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On the Representation of Triangulated Graphs in Trees

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The intersection graph of a family of subtrees of a tree is always a triangulated graph, and vice versa every finite triangulated graph can be represented in this way. The analogous statement is in general not true for infinite triangulated graphs. A characterization of the graphs which are "tree-representable" in this sense is given by means of simplicial decompositions.

1. INTRODUCTION

Let us call a graph G *tree-representable* if there exist a tree T and a family T_i $(i \in I)$ of subtrees of T such that G is isomorphic to the intersection graph of this family. It is easy to see that every tree-representable graph G is triangulated, i.e. there is no circuit of length $\geqslant 4$ which is an induced subgraph of G.* In the case of finite G a well-known theorem, first announced by L. Surányi (see [5], p. 584) and independently proven by several authors ([1], [3], [9]), says that vice versa G must be tree-representable if it is triangulated (see also [4], chap. 4 for further reference).

In the present note the same problem is treated if the hypothesis that G be finite is omitted. We shall see that in general an infinite triangulated graph is not tree-representable. In our main theorem we shall characterize the tree-representable graphs as those graphs G which have a simplicial decomposition with cliques of G as its members such that a certain condition on the simplices of attachment is fulfilled (see Section 2 for definitions). The main idea of our argumentation is to make use of the tree structure which is naturally associated with a simplicial decomposition. As a corollary we get the extension of L. Surányi's result to all graphs which do not contain an infinite complete subgraph.

Graphs considered in this paper may be infinite, but are undirected and do not contain loops or multiple edges. Instead of "complete graph" we say "simplex". A *clique* of a graph is a maximal (with respect to inclusion) simplex contained in G. Subgraphs of G induced by certain vertices $x, y, z, \dots$ are denoted by $G[x, y, z, \dots]$.

2. SIMPLICIAL DECOMPOSITIONS

In this section we give a brief description of the main tool in our argumentations, namely the simplicial decompositions and prime graph decompositions of graphs. For details and proofs see [6] or [8], chap. X.

Let G be a graph, $\sigma > 0$ an ordinal, and assume, for each ordinal $\lambda < \sigma$, G_λ to be a subgraph of G. Then the family of the G_λ (in this indexing) forms a *simplicial decomposition* of G, if the following conditions are fulfilled:

(a) G is the union of the G_λ;

(b) For every τ, $0 < \tau < \sigma$, $S_\tau := (\bigcup_{\lambda<\tau} G_\lambda) \cap G_\tau$ is a simplex;

(c) S_τ is properly contained in both $\bigcup_{\lambda<\tau} G_\lambda$ and in G_τ (for all τ, $0 < \tau < \sigma$).

The S_τ are called the *simplices of attachment* of the given decomposition.

It follows that each member of a simplicial decomposition of G must be an induced subgraph of G, further that each S_τ $(0 < \tau < \sigma)$ separates in G each vertex of $(\bigcup_{\lambda<\tau} G_\lambda) - S_\tau$ from each vertex of $G_\tau - S_\tau$.

* In his poineering paper [2] Dirac calls these graphs "rigid circuit graphs". Also the notion "chordal graph" is used.

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A graph is called *prime* if it has no simplicial decomposition with at least two members or, equivalently, if it is not separated by a simplex. Each prime induced subgraph P of a graph G is contained in a maximally prime induced subgraph $\hat{P}$ of G (by Zorn's Lemma). A simplicial decomposition is a *prime graph decomposition* (briefly: pgd) if each of its members is prime.

(1) *A graph G is triangulated if and only if each prime induced subgraph of G is a simplex (or, equivalently, if and only if the cliques are exactly the maximally prime induced subgraphs of G).*

Not every triangulated graph has a pgd (see [8], p. 164 for an example). Since every graph without an infinite simplex admits a pgd ([8], p. 166) we have

(2) *Every triangulated graph without an infinite simplex has a pgd which has just its cliques as its members.*

In general an infinite triangulated graph may have more cliques than vertices (in cardinality). For example the comparability graph of a rooted tree which is regular of finite degree $r \geq 3$ has uncountably many cliques and uncountably many pgd's which have pairwisely no member in common.

