J. Math. Anal. Appl. 388 (2012) 480–489

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Essential selfadjointness of singular magnetic Schrödinger operators on Riemannian manifolds

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ARTICLE INFO

Article history:
Received 6 December 2010
Available online 10 October 2011
Submitted by H.R. Parks

Keywords:
Essential selfadjointness
Magnetic Schrödinger operator
Riemannian manifold

ABSTRACT

In this paper we extend the well-known Leinfelder–Simader theorem on the essential selfadjointness of singular Schrödinger operators to arbitrary complete Riemannian manifolds. This improves some earlier results of Shubin, Milatovic and others.

1. Introduction

The analysis of Schrödinger operators has been in the focus of many mathematical physicists for almost a century. In the last decade there has been a lot of interest in properties of Schrödinger operators on Riemannian manifolds (see e.g. [1,4,16,8,20]). In these works the authors extend several results which are known in the euclidean case to larger classes of manifolds. One of the most fundamental questions concerning Schrödinger operators is the question of essential selfadjointness. In the euclidean case very general and in some respect optimal results ensuring essential selfadjointness of magnetic Schrödinger operators have been derived in the work [15] of Leinfelder and Simader. In contrast to the euclidean case existing results concerning essential selfadjointness on Riemannian manifolds are still incomplete. In particular the analogue of the result of Leinfelder and Simader is still missing. The most general result seems to go back to M. Shubin (see [24]), where he proves essential selfadjointness of semibounded magnetic Schrödinger operators under the assumption that the magnetic potential is continuously differentiable. This type of result has later been generalized in [1] to operators of Schrödinger type acting in sections of vector bundles and several further results have been established in [17–19]. In all these results the 'magnetic potential' is assumed to be at least Lipschitz. In this work we prove an extension of the famous result of Leinfelder and Simader on the essential selfadjointess of magnetic Schrödinger operators in $\mathbb{R}^n$. In the euclidean case the result of Leinfelder and Simader asserts that essential selfadjointness holds e.g. as long as the potential satisfies a certain lower bound and the coefficients satisfy certain minimal local regularity requirements much weaker than continuous differentiability. Due to the local character of the regularity requirements on the coefficients it is tempting to believe that the result carries over without essential changes to general complete Riemannian manifolds. Thus in particular the differentiability of the vector potential, as supposed by Shubin in [24], should not be necessary. A careful look at the original work of Leinfelder and Simader demonstrates that a sequence of cut-off functions with uniformly bounded derivatives of first and second order plays an important role in their approach. For arbitrary complete Riemannian manifolds it seems to be unknown whether such a sequence of cut-off functions exists. This technical problem might explain why the analogue of

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.09.060

the result of Leinfelder and Simader for Riemannian manifolds has not been established, yet. In order to avoid this problem we combine ideas of Leinfelder and Simader with an old idea of Chernoff. This will allow us to give a rather transparent proof of a general result of Povzner–Wienholtz–Simader-type under more or less minimal local requirements. Our result demonstrates in contrast to the main result of [24] that essential selfadjointness on manifolds holds under conditions which parallel those of the euclidean case. Moreover, our proof is even more elementary than Shubin's since we can avoid the use of a non-trivial result due to Karcher concerning the existence of smooth cut-off functions with uniformly bounded gradients. Given the work of Leinfelder and Simader our result is not at all surprising, it is much more surprising that only weaker results can be found in the literature. Let us finally remark that Shubin's result was used in [4] in order to derive continuity properties of functions of the magnetic Schrödinger operator on manifolds of bounded geometry. Our result can now be used in order to improve this result. But we should point out that the well-known probabilistic approach via the Feynman–Kac–Itô formula was used by the present author in [14] in order to derive quite sharp results which extend the results of the papers [2] and [3] from the euclidean setting to the case of manifolds with lower bounded Ricci curvature. Indeed once the necessary results concerning essential selfadjointness are established, such an extension to more general manifolds follows closely the approach used in the euclidean case in [2] and [3].

2. The main result

Let (M,g) be a complete Riemannian manifold of dimension n. The metric g induces in a canonical way a measure μ which in local coordinates is given by $d\mu(x)=\sqrt{g}\,dx_1\dots dx_n$. Denote by $TM=(T_xM)_{x\in M}$ and $T^*M=(T_x^*M)_{x\in M}$ the tangent and cotangent bundle, respectively. We will work with complex-valued sections of TM and T^*M and therefore have to complexify the inner product spaces T_xM and T_x^*M in the usual way. The scalar product in T_xM and T_x^*M will be denoted by $\langle\cdot,\cdot\rangle$ with induced norm $|\cdot|$. The Riemannian measure μ induces the Lebesgue spaces $L^p(M)$, $L^p(TM)$ and $L^p(T^*M)$ consisting of pth power integrable functions, vectorfields and 1-forms, respectively. By grad and d we will denote the Riemannian gradient and the exterior derivative, respectively. We will also use the weak adjoint d^* of the exterior derivative d. Moreover, we have to introduce the Sobolev space $H_0^1(\Lambda)$, where $\Lambda\subset M$ is an arbitrary open subset. As usual $H_0^1(\Lambda)$ is defined as the closure of $C_c^\infty(\Lambda)$ with respect to the norm $\|\cdot\|_{L^2(\Lambda)}+\|\mathrm{grad}\cdot\|_{L^2(\Lambda)}$. Every Riemannian manifold admits a unique torsion free connection, called the Levi–Civita connection. The Levi–Civita connection is denoted by ∇. Moreover we need the Ricci curvature, which we denote by Ric. Recall that Ric is by definition a smooth section in $T^*(M)\otimes T^*(M)$ such that for every $x\in M$ the bilinear form Ric_x on the vector space T_xM is symmetric.

