.1, so that Parts (1) and (2) apply directly to the solutions of the system (2.11)–(2.12) constructed in that proposition. In Part (3), the assumption (7.9) is that required in Proposition 5.2 to ensure the existence and appropriate estimates of ϕ_p, ϕ'_p. That assumption, together with the existence of the limits (7.10), is ensured under the assumptions of Proposition 5.3, namely if (k, ℓ) satisfy in addition (5.24) for some k_0 satisfying (5.25) and if $(p+2)\,\gamma > 1$ and $P_p(1) < \infty$.

Proposition 7.2 prompts us to make the following definition.

DEFINITION 7.3. Two pairs of asymptotic states (w_+, ψ_+) and (w'_+, ψ'_+) are gauge equivalent if $w_+ \exp(-i\psi_+) = w'_+ \exp(-i\psi'_+)$.

With this definition, Proposition 7.2 implies that two gauge equivalent solutions of the system (2.11)–(2.12) in $\mathscr{R}(\Omega_0)$ are images of two gauge equivalent pairs of asymptotic states. One should however not overlook the following fact. The wave operator Ω_0 is defined through Proposition 6.5 which uses an *increasing* ρ defined by (6.1) whereas Proposition 7.2 uses a *decreasing* ρ (essentially in the proof of (7.12)). In order to apply Proposition 7.2 to the solutions constructed in Proposition 6.5 with increasing ρ, one has therefore to take some large t_0, to define

$$\tilde{\rho}(t) = \rho(t_0) - \int_{t_0}^{t} \rho'(t_1)\, dt_1$$

and to apply Proposition 7.2 with that new $\tilde{\rho}$, thereby ending up with information on $(\omega, w_+, w'_+, \quad \psi_+, \psi'_+)$ in spaces K or Y associated with $\tilde{\rho}(\infty) = \rho(\infty) - 2\int_{t_0}^{\infty} \rho'(t)\, dt < \rho(\infty)$. That fact of course does not impair the algebraic relations expressing gauge invariance.

We now turn to the converse result, namely to the fact that gauge equivalent asymptotic states generate gauge equivalent solutions through Ω_0.

PROPOSITION 7.3. *Let* (k, ℓ) *and* (k_+, ℓ_+) *satisfy the assumptions of Proposition* 6.5, *namely* (3.48), $k \geqslant 1 - v/2$ *and* (6.38). *Let* ρ *and the estimating functions of time also satisfy the assumptions of Proposition* 6.5. *Let* $(w_+, \psi_+), (w'_+, \psi'_+) \in K^{k_+}_{\rho_\infty} \oplus Y^{\ell_+}_{\rho_\infty}$ *be gauge equivalent and let* $(w, \varphi), (w', \varphi')$ *be their images under* Ω_0. *Then* (w, φ) *and* (w', φ') *are gauge equivalent.*

Proof. The proof is identical with that of Proposition II.7.3. We reproduce it for completeness.

Let t_0 be sufficiently large and let (w_{t_0}, φ_{t_0}) and $(w'_{t_0}, \varphi'_{t_0})$ be the solutions of the system (2.11)–(2.12) constructed by Proposition 6.4. From the initial conditions

$$w_{t_0}(t_0) = V(t_0), \qquad w'_{t_0}(t_0) = V'(t_0),$$
$$\varphi_{t_0}(t_0) = \phi_p(t_0) + \chi(t_0), \qquad \varphi'_{t_0}(t_0) = \phi'_p(t_0) + \chi'(t_0),$$

from the fact that $\phi_p = \phi'_p$ by Proposition 5.2, part (2) and that $V \exp(-i\chi) = V' \exp(-i\chi')$ by Proposition 6.2, part (4), it follows that

$$w_{t_0}(t_0) \exp(-i\varphi_{t_0}(t_0)) = w'_{t_0}(t_0) \exp(-i\varphi'_{t_0}(t_0))$$

and therefore by Lemma 7.2, (w_{t_0}, φ_{t_0}) and $(w'_{t_0}, \varphi'_{t_0})$ are gauge equivalent, namely

$$w_{t_0}(t) \exp(-i\varphi_{t_0}(t)) = w'_{t_0}(t) \exp(-i\varphi'_{t_0}(t)) \tag{7.15}$$

for all t for which both solutions are defined.