In a simplicial decomposition G_λ $(\lambda < \sigma)$ some G_λ may be contained in some G_τ with $\lambda \neq \tau$; then necessarily $\lambda < \tau$ and $G_\lambda \subseteq S_\tau$. If $\lambda \neq \tau$ implies $G_\lambda \not\subseteq G_\tau$, for all $\lambda, \tau < \sigma$, then this simplicial decomposition is called *reduced*. It has been shown in [8], p. 162:

(3) *If a graph G has a pgd then it also has a reduced pgd.*

A pgd G_λ $(\lambda < \sigma)$, with simplices of attachment S_λ, will be called *strict* if for every τ, $0 < \tau < \sigma$, there is an $f(\tau) < \tau$ such that $S_\tau \subseteq G_{f(\tau)}$. It follows easily (see [8], p. 152, (1.2)):

(4) *If all the simplices of attachment of a pgd are finite, then this pgd is strict.*

Furthermore the construction in the proof of (3) in [8], p. 162 yields

(5) *If a graph has a strict pgd then it also has a strict reduced pgd.*

Let G be a graph having a pgd G_λ $(\lambda < \sigma)$, and let H be an induced subgraph of G. Put $H_\lambda = G_\lambda \cap H$. If H is not a simplex there is a smallest τ such that the H_λ $(\lambda \leq \tau)$ do not form a chain. Then $\bigcup_{\lambda < \tau} H_\lambda$, followed by the H_λ with $\lambda \geq \tau$, form a simplicial decomposition of H if those H_λ are omitted which are contained in an "earlier" member of this family. If G is triangulated each H_λ is a simplex, and therefore we have again a pgd. If the given pgd of G is strict then the latter pgd of H is strict too. Thus we can state

(6) *If a triangulated graph G has a strict pgd then also every induced subgraph of G has a strict (reduced) pgd.*

3. The Helly Property

THE HELLY PROPERTY. *A family G_i $(i \in I)$ of graphs (or sets) is said to have the Helly property (H) if for each $J \subseteq I$ holds:*
If $G_i \cap G_j \neq \emptyset$ for any two $i, j \in J$, then $\bigcup_{i \in J} G_i \neq \emptyset$.

It is well-known that for a finite tree T every family of subtrees of T enjoys the Helly property. The analogue is in general not true if T is infinite: For example, in the family of all infinite subpaths of a one-sided infinite path among any two members one contains the other, but the intersection of the entire family is empty. We will however give a compensation for (H) by considering the so-called ends of the infinite tree in question. First we state a Lemma.

(7) *Let G be a graph without an infinite path. Let Z_i $(i \in I)$ be a chain of connected non-empty subgraphs of G (i.e. I can be considered to be linearly ordered in such a way that $i<j \Rightarrow Z_i \supseteq Z_j$). Then $\bigcap_{i \in I} Z_i \neq \emptyset$.*

Proof. Assume, on the contrary, $\bigcap_{i\in I} Z_i = \emptyset$. Choose $i_0 \in I$ and $x_0 \in V(Z_{i_0})$. By assumption there exists $i_1 > i_0$ with $x_0 \notin V(Z_{i_1})$. By the connectedness of Z_{i_0} there is a path P_1 in Z_{i_0} connecting x_0 with some $x_1 \in V(Z_{i_1})$ and having only x_1 in common with Z_{i_1}. Analogously there is $i_2 > i_1$ with $x_1 \notin V(Z_{i_2})$; choose a path P_2 in Z_{i_1} connecting x_1 with some $x_2 \in V(Z_{i_2})$ and having only x_2 in common with Z_{i_2}. This procedure can be repeated infinitely often, resulting in an infinite sequence of paths $P_1, P_2, P_3, \ldots$ whose union would form an infinite path in G, with contradiction.

