Multiplications in solenoids as hyperbolic attractors

Floris Takens

Grotestraat 31, Bedum 9781 HB, Netherlands

Dedicated to Jan M. Aarts on the occasion of his retirement

Abstract

The solenoid was first introduced by Vietoris, motivated by questions from algebraic topology. It later appeared in the study of dynamical systems. This paper discusses the history of solenoids and settles an isomorphism problem.

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1. Introduction

The solenoid has an interesting history in geometry and dynamics.

It was first introduced by Vietoris in [9] as part of the construction of an example of a continuum for which the fundamental group, in the sense of Vietoris, depends on the base point. We remind here that in this paper Vietoris introduced a special type of homology, cohomology, and fundamental groups for compact metric spaces. This construction was generalized and modified by various authors, see [3] for an account of the history. The resulting homology and cohomology groups are now named after Čech; the fundamental groups have been forgotten. In the notation of our Section 3 Vietoris constructed in this paper the solenoid $\Sigma_2$. In the following, if we speak of the solenoid, we mean $\Sigma_2$. 

E-mail address: f.takens@math.rug.nl (F. Takens).

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Independently and around the same time Van Danzig made an extensive study of these solenoids [2] as a class of spaces with very strong homogeneity properties without being manifolds. This homogeneity is due to the fact that these spaces can be given the structure of a topological group. In our notation, Van Danzig restricted himself in this publication to the solenoids $\Sigma_n$, for $n \in \mathbb{N}$ (in the notation of our Section 3). One of his results was a topological classification of these spaces. The topological classification of the more general solenoids, was completed by McCord [5] and simplified by Aarts and Fokkink [1].

In dynamics it first appeared as an example, due to Nemytskii, of the closure of an almost periodic motion which is not quasiperiodic. This example can be found in [6, Chapter V, Section 8] which is a translation of Russian publications in the period 1947–1952. The relation between the group structure and the dynamical structure is the following. The solenoid contains a dense subgroup which is isomorphic to the additive group of the reals $\mathbb{R}$ (and, as a subspace, is an immersion of $\mathbb{R}$). If we identify the reals with this subgroup, then the time evolution over an interval $t$ corresponds to the translation $x \mapsto x + t$. As far as I know, it is not known which solenoids are conjugated to the closure of an orbit of a smooth dynamical system. We recall that the closure of a quasiperiodic motion is not only a topological group but even a torus group.

Later the solenoid was used by Williams, first as an example of a space on which an ‘unstable’ homeomorphism could be defined (this is what one calls nowadays an expansive homeomorphism) in [10] and later as one of the first examples of a hyperbolic, chaotic, and fractal attractor in [11]. We describe this example in detail in our Section 2.

It turns out that the dynamics in $\Sigma_2$, introduced by Williams, can be easily described in terms of the group structure. It is just multiplication by 2, i.e., the map sending $x$ to $x + x$. In this paper we give a classification of those multiplications in general solenoids which are conjugated to a hyperbolic attractor.

2. The solenoid as attractor of a diffeomorphism

We consider a solid torus $T = S^1 \times D^2$ in dimension 3 and a diffeomorphism of $T$ to a subset of $T$ such that in the $S^1$ direction we have an expanding map of degree 2 and in the $D^2$ direction we have a (strong) contraction. Taking $s \in \mathbb{R}$ mod 1 as a coordinate on $S^1$ and $x, y$, with $x^2 + y^2 \leq 1$, as coordinates on $D^2$, an example of such a map is given by:

$$\varphi(s, x, y) = (2s \bmod 1, c_1 \cos(2\pi s) + c_2 x, c_1 \sin(2\pi s) + c_2 y),$$

with $0 < c_2 < c_1$ and $c_1 + c_2 < 1$. The solenoid is then defined as $S = \bigcap \varphi^i(T)$. The dynamics on $S$ is given by $\varphi|_S$.