We now take the limit $t_0 \to \infty$ for fixed t in (7.15). By Proposition 6.5, part (2), for fixed t, (w_{t_0}, φ_{t_0}) and $(w'_{t_0}, \varphi'_{t_0})$ converge respectively to (w, φ) and (w', φ') in $K^{k'}_{\rho} \oplus Y^{\ell'}_{\rho}$. By Lemma 3.3, one can take the limit $t_0 \to \infty$ in

(7.15), thereby obtaining (7.5), so that (w, φ) and (w', φ') are gauge equivalent. ▮

We can now define the (local) wave operators (at infinity) for u.

DEFINITION. The wave operator Ω is defined as the map

$$\Omega\colon u_+ \to u = (\Lambda \circ \Omega_0)(Fu_+, 0), \tag{7.16}$$

where Ω_0 and Λ are defined by Definition 7.1 and by (7.4).

We collect in the following proposition the information on Ω that follows from the previous study, in particular from Propositions 6.5 and 7.3.

PROPOSITION 7.4. *Let $p \geqslant 0$ be an integer with $(p+2)\gamma > 1$. Let ρ (defined by* (6.1)*) and the estimating functions of time satisfy the assumptions of Proposition* 6.5 *(see especially* (6.39) (6.40) *and Remark* 6.4*). Let*

$$\lambda = \mu - n + 2 \leqslant 2\nu, \tag{7.17}$$

$$k \geqslant 1 - \nu/2, \qquad k > 1 - \nu + \mu/2. \tag{7.18}$$

Let k_+ satisfy (6.38) *for some ℓ satisfying $\ell > n/2 - \nu$, $\ell \geqslant k - \nu$. Then*

(1) *The wave operator Ω maps $FK^{k_+}_{\rho\infty}$ to $X^k_\rho([T, \infty))$ for some T, $1 \leqslant T < \infty$. (Actually T depends on u_+).*

(2) *Ω is injective.*

Proof. *Part* (1) follows from the definitions, from Lemma 7.1 and from Proposition 6.5 with $\psi_+ = 0$. The only point to be checked is the fact that (7.17) (7.18) imply the existence of ℓ such that (k, ℓ) satisfies (3.48). Now the ℓ dependent part of (3.48) reduces to

$$k - \nu \leqslant \ell \leqslant k + \nu - \lambda \tag{7.19}$$

$$n/2 - \nu < \ell < 2k - \lambda + \nu - n/2 \tag{7.20}$$

and the compatibility of (7.19) (7.20) for ℓ is easily seen to reduce to (7.17) and to the second inequality in (7.18).

Part (2) follows from Proposition 7.2 and from the fact that a gauge equivalence class of asymptotic states contains at most one element with $\psi_+ = 0$. ▮

Remark 7.2. One may wonder whether the restriction to asymptotic states with $\psi_+ = 0$ in (7.16) restricts the range of Ω as compared with that of $\Lambda \circ \Omega_0$. From Proposition 7.3, it follows that

$$(\Lambda \circ \Omega_0)(w_+, \psi_+) = (\Lambda \circ \Omega_0)(w_+ \exp(-i\psi_+), 0)$$

in so far as $w_+ \exp(-i\psi_+)$ has the regularity needed to apply Proposition 6.5. This is the case if $\ell_+ \geqslant k_+ - 2$ by Lemma 3.3. That condition however is significantly stronger than the condition on ℓ_+ contained in (6.38), especially for large p, i.e. for small γ. Therefore, there is actually a restriction of the range for regularity reasons, in spite of the (algebraic) gauge invariance of the construction expressed by Proposition 7.3.

We next collect the information and in particular the asymptotic estimates obtained for the solutions of the equation (1.1) in $\mathscr{R}(\Omega)$.

PROPOSITION 7.5. *Let* $0 < \mu \leqslant n - 2 + 2\nu \leqslant n$. *Let* $0 < \gamma \leqslant 1$ *and let* $p \geqslant 0$ *be an integer with* $(p+2)\gamma > 1$. *Let* ρ *(defined by* 6.1*) and the estimating functions of time satisfy the assumptions of Proposition* 6.5 *(especially* (6.39) (6.40)*). Let* (k, ℓ, k_+) *satisfy* (7.18) (3.48) (6.38). *Let* $u_+ \in FK^{k_+}_{\rho\infty}$ *and* $a_+ = \| Fu_+; K^{k_+}_{\rho\infty} \|$. *Then*

(1) *There exists* T, $1 \leqslant T < \infty$, *and there exists a unique solution* $u \in X^k_\rho([T, \infty))$ *of the equation* (1.1) *which can be represented as*

$$u = M\, D \exp(-i\varphi)\, w$$

where (w, φ) *is a solution of the system* (2.11)–(2.12) *such that* $(w, \bar{h}_0^{-1}\varphi) \in \mathscr{X}^{k,\ell}_\rho([T, \infty))$ *and such that*

$$|w(t) - Fu_+|_{k-1+\nu}\, h_0(t) \to 0 \tag{7.21}$$

$$|\varphi(t) - \phi_p(t)|_{\ell-1+\nu} \to 0 \tag{7.22}$$

when $t \to \infty$. *The time* T *can be defined by a condition of the type* (6.41) *with* $b_+ = 0$.

