Feasibility conditions for non-symmetric 3-class association schemes

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Abstract

In this paper a set of necessary conditions for the existence of a non-symmetric 3-class association scheme expressed in terms of its symmetric closure will be given.

1. Introduction

In [1, p. 58] the question is posed when a symmetric association scheme can be the symmetric closure (the "symmetrization" in the terminology of [1]) of a non-symmetric association scheme. In this vein we consider in this paper a symmetric 2-scheme (we shall call an n-class association scheme briefly an \textit{n-scheme}) and ask under what conditions on the parameters it is feasible that such a 2-scheme is the symmetric closure of a non-symmetric 3-scheme. This leads to the feasibility conditions in Section 4.

This paper is one of our on-going study on the existence and the construction of non-symmetric 3-schemes. Throughout this study we utilize the fact that the existence of a non-symmetric 3-scheme implies the existence of a symmetric 2-scheme (its symmetric closure).

We show in this paper also the surprising fact that the parameters of a non-symmetric 3-scheme are in essence completely determined by the ones of its symmetric closure. It appears that the parameters of a non-symmetric 3-scheme are determined once \(v, v_1, p_{11}^1\) and \(p_{11}^2\) are given.

Non-symmetric 3-schemes do exist. Liebler and Mena mention in [11] an infinite class of distance regular digraphs which are immediately seen to be equivalent to \textit{primitive} non-symmetric 3-schemes; see also [5]. In [10] we construct a primitive
scheme on 36 elements. Imprimitive schemes are constructed in [8], and in [6] we discussed, among other things, the non-symmetric 3-schemes, which can be formed over finite commutative rings. The results we have found so far seem to indicate that the chances that a symmetric 2-scheme is the symmetric closure of a non-symmetric 3-scheme are not very high.

For details of several of the proofs given in this paper we refer to the report [7]. We shall use the notation of Delsarte as it was introduced for association schemes in [4]. This implies the use of a few peculiar notations: if \( P \) is any complex entity (number, vector, etc.) then \( P^* \) denotes the complex conjugate of \( P \) and if \( S \) is a set then \( S^* \) denotes the set of all complex conjugates of the elements of \( S \).

2. Preliminaries

**Definition 2.1.** Let \( X \) be a set with \( v \) elements. Let \( R = \{R_0, R_1, \ldots, R_n\} \) be a family of \( n+1 \) binary relations on \( X \). The pair \((X, R)\) will be called an association scheme with \( n \) classes (also called an \( n \)-scheme) if the following conditions are satisfied:

1. the family \( R \) is a partition of \( X^2 \) and \( R_0 \) is the diagonal (equality) relation;
2. for any \( i \in \{0, 1, \ldots, n\} \) the inverse \( R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\} \) of the relation \( R_i \) belongs to \( R \) (the index of the relation \( R_i^{-1} \) is denoted by \( i_R \));
3. for \( i, j, k \in \{0, 1, \ldots, n\} \) the so-called intersection numbers

\[ p^k_{ij} = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}| \]

are independent of the choice of \( (x, y) \in R_k \);
4. for all \( i, j, k \in \{0, 1, \ldots, n\} \) we have \( p^k_{ij} = p^k_{ji} \).

For every \( i \) the number \( p^0_{ii} \) is called the valency of \( R_i \) and is denoted by \( v_i \). An association scheme \((X, R)\) is called symmetric if all its relations are symmetric, i.e. \( i = i_R \) for all \( i \), otherwise it is called non-symmetric. We denote the symmetric closure of an \( n \)-scheme \((X, R)\) by \((X, \overline{R})\) (here \( \overline{R} = \{R \cup R^{-1} \mid R \in R\} \)). The adjacency matrix of the relation \( R_i \) is denoted by \( D_i \), while the \( n+1 \) maximal common eigenspaces of \((X, R)\) are denoted by \( V_k \). The eigenvalue of \( D_i \) on \( V_k \) is denoted by \( \lambda_i(k) \), and we denote \( \dim(V_k) \) by \( \mu_k \): the multiplicities of \((X, R)\). The co-intersection numbers (or Krein parameters) are denoted by \( q^k_{ij} \). \( P \) is the matrix with \((i, j)\)-entry \( P_{ij}(t) \) and if \( PQ = vI \) then \( Q \) has \((i, j)\)-entry \( Q_{ij}(t) \). \( L_i \) is the matrix with \((k, j)\)-entry \( p^k_{ij} \) and \( M_i \) one with \((k, j)\)-entry \( q^k_{ij} \).