We consider the class $\mathcal{M}(M)$ of admissible magnetic potentials given by

$$\mathcal{M}(M)=\left\{A\in L^4_{loc}\left(T^*M\right)\;\middle|\;d^*A\in L^2_{loc}(M)\right\}.$$

Note that in this section $\mathcal{M}$ is used differently to Section 3. Assume that for $A\in\mathcal{M}(M)$ and $V\in L^2_{loc}(M)$ the quadratic form

$$\begin{aligned}&q[A,V]\restriction C_c^\infty(M):C_c^\infty(M)\times C_c^\infty(M)\to\mathbb{C},\\&(\varphi,\psi)\mapsto\int_M\langle d\varphi-iA\varphi,d\psi-iA\psi\rangle\,d\mu+\int_M V\varphi\overline{\psi}\,d\mu\end{aligned}$$

is semibounded. In fact, as every quadratic form semibounded from below becomes non-negative by adding a constant, we will assume that $q[A,V]\restriction C_c^\infty(M)$ is non-negative. Then it is closable since it is the form of a semibounded symmetric operator $\tilde H(A,V)$, which on $C_c^\infty(M)$ is defined by

$$\tilde H(A,V)\psi=-\Delta\psi-2i\langle A,d\psi\rangle+\left(id^*A+|A|^2+V\right)\psi.$$

The closure of the form $q[A,V]\restriction C_c^\infty(M)$ will be denoted by $q[A,V]$ and the uniquely associated selfadjoint operator by $H(A,V)$. The operator $H(A,V)$ is given by

$$\begin{aligned}\mathcal{D}\big(H(A,V)\big)&=\left\{u\in\mathcal{D}\big(q[A,V]\big)\;\middle|\;\exists v\in L^2(M)\;\forall w\in\mathcal{D}\big(q[A,V]\big)\colon q[A,V](u,w)=(v,w)_{L^2(M)}\right\}\\&=\left\{u\in\mathcal{D}\big(q[A,V]\big)\;\middle|\;\exists v\in L^2(M)\;\forall w\in C_c^\infty(M)\colon q[A,V](u,w)=(v,w)_{L^2(M)}\right\},\\H(A,V)u&=v.\end{aligned}$$

Thus $H(A,V)$ is the Friedrichs extension of $\tilde H(A,V)$ and $H(A,V)\restriction C_c^\infty(M)=\tilde H(A,V)$. We will also need magnetic Schrödinger operators, which are defined on open subsets of M and satisfy Dirichlet boundary conditions. Let $\Lambda\subset M$ be an open subset and assume that the quadratic form $q_\Lambda[A,V]$

$$\begin{aligned}&q_\Lambda[A,V]:C_c^\infty(\Lambda)\times C_c^\infty(\Lambda)\to\mathbb{C},\\&(\varphi,\psi)\mapsto\int_M\langle d\varphi-iA\varphi,d\psi-iA\psi\rangle\,d\mu+\int_M V\varphi\overline{\psi}\,d\mu\end{aligned}$$

is closable with a closure which is denoted by $q_\Lambda[A, V]$ then the uniquely associated selfadjoint operator $H_\Lambda(A, V)$ is called the magnetic Schrödinger operator in Λ satisfying Dirichlet conditions. If $\Lambda = B(o, r)$ with $o \in M$ fixed we also write $H_r(A, V)$ instead of $H_{B(o,r)}(A, V)$.

Recall that the Kato-class $\mathcal{K}(M)$ (compare Section E.3 and Definition 2.1 in [7]) on M consists of all function $q : M \to \mathbb{R}$ such that

$$\lim_{t\to 0} \sup_{x\in M} \int_0^t \int_M p(s, x, y)\big|q(y)\big|\, d\mu(y)\, ds = 0, \tag{2.1}$$

where $p(t, x, y)$ denotes the heat kernel associated to the Laplacian on the Riemannian manifold M. We define the local Kato-class $\mathcal{K}_{loc}(M)$ to consist of all functions $q : M \to \mathbb{R}$ such that for all compact $K \subset M$ $\mathbf{1}_K q \in \mathcal{K}(M)$. As is shown on p. 57 in [7] Kato-class potentials are infinitesimally $-\Delta$-form bounded. This allows us to define several quadratic forms using the KLMN-theorem. Moreover Kato-class potentials preserve several important mapping properties of the free heat semigroup.

Theorem 1. *Let (M, g) be an arbitrary complete Riemannian manifold and let $A \in \mathcal{M}(M)$ and $V \in L^2_{loc}(M)$ be given. Assume that $V_- \in \mathcal{K}_{loc}(M)$ and that for some $C \in \mathbb{R}$*

$$\forall \varphi \in C_c^\infty(M): \quad \int_M \big|(d - iA)\varphi\big|^2\, d\mu + \int_M V|\varphi|^2\, d\mu \geqslant C\|\varphi\|^2_{L^2(M)}.$$

Then $H(A, V) \restriction C_c^\infty(M)$ is essentially selfadjoint.

Theorem 1 might be called a result of Povzner–Wienholtz–Simader type, since it asserts that a semibounded Schrödinger operator is already essentially selfadjoint. The above result is in some respect optimal. Observe e.g. that the formal differential operator $H(A, V)$ maps every smooth function with compact support to an element of $L^2(M, \mu)$ if and only if $|A|^2 \in L^2_{loc}(M, \mu)$, $d^*A \in L^2_{loc}(M, \mu)$ and $V \in L^2_{loc}(M, \mu)$. Thus the local requirements on A and V_+ are minimal.

We want to stress that the same type of condition occurs in a quite different situation. Consider the diffusion operator $D := -\Delta + \operatorname{grad} \ln F \cdot \nabla =: -\Delta + b \cdot \nabla$ in the case $M = \mathbb{R}^n$. Under some weak conditions on the non-negative function F this operator D is symmetric in $L^2(\mathbb{R}^n, F(x)\, dx)$. In [9] it is shown that the condition $|b|^2 \in L^2_{loc}(\mathbb{R}^n, F(x)\, dx)$ is necessary in order to have essential selfadjointess of $D \restriction C_c^\infty(\mathbb{R}^n)$. In [13] we proved that under some additional assumptions on F this is also sufficient. This indicates that the L^4_{loc}-condition in Theorem 1 does not only occur because of the multiplicative term $|A|^2$ but more importantly because of the first order term $\langle A, d\varphi\rangle$. This will also become clear during the proof of Theorem 1.

