Europ. J. Combinatorics (1991) **12**, 129–145

P.L. Homeomorphic Manifolds are Equivalent by Elementary Shellings†

UDO PACHNER

Shellability of simplicial complexes has been a powerful concept in polyhedral theory, in p.l. topology and recently in connection with Cohen–Macaulay rings and toric varieties. It is well known that all 2-spheres and all boundary complexes of convex polytopes are shellable, but the analogous theorem fails for general simplicial balls and spheres. In this paper we study transformations of simplicial p.l. manifolds by elementary boundary operations (shellings and inverse shellings). As the main result we shall show that a simplicial p.l. manifold $\mathcal{M}$ can be transformed to any other simplical p.l. manifold $\mathcal{M}'$ homeomorphic to $\mathcal{M}$ using these elementary operations. The tools we need and related results are summarized. In the last part we study generalized shellings of totally strongly connected simplicial complexes and the effects on the face numbers of the complex.

1. Introduction

The concept of stellar subdivision and shellability has an interesting history going back to the 19th century. The early 'proofs' of the Euler relation for convex polytopes (Schläfli, 1852) were based on the then unproved assumption that boundary complexes of convex polytopes are shellable (see [20]). This incompleteness was rectified 120 years later by Bruggesser and Mani [8]. Shellability then played a key role in the first complete proof of the upper-bound conjecture (Motzkin, 1957) by McMullen [29], which provides a tight upper bound on the number of faces of a convex d-polytope with n vertices.

The study of convex polytopes and polyhedral sets has been stimulated since the early 1950s by many problems arising from linear programming. In the past 15 years the interest in convex polytopes and simplicial manifolds has advanced greatly by the development of strong connections to Cohen–Macaulay rings and toric varieties. The proof of McMullen's g-conjecture—the complete characterization of the face numbers of simplicial polytopes—is one of the fundamental results based on this theory [5, 43]. More detailed background and motivation is presented in the following sections.

2. Basic Concepts

Let P be a convex polytope. The boundary complex of P is denoted by $\mathcal{B}(P)$ and $\mathcal{F}(P) := \mathcal{B}(P) \cup \{P\}$. For a single point p we write $\mathcal{F}(\{p\}) =: \bar{p}$. For more information about polytopes the reader is referred to [20]. In the sequel T^d always denotes a d-dimensional simplex.

A finite simplicial complex $\mathcal{C}$ is defined in the usual way in an abstract sense. Nevertheless, we also use notation and constructions arising from geometrical realizations of simplicial complexes. The members of $\mathcal{C}$ are the *faces* of $\mathcal{C}$, and $\dim A$ denotes the dimension of a face A of $\mathcal{C}$. $\mathcal{C}$ is a simplicial n-complex if n is the

† This paper is dedicated to Professor Günter Ewald on the occasion of his 60th birthday.

0195-6698/91/020129 + 17 $02.00/0

maximum dimension of its faces. We use the following notation:

$\mathrm{st}(A; \mathscr{C}) := \{B \in \mathscr{C}: A \subseteq B\}$ '*(open) star*',

$\mathrm{clst}(A; \mathscr{C}) := \bigcup \{\mathscr{F}(B): B \in \mathrm{st}(A; \mathscr{C})\}$ '*(closed) star*',

$\mathrm{ast}(A; \mathscr{C}) := \{B \in \mathscr{C}: B \cap A = \emptyset\}$ '*antistar*',

$\mathrm{link}(A; \mathscr{C}) := \mathrm{ast}(A; \mathscr{C}) \cap \mathrm{clst}(A; \mathscr{C})$,

$\Delta_k(\mathscr{C}) := \{A \in \mathscr{C}: \dim A = k\}$,

$\mathrm{skel}_k(\mathscr{C}) := \{A \in \mathscr{C}: \dim A \leqslant k\}$ '*k-skeleton*',

$\mathrm{vert}(\mathscr{C}) := \Delta_0(\mathscr{C})$ '*vertices*',

$|\mathscr{C}| := \bigcup \mathscr{C}$ '*underlying polyhedron (topological space)*'.