From now on \((X, \overline{R})\) in this paper denotes a symmetric 2-scheme and its parameters are provided with a bar. \((X, R)\) denotes, unless otherwise stated, a non-symmetric 3-scheme. We suppose throughout this paper that \( R_2 = R_2^{-1} \) and \( V_2^* = V_1 \). In this paper we shall use the following shorthand notation for the parameters of \((X, R)\):

\[ u = v_1/v_3, \ u' = \mu_1/\mu_3 \]
\( \alpha = p_{11}^1, \beta = p_{22}^2, \gamma = p_{33}^3, \delta = p_{13}^1, \epsilon = p_{23}^1, \)
\( \lambda = p_{33}^3, \Lambda = P_{1}(1), \Phi = P_{3}(1), \Psi = P_{1}(3), \Omega = P_{3}(3). \)
\( \alpha' = q_{11}^1, \beta' = q_{22}^2, \gamma' = q_{33}^3, \delta' = q_{13}^1, \epsilon' = q_{23}^1, \)
\( \lambda' = q_{33}^3, \Lambda' = Q_{1}(1), \Phi' = Q_{3}(1), \Psi' = Q_{1}(3), \Omega' = Q_{3}(3). \)

**Theorem 2.2.** The intersection matrices and the first eigenvalue matrix of \((X,R)\) have the following form. \( L^0 = I \) and

\[
L_1 = \begin{pmatrix}
0 & 0 & v_1 & 0 \\
1 & \alpha & \alpha & \delta \\
0 & \beta & \alpha & \epsilon \\
0 & \mu & \delta & \epsilon \\
\end{pmatrix}, \quad L_2 = \begin{pmatrix}
0 & v_1 & 0 & 0 \\
0 & \alpha & \beta & \epsilon \\
1 & \alpha & \alpha & \delta \\
0 & \mu & \delta & \epsilon \\
\end{pmatrix}, \\
L_3 = \begin{pmatrix}
0 & 0 & 0 & v_3 \\
0 & \delta & \epsilon & \gamma \\
0 & \epsilon & \delta & \gamma \\
1 & \mu & \gamma & \lambda \\
\end{pmatrix}, \quad P = \begin{pmatrix}
1 & v_1 & v_1 & v_3 \\
1 & \Lambda & \Lambda^* & \Phi \\
1 & \Lambda^* & \Lambda & \Phi \\
1 & \Psi & \Psi & \Omega \\
\end{pmatrix}.
\]

It holds that \( \Lambda \in \mathbb{C} \setminus \mathbb{R} \) while the other first eigenvalues are real.

For \( i = 0, 1, 2, 3 \) the matrix \( M_i \) can be found from the matrix \( L_i \) by replacing \( v_i \) by \( \mu_i \) and by providing the respective intersection numbers with an accent. The matrix \( Q \) can be derived from \( P \) by replacing, for \( i = 1, 3 \), the \( v_i \) by \( \mu_i \) and by providing the eigenvalues \( \Lambda, \Phi, \Psi \) and \( \Omega \) with an accent. Again \( \Lambda' \in \mathbb{C} \setminus \mathbb{R} \) and the other second eigenvalues are real.

**Proof.** The form of the matrices \( L_i \) and \( M_j \) can be derived from \( L_iL_j = L_jL_i \) and \( M_iM_j = M_jM_i \), respectively.

Since \( V_2 = V_1^* \) we have \( P_1(2) = P_1^*(1) \). \( R_2 = R_1^{-1} \) and \( V_2 = V_1^* \) now easily implies the theorem. \( \square \)

From \( L_iL_j = L_jL_i \) we derive

\[(\alpha - \beta)\varepsilon = \mu\gamma(\epsilon - \delta) \quad \text{and} \quad \alpha - \beta + \delta - \varepsilon = -1. \tag{1}\]

### 3. The splitting of symmetric 2-schemes

**Definition 3.1.** Let \((X,R)\) be such that \((X,\overline{R})\) is its symmetric closure then we call \((X,\overline{R})\) a splitting of \((X,\overline{R})\). If \( \overline{R}_0 = R_1 \cup R_2 \) and \( \overline{V}_S = \overline{V}_1 \oplus \overline{V}_2 \) then we say that

- the splitting is according to case I if \( s = S = 1 \),
- the splitting is according to case II if \( s = 1 \) and \( S = 2 \),
- the splitting is according to case III if \( s = 2 \) and \( S = 1 \),
- the splitting is according to case IV if \( s = S = 2 \).