Two one-sided infinite paths U, V in a graph G are called *equivalent* in G, briefly: $U \sim_G V$, if there is a third one-sided infinite path W meeting both U and V in an infinite number of vertices. $\sim_G$ is an equivalence relation in the set of one-sided infinite paths of G, and each equivalence class with respect to $\sim_G$ is called an *end* of G (see [7]). If T is a tree and r a fixed vertex of T, then the ends of T are in natural one-to-one correspondence with the infinite paths starting in r. For an end $\mathfrak{E}$ of T the path of $\mathfrak{E}$ starting in r is called the r, $\mathfrak{E}$-path of T.

Let S be a subtree of the tree T. An end $\mathfrak{E}$ of T is said to belong to S if there is an infinite subpath of S which lies in $\mathfrak{E}$. (Thus S is considered to be completed by adding all those ends $\mathfrak{E}$ of T as "figurative vertices" such that an infinite subpath of S lies in $\mathfrak{E}$.) Then we have:

(8) *Any family of subtrees T_i $(i \in I)$ of a tree T satisfies the following modified Helly condition* (H*): *If, for some $J \subseteq I$, $T_i \cap T_j \neq \emptyset$ for all $i, j \in J$, then either $\bigcap_{j \in J} T_j$ is a (non-empty) subtree of T or the T_j $(j \in J)$ have exactly one end of T in common.*

Proof. We assume, on the contrary, that $\bigcap_{j\in J} T_j = \emptyset$ and that no end belongs to all T_j $(j \in J)$. We fix a vertex r of T as a root. Then to every end $\mathfrak{E}$ of T there is a $j_{\mathfrak{E}} \in J$ such that $T_{j_{\mathfrak{E}}}$ does not share an infinite path with $\mathfrak{E}$; this means that on the r, $\mathfrak{E}$-path P of T there is a vertex $x_{\mathfrak{E}} \neq r$ such that the infinite subpath of P starting in x has empty intersection with $T_{j_{\mathfrak{E}}}$. Let U be the union of all paths starting in r and containing no $x_{\mathfrak{E}}$ as an internal vertex.

By construction U is a tree without infinite subpath. We put $U \cap T_j = U_j$ for $j \in J$. If we had $U_j \cap U_h = \emptyset$ for some $j, h \in J$, then there would be an $x_{\mathfrak{E}} \in V(U)$ such that $T_j \cap T_h$ lies completely behind $x_{\mathfrak{E}}$ (with respect to the order of the rooted tree (T, r)). Since T_j is connected, by the choice of $T_{j_{\mathfrak{E}}}$, and because of $T_j \cap T_{j_{\mathfrak{E}}} \neq \emptyset$, it follows $x_{\mathfrak{E}} \in V(T_j)$. Analogously $x_{\mathfrak{E}} \in V(T_h)$. But then $U_j \cap U_h \neq \emptyset$.

Hence the U_j have pairwisely non-empty intersection. Let $<$ be a well order of J, and put $Z_j = \bigcap_{i \leq j} U_i$ for each $j \in J$. Then $i<j \Rightarrow Z_i \supseteq Z_j$, and $Z_j = \bigcap_{i<j}(Z_i \cap U_j)$ for each $j \in J$. If all Z_j are non-empty, then by (7) $\bigcap_{j\in J} Z_j = \bigcap_{j \in J} U_j \neq \emptyset$. Otherwise there is a smallest j with $Z_j = \emptyset$. Again by (7) there is a smallest $i<j$ with $Z_i \cap U_j = \emptyset$, or $\emptyset = (\bigcap_{k<i} Z_k \cap U_i) \cap U_j = \bigcap_{k<i}(Z_k \cap U_i \cap U_j)$. The $D_k := Z_k \cap U_i U_j$ $(k<i)$ are all non-empty (since any two of the graphs Z_k, U_i, U_j have non-empty intersection, by the choice of i, j); hence by (7) we had $\bigcap_{k<i} D_k \neq \emptyset$. In any case our assumption at the beginning of the proof leads to a contradiction.

Of course, if two distinct ends $\mathfrak{E}, \mathfrak{E}'$ are common to all T_j, then all the T_j contain the (uniquely determined) two-sided infinite path in T which has subpaths in $\mathfrak{E}$ and in $\mathfrak{E}'$. Hence if $\bigcap_{j\in J} T_j = \emptyset$, then the T_j share exactly one end. This completes the proof.