3. Proof of Theorem 1

We will need the fact that a magnetic potential $A \in \mathcal{M}(M)$ can be suitably approximated by smooth compactly supported 1-forms. For an open subset $\Lambda \subset M$ and two smooth compactly supported 1-forms α and β define

$$l_\Lambda(\alpha, \beta) = \left(\int_\Lambda |\alpha - \beta|^4\, d\mu\right)^{\frac{1}{4}} + \left(\int_\Lambda \big|d^*\alpha - d^*\beta\big|^2\, d\mu\right)^{\frac{1}{2}}.$$

Let $(\Lambda_n)_{n\in\mathbb{N}}$ be an exhaustion of M by bounded open subsets with smooth boundary then we set

$$l(\alpha, \beta) = \sum_{n=1}^{\infty} \min\left(\frac{1}{2^n}, l_{\Lambda_n}(\alpha, \beta)\right).$$

It is easy to see that l defines a metric on the space $\Gamma_{comp}(T^*M)$ of smooth compactly supported 1-forms. Let $\tilde{\mathcal{M}}(M)$ be the completion of $\Gamma_{comp}(T^*M)$ with respect to the metric l.

Lemma 1. $\mathcal{M}(M) = \tilde{\mathcal{M}}(M)$.

Proof. Because of the local character of the convergence this is a rather direct consequence of the Friedrichs mollification. □

The following lemma is a geometric version of the well-known Gagliardo–Nirenberg inequality

$$\forall \varphi \in C_c^\infty\big(\mathbb{R}^n\big): \quad \|\operatorname{grad} \varphi\|^2_{L^4(\mathbb{R}^n)} \leqslant C_n\|\varphi\|_\infty\|\Delta\varphi\|_{L^2(\mathbb{R}^n)} \tag{3.1}$$

used in [15] as a fundamental tool. A closer look at the proof given in [15] shows that Leinfelder and Simader first establish the inequality

$$\forall\varphi \in C_c^\infty\left(\mathbb{R}^n\right):\quad \|\operatorname{grad}\varphi\|^2_{L^4(\mathbb{R}^n)} \leqslant C_n\|\varphi\|_\infty\left(\sum_{i,j=1}^n \int_{\mathbb{R}^n} |\partial_{ij}\varphi|^2\,dx\right)^{\frac{1}{2}} \tag{3.2}$$

which involves the Hessian of φ and then use the elementary equation

$$\sum_{i,j=1}^n \|\partial_{ij}\varphi\|^2_{L^2(\mathbb{R}^n)} = \|\Delta\varphi\|^2_{L^2(\mathbb{R}^n)}$$

in order to derive inequality (3.1). Our proof of the analogue of Eq. (3.2) is a coordinate free version of the one given in [15]. In the proof we use the Einstein summation convention and sum over repeated indices.

Lemma 2. *Let $\nabla d\psi$ denote the Hessian of a smooth function ψ. For all bounded $\varphi \in \mathcal{D}(-\Delta)$ with compact support one has*

(1) $\|\operatorname{grad}\varphi\|_4^2 \leqslant C\|\varphi\|_\infty\|\nabla d\varphi\|_2$,
(2) $\|\operatorname{grad}\varphi\|_4^2 \leqslant C_\varphi\|\varphi\|_\infty(\|\Delta\varphi\|_2 + \|\varphi\|_2)$,

where C depends only on the dimension of M and C_φ depends only on the lower bound of the Ricci curvature on the support of φ.

Proof. It is enough to prove the assertions for real-valued functions. For $\varphi \in C_c^\infty(M;\mathbb{R})$ we have

$$\begin{aligned}\int_M |\operatorname{grad}\varphi|^4\,d\mu &= \int_M \langle\operatorname{grad}\varphi,\operatorname{grad}\varphi\rangle\langle\operatorname{grad}\varphi,\operatorname{grad}\varphi\rangle\,d\mu\\ &= \int_M \big\langle\operatorname{grad}\varphi, \langle\operatorname{grad}\varphi,\operatorname{grad}\varphi\rangle\operatorname{grad}\varphi\big\rangle\,d\mu\\ &= -\int_M \langle\operatorname{grad}\varphi,\operatorname{grad}\varphi\rangle\langle\operatorname{div}\operatorname{grad}\varphi\rangle\varphi\,d\mu - \int_M \big\langle\operatorname{grad}|\operatorname{grad}\varphi|^2,\operatorname{grad}\varphi\big\rangle\varphi\,d\mu\\ &= -\int_M |\operatorname{grad}\varphi|^2(\Delta\varphi)\varphi\,d\mu - \int_M \big\langle\operatorname{grad}|\operatorname{grad}\varphi|^2,\operatorname{grad}\varphi\big\rangle\varphi\,d\mu.\end{aligned}$$

Using the definition of the Hessian $\nabla d\varphi$ (see [12, Definition 7.106]) we have

$$\partial_l\langle\operatorname{grad}\varphi,\operatorname{grad}\varphi\rangle = 2\langle\nabla_{\partial_l}\operatorname{grad}\varphi,\operatorname{grad}\varphi\rangle = 2(\nabla d\varphi)(\partial_l,\operatorname{grad}\varphi)$$

and therefore we conclude that

$$\begin{aligned}\left|\int_M \big\langle\operatorname{grad}|\operatorname{grad}\varphi|^2,\operatorname{grad}\varphi\big\rangle\varphi\,d\mu\right| &= 2\left|\int_M \big\langle g^{kl}\partial_l|\operatorname{grad}\varphi|^2\partial_k,\operatorname{grad}\varphi\big\rangle\varphi\,d\mu\right|\\ &\leqslant 2\|\varphi\|_\infty\int_M |\nabla d\varphi||\operatorname{grad}\varphi|^2\,d\mu\\ &\leqslant 2\|\varphi\|_\infty\left(\int_M |\nabla d\varphi|^2\,d\mu\right)^{\frac{1}{2}}\left(\int_M |\operatorname{grad}\varphi|^4\,d\mu\right)^{\frac{1}{2}}.\end{aligned}$$