\( n \) is the index such that \( \overline{R}_n = R_3 \) and \( N \) is the index such that \( \overline{V}_N = \overline{V}_3 \).
Lemma 3.2. The following hold.

1. \( \bar{v}_s = 2v_1 = 2v_2 \),
2. \( \bar{\mu}_S = 2\mu_1 = 2\mu_2 \),
3. \( \bar{P}_{SS} = 3\alpha + \beta \),
4. \( \bar{P}_{nn} = \delta + \epsilon \),
5. \( \bar{P}_{nn} = \gamma \),
6. \( \bar{P}_{nn} = \lambda \),
7. \( \bar{P}_S(S) = \Lambda + \Lambda^* = \alpha - \beta \),
8. \( \bar{P}_n(N) = 2\Psi \).

Proof. 1 and 2 are immediate. Obviously \( \bar{D}_0 = D_0, \bar{D}_s = D_1 + D_2 \), and \( \bar{D}_n = D_3 \) and expanding \( \bar{D}_s^2, \bar{D}_n^2, D_1^2 D_2 \) and \( D_3^2 \) (using Theorem 2.2) we find 3–6.

Plainly \( \bar{P}_S(S) = \Lambda + \Lambda^* \) and \( \bar{P}_n(N) = 2\Psi \). Calculating the traces of \( L_1 \) and \( \bar{L}_s \) one finds \( \bar{P}_s(S) = \alpha - \beta \). □

Theorem 3.3. The parameters of a non-symmetric 3-scheme are determined once \( v, v_1, \alpha \) and \( \beta \) are given. With \( A = \alpha - \beta \) and \( U = \sqrt{\frac{vv_1}{\mu_1}} \) they can be computed as follows.

1. \( v_2 = v_1, v_3 = v - 2v_1 - 1 \) and \( u = v_1/v_3 \).
2. \( \delta = v_1 - 2x - 1, \epsilon = v_1 - \alpha - \beta, \gamma = v_3 - \delta - \epsilon \) and \( \lambda = v_3 - 2v\gamma - 1 \).
3. \( \mu_3 = \frac{[v_1 + (v_2 - v_1)]}{2}, \mu_2 = \frac{1}{2}\mu_3 \).
4. \( \alpha' = \frac{1}{2}\frac{v_2}{v_1} \left[ 1 \left( A(A^2 + U^2)/4v_1^2 \right) - \left( (1 + A)^2/v_1^2 \right) \right], \beta' = \alpha' - (\mu_1/v_1)A, \delta' = \mu_1 - 2\alpha - 1, \epsilon' = \mu_1 - \alpha' - \beta', \gamma' = \mu_3 - \delta' - \epsilon', \lambda' = \mu_3 - 2u\gamma' - 1 \).
5. \( \Lambda = \frac{1}{2}(A + \mu U), \Phi = -(1 + A), \Psi = \omega \gamma - \epsilon, \Omega = 2\epsilon - 2u\gamma - 1 \).

Proof. The intersection numbers follow from the \( L_e \)-matrices. It is well known that \( \bar{\mu}_S(P_n(N) - \bar{P}_S(S)) = \bar{P}_s(N)(v - 1) + \bar{v}_s \). Computing the trace of \( L_1 \) and using Lemma 3.2 we get \( \frac{1}{2}\bar{P}_n(N) = \Psi = \omega \gamma - \epsilon \). Using this the given formulas for the \( \mu_1 \) are easily derived.

Theorem II.3.6 in [1] yields the formula for \( \alpha' \). In an analogous way as \( \bar{P}_n(S) = \alpha - \beta \) is derived one finds \( \bar{D}_S(s) = \alpha' - \beta' \). From \( \bar{\mu}_S \bar{P}_n(S) = \bar{v}_s \bar{D}_S(s) \) one now derives the formula for \( \beta' \).