The maximal faces of $\mathscr{C}$ are the *facets* of $\mathscr{C}$. $\mathscr{C}$ is *pure* provided that all facets have the same dimension. A pure simplicial complex $\mathscr{C}$ is *strongly connected* provided that every two facets F, F' of $\mathscr{C}$ can be linked together by a *path of facets* $F = F_0, \ldots, F_r = F'$, i.e. $F_{i-1} \cap F_i$ is a common facet of F_{i-1}, F_i for $i = 1, \ldots, r$. A *missing face* of $\mathscr{C}$ is a simplex $D \notin \mathscr{C}$ with $\mathscr{B}(D) \subseteq \mathscr{C}$, $\dim D \geqslant 1$. A subcomplex $\mathscr{C}'$ of $\mathscr{C}$ is *full* in $\mathscr{C}$ provided that $A \in \mathscr{C}$; $\mathrm{vert}(A) \subseteq \mathscr{C}'$ implies $A \in \mathscr{C}'$. A simplicial n-complex $\mathscr{M}$ is called a *simplicial n-ball, sphere or manifold* if $|\mathscr{M}|$ is a ball, a sphere or a manifold, respectively.

All balls, spheres, manifolds and homeomorphisms to be considered are piecewise linear.

Bd$(\mathscr{C})$ denotes the *boundary complex* of a pure simplicial n-complex $\mathscr{C}$. This is the subcomplex of $\mathscr{C}$ which has as facets those $(n-1)$-faces of $\mathscr{C}$ which are contained in only one facet of $\mathscr{C}$. The set of the *interior faces* of $\mathscr{C}$ is denoted by $Int(\mathscr{C}) := \mathscr{C} \backslash Bd(\mathscr{C})$. We use '$\cong$' for homeomorphic polyhedra and '$\approx$' for isomorphic complexes. However, because additional isomorphisms are always allowed (and often necessary) we shall mostly write '$=$' instead of '$\approx$'.

The *join* of simplicial complexes $\mathscr{C}, \mathscr{C}'$ is defined by $\mathscr{C} \cdot \mathscr{C}' := \{A \cdot A': A \in \mathscr{C}, A' \in \mathscr{C}'\}$, where $A \cdot A' := A \cup A'$ if the complexes are considered as abstract complexes. A realization in euclidean space is given by the convex hull $A \cdot A' := \mathrm{conv}(A \cup A')$. Here it is always assumed that $|\mathscr{C}|, |\mathscr{C}'|$ are joinable (see [19, 22]). This is, for instance, the case if $|\mathscr{C}|, |\mathscr{C}'|$ are embedded into disjoint affine subspaces containing no parallel lines. The join of subsets of joinable complexes is defined in the obvious way. We write $\mathscr{C} \cdot A$ for short instead of $\mathscr{C} \cdot \{A\}$. But it is important to realize that one has to distinguish between the join of $\mathscr{C}$ with the empty simplex ($\mathscr{C} \cdot \{\emptyset\} = \mathscr{C}$) and the join with the empty complex ($\mathscr{C} \cdot \emptyset = \emptyset$).

(2.1) Definition. (1) Let $\mathscr{M}$ be a simplicial n-manifold, and let $F = A \cdot B$ be a facet of $\mathscr{M}$ such that $A \in Int(\mathscr{M})$, $\mathscr{B}(A) \cdot B \subseteq Bd(\mathscr{M})$ and $\dim A$, $\dim B \geqslant 0$. Then we call

$$\mathscr{M}' := \rho_{-F}\mathscr{M} := \mathscr{M} \backslash \mathscr{F}(A) \cdot B$$

an (*elementary*) *k-shelling* of $\mathscr{M}$, where $k := \dim B$. The inverse operation is denoted by $\rho_{+F}\mathscr{M} := \rho_{-F}^{-1}\mathscr{M}$, and $\rho^{\pm}$ represents an *elementary boundary operation* which is a shelling or an inverse shelling.

(2) For simplicial n-manifolds $\mathscr{M}, \mathscr{M}'$ we define:

$$\mathscr{M} \xrightarrow{\text{sh}} \mathscr{M}' :\Leftrightarrow \mathscr{M}' = \rho_r \cdots \rho_1 \mathscr{M},$$

$$\mathscr{M} \approx_{\text{sh}} \mathscr{M}' :\Leftrightarrow \mathscr{M}' = \rho_r^{\pm} \cdots \rho_1^{\pm} \mathscr{M}.$$

(3) For a simplicial n-ball $\mathcal{K}$ we say:

$$\mathcal{K} \text{ is } \textit{shellable} :\Leftrightarrow \mathcal{K} \xrightarrow{\text{sh}} \mathcal{F}(T^n).$$

A simplicial n-sphere $\mathcal{S}$ is called shellable if there exists a facet F of $\mathcal{S}$ such that $\mathcal{S}\setminus\{F\}$ is a shellable n-ball.