Hence we see that

$$\int_M |\operatorname{grad}\varphi|^4\,d\mu \leqslant (\sqrt{n}+2)\|\varphi\|_\infty\|\operatorname{grad}\varphi\|_4^2\|\nabla d\varphi\|_2.$$

By the Bochner–Weitzenböck formula (see e.g. formula (1.1) in [23]) we have for smooth functions with compact support $f \in C_c^\infty(M)$

$$\frac{1}{2}\Delta|\operatorname{grad} f|^2 - \langle\operatorname{grad}\Delta f,\operatorname{grad} f\rangle = |\nabla df|^2 + \operatorname{Ric}(\operatorname{grad} f,\operatorname{grad} f).$$

Using

$$\int_M \Delta|\operatorname{grad} f|^2\,d\mu = 0$$

we get after integrating by parts

$$\int_M |\nabla df|^2\,d\mu = \int_M |\Delta f|^2\,d\mu - \int_M \operatorname{Ric}(\operatorname{grad} f, \operatorname{grad} f)\,d\mu.$$

The desired result is now proved for smooth functions $\varphi \in C_c^\infty(M)$. The extension to $\varphi \in H^2(M) \cap L^\infty_{comp}(M)$ is straightforward. □

Remark 2. Observe that without lower bound on the Ricci curvature one cannot estimate the L^4-norm of the gradient of smooth test functions u by the L^2-norms of the Laplacian of u and the function u itself. This is another difference to the euclidean case (3.1), which makes the general case somewhat more complicated.

The following theorem constitutes the core of the hyperbolic approach. The finite speed of propagation property of solutions of many wave equations can effectively be used in a localization of the problem. This fact was noticed by Chernoff in [5]. The next theorem is in some sense an abstract version of his argument and was already applied in this form e.g. in [13].

Theorem 3. *Let $(N, \mathfrak{M}, \nu)$ be a σ-finite measure space on a locally compact space N and let $(S, \mathcal{D}(S))$ be a selfadjoint operator in $L^2(N, \nu)$ which is semibounded from below. Assume that the set of functions in $\mathcal{D}(S)$ with compact support is dense in $L^2(N, \nu)$ and that for every $v \in \mathcal{D}(S)$ with compact support the solution $u(t) = \cos(t\sqrt{S})v$ to the abstract wave equation*

$$\frac{d^2u}{dt^2}(t) = -Su(t), \qquad u(0) = v \tag{3.3}$$

has compact support, then the restriction of S to the class $\mathcal{D}(S)_{comp}$ of functions in $\mathcal{D}(S)$ with compact support is essentially selfadjoint.

For convenience of the reader we give Chernoff's proof (see [5]) in slightly altered form so that it fits to our situation.

Proof. Let $\hat{S}$ denote the restriction of S to $\mathcal{D}(S)_{comp}$. The operator $\hat{S}$ is essentially selfadjoint, if the deficiency indices $\gamma_\pm(\hat{S}) := \dim \operatorname{Ker}(\hat{S}^* \mp i)$ satisfy $\gamma_\pm(\hat{S}) = 0$ (see Chapter 10.1 in [25]).

Suppose there is a $\psi \in \mathcal{D}(\hat{S}^*)$ such that $\hat{S}^*\psi = -i\psi$ and define $g(t) = (u(t), \psi)_{L^2(N,\nu)} = (\cos(t\sqrt{S}) f, \psi)_{L^2(N,\nu)}$ for any $f \in \mathcal{D}(S)_{comp}$. Then g is uniformly bounded because $\cos(t\sqrt{S})$ is unitary and

$$\frac{d^2g}{dt^2} = -\big(S\cos(t\sqrt{S}) f, \psi\big)_{L^2(N,\nu)} = -\big(\hat{S}\cos(t\sqrt{S}) f, \psi\big)_{L^2(N,\nu)} = -\big(\cos(t\sqrt{S}) f, \hat{S}^*\psi\big)_{L^2(N,\nu)} = ig(t).$$

Hence, there are $a, b \in \mathbb{C}$ such that $g(t) = ae^{z_0 t} + be^{-z_0 t}$, where z_0 satisfies $z_0^2 = i$. Since z_0 is not purely imaginary this contradicts the uniform boundedness of g if $a, b \neq 0$. So $g = 0$ and in particular $g(0) = (f, \psi)_{L^2(N,\nu)} = 0$. As the functions $f \in \mathcal{D}(S)_{comp}$ are dense in $L^2(N, \nu)$, we must have $\psi \equiv 0$ and therefore $\gamma_-(\hat{S}) = 0$. The same argument applies to $\gamma_+(\hat{S}) = 0$. □

The next lemma is known (see [5] for related results). A very interesting approach allowing to establish finite propagation for rather general wave equations is presented in [6]. In this work the authors prove in a very general Dirichlet form setting that the property of finite propagation speed for the wave equation is equivalent to the so called Davies' bounds for the parabolic equation. Their results can also be used in order to deduce the following lemma.

Lemma 3. *Let $A \in \Gamma(T^*M)$ be a bounded smooth 1-form and let $V \in C^\infty(M)$ be a smooth potential, which is semibounded from below. Let $H(A, V)$ denote the Friedrichs extension of $\tilde{H}(A, V)$. Then for all open sets Ω_i $(i = 1, 2)$ and $u_i \in L^2(M, \mu)$ with $\operatorname{supp}(u_i) \subset \Omega_i$ $(i = 1, 2)$ we have*

$$\big(\cos\big(t\sqrt{H(A, V)}\big)u_1, u_2\big)_{L^2(M,\mu)} = 0$$

whenever $0 < t < \operatorname{dist}(\Omega_1, \Omega_2) = R$.