4. An Example

We now construct an explicit example of a triangulated graph G, with one infinite clique, which is not tree-representable.

Let $u; t_1, t_2, \ldots; s_1, s_2, \ldots$ be the vertices of G; further let $[t_1, t_2], [t_2, t_3], [t_3, t_4], \ldots$, all $[s_i, s_j]$ with $i \neq j$, all $[s_i, t_j]$ with $i \leq j$, and all $[u, s_i]$ $(i = 1, 2, \ldots)$ be its edges. Then the cliques of G are $C_i := G[t_i, t_{i-1}, s_1, \ldots, s_{i-1}]$ for $i = 2, 3, \ldots$ and $C_\infty := G[u, s_1, s_2, \ldots]$. It is easily seen that G is triangulated and its cliques $C_2, C_3, \ldots, C_\infty$ form, in this order, a pgd of G.

Assume G to have a representation on a tree T where, say, subtrees t_i, S_i, U of T correspond to the vertices t_i, s_i, u respectively. Let $T_i \cap T_{i+1} =: D_i$ $(i = 1, 2, \ldots)$; then each $D_i \neq \emptyset$, but $i \neq j \Rightarrow D_i \cap D_j = \emptyset$ (because of $T_i \cap T_j \neq \emptyset \Leftrightarrow [t_i, t_j] \in E(G) \Leftrightarrow |i-j| = 1$). It follows that $T_1 \cup T_2 \cup T_3 \cup \ldots$ contains a one-sided infinite path P which meets $D_1, D_2, D_3, \ldots$ (in this order). For fixed i we have $S_i \cap T_j \neq \emptyset$ for all $j \leq i$ and $S_i \cap T_j = \emptyset$ for $j < i$; hence S_i meets $D_i, D_{i+1}, D_{i+2}, \ldots$, but not $D_{i-1}, D_{i-2}, \ldots, D_1$. Therefore the S_i have exactly the end $\mathfrak{E}$ to which P belongs in common. We conclude that $\mathfrak{E}$ must be the intersection of the subtrees corresponding to C_∞ (according to (H*)). Therefore U must contain a path $\in \mathfrak{E}$, i.e. an infinite subpath P' or P. Then, however, U would meet infinitely many of the D_i, and u would be adjacent in G to infinitely many of the t_i, which is a contradiction. Hence G is not tree-representable.

One sees further that C_∞ must be the last element in every pgd of G and that for that reason G cannot admit a strict pgd. That this fact is the turning point in our representation problem will become apparent in our main theorem.

5. Main Result

We now characterize the structure of the tree-representable graphs as follows:

Theorem. *A graph G is tree-representable if and only if it has a strict pgd in which all members are cliques of G.*

Proof. We assume first that G has a strict pgd G_λ $(\lambda < \sigma)$ where all G_λ are cliques of G. For the simplices of attachment S_λ we have by assumption: To every τ, $0 < \tau < \sigma$, there is an $f(\tau) < \tau$ with $S_\tau \subset G_{f(\tau)}$. Let now T be the graph with the ordinals $<\sigma$ as vertices and the edges $[\tau, f(\tau)]$ $(0 < \tau < \sigma)$. T is connected, because any sequence τ, $f(\tau), f^2(\tau), f^3(\tau), \ldots$ ends in 0 after a finite number of steps. T does not contain a circuit; for if Z is a connected finite subgraph of T with the vertices $\lambda_1 < \lambda_2 < \cdots < \lambda_k$, then λ_k has degree ≤ 1 with respect to Z. Thus T is a tree.

For each $x \in V(G)$ let T_x be the subgraph of T induced by $\{\lambda \mid x \in V(G_\lambda)\}$; of course $T_x \neq \emptyset$. We show that each T_x is connected, i.e. a tree. Assume $\rho, \tau \in V(T_x)$, say $\rho < \tau$. Then the ρ, τ-path in T contains $f(\tau)$ (also if ρ, τ are not comparable in T with respect to the root 0). Furthermore x lies in $G_\tau \cap \bigcup_{\lambda<\tau} G_\lambda = S_\tau \subseteq G_{f(\tau)}$, hence $f(\tau) \in V(T_x)$. $f(\tau), \rho$ have shorter distance in T than τ, ρ. We see, by induction on their distance in T, that with any ρ, τ also the ρ, τ-path $\subseteq T$ belongs to T_x.