(2) *The solution is obtained as* $u = \Omega u_+$, *following Definition* 7.4.

(3) *The map* Ω *is continuous from* $FK^{k_+}_{\rho\infty}$ *to the norm topology of* $X^{k'}_\rho(I)$ *for* $k' < k$ *and to the weak-$*$ topology of* $X^k_\rho(I)$ *for any compact interval* $I \subset [T, \infty)$, *and to the weak topology of* MDK^k_ρ *pointwise in* t.

(4) *The solution* u *satisfies the following estimate for* $t \geqslant T$:

$$\| \langle J(t) \rangle^k f(J(t))(\exp[i\phi_p(t, x/t)]\, u(t) - M(t)\, D(t)\, Fu_+)\|_2 \leqslant A(a_+)\, h_3(t) \tag{7.23}$$

for some estimating function $A(a_+)$.

(5) *Let $2 \leqslant r < \infty$. The solution u satisfies the following estimate:*

$$\|u(t) - \exp[-i\phi_p(t, x/t)]\, M(t)\, D(t)\, Fu_+\|_r \leqslant A(a_+)\, \rho(t)^{-\beta}\, t^{-\delta(r)} h_3(t), \tag{7.24}$$

where $\delta(r) = n/2 - n/r$, $\beta = \nu^{-1}(\delta(r) - k) \vee 0$ if $r < \infty$ or $k > n/2$, $\beta = \nu^{-1}(n/2 - k + \varepsilon)$ if $r = \infty$ and $k \leqslant n/2$.

Proof. Parts (1) (2) (3) follow from Propositions 6.5 and 4.3, from Definition 7.4 and from Proposition 7.4.

Part (4). From the commutation relation (7.2), from Remark 3.1 and from Lemma 3.3, it follows that the LHS of (7.23) is estimated by

$$\begin{aligned}
\|\cdot\|_2 &\leqslant C\, |\exp(i(\phi_p - \varphi))\, w - Fu_+|_k \\
&\leqslant C\{|w - Fu_+|_k + |\exp(i(\phi_p - \varphi)) - 1|_\ell\, |w|_k\} \\
&\leqslant C\{|w - W_p|_k + |W_p - w_+|_k + \exp(C\, |\phi_p - \varphi|_\ell)\, |\phi_p - \varphi|_\ell\, |w|_k\}
\end{aligned}$$

and the result follows from the estimates (6.72) (5.9) (6.73) (6.74).

Part (5) follows from Part (4) and from the inequality

$$\begin{aligned}
\|v\|_r &= t^{-\delta(r)}\, \|D^* M^* v\|_r \leqslant C t^{-\delta(r)}\, \|\langle \nabla \rangle^m\, D^* M^* v\|_2 \\
&= C t^{-\delta(r)}\, \|\langle J(t) \rangle^m\, v\|_2
\end{aligned}$$

which follows from (7.2) and Sobolev inequalities with $m = \delta(r)$ if $r < \infty$, $m = n/2 + \varepsilon$ if $r = \infty$, together with the fact that

$$|\xi|^{m-k}\, f(\xi)^{-1} \leqslant C\rho^{-\beta}. \quad \blacksquare$$

Remark 7.3. In (7.23) and (7.24), one could replace $MDFu_+$ by $U(t)\, u_+$ since the difference is small in the relevant norms. One could also replace Fu_+ by W_p, but this would not produce any improvement in the estimates, since the main contribution comes from the phase. The estimate (7.24) is a rather weak one, since we have omitted the function f which generates the Gevrey regularity. It is only one example of a large number of similar estimates exhibiting the typical $t^{-\delta(r)}$ decay associated with L^r norms.

The final step of the standard construction of the wave operators for the equation (1.1) would consist in extending the solutions u to arbitrary finite times, and defining the maps $\Omega_1 \colon u_+ \to u(1)$ where $u = \Omega u_+$. In order not to waste the Gevrey regularity of the local solutions at infinity, this would require a treatment of the global Cauchy problem at finite times for arbitrarily large data in the same Gevrey framework. This is a rather different problem and we shall refrain from considering it here.