Now let for some constant $c \in \mathbb{R}$ the potentials $V_n, V \in L^2_{loc}(M)$ satisfy $V_n, V > c$. Exactly as in Lemma 5 of [15] one shows that $A_n \to A$ in $L^2_{loc}(T^*M)$ and $V_n \to V$ in $L^2_{loc}(M)$ imply that the operators $H(A_n, V_n)$ converge to $H(A, V)$ in the strong resolvent sense. An application of Lemma 3 together with the just mentioned approximation result allows us to conclude that for every $A \in L^2_{loc}(T^*M)$ and every $W \in L^2_{loc}(M)$ which is semibounded from below

$$\left(\cos\left(t\sqrt{H(A,W)}\right)u, v\right)_{L^2(M)} = 0 \tag{3.4}$$

if $u, v \in L^2(M, \mu)$ with $t < \operatorname{dist}(\operatorname{supp}(u), \operatorname{supp}(v))$. In the next lemma we remove the restriction that the potential is semibounded from below. We will see that not semiboundedness of the potential but semiboundedness of the operator matters.

Lemma 4. *Let $H(A,V)$ be as in Theorem 1. For every $u \in L^2(M)$ with compact support in $B_r(o)$ we have*

$$\forall v \in \left\{w \in L^2(M) \mid w \restriction B_{r+t}(o) = 0 \text{ a.e.}\right\}: \quad \left(\cos\left(t\sqrt{H(A,V)}\right)u, v\right)_{L^2(M)} = 0.$$

Proof. Recall that $q[A,V]$ is assumed to be non-negative. Now, consider the sequence $(q_n, \mathcal{D}(q_n))$ of closed non-negative quadratic forms given by $q_n = q[A, V_+ - \min(V_-, n)]$ with $\mathcal{D}(q_n) = \mathcal{D}(q[A, V_+ - \min(V_-, n)]) = \mathcal{D}(q[A, V_+])$. By monotone form convergence one concludes that the sequence of operators $H(A, V_n) = H(A, V_+ - \min(V_-, n))$ converges to the operator H, which is associated to the regular part q^r of the quadratic form q (see [22]) given by

$$q(\varphi) = \inf_n q\left[A, V_+ - \min(V_-, n)\right](\varphi), \qquad \mathcal{D}(q) = \mathcal{D}\left(q[A, V_+]\right)$$

in strong resolvent sense. Recall that the regular part q^r of a quadratic form q is by definition the largest closable quadratic form which is smaller than q. Thus it is enough to show that in our case q is actually closable and that the closure of q coincides with the closure of $q[A,V] \restriction C_c^\infty(M)$. For every $u \in \mathcal{D}(q[A, V_+])$ there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \subset C_c^\infty(M)$ such that $\varphi_n \to u$ as $n \to \infty$ with respect to the norm $\sqrt{q[A, V_+](\cdot) + \|\cdot\|^2_{L^2(M)}}$. By assumption we have

$$\forall n \in \mathbb{N}: \quad \int_M \left|(d - iA)\varphi_n\right|^2 d\mu + \int_M V_+|\varphi_n|^2 d\mu - \int_M V_-|\varphi_n|^2 d\mu \geqslant 0.$$

Thus

$$\forall n \in \mathbb{N}: \quad \int_M \left|(d - iA)\varphi_n\right|^2 d\mu + \int_M V_+|\varphi_n|^2 d\mu \geqslant \int_M V_-|\varphi_n|^2 d\mu$$

and therefore

$$\limsup_{n\in\mathbb{N}} \int_M V_-|\varphi_n|^2 d\mu < \infty.$$

This implies that $q[A,V](\varphi_n - u) + \|\varphi_n - u\|^2_{L^2(M)} \to 0$ as $n \to \infty$. The form domain $\mathcal{D}(q)$ is therefore contained in $\overline{C_c^\infty(M)}^{q[A,V]}$. Thus q is closable and its closure coincides with the closure of $q[A,V] \restriction C_c^\infty(M)$. Hence we get $H(A, V_n) \to H(A,V)$ with $V_n = V_+ - \min(V_-, n)$ in the strong resolvent sense. And since the V_n are semibounded from below, we conclude that Eq. (3.4) holds with $H(A,W)$ replaced by $H(A,V)$. □

This gives us directly the following assertion

Corollary 1. *Let A and V be as in Theorem 1 then the subspace $\mathcal{D}_{comp}$ consisting of all compactly supported functions in the domain $\mathcal{D}(H(A,V))$ of $H(A,V)$ forms an operator core for $H(A,V)$.*

Remark 4. The above approach to the localization of the problem is also applicable to operators of Schrödinger type acting on more general vector bundles. We will apply this approach to the setting considered in [1] in a subsequent project.

For the proof of Theorem 1 it is essential to show that bounded compactly supported functions form an operator core. This will be achieved in the Lemma 5 below. For $r > 0$ let $H_r(A,V)$ denote the operator $H(A,V)$ in the ball $B_r(o)$ with Dirichlet boundary condition, i.e. $H_r(A,V)$ is generated by the closure of $q_r[A,V]$ of the quadratic form $q[A,V] \restriction C_c^\infty(B_r(o))$. The domain $\mathcal{D}(q_r[A,V])$ of the form $q_r[A,V]$ is of course contained in $\mathcal{D}(q[A,V])$. Observe that for $u \in \mathcal{D}(H(A,V))$ with support contained in $B_r(o)$ we also have $u \in \mathcal{D}(H_r(A,V))$ and $H_r(A,V)u = H(A,V)u$. On the other hand if $u \in \mathcal{D}(H_r(A,V))$ with $\operatorname{supp}(u) \subset B_r(o)$ we also have $u \in \mathcal{D}(H(A,V))$ with $H(A,V)u = H_r(A,V)u$. We further set for some fixed $T > 0$

$$\mathcal{C} = \left\{\varphi e^{-tH_{R+3}(A,V)}u \mid 0 < t < T,\ u \in \mathcal{D}\left(H(A,V)\right) \text{ with } \operatorname{supp}(u) \subset B_R \text{ for some } R > 0,\ \varphi \in C_c^\infty\left(B_{R+2}(o)\right),\ 0 \leqslant \varphi \leqslant 1,\ \varphi \restriction B_R = 1\right\}.$$

Lemma 5. *Let A and V be as in Theorem 1. Then the set $\mathcal{C} \subset \mathcal{D}(H(A,V)) \cap L^\infty_{comp}(M)$ forms an operator core for $H(A,V)$.*