The results which are found up to now in this proof combined with \( \mu_i P_i(i) = v_i Q_i^*(j) \) yield the rest of the formulas apart from the formula for \( \Lambda \). \( \Re(A) = \frac{1}{2}(\alpha - \beta) \) comes from Lemma 3.2. Computing the traces of \( D_1^2 \) and \( D_1 D_2 \) and subtracting leads to \( \mu_1(A - A^*)^2 = -vv_1 \), which implies \( \Im(A) = \frac{1}{2}U \). □

Theorem 3.4. Let \((X, R)\) be a non-symmetric 3-scheme which is a splitting of the symmetric 2-scheme \((X, \overline{R})\). The parameters of \((X, R)\) expressed in those of \((X, \overline{R})\) in the case that \((X, \overline{R})\) is split according to one of the cases are as follows.

- \( v_1 = \frac{1}{2}\bar{v}_s, v_3 = \bar{v}_n, \mu_1 = \frac{1}{2}\bar{\mu}_S, \mu_3 = \bar{\mu}_N, \)
- \( \alpha = \frac{1}{4}(\bar{P}_{SS} + \bar{P}_S(S)), \beta = \frac{1}{4}(\bar{P}_{SS} + 3\bar{P}_S(S)), \gamma = \bar{P}_{nn}, \)
- \( \delta = \frac{1}{4}(\bar{P}_{SS} + \bar{P}_n(S)), \epsilon = \frac{1}{4}(\bar{P}_{nn} + \bar{P}_n(S)), \lambda = \bar{P}_{nn}, \)
- \( \alpha' = \frac{1}{4}(\bar{Q}_{SS} + \bar{Q}_S(s)), \beta' = \frac{1}{4}(\bar{Q}_{SS} + 3\bar{Q}_S(s)), \gamma' = \bar{Q}_{NN}, \)
- \( \delta' = \frac{1}{4}(\bar{Q}_{SS} + \bar{Q}_n(s)), \epsilon' = \frac{1}{4}(\bar{Q}_{nn} + \bar{Q}_n(s)), \lambda' = \bar{Q}_{NN}. \)
A = \frac{1}{2} \left( \mathcal{P}_3(\mathcal{S}) + \sqrt{\mathcal{V}_3/\mathcal{Q}_3} \right), \quad \Psi = \mathcal{P}_n(\mathcal{S}), \quad \Omega = \mathcal{P}_n(\mathcal{N}).

A' = \frac{1}{2} \left( \mathcal{Q}_3(\mathcal{S}) - \sqrt{\mathcal{V}_3/\mathcal{Q}_3} \right), \quad \Psi' = \mathcal{Q}_n(\mathcal{S}), \quad \Omega' = \mathcal{Q}_n(\mathcal{N}).

The above theorem implies that the parameters of \((X, R)\) are determined by those of \((X, \bar{R})\) once it is known according to which case \((X, R)\) has been split to form \((X, R)\).

4. The feasibility conditions

We give in this section necessary conditions (the so-called feasibility conditions) on the parameters of a symmetric 2-scheme \((X, R)\) in order that \((X, R)\) is the symmetric closure of a non-symmetric 3-scheme. The conditions are not sufficient, as we shall show in due course.

Lemma 4.1. If \((X, \bar{R})\) is the symmetric closure of \((X, R)\) then

1. \(\overline{P}_i(j) \in \mathbb{Z}\) for all \(i\) and \(j\),
2. \(\overline{P}_3(N) \equiv 0 \pmod{2}\),
3. \(\overline{P}^n_{ss} \equiv 0 \pmod{2\overline{P}_n(N)}\).