APPENDIX A

In this appendix we derive a number of properties of the function $\tilde{f}$ defined by (3.2), which make it a possible substitute for the function f_0 defined by (3.1) in the definition of the spaces $\mathscr{X}_\rho^{k,\ell}$. In all this appendix, we assume $0 < \nu \leqslant 1$ and we take $\rho = 1$. The parameter ρ can be reintroduced easily by scaling. Accordingly we define

$$\tilde{f}(\xi) = \sum_{j \geqslant 0} (j!)^{-1/\nu} |\xi|^j, \tag{A.1}$$

$$F(\xi) = \sum_{j \geqslant 0} (j+1)^{-1} (j!)^{-1/\nu} |\xi|^{j+1}. \tag{A.2}$$

We also use (A.1) and (A.2) to define $\tilde{f}$ and F when applied to $\xi \in \mathbb{R}^+$. With that convention, we have $\tilde{f}(\xi) = \tilde{f}(|\xi|)$ and $F(\xi) = F(|\xi|)$ for all $\xi \in \mathbb{R}^n$. Furthermore $\tilde{f} = dF/d|\xi|$.

We first derive some preliminary estimates which allow in particular for a comparison of $\tilde{f}$ and f_0.

Lemma A.1. *The following estimates hold for all $\xi \in \mathbb{R}^n$:*

$$\sum_{j \geqslant 1} j(j!)^{-1/\nu} |\xi|^j \leqslant |\xi|^\nu \sum_{j \geqslant 0} (j!)^{-1/\nu} |\xi|^j \leqslant \sum_{j \geqslant 0} (j+1)(j!)^{-1/\nu} |\xi|^j, \tag{A.3}$$

$$\sum_{j \geqslant 0} (j+1)^{-1} (j!)^{-1/\nu} |\xi|^{j+1} \leqslant |\xi|^{1-\nu} \sum_{j \geqslant 0} (j!)^{-1/\nu} |\xi|^j, \tag{A.4}$$

$$\tilde{f}^{-1}(d\tilde{f}/d|\xi|) \leqslant |\xi|^{\nu-1} \leqslant F^{-1}\tilde{f} = F^{-1}(dF/d|\xi|), \tag{A.5}$$

$$|\xi|^\nu \tilde{f} \leqslant d(|\xi| \tilde{f})/d|\xi|, \tag{A.6}$$

$$F(a)(|\xi| \vee a)^{\nu-1} \exp(\nu^{-1}(|\xi|^\nu - a^\nu)) \leqslant \tilde{f}(\xi) \leqslant \exp(\nu^{-1} |\xi|^\nu) \tag{A.7}$$

for all $a > 0$,

$$\tilde{f}(\xi) = (2\pi)^{(\nu-1)/2\nu} \nu^{1/2} |\xi|^{(\nu-1)/2} \exp(\nu^{-1} |\xi|^\nu)(1 + o(1)) \qquad \textit{when} \quad |\xi| \to \infty. \tag{A.8}$$

Proof. (A.3a) and (A.4) follow from the Hölder inequality on $\mathbb{Z}^+$ with the measure $(j!)^{-1/\nu} |\xi|^j$ and the exponents $1/\nu$ and $1/(1-\nu)$, applied respectively to the pairs of functions $(j, 1)$ and $((j+1)^{-1}, 1)$.

(A.3b) follows similarly from the Hölder inequality with the measure $(j+1)(j!)^{-1/\nu} |\xi|^j$ and the exponents $\nu^{-1}(1+\nu)$ and $1+\nu$, applied to the functions $(j+1)^{-1}$ and 1.

(A.5) is a rewriting of (A.3a) and (A.4), while (A.6) is a rewriting of (A.3b).

(A.7) follows from (A.5). In fact (A.5) states that the functions $\tilde{f}(\xi)\exp(-|\xi|^\nu/\nu)$ and $F(\xi)\exp(-|\xi|^\nu/\nu)$ are respectively decreasing (and therefore less than one) and increasing in $|\xi|$. The first fact yields (A.7b) while both of them together with $\tilde{f} \geqslant |\xi|^{\nu-1} F$ yield (A.7a). Note also that (A.7b) follows directly from the definition (A.1) and from the fact that

$$\tilde{f}(\xi) = \|(j!)^{-1} |\xi|^{\nu j}; \ell^{1/\nu}\|^{1/\nu} \leqslant \|(j!)^{-1} |\xi|^{\nu j}; \ell^1\|^{1/\nu} = \exp(\nu^{-1} |\xi|^\nu)$$

by the standard embedding of ℓ^p spaces.