Proof. We already know that the class of functions in $\mathcal{D}(H(A,V))$ with compact support form an operator core. Thus it is enough to prove that every $u \in \mathcal{D}(H(A,V))$ having compact support in the ball $B_R(o)$ of radius R with center o can be approximated by bounded compactly supported functions with respect to the graph norm. Let $\varphi \in C_c^\infty(M)$ be a smooth function with $0 \leqslant \varphi \leqslant 1$, $\varphi(x) = 1$ for $d(x,o) < R+1$ and $\varphi(x) = 0$ for $d(x,o) > R+2$. By the diamagnetic inequality we have for every $f \in L^2(B_{R+3})$ the domination property $|e^{-tH_{R+3}(A,V)} f| \leqslant e^{-tH_{R+3}(0,V)}|f|$. By the Feynman–Kac Formula and the condition $V \in \mathcal{K}_{loc}(M)$ one concludes that the semigroup $e^{-tH_{R+3}(0,V)}$ consists of integral operators with integral kernels $p(t,x,y)$ satisfying $\sup_{x,y \in B_{R+3}(o)} p(t,x,y) \leqslant c(t) < \infty$. In order to see this just recall that for $f \in L^2(M,\mu)$

$$e^{-tH_{R+3}(0,V)} f(x) = \mathbb{E}_x\left[e^{-\int_0^t V(X_s)\,ds} f(X_t),\ t < \tau_{B_{R+3}(o)}\right],$$

where the expectation is with respect to Brownian motion on M killed by exiting $B_{R+3}(o)$. Now we can use the assumption $V_- \in \mathcal{K}_{loc}(M)$ and the results in [7] (in particular Theorem 2.21 and Corollary 3.6 in [7]) in order to conclude that $e^{-tH_{R+3}(0,V)}$ has a bounded integral kernel given by

$$p(t,x,y) = \mathbb{E}_{0,x}^{t,y}\left[e^{-\int_0^t V(X_s)\,ds},\ t < \tau_{B_{R+3}(o)}\right],$$

where $\mathbb{E}_{0,x}^{t,y}$ denotes expectation with respect to the unnormalized Brownian bridge measure and $\tau_{B_{R+3}(o)} = \inf\{t > 0 \mid d(o,X_t) \geqslant R+3\}$ the first exit time from the ball $B_{R+3}(o)$. This property gives exactly as in the euclidean case $e^{-tH_{R+3}(0,V)}u \in L^\infty_{loc}(M)$. For $\psi_t = \varphi e^{-tH_{R+3}(A,V)}u = \varphi u_t \in L^\infty(M)$ we get

$$L^2 - \lim_{t\to 0} \psi_t = \varphi u = u$$

and

$$\begin{aligned}
&\left(\int_M \left|H(A,V)(\psi_t - u)\right|^2 d\mu\right)^{\frac12} \\
&\quad= \left(\int_M \left|H(A,V)(\psi_t - \varphi u)\right|^2 d\mu\right)^{\frac12} \\
&\quad= \left(\int_M \left|\varphi H_{R+3}(A,V)(u_t - u) - 2i\langle d_A(u_t-u), d\varphi\rangle - (u_t-u)\Delta\varphi\right|^2 d\mu\right)^{\frac12} \\
&\quad\leqslant \left\|\varphi H_{R+3}(A,V)(u_t-u)\right\|_{L^2(B_{R+3}(o))} + 2\left(\int_{B_{R+3}(o)} \left|\langle d_A(u_t-u), d\varphi\rangle\right|^2 d\mu\right)^{\frac12} + \left\|(u_t-u)\Delta\varphi\right\|_{L^2(B_{R+3}(o))} \\
&\quad\leqslant \left\|H_{R+3}(A,V)(u_t-u)\right\|_{L^2(B_{R+3}(o))} + c_1\left\|(d-iA)(u_t-u)\right\|_{L^2(B_{R+3}(o))} + c_2\|u_t-u\|_{L^2(B_{R+3}(o))}.
\end{aligned}$$

All summands on the right-hand side converge to 0 as $t \to 0$. To see that for the first summand, notice that due to the spectral theorem

$$\begin{aligned}
\left\|H_{R+3}(A,V)(u_t-u)\right\|_{L^2(B_{R+3}(o))} &= \left\|H_{R+3}(A,V)\left(e^{-tH_{R+3}(A,V)}u - u\right)\right\|_{L^2(B_{R+3}(o))} \\
&= \left(\int_{\sigma(H_{R+3}(A,V))} \lambda^2\left(e^{-t\lambda}-1\right)^2 d\|E_\lambda u\|^2\right)^{1/2},
\end{aligned}$$

where $(E_\lambda)_\lambda$ denotes the spectral resolution of $H_{R+3}(A,V)$. Using the theorem of dominated convergence, one can show that $\|H_{R+3}(A,V)(u_t-u)\|_{L^2(B_{R+3}(o))}$ converges to 0 as $t \to 0$. To see that $\|(d-iA)(u_t-u)\|_{L^2(B_{R+3}(o))}$ converges to 0, notice that

$$\begin{aligned}
\left\|(d-iA)(u_t-u)\right\|^2_{L^2(B_{R+3}(o))} &= q_{R+3}[A,0](u_t-u) \\
&= q_{R+3}[A,V](u_t-u) - \int_{B_{R+3}(o)} V|u_t-u|^2\,d\mu \\
&\leqslant q_{R+3}[A,V](u_t-u) + \int_{B_{R+3}(o)} V_-|u_t-u|^2\,d\mu,
\end{aligned}$$

where q_{R+3} denotes the quadratic form generated by H_{R+3}. Now, recall that the potential $\mathbf{1}_{B_{R+3}(o)}V_- \in \mathcal{K}(M)$ and that Kato-class potentials are infinitesimally $-\Delta$-form bounded [7, p. 57], i.e. for every $\varepsilon > 0$ there is $b_\varepsilon > 0$ such that for all $w \in H_0^2(B_{R+3}(o))$

$$\|\sqrt{V_-}w\|_{L^2(B_{R+3}(o))} \leqslant \varepsilon \|\operatorname{grad} w\|_{L^2(B_{R+3}(o))} + b_\varepsilon \|w\|_{L^2(B_{R+3}(o))}.$$