This diffeomorphism is (structurally) stable in the sense that if $\tilde{\varphi} : T \to T$ is $C^1$ sufficiently close to $\varphi$, then there is a homeomorphism $h : T \to T$ such that $h\varphi = \tilde{\varphi}h$. This stability is a consequence of the fact that $\varphi$ is hyperbolic in the sense that for each $q \in S$ there is a splitting $T_q = E^s(q) + E^u(q)$ of the 3-dimensional space of tangent vectors at $q$ such that, under repeated application of $\varphi$, vectors in $E^s(q)$, respectively $E^u(q)$, are exponentially contracted, respectively expanded; in fact the contracting vectors are tangent to the discs $\{x\} \times D^2$. For the theory of hyperbolic dynamical systems and their structural stability see, e.g., [8].
The attractor is chaotic in the sense that arbitrarily small differences in initial states (in the $S^1$ direction) grow exponentially under repeated application of the map $\varphi$; for a complete definition we also refer to [8].

Finally, $S$ is not a manifold: for each $s \in \mathbb{R}$, $S \cap \{s\} \times D^2$ is a Cantor set. Therefore we call $S$ a fractal set.

We note that this example can easily be generalized: we can take on $S^1$ an expanding map of degree $k > 2$ (instead of degree 2). In the definition of $\varphi$ this means that we have to replace ‘$2s \bmod 1$’ by ‘$ks \bmod 1$’ (in order to keep this map injective one has to further restrict $c_2$); this defines the map $\varphi_k$ (so $\varphi = \varphi_2$). The attractor $\bigcap_k \varphi_k(T)$ is denoted by $S_k$. For the sake of completeness we note that we can even make orientation reversing attractors by taking negative values for $k$.

Note that, in the theory of dynamical systems the notion of solenoid has been generalized to include all the attractors which are locally homeomorphic to the product of a Cantor set and an interval, see in particular [7]. These attractors, which are not related with topological groups will not be considered here.

3. Cantor groups and (generalized) solenoids as topological groups

We give here definitions and a few of the elementary properties. We mainly follow [4]. Let $\mathbf{a} = (a_0, a_1, \ldots)$ be an infinite sequence of integers bigger than 1. This defines a metric $\rho_\mathbf{a}$ on the natural numbers $\mathbb{N} = \{0, 1, \ldots\}$:

$$\rho_\mathbf{a}(n, m) = 2^{-i} \text{ if and only if }$$

$$|n - m| \text{ is a multiple of } \prod_{k=0}^{i-1} a_i \text{ but not of } \prod_{k=0}^{i} a_i,$$

where the empty product is interpreted as 1. We define $\Delta_\mathbf{a}$ as the completion of $\mathbb{N}$ with respect to this metric. It is a compact metric space; in fact it is a Cantor set. The elements of $\Delta_\mathbf{a}$ can be represented by infinite sequences $(n_0, n_1, \ldots)$ with $0 \leq n_i < a_i$; such a sequence represents the following limit with respect to the metric $\rho_\mathbf{a}$:

$$\lim_{k \to \infty} \sum_{i=0}^{k} n_i \cdot a_{i-1} \cdot a_{i-2} \cdots a_0.$$

Note that $\Delta_\mathbf{a}$ is a topological group, i.e., the addition on the natural numbers can be extended in a continuous way and for each $x \in \Delta_\mathbf{a}$ there is an additive inverse $-x$: if $x$ is represented by the sequence $(n_0, n_1, \ldots)$ then $-x$ is represented by the sequence $(\bar{n}_0, \bar{n}_1, \ldots)$ such that for each $k$:

$$- \sum_{i=0}^{k} \bar{n}_i \cdot a_{i-1} \cdots a_0 = \sum_{i=0}^{k} n_i \cdot a_{i-1} \cdots a_0 \mod a_k \cdots a_0.$$

Note that $x \mapsto -x$ is an isometry. We call the groups which can be constructed in this way Cantor groups.
If no confusion is possible we will not distinguish between an element of $\Delta_a$ and the corresponding sequence. We define $Z^j_a \subset \mathbb{N}$ as the set of multiples of $\prod_{j=0}^{m-1} a_j$. These sets define a basis of neighbourhoods of 0 in $\mathbb{N}$ with respect to the metric $\rho_a$. Clearly there is a homeomorphism $h: \Delta_a \rightarrow \Delta_b$, which is also a group isomorphism sending the element represented by $(1, 0, 0, \ldots)$ in $\Delta_a$ to the corresponding element in $\Delta_b$, if and only if there is for each $i$ some $j$ such that $Z^j_a \subset Z^j_b$ and $Z^j_b \subset Z^j_a$. In this case we call the Cantor-groups $\Delta_a$ and $\Delta_b$ equivalent.