(A.8) is proved in [15] (see (8.07) p. 308) in the special case $\nu \geqslant 1/4$, but the proof extends easily to the whole range $0 < \nu \leqslant 1$. ▮

The estimates (A.7) and the asymptotic property (A.8) compare $\tilde{f}$ with f_0 by stating that essentially $\tilde{f} \sim f_0^{1/\nu}$. On the other hand (A.6) is the analogue of the fact that $df_0/d|\xi| = \nu |\xi|^{\nu-1} f_0$ and allows for the construction of function space norms satisfying an inequality which can replace (3.12) in the subsequent estimates.

We next show that $\tilde{f}$ satisfies estimates similar to those of Lemma 3.1.

Lemma A.2. *Let $(\xi, \eta) \in \mathbb{R}^n$. Then $\tilde{f}$ satisfies the estimates*

$$\tilde{f}(\xi) \leqslant \tilde{f}(\xi-\eta)\,\tilde{f}(\eta) \qquad \textit{for all } (\xi,\eta), \tag{A.9}$$

$$\tilde{f}(\xi) \leqslant \tilde{f}(\xi-\eta)\exp(|\eta|^\nu) \qquad \textit{for} \quad |\xi| \wedge |\eta| \leqslant |\xi-\eta|, \tag{A.10}$$

$$(\tilde{f}(\eta)-\tilde{f}(\xi))\,|\eta|^{1-\nu} \leqslant |\xi-\eta|\,\tilde{f}(\eta) \qquad \textit{for} \quad |\xi| \leqslant |\eta|, \tag{A.11}$$

$$|\tilde{f}(\xi)-\tilde{f}(\eta)|\,|\eta|^{1-\nu} \leqslant |\xi-\eta|^{1-\nu}\,\tilde{f}(\xi-\eta)\,\tilde{f}(\eta) \qquad \textit{for all } (\xi,\eta), \tag{A.12}$$

$$|\tilde{f}(\xi)-\tilde{f}(\eta)|\,|\eta|^{1-\nu} \leqslant |\xi-\eta|^{1-\nu}(\exp(|\xi-\eta|^\nu)-1)\,\tilde{f}(\eta) \qquad \textit{for} \quad |\xi| \wedge |\xi-\eta| \leqslant |\eta|, \tag{A.13}$$

$$|\tilde{f}(\xi)-\tilde{f}(\eta)|\,|\eta|^{1-\nu} \leqslant C\,|\xi-\eta|^{1-\nu}(\tilde{f}(\xi-\eta)-1)\exp(|\eta|^\nu) \qquad \textit{for} \quad |\xi| \wedge |\eta| \leqslant |\xi-\eta|. \tag{A.14}$$

In (A.14) *one can take $C = 1$, except in the region $|\xi| \leqslant |\xi-\eta| \leqslant |\eta|$ where $C = 2^{1-\nu}$.*

The function $\tilde{f}(\xi) \vee \tilde{f}(1)$ satisfies the same estimates as $\tilde{f}(\xi)$.

Proof. (A.9) is trivial except if $|\xi| \geqslant |\eta| \vee |\xi-\eta|$. In all cases, we estimate

$$\begin{aligned}\tilde{f}(\xi) &\leqslant \sum_{j,k \geqslant 0} (j!)^{-1}(k!)^{-1}((j+k)!)^{1-1/\nu} |\xi-\eta|^j |\eta|^k \\ &\leqslant \sum_{j,k \geqslant 0} (j!)^{-1/\nu}(k!)^{-1/\nu} |\xi-\eta|^j |\eta|^k = \tilde{f}(\xi-\eta)\,\tilde{f}(\eta)\end{aligned} \tag{A.15}$$

since $(j+k)! \geqslant j!k!$.

(A.10) is trivial in the allowed region except if $|\eta| \leqslant |\xi-\eta| \leqslant |\xi|$. In that case, we rewrite the first inequality in (A.15) as

$$\begin{aligned}\tilde{f}(\xi) &\leqslant \sum_{k\geqslant 0}\left\{\sum_{j\geqslant 0}((j!)^{-1/\nu}|\xi-\eta|^j)^\nu(((j+k)!)^{-1/\nu}|\xi-\eta|^{j+k})^{1-\nu}\right\}\\ &\quad\times(k!)^{-1}|\eta|^k|\xi-\eta|^{(\nu-1)k}\\ &\leqslant \tilde{f}(\xi-\eta)\exp(|\eta|\,|\xi-\eta|^{\nu-1})\leqslant \tilde{f}(\xi-\eta)\exp(|\eta|^\nu)\end{aligned} \tag{A.16}$$

by the Hölder inequality applied to the sum over j for fixed k and the fact that $|\eta| \leqslant |\xi-\eta|$.