By the quadratic form version of the diamagnetic inequality (see Chapter 2.3 of [21]) this gives for every $w \in H_0^1(B_{R+3}(o))$

$$\|\sqrt{V_-}w\|_{L^2(B_{R+3}(o))} \leqslant \varepsilon \big\|(d - iA)w\big\|_{L^2(B_{R+3}(o))} + b_\varepsilon \|w\|_{L^2(B_{R+3}(o))}.$$

Hence,

$$\begin{aligned}\int\limits_{B_{R+3}(o)} V_-|u_t - u|^2\,d\mu &\leqslant \varepsilon q_{R+3}[A,0](u_t - u) + b_\varepsilon \|u_t - u\|^2_{L^2(B_{R+3}(o))} \\ &\leqslant \varepsilon q_{R+3}[A,V_+](u_t - u) + b_\varepsilon \|u_t - u\|^2_{L^2(B_{R+3}(o))} \end{aligned} \tag{3.5}$$

and this implies

$$\big\|(d - iA)(u_t - u)\big\|^2_{L^2(B_{R+3}(o))} \leqslant q_{R+3}[A,V](u_t - u) + \varepsilon q_{R+3}[A,V_+](u_t - u) + b_\varepsilon \|u_t - u\|^2_{L^2(B_{R+3}(o))}.$$

With the help of Eq. (3.5) we also find that

$$\begin{aligned} q_{R+3}[A,V](u_t - u) &= q_{R+3}[A,V_+](u_t - u) - \int\limits_{B_{R+3}(o)} V_-|u_t - u|^2\,d\mu \\ &\geqslant (1-\varepsilon) q_{R+3}[A,V_+](u_t - u) - b_\varepsilon \|u_t - u\|^2_{L^2(B_{R+3}(o))}, \end{aligned}$$

which shows

$$\big\|(d - iA)(u_t - u)\big\|^2_{L^2(B_{R+3}(o))} \leqslant \frac{1}{1-\varepsilon} q_{R+3}[A,V](u_t - u) + \frac{b_\varepsilon}{1-\varepsilon}\|u_t - u\|^2_{L^2(B_{R+3}(o))}.$$

By the spectral theorem and dominated convergence, we conclude that $\|(d - iA)(u_t - u)\|_{L^2(B_{R+3}(o))}$ tends to 0 as $t \to 0$. Putting everything together we see that

$$\lim_{t\to 0}\bigg(\int\limits_M \big|H(A,V)(\psi_t - u)\big|^2\,d\mu\bigg)^{\frac{1}{2}} = 0.$$

This finishes the proof. □

The Gagliardo–Nirenberg inequality allows us to deduce some regularity properties of functions f belonging to $\mathcal{C}$. First observe that exactly as in Lemma 8 of [15] one deduces

Lemma 6. *Let $\Omega \subset M$ be a bounded open set and let $c > 0$. There exists a constant $b > 0$ such that for all $u \in H^2(M) \cap L^\infty(M)$ with* $\operatorname{supp}(u) \subset\subset \Omega$ *and all vector potentials $A \in \mathcal{M}(M)$ satisfying the conditions of Theorem* 1 *and $\|d^*A\|_{L^2(\Omega,\mu)} + \||A|^2\|_{L^2(\Omega,\mu)} \leqslant c$ the inequality*

$$\|\Delta u\|_{L^2(M)} \leqslant \big\|H(A,V)u\big\|_{L^2(M)} + b\|u\|_{L^\infty}$$

holds true.

This gives

Lemma 7. *Let A and V be as in Theorem* 1*. Then we have*

$$\mathcal{C} \subset H^2(M) \cap L^\infty_{comp}(M).$$

In particular grad $f \in L^4(M)$ *for every* $f \in \mathcal{C}$.

Proof. Our proof is a modification of the proof of Lemma 9 in [15]. Let $u \in \mathcal{D}(H(A,V))$ with $\operatorname{supp}(u) \subset B_R(o)$ and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of smooth 1-forms such that

$$\int\limits_{B_{R+3}(o)} |A_n - A|^4\,d\mu + \int\limits_{B_{R+3}(o)} \big|d^*A_n - d^*A\big|^2\,d\mu \to 0$$

as $n\to\infty$. Then $e^{-tH_{R+3}(A_n,V)}\to e^{-tH_{R+3}(A,V)}$ as $n\to\infty$ with respect to strong operator convergence (this assertion directly follows from the Feynman–Kac–Itô representation of the semigroup $e^{-tH_{R+3}(A,V)}$). Set $u_n=e^{-tH_{R+3}(A_n,V)}u$.

The assumption $V_-\in\mathcal{K}_{loc}(M)$ implies (see proof of Lemma 5) that there exists a $C>0$ such that for all $w\in C_c^\infty(B_{R+3}(o))$ and every $n\in\mathbb{N}$

$$\big\|(d-iA_n)w\big\|^2_{L^2(B_{R+3}(o))}+\|w\|^2_{L^2(B_{R+3}(o))}\leqslant C\big(\big\|(d-iA_n)w\big\|^2_{L^2(B_{R+3}(o))}+\|w\|^2_{L^2(B_{R+3}(o))}-\|\sqrt{V_-}w\|^2_{L^2(B_{R+3}(o))}\big).$$

In particular, there is a constant $\tilde{C}$ such that for all $n\in\mathbb{N}$

$$\big\|(d-iA_n)u_n\big\|^2_{L^2(B_{R+3}(o))}+\|u_n\|^2_{L^2(B_{R+3}(o))}\leqslant\tilde{C}\left(\big\|(d-iA_n)u_n\big\|^2_{L^2(B_{R+3}(o))}+\|u_n\|^2_{L^2(B_{R+3}(o))}+\int\limits_{B_{R+3}(o)}V|u_n|^2\,d\mu\right).$$