Let now $a = (a_0, a_1, \ldots)$ be a sequence defining a Cantor group. By the following two constructions we obtain sequences defining equivalent Cantor groups. First we replace $a$ by its sequences of prime decompositions, i.e., by the sequence $pa = (pa_0, 1, pa_0, 2, \ldots, pa_0, m_a, pa_1, 1, \ldots, pa_1, m_1, \ldots)$, where all elements in the sequence $pa$ are prime and where $\prod_{i=1}^{m_i} pa_i, i = a_i$. The fact that we obtain in this way an equivalent Cantor group means that we might restrict to sequences with only prime numbers. The second construction consists of applying an arbitrary permutation to the elements $a_i$ of the sequence $a$.

Since we will often make use of sequences, defining Cantor groups, with all elements equal we introduce the following notation. For $i \in \mathbb{N}$, $i$ denotes the sequence $i = (i, i, \ldots)$. In order to define the generalized solenoid, corresponding to a sequence $a$ as above, we take the product of $\mathbb{R}$, as an additive topological group and $\Delta_a$ and divide it out by the cyclic group generated by the element $(-1, (1, 0, 0, \ldots))$. We denote this topological group by $\Sigma_a$. In the next section we show how $\Sigma_b$ and $\Sigma_a$ are related.

As a topological space $\Sigma_a$ can also be obtained from $[0, 1] \times \Delta_a$ by the identification of $(1, x)$ with $(0, P_1(x))$ for each $x \in \Delta_a$, where $P_1$ stand for ‘plus 1’, i.e., for the map which sends each $x \in \Delta_a$ to $x + (1, 0, 0, \ldots)$. We remind that the first example, due to Vietoris, of a solenoid is in our present notation $\Sigma_2$; the other solenoids are referred to as generalized solenoids.

Note that for the sequences $a = (a_0, a_1, \ldots)$ and $\vec{a} = (a_1, a_2, \ldots)$ the topological groups $\Sigma_a$ and $\Sigma_{\vec{a}}$ are isomorphic, i.e., they admit a group isomorphism which is also a homeomorphism. In order to see this we identify $\Sigma_{\vec{a}}$ with the subgroup of those elements of $\Sigma_a$ which are represented by sequences starting with at least one zero (the closure of the natural numbers which are a multiple of $a_0$). In this way we can define a homeomorphism, preserving the group operation, from $\Sigma_{\vec{a}}$ to $\Sigma_a$ by taking $h([t, x]) = [a_{0t}, x]$ for $x \in \Delta_{\vec{a}}$. In order to see that this confusing definition is correct, it is important to note that the maps $P_1$ in $\Delta_a$ and in $\Delta_{\vec{a}}$ do not agree: ‘plus 1’ in $\Delta_{\vec{a}}$ corresponds to $a_0t$ times ‘plus 1’ in $\Delta_a$.

We conclude that $\Sigma_a$ and $\Sigma_b$ are isomorphic as topological groups if the sequences $a$ and $b$ can be transformed into each other by the following operations, and they inverses:

- replacing a sequence by its sequence of prime decompositions;
- permuting the elements of a sequence;
- deleting a finite number of elements of a sequence.

Due to the topological classification of solenoids, see [5] and [1], we can replace in the above statement ‘if’ by ‘if and only if’. 
4. The algebraic and the dynamical solenoid

In this section we provide a homeomorphism between $S^2$ and $\Sigma^2$ and show how their structures are related. A corresponding homeomorphism between $S^n$ and $\Sigma^n$, for any $n > 2$, can be constructed in a similar way.

In both versions of the solenoid we have one ‘component’ which is a circle ($S^1$ or $\mathbb{R}$ mod 1). On that component we take the identity. So we only have to construct a homeomorphism between $\Delta^2$ and $S^2 \cap \{0\} \times D^2$.