Proof. Theorem 3.4 implies 1 and 2.

\[
\overline{P}^n_{ss} = \delta + \varepsilon, \quad \text{so} \quad \overline{P}_s \overline{P}^n_s = \overline{V}_s \overline{P}^n_{ss} \implies \overline{P}^n_{ss} = 2u(\delta + \varepsilon).
\]

\[
\Psi = u\varepsilon - \varepsilon \quad \text{and} \quad \Phi = -1 - (\varepsilon - \beta).
\]

Using (1) one finds \(\varepsilon = \Phi\Psi\) and \(\Phi = \delta - \varepsilon\). Since \(u(\delta - \varepsilon) \in \mathbb{Z}\), one finds \(u\Phi \in \mathbb{Z}\). Hence

\[
\overline{P}^n_{ss} = u\Phi(1 + 2\Psi) = -2u\Phi\Omega = -2u\Phi\overline{P}_n(N).
\]

This implies 3. \(\Box\)

Theorem 4.2. Let \((X, \bar{R})\) be a primitive symmetric 2-scheme then the conditions stated below are necessary conditions in order that \((X, \bar{R})\) can be split into a (primitive) non-symmetric 3-scheme.

1. \(\overline{P}_i(j) \in \mathbb{Z} \setminus \{0\}\) for all \(i\) and \(j\).
2. \(\overline{V}_s \equiv 0 \pmod{2}\).
3. \(\overline{Q}_3(S) \equiv 0 \pmod{2}\).
4. \(\overline{P}_3(N) \equiv 0 \pmod{2}\).
5. \(\overline{P}^n_{ss} \equiv 0 \pmod{2\overline{P}_n(N)}\).
6. \(\overline{P}^n_{ss} + \overline{P}_3(S) \equiv 0 \pmod{4}\).
7. \(-\overline{P}^n_{ss} \leq \overline{P}_s(S) \leq \frac{1}{2}\overline{P}^n_{ss}\).
8. \(-\overline{Q}^n_{ss} \leq \overline{Q}_3(S) \leq \frac{1}{2}\overline{Q}^n_{ss}\).
9. \(-\overline{Q}^n_{ss} \leq \overline{Q}_n(N) \leq \overline{Q}^n_{ss}\).
10. If $-\bar{q}_{SS} < \bar{Q}_S(s) < \frac{1}{2} \bar{q}_{SS}$ then $\bar{\mu}_N \leq \frac{1}{8}(\bar{\mu}_S^2 - 6\bar{\mu}_S)$.

11. If either $\bar{q}_{SS} = -\bar{Q}_S(s)$ or $\bar{q}_{SS} = 3\bar{Q}_S(s)$ then $\bar{\mu}_N \leq \frac{1}{8}(\bar{\mu}_S^2 - 2\bar{\mu}_S)$.

If $(X, \bar{R})$ is imprimitive then only the conditions 2–9 apply.

**Proof.** As is well known the scheme $(X, \bar{R})$ is primitive if and only if $\bar{P}_1(1)\bar{P}_2(2)\bar{P}_1(1)\bar{P}_2(2) 
eq 0$. But this and Lemma 4.1 imply 1.

Conditions 2, 3, 6 follow directly from Theorem 3.4 and from Lemma 4.1 we derive 4 and 5, while $\alpha \geq 0$ and $\beta \geq 0$ imply condition 7.

According to Theorem (3.5.11) in [3] the conditions 8 and 9 are what remain of the 64 Krein conditions for $(X, R)$, if the trivial fulfilled ones and those which are implied by the existence of $(X, \bar{R})$ are left out.

The conditions 10 and 11 are the adaptation of the so-called Neumaier conditions (which can be found in [1, Theorem II.4.8]) to the present situation. (Several co-intersection numbers are not 0: since $(X, \bar{R})$ is primitive, $(\alpha' - \beta')\gamma' \delta' \neq 0$ as is shown in [8] and also not both $\alpha' = 0$ and $\delta' = 0$ (using the Neumaier conditions it is easy to show that for primitive non-symmetric 3-schemes $\mu_i \geq 2$ holds).)

The conditions which can be derived from $L_i L_j = L_j L_i$ and $PQ = vI$ are consequences of the existence of $(X, \bar{R})$, while neither from $v v_1 / \mu_i \in \mathbb{N}$ nor from the Frame quotient new conditions can be derived.

Note that in the next definition we distinguish between the feasibility of a splitting of $(X, \bar{R})$ and the feasibility of the existence of $(X, R)$.

**Definition 4.3.** Let $(X, \bar{R})$ be a symmetric 2-scheme then it is said that the splitting of $(X, \bar{R})$ into a non-symmetric 3-scheme is feasible if the parameters of $(X, \bar{R})$ satisfy the conditions mentioned in Theorem 4.2.