Further we have for any distinct $x, y \in V(G)$: $[x, y] \in E(G) \Leftrightarrow \exists \lambda < \sigma$ with $x, y \in V(G_\lambda) \Leftrightarrow \exists \lambda < \sigma$ with $\lambda \in V(T_x)$ and $\lambda \in V(T_y) \Leftrightarrow T_x \cap T_y \neq \emptyset$.

Thus $x \to T_x$ (for $x \in V(G)$) is a representation of G in the tree T.

Now let us assume that, on the other hand, G is tree-representable, i.e. there exists a tree T and a mapping $x \to T_x$ such that each T_x is a subtree of T and $T_x \cap T_y \neq \emptyset \Leftrightarrow [x, y] \in E(G)$.

For every simplex $S \subseteq G$ we have by (H*):

$$T_S := \bigcap_{x \in V(S)} T_x$$

either is a non-empty subtree of T or consists of a single end of T.

For each vertex t of T which is in no T_C for any clique C of G we add a new vertex v_t to G, connecting it by edges to all $x \in V(G)$ with $t \in V(T_x)$. Of course for each such t these x form a simplex in G and together with v_t a clique in the graph $\hat{G}$ arising from G by the described extension. The other cliques of $\hat{G}$ coincide with those of G.

We represent, in addition, each v_t by the trivial tree consisting of t only. Then $\hat{G}$ is represented on T; if we find a strict pgd of $\hat{G}$ in cliques then by (6) we also have such a decomposition for G. Therefore we may assume without loss of generality that $\hat{G} = G$, i.e. that the T_C, C any clique of G, cover all the vertices of T.

We now choose a root r of T and well-order the cliques C of G "in accordance" with the order in which the T_C appear on the rooted tree (T, r). More precisely: We let C_0 be the clique of G such that r is in T_{C_0}. If, for some ordinal $\tau > 0$, all C_λ with $\lambda < \tau$ are already determined and there is a vertex t of T not contained in a T_{C_λ} $(\lambda < \tau)$, then passing on T from r to t we meet a first T_C which is not among the T_{C_λ} $(\lambda < \tau)$ and of course disjoint from all of them. We choose this C as C_τ. If, on the other hand, all the vertices of T are in $\bigcup_{\lambda<\tau} T_{C_\lambda}$, we finish the procedure.

In this way by transfinite induction a well-ordering C_λ $(\lambda < \sigma)$ of all those cliques of G which do not correspond to a single end of T is obtained. We assert that the C_λ $(\lambda < \sigma)$ form a strict pgd of G.

Of course each vertex v and each edge $[v, w]$ of G is contained in some C_λ. (v lies in each C with $T_C \subseteq T_v$, and $[v, w]$ is in each C with $T_C \subseteq T_v \cap T_w \neq \emptyset$.) Hence G is the union of the C_λ $(\lambda < \sigma)$.

If $0 < \tau < \sigma$, then T_{C_τ} has an immediate predecessor T_{C_ρ} on T with respect to the root r; $\rho < \tau$ by the choice of our well-ordering. If the vertex x is in $D := C_\tau \cap \bigcup_{\lambda<\tau} C_\lambda$, then T_x meets T_{C_τ} and some T_{C_λ} with $\lambda < \tau$. By the choice of ρ and the properties of our well-ordering it follows that T_x must also meet T_{C_ρ}, i.e. $x \in V(C_\rho)$. Hence D is a simplex properly contained in C_ρ and in C_τ, and we see that the C_λ indeed form a strict pgd of$G$. This completes the proof.

With (2) and (4) we get as an immediate consequence

COROLLARY. *A graph without an infinite simplex is tree-representable if and only if it is triangulated.*

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