(A.11). We estimate

$$\begin{aligned}\tilde{f}(\eta)-\tilde{f}(\xi) &= \sum_{j\geqslant 1}(j!)^{-1/\nu}(|\eta|^j-|\xi|^j)\leqslant|\xi-\eta|\sum_{j\geqslant 1}(j!)^{-1/\nu}\,j\,|\eta|^{j-1}\\ &\leqslant|\xi-\eta|\,|\eta|^{\nu-1}\,\tilde{f}(\eta)\end{aligned}$$

by (A.3a).

(A.12) follows from (A.11) and from $|\xi-\eta|^\nu \leqslant \tilde{f}(\xi-\eta)$ if $|\xi| \leqslant |\eta|$. If $|\xi| \geqslant |\eta|$, we estimate

$$\begin{aligned}\tilde{f}(\xi)-\tilde{f}(\eta) &\leqslant \sum_{j\geqslant 1,\,k\geqslant 0}(j!)^{-1}(k!)^{-1}((j+k)!)^{1-1/\nu}|\xi-\eta|^j|\eta|^k\\ &= \sum_{j,\,k\geqslant 0}((j+1)!)^{-1}(k!)^{-1}((j+k+1)!)^{1-1/\nu}|\xi-\eta|^{j+1}|\eta|^k\\ &\leqslant \sum_{j,\,k\geqslant 0}(j+1)^{-1}(j!)^{-1/\nu}|\xi-\eta|^{j+1}(k+1)((k+1)!)^{-1/\nu}|\eta|^k\end{aligned}$$

since $(j+k+1)! \geqslant j!(k+1)!$,

$$\cdots\leqslant|\xi-\eta|^{1-\nu}\,\tilde{f}(\xi-\eta)\,|\eta|^{\nu-1}\,\tilde{f}(\eta)$$

by (A.3a) and (A.4).

(A.13) follows from (A.11) and $|\xi-\eta|^\nu \leqslant (\exp(|\xi-\eta|^\nu)-1)$ if $|\xi| \leqslant |\eta|$. If $|\xi-\eta| \leqslant |\eta| \leqslant |\xi|$, we estimate in the same way as in (A.16)

$$\begin{aligned}(\tilde{f}(\xi)-\tilde{f}(\eta))\,|\eta|^{1-\nu} &\leqslant |\eta|^{1-\nu}\sum_{j\geqslant 1}\left\{\sum_{k\geqslant 0}((k!)^{-1/\nu}|\eta|^k)^\nu(((j+k)!)^{-1/\nu}|\eta|^{j+k})^{1-\nu}\right\}\\ &\quad\times(j!)^{-1}|\xi-\eta|^j|\eta|^{(\nu-1)j}\\ &\leqslant \tilde{f}(\eta)\{|\eta|^{1-\nu}(\exp(|\xi-\eta|\,|\eta|^{\nu-1})-1)\}\end{aligned} \tag{A.17}$$

by the Hölder inequality applied to the sum over k for fixed j. Now for fixed $|\xi-\eta|$, the last bracket in (A.17) is a decreasing function of $|\eta|$ and for

$|\eta|\geqslant|\xi-\eta|$ is therefore less than its value for $|\eta|=|\xi-\eta|$, which yields (A.13) in that case.

(A.14). If $|\xi|\vee|\eta|\leqslant|\xi-\eta|$, (A.14) with $C=1$ follows from $|\eta|\leqslant|\xi-\eta|$ and from

$$|\tilde{f}(\xi)-\tilde{f}(\eta)|\leqslant\tilde{f}(\xi-\eta)-1.$$

If $|\eta|\leqslant|\xi-\eta|\leqslant|\xi|$, (A.14) with $C=1$ follows from $|\eta|\leqslant|\xi-\eta|$ and from a minor variant of (A.16) with the sum over j restricted to $j\geqslant 1$, so that

$$\tilde{f}(\xi)-\tilde{f}(\eta)\leqslant(\tilde{f}(\xi-\eta)-1)\exp(|\eta|^{\nu}). \tag{A.18}$$

If $|\xi|\leqslant|\xi-\eta|\leqslant|\eta|$, (A.14) with $C=2^{1-\nu}$ follows from $|\eta|\leqslant 2\,|\xi-\eta|$ and from (A.18) with ξ and η interchanged.