It is said that the existence of a non-symmetric 3-scheme $(X, R)$ is feasible if it has not yet been shown that the symmetric closure of $(X, R)$ cannot correspond to a symmetric 2-scheme $(X, \bar{R})$, and if the splitting of $(X, \bar{R})$ into $(X, R)$ is feasible.

Let $(X, \bar{R})$ be a symmetric 2-scheme then it is said that the splitting of $(X, \bar{R})$ into a non-symmetric 3-scheme $(X, R)$ is realizable if $(X, R)$ exists.

The conditions mentioned in Theorem 4.2 are called the feasibility conditions.

**Lemma 4.4.** Let $(X, R)$ be an $n$-scheme and suppose $p_{is}^l \neq 0$ for some fixed $l$, $s$ and $t$. Then $p_{ij}^l \leq \sum_{r=0}^{n} \min \{ p_{ir}^l, p_{jr}^l \}$ for all $i$ and $j$.

Lemma 4.4 is an adaptation of a lemma given in [2] to the case of non-symmetric $n$–schemes. We did not include the conditions implied by Lemma 4.4 in the feasibility conditions. However, once the numerical values of a given scheme are known the conditions can be easily checked, using a computer. It has been done for the schemes mentioned in Table 1, and no new restrictions have been found.

**Theorem 4.5.** The following hold.
1. If \((X, R)\) is pseudo-cyclic then the splitting of \((X, R)\) is never feasible.

2. If \((X, R)\) is a triangular scheme \(\Delta(t) = J(t, 2)\) with the relations and eigenspaces numbered according to (2), then the splitting of \((X, R)\) is feasible if
   
   (a) \(t = 4\) and the splitting is according to case II, or
   
   (b) \(t \equiv 7 \pmod{8}\) and the splitting is according to case IV.

**Proof.** \((X, R)\) is pseudo-cyclic if and only if \(\mu_1 = \mu_2 = \bar{v}_1 = \bar{v}_2 = \frac{1}{2}(v-1)\). In that case \(\overline{P}_{11} = \overline{P}_{22} = \frac{1}{4}(v-1), \overline{P}_1(1) = \overline{P}_2(2) = \frac{1}{2}(-1 + \sqrt{v})\) and \(\overline{P}_1(2) = \overline{P}_2(1) = \frac{1}{2}(-1 - \sqrt{v})\). However, by Theorem 4.2, \(\overline{P}_i(j) \in \mathbb{Z}\), so \(v = w^2\) for \(w \in \mathbb{N}\). Now condition 6 of Theorem 4.2 implies \(w^2 \pm 2w - 7 \equiv 0 \pmod{16}\), which is not possible.

If \((X, R)\) is a \(\Delta(t)\) then

\[
\overline{P} = \begin{pmatrix}
1 & 2t - 4 & \frac{1}{2}(t - 2)(t - 3) \\
1 & t - 4 & 3 - t \\
1 & -2 & 1
\end{pmatrix}.
\]  

This determines the numbering of the cases. Since \(\overline{P}_2(2) = 1\), it is immediate that the splitting of case III is not feasible. The calculations for the other cases are somewhat more complicated. We leave this to the reader. \(\square\)

Using [1, Theorem II.4.2] it is not difficult to show that if for a symmetric 2-scheme \(v\) is prime then that scheme is pseudo-cyclic. Hence Theorem 4.5 implies that for a non-symmetric 3-scheme, \(v\) cannot be prime.

The splitting of \(\Delta(4)\) is realizable. \(\Delta(4)\) is imprimitive, hence its splitting is also imprimitive. The graph of the first relation of the splitting can be represented as follows.

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**Table 1**

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<tr>
<th>No.</th>
<th>(v)</th>
<th>(v_1)</th>
<th>(P_{11})</th>
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<th>(\beta)</th>
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In [5] it has been shown that $\Delta(7)$ cannot be split.

There are several well-known necessary conditions for the existence of a symmetric 2-scheme and so one can set up a list with parameter sets possibly corresponding to symmetric 2-schemes and with the property that no other parameter sets need to be considered.

Applying our feasibility conditions to such a list with $v \leq 81$ one finds the following remarkably short list of parameter sets corresponding to primitive symmetric 2-schemes of which the splitting is feasible.

References