We first introduce some notation. For $s$ in $[0, 1)$, or in $\mathbb{R}$ mod 1, we denote $s \times D^2$ by $Ds \subset T$. For each $k \geq 0$, $\varphi^k(T) \cap D_0$, where $\varphi$ is as defined in Section 2, consists of $2^k$ discs which are of the form $\varphi^k(D_i/2^k)$ for $i = 0, 1, \ldots, 2^k - 1$. We code the $i$th disc by the dyadic expression for $i$ in reverse order (the reason for this reverse order is to obtain consistency with the sequences used to label the elements of $\Delta^2$).

With this notation it is clear that if we apply the map $\varphi$ to the disc with code $(m_0, \ldots, m_{k-1})$ we obtain the disc with code $(0, m_0, \ldots, m_{k-1})$; the disc with code $(m_0, \ldots, m_{k-1})$ is contained in the disc with code $(m_0, \ldots, m_{k-1} - 1)$. On these sequences of zeros and ones of length $k$, which are the codes of the discs in $\varphi^k(T) \cap D_0$, we define the map $P^{(k)}_1$ by:

$$P^{(k)}_1(m_0, \ldots, m_{k-1}) = (\tilde{m}_0, \ldots, \tilde{m}_{k-1})$$

if

$$\left( \sum_{i=0}^{k-1} m_i 2^i \right) + 1 = \left( \sum_{i=0}^{k-1} \tilde{m}_i 2^i \right) \mod 2^k.$$

So $P^{(k)}_1$ can be interpreted as the dyadic form of the ‘plus 1’ map modulo $2^k$. If we now start with a disc of $\varphi^k(T) \cap D_0$ with coding $m_0, \ldots, m_{k-1}$ and follow the continuation of this disc in $\varphi^k(T) \cap D_s$ for $s$ going from 0 to 1, we will end up in the disc with coding $P^{(k)}_1(m_0, \ldots, m_{k-1})$.

Now we come to the definition of the homeomorphism between $S^2 \cap D_0$ and $\Delta^2$. The points of $\Sigma^2 \cap D_0$ are labelled by infinite sequences of zeros and ones: the point corresponding to $(m_0, m_1, \ldots)$ is contained in the discs with label $(m_0, m_1, \ldots)$ for each $l$. In this way we obtain a homeomorphism between $S^2 \cap D_0$ and $\Delta_2$ by sending each point to the point with the same sequence. This homeomorphism can be extended in a unique way to a homeomorphism between $S^2$ and $\Sigma^2$ in such a way that the circle, or $\mathbb{R}$ mod 1, coordinate is preserved.

It is straightforward to verify that the map $\varphi_2|S^2$ corresponds in $\Sigma^2$ to multiplication by 2, i.e., to the map $x \mapsto x + x$.

5. Generalized solenoids

In this final section we treat the following problem. Suppose we are given a sequence $a$ defining a solenoid and an integer $n$ greater than 1. Is the multiplication by $n$ in $\Sigma^a$ conjugated to a hyperbolic attractor? It turns out that, up to isomorphism, the only solenoids
with multiplication which are conjugated to a hyperbolic attractor are the solenoids which we saw in the previous section, namely \( \Sigma \) with multiplication by \( n \).

We denote by \( M_n \) the multiplication by \( n \); it is a map which is defined in any group. If no confusion is possible, we denote the subgroup \( \{0\} \times \Delta_a \) of \( \Sigma \) by \( \Delta_a \). The relation between the maps \( M_n \) in \( \Sigma \) and in \( \Delta_a \) is the following: for \( k = 0, 1, \ldots, n - 1 \), \( M_n([\frac{k}{n}] \times \Delta_a) \) equals \( P_1^k(M_n(\Delta_a)) \), where \( P_1 \) is as in Section 3 the ‘plus 1’ map. Observe that each \( P_1^k(M_n(\Delta_a)) \) is closed and that all the ‘ordinary’ natural numbers, i.e., all elements of \( \Delta_a \) which are represented by finite sequences, are contained in at least one of the sets \( P_1^k(M_n(\Delta_a)) \), \( k = 0, 1, \ldots, n - 1 \). So the union of these sets equals \( \Delta_a \).