This gives

$$\begin{aligned}\big\|(d-iA_n)u_n\big\|^2_{L^2(B_{R+3}(o))}+\|u_n\|^2_{L^2(B_{R+3}(o))}&\leqslant\tilde{C}\big(H_{R+3}(A_n,V)u_n,u_n\big)_{L^2(B_{R+3}(o))}\\&\leqslant\tilde{C}\big\|H_{R+3}(A_n,V)u_n\big\|_{L^2(B_{R+3}(o))}\|u_n\|_{L^2(B_{R+3}(o))},\end{aligned}$$

which implies

$$\|u_n\|_{L^2(B_{R+3}(o))}\leqslant C\big\|H_{R+3}(A_n,V)u_n\big\|_{L^2(B_{R+3}(o))}$$

and

$$\big\|(d-iA_n)u_n\big\|_{L^2(B_{R+3}(o))}\leqslant C\big\|H_{R+3}(A_n,V)u_n\big\|_{L^2(B_{R+3}(o))}.$$

Now, set $v_n=\varphi u_n$ with φ as in the definition of the set $\mathcal{C}$. Then $v_n\in\mathcal{D}(H_{R+3}(A_n,V))\cap L^\infty$ and in particular $H_{R+3}(A_n,V)v_n\in L^2(B_{R+3}(o))$. Hence due to $Vv_n\in L^2(B_{R+3}(o))$,

$$\big\|-H_{R+3}(A_n,0)v_n\big\|_{L^2(B_{R+3}(o))}=\big\|H_{R+3}(A_n,V)v_n-Vv_n\big\|_{L^2(B_{R+3}(o))}<\infty.$$

Observing that $Vv_n\in L^2$ general results concerning elliptic regularity (see [10] or [11]) then imply that $v_n\in H^2(M)$. This allows us to apply Lemma 6, which shows that there is a constant $c>0$ such that

$$\|\Delta v_n\|_{L^2(B_{R+3}(o))}\leqslant\big\|H_{R+3}(A_n,V)v_n\big\|_{L^2(B_{R+3}(o))}+c\|v_n\|_{L^\infty(B_{R+3}(o))}.$$

Since in weak sense

$$H_{R+3}(A_n,V)v_n=\varphi H_{R+3}(A_n,V)u_n-2\langle du_n-iA_nu_n,d\varphi\rangle-u_n\Delta\varphi$$

(this can be proved exactly as Lemma 5 in [15]), we see that there are constants $c_1,c_2,c_3,a,b>0$ such that

$$\begin{aligned}\|\Delta v_n\|_{L^2(B_{R+3}(o))}&\leqslant c_1\big\|H_{R+3}(A_n,V)u_n\big\|_{L^2(B_{R+3}(o))}+c_2\big\|(d-iA_n)u_n\big\|_{L^2(B_{R+3}(o))}+c_3\|u_n\|_{L^\infty(B_{R+3}(o))}\\&\leqslant a+b\big\|e^{-tH_{R+3}(0,V)}|u|\big\|_{L^\infty(B_{R+3}(o))},\end{aligned}$$

where the diamagnetic inequality

$$\big|e^{-tH_{R+3}(A_n,V)}u\big|\leqslant e^{-tH_{R+3}(0,V)}|u|$$

and the spectral theorem were used in the last step to show that there is an n independent upper bound. Observe that – as used already above – the operator $e^{-tH_{R+3}(0,V)}$ is ultracontractive due to the assumption $V_-\in\mathcal{K}_{loc}(M)$. Hence $e^{-tH_{R+3}(0,V)}u\in L^\infty(B_{R+3}(o))$.

The unit ball in the Hilbert space $L^2(B_{R+3}(o))$ is weakly compact. Thus, there exists a weakly convergent subsequence of $(\Delta v_n)_{n\in\mathbb{N}}$. Since $v_n\to\varphi e^{-tH_{R+3}(A,V)}u$ in $L^2(B_{R+3}(o))$ we conclude that $\varphi e^{-tH_{R+3}(A,V)}u\in H^2(M)$. Thereby we also conclude that $\operatorname{grad} f\in L^4(M)$ for every $f\in\mathcal{C}$ using Lemma 2. □

Now we can easily complete the proof of Theorem 1.

Proof of Theorem 1. We have already shown that the set $\mathcal{D}(H(A,V))\cap L^\infty_{comp}(M)$ builds an operator core for $H(A,V)$. Thus it remains to show that every function $f\in\mathcal{D}(H(A,V))\cap L^\infty_{comp}(M)$ can be approximated by smooth functions $\varphi_n\in C_c^\infty(M)$ with respect to the operator norm. By Lemma 6 we conclude that $\Delta f\in L^2(M)$ and consequently by Lemma 2 $\operatorname{grad} f\in L^4(TM)$. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence obtained from f by Friedrichs mollification. Since the support of f is a compact subset of M the Friedrichs mollifiers can be constructed in the standard way in local coordinates. Then we have $f_n\in C_c^\infty(U)$ for any $n\in\mathbb{N}$ and some bounded open set $U\subset M$, $f_n\to f$ in $L^\infty(M)$ and

$$\lim_{n\to\infty}\big\|\operatorname{grad}(f-f_n)\big\|_{L^4(TM)}\quad\text{and}\quad\lim_{n\to\infty}\big\|\Delta(f_n-f)\big\|_{L^2(M)}=0.$$

Therefore we get

$$\begin{aligned}&\big\|H(A,V)f-H(A,V)f_n\big\|_{L^2(M)}\\&\leqslant\big\|\Delta(f-f_n)\big\|_{L^2(M)}+2\big\|\langle A,(f-f_n)\rangle\big\|_{L^2(M)}+\big\|\big(id^*A+|A|^2+V\big)(f-f_n)\big\|_{L^2(M)}\\&\leqslant\big\|\Delta(f-f_n)\big\|_{L^2(M)}+2\big\||A|\big\|_{L^4(U)}\big\|\mathrm{grad}(f-f_n)\big\|_{L^4(TU)}+\big\|\big(id^*A+|A|^2+V\big)\big\|_{L^2(U)}\|f-f_n\|_{L^\infty}\to 0\end{aligned}$$

as $n\to\infty$. □

Acknowledgments

The authors would like to thank the referee for his very careful and helpful report.

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