The last statement of Lemma A.2 is obvious as regards (A.9) (A.10) and follows from the fact that

$$|\tilde{f}(\xi)\vee a-\tilde{f}(\eta)\vee a|\leqslant|\tilde{f}(\xi)-\tilde{f}(\eta)|$$

for all ξ, η and $a>0$ as regards (A.11)–(A.14), in the same way as in Lemma 3.1. ∎

It follows from Lemma A.2 that $\tilde{f}(\xi)$ and $\tilde{f}(\xi)\vee\tilde{f}(1)$ satisfy the basic estimates (A.9) and (A.12) which are used throughout this paper, thereby making those functions into suitable substitutes for f_0 and f in the definition of the spaces. Note also that by (A.7) (A.8), $\exp(|\eta|^{\nu})\sim\tilde{f}(\eta)^{\nu}$ (up to a small power of η), which makes (A.10) (A.13) (A.14) into close analogues of (3.4) (3.6) (3.7).

We finally use the function $\tilde{f}$ to relate the definition of spaces such as K^k_ρ or Y^ℓ_ρ to more standard definition of Gevrey spaces. Since this part is meant to be only illustrative, we restrict our attention to space dimension $n=1$. The Gevrey class $G_{1/\nu}$ can be defined as the vector space of $\mathscr{C}^\infty$ functions u such that there exists a constant C such that

$$c_j C^{-j}(j!)^{-1/\nu}\,\partial^j u\in\ell^p_j(L^q_x), \tag{A.19}$$

where $1\leqslant p,\,q\leqslant\infty$ and $\{c_j\}$ is a sequence of positive numbers with at most polynomial increase or decrease at infinity. Of course if one allows for all possible $C>0$, the parameters p, q, and $\{c_j\}$ are irrelevant, and one can take $p=q=\infty$ and $c_j\equiv 1$, which yields the standard definition. Here however we fix C, in fact $C=\rho^{-1/\nu}$, so as to obtain a Banach space, and those parameters become important. Since moreover we want a Hilbert space in order to apply the energy method in a convenient way, we take $p=q=2$, thereby obtaining the Gevrey Hilbert space X with norm

$$\|u;X\|=\|b_j\rho^{j/\nu}\partial^j u;\,\ell^2_j(L^2_x)\|, \tag{A.20}$$

where $b_j = c_j(j!)^{-1/\nu}$. Let now

$$f(\xi) = f(|\xi|) = \sum_{j \geqslant 0} a_j \, |\xi|^j = f_+(\xi) + f_-(\xi),$$

where $\{a_j\}$ is a sequence of positive numbers and where f_+ and f_- denote the sums over even and odd j respectively. We claim that for a suitable relation between $\{a_j\}$ and $\{b_j\}$, the norm in X is equivalent to the norm

$$\|u\|_* = \|f(\rho^{1/\nu}\xi)\, \hat{u}\|_2. \tag{A.21}$$

In fact

$$f_+(\xi)^2 + f_-(\xi)^2 \leqslant f(\xi)^2 \leqslant 2(f_+(\xi)^2 + f_-(\xi)^2)$$

so that if we define $\{b_j\}$ by

$$f_+(\xi)^2 + f_-(\xi)^2 = \sum_{j \geqslant 0} b_j^2 \, |\xi|^{2j} \tag{A.22}$$

then

$$\|u; X\| \leqslant \|u\|_* \leqslant \sqrt{2}\, \|u; X\| \tag{A.23}$$

which proves the equivalence. The relation (A.22) can be rewritten as

$$b_k^2 = \sum_{0 \leqslant j \leqslant 2k} a_j a_{2k-j}.$$

In the special case of $\tilde{f}$, where $a_j = (j!)^{-1/\nu}$, it is obvious that

$$a_j \leqslant b_j \leqslant \sqrt{2j+1}\, a_j$$

and one can show that actually when $j \to \infty$

$$b_j = a_j(\pi \nu j)^{1/4}\,(1 + o(1)) \tag{A.24}$$

which together with (A.8) makes it possible to define equivalent norms of the form (A.20) for the spaces K_ρ^k and Y_ρ^ℓ.

APPENDIX B

In this appendix we prove a Lemma which exemplifies the fact that for $\rho > 0$ and $\nu < 1$, the lower condition $\ell + 2 > n/2$ on ℓ needed in Lemma 3.3 in order to make Y_ρ^ℓ into an algebra can be relaxed by using the estimate

(3.4) instead of (3.3) (but not uniformly in ν and ρ). Following Remark 3.1, we consider the Hilbert space K with norm

$$\|u; K\| = \|\bar{f}\hat{u}\|_2$$

where $\bar{f}$ is either ff_1 or $f_0 f_1$ with f_0, f defined by (3.1) and f_1 defined by (3.10) for some $k_<, k_> \in \mathbb{R}^+$.