If \((\Sigma_a, M_n)\) is conjugated to a hyperbolic attractor, then \( M_n \) should be at least a homeomorphism in \( \Sigma_a \), and hence \( \Delta_a \) should be the disjoint union of the sets \( P_1^k(M_n(\Delta_a)) \). If \( M_n \) is a homeomorphism in \( \Sigma_a \) then this also holds for any power \( M_n^l = M_n\) of \( M_n \), so \( \Delta_a \) is also the disjoint union of the sets \( P_1^k(M_n^{\prime l}(\Delta_a)) \) for \( l = 0, 1, \ldots, n^l - 1 \).

Now we derive from this that all the prime factors of \( n \) must occur infinitely often in \( \text{pa} \), the sequence of prime decompositions of \( a \) as defined in Section 3. Suppose this is not the case. Then there is a prime factor \( p \) of \( n \) which only occurs less than \( N \) times in \( \text{pa} \). In that case any neighbourhood of zero contains ‘ordinary’ natural numbers which are not divisible by \( p^N \). Consider now \( M_n^N(\Delta_a) \). It should be open, since it is the complement of the closed sets \( P_1^l(M_n^N(\Delta_a)), l = 1, \ldots, n^N - 1 \). This means that it contains an ‘ordinary’ natural number which is not divisible by \( p^N \). But such an ‘ordinary’ natural number must belong to one of the other sets \( P_1^l(M_n^N(\Delta_a)), l = 1, \ldots, n^N - 1 \). This is the required contradiction.

So we proved:

If \( M_n \) defines a homeomorphism in \( \Sigma_a \), then each prime factor of \( n \) occurs infinitely often in the sequence \( \text{pa} \) of prime decompositions of \( a \).

Next we observe that if \((\Sigma_a, M_n)\) is conjugated to a hyperbolic attractor, the map \( M_n \) in \( \Delta_a \), is, at least in a neighbourhood of \( 0 \), a contraction. In order to see this we have to come back to the hyperbolic splitting in contracting and expanding vectors mentioned already in the introduction. In general for a hyperbolic attractor, the dimension of the expanding part of the splitting is at most equal to the topological dimension of the attractor, e.g., see [8]. This topological dimension is here equal to 1. Now we consider the situation in a neighbourhood of the point 0 element in \( \Sigma_a \) which we may take as \((-\varepsilon, \varepsilon) \times \Delta_a\). In the first coordinate we have the expansion (by a factor \( n \)), so in the complementary directions we must have contractions.

In fact, the point 0 is a fixed point of \( M_n \). If \( M_n \) has an extension to a diffeomorphism for which \( \Sigma_a \) is a hyperbolic attractor, then 0 is a saddle point with a 1-dimensional unstable manifold and the intersection of the stable manifold of this saddle point with \( \Sigma_a \) is locally equal to \( \{0\} \times \Delta_a \).

For \( i \) sufficiently big, \( M_i \) is a contraction on the neighbourhood of \( 0 \) in \( \Delta_a \) consisting of elements which are represented by sequences which start with at least \( i \) zeros. This means that each number of the form \( \prod_{l=1}^{m} a_l \) divides a power of \( n \). This implies that in the sequence of prime decompositions of \((a_i, a_{i+1}, \ldots)\) there are only prime factors of \( n \).
So we proved:

If \((\Sigma_a, \mathcal{M}_n)\) is conjugated to a hyperbolic attractor then, up to a finite number of exceptions, the sequence \(p\alpha\) of prime decompositions of \(\alpha\) contains only the primes which divide \(n\) and it contains each of these primes infinitely often.

Now we come back again to the classification of generalized solenoids in Section 3 and conclude:

If \((\Sigma_a, \mathcal{M}_n)\) is conjugated to a hyperbolic attractor, then \(\Sigma_a\) is as a topological group isomorphic to \(\Sigma_n\).

Final remark. Up to now we only considered multiplication with positive integers. Composing with the map \(x \mapsto -x\) we obtain negative multiplications; they correspond to the orientation reversing solenoid attractors mentioned in Section 2.

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References