LEMMA B.1. *Let K be as above, with $0<\nu<1$, $\rho>0$, $k_> \geqslant 0$ and $0 \leqslant k_< < n/2$. Then K is an algebra, namely there exists a constant C such that for all $u_1, u_2 \in K$*

$$\|u_1 u_2; K\| \leqslant C \, \|u_1; K\| \, \|u_2; K\|. \tag{B.1}$$

One can take

$$C^2 = \int d\eta \; \bar{f}(\eta)^{-2} \, (1 + 2^{2k} f_0(\eta)^{2\nu}) \tag{B.2}$$

where $k = k_< \vee k_>$ and the integral converges under the assumptions made on ν, ρ and $k_<$.

Proof. By the Schwarz inequality, we estimate

$$\begin{aligned}
\|\bar{f}\,\widehat{u_1 u_2}\,\|_2^2 &= \int d\xi \; \bar{f}(\xi)^2 \left| \int d\eta \; \hat{u}_1(\eta) \; \hat{u}_2(\xi-\eta) \right|^2 \\
&\leqslant \int d\xi \; \bar{f}(\xi)^2 \left\{ \int d\eta \; \bar{f}(\eta)^{-2} \; \bar{f}(\xi-\eta)^{-2} \right\} \\
&\quad \times \left\{ \int d\eta \; \bar{f}(\eta)^2 \; \bar{f}(\xi-\eta)^2 \; |\hat{u}_1(\eta)|^2 \; |\hat{u}_2(\xi-\eta)|^2 \right\} \leqslant C^2 \; \|\bar{f}\hat{u}_1\|_2^2 \; \|\bar{f}\hat{u}_2\|_2^2
\end{aligned}$$

where

$$\begin{aligned}
C^2 &= \operatorname*{Sup}_{\xi} \; \bar{f}(\xi)^2 \int d\eta \; \bar{f}(\eta)^{-2} \; \bar{f}(\xi-\eta)^{-2} \\
&= 2 \operatorname*{Sup}_{\xi} \; \bar{f}(\xi)^2 \int_{|\eta| \leqslant |\xi-\eta|} \bar{f}(\eta)^{-2} \; \bar{f}(\xi-\eta)^{-2}.
\end{aligned}$$

Now for $|\xi| \leqslant |\xi-\eta|$, $\bar{f}(\xi) \leqslant \bar{f}(\xi-\eta)$ while for $|\eta| \leqslant |\xi-\eta| \leqslant |\xi|$ one has $|\xi| \leqslant 2\,|\xi-\eta|$ so that

$$f_1(\xi) \leqslant 2^{k_< \vee k_>} f_1(\xi-\eta) = 2^k f_1(\xi-\eta),$$

and

$$f_{(0)}(\xi) \leqslant f_{(0)}(\xi-\eta)\, f_0(\eta)^{\nu}$$

by (3.4). Therefore

$$C^2 \leqslant 2 \operatorname*{Sup}_{\xi} \left\{ \int_{|\eta| \vee |\xi| \leqslant |\xi-\eta|} d\eta\, \bar{f}(\eta)^{-2} + \int_{|\eta| \leqslant |\xi-\eta| \leqslant |\xi|} d\eta\, \bar{f}(\eta)^{-2}\, 2^{2k} f_0(\eta)^{2\nu} \right\}.$$

The Supremum over ξ is easily seen to be the limit $|\xi| \to \infty$, namely

$$C^2 \leqslant 2 \left\{ \int_{\xi\cdot\eta \leqslant 0} d\eta\, \bar{f}(\eta)^{-2} + \int_{\xi\cdot\eta \geqslant 0} d\eta\, \bar{f}(\eta)^{-2}\, 2^{2k} f_0(\eta)^{2\nu} \right\}$$

$$= \int d\eta\, \bar{f}(\eta)^{-2}\, (1 + 2^{2k} f_0(\eta)^{2\nu})$$

and the integral converges for small η by the condition $k_< < n/2$ and for large η by the conditions $\rho > 0$ and $\nu < 1$. ∎

The same result with essentially the same proof holds if one replaces f_0 by $\tilde{f}$ in the definition of K. One then has to replace (3.4) by (A.10) in the proof. The same result also holds for arbitrary $k_> \in \mathbb{R}$ (still with $0 \leqslant k_< < n/2$), but the proof is more cumbersome for $k_> < 0$.

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