Remarks and Additional Notation. (1) It can happen that there exists a face $A \in Int(\mathcal{M})$ and different faces B_1, B_2 such that $\mathcal{B}(A)\cdot B_1$, $\mathcal{B}(A)\cdot B_2 \subseteq Bd(\mathcal{M})$ and $A\cdot B_1$, $A\cdot B_2$ are both facets of $\mathcal{M}$. But for every $B \in Bd(\mathcal{M})$ there exists at most one $A \in Int(\mathcal{M})$ such that $A\cdot B$ is a facet of $\mathcal{M}$ with $\mathcal{B}(A)\cdot B \subseteq Bd(\mathcal{M})$. Hence ρ_{-F} is uniquely determined by B and we write $\rho_{-F} =: \rho_B^-$. Conversely, we write ρ_A^+ for an inverse elementary shelling. This implies that $A \in Bd(\mathcal{M})$ and $\text{link}(A; Bd(\mathcal{M})) = \mathcal{B}(B)$ for a missing face B of $\mathcal{M}$.

(2) $\mathcal{M} \xrightarrow{\text{sh}} \mathcal{M}'$ as well as $\mathcal{M}' \approx_{\text{sh}} \mathcal{M}'$ imply $|\mathcal{M}| \cong |\mathcal{M}'|$.

Obviously '$\approx_{\text{sh}}$' is an equivalence relation.

There is a strong connection between shellings and certain stellar operations.

(2.2) Definition. Let $\mathcal{M}$ be a simplicial n-manifold and let $\varnothing \neq A \in \mathcal{M}$ such that $\text{link}(A; \mathcal{M}) = \mathcal{B}(B)\cdot\mathcal{L}$, where $B \neq \varnothing$ is a simplex not contained in $\mathcal{M}$. Then we call

$$\kappa_{(A,B)}\mathcal{M} := (\mathcal{M}\setminus A\cdot\mathcal{B}(B)\cdot\mathcal{L}) \cup \mathcal{B}(A)\cdot B\cdot\mathcal{L}$$

a *stellar exchange*.

Remarks, Examples and Additional Notation. (1) Clearly $\kappa_{(A,B)}\mathcal{M}$ is again a simplicial n-manifold with $|\kappa_{(A,B)}\mathcal{M}| \cong |\mathcal{M}|$. Obviously $\kappa_{(A,B)}^{-1} = \kappa_{(B,A)}$ holds. The equivalence of simplicial manifolds by stellar exchanges is denoted by '$\approx_{\text{stex}}$'.
(2) In the case of $\dim B = 0$, i.e. $B = \{b\}$ is a (new) vertex, the operations $\kappa_{(A,B)} =: \sigma_{(A,B)} =: \sigma_A$ are well known as *stellar subdivisions* (see [19, 22]). Here $A \in Bd(\mathcal{M})$ or $A \in Int(\mathcal{M})$ respectively, depending on whether $\mathcal{L}$ is a ball or a sphere. Conversely, $\kappa_{(A,B)}^{-1} = \sigma_B^{-1}$ is an *inverse stellar subdivision* in the case of $\dim A = 0$. Clearly the definitions of stellar subdivisions and their inverses are still applicable to arbitrary simplicial complexes (and even to more general complexes). To conform with the former notation $\mathcal{C} \xrightarrow{\text{st}} \mathcal{C}'$ means that $\mathcal{C}'$ is obtainable from $\mathcal{C}$ by stellar subdivisions and '$\approx_{\text{st}}$' denotes the stellar equivalence using both stellar and inverse stellar subdivisions.
(3) $\kappa(A, B) = \sigma_B^{-1}\sigma_A$ holds.
(4) If $\dim A + \dim B = n$ (i.e. $\mathcal{L} = \{\varnothing\}$) then $\kappa_{(A,B)} =: \chi_{(A,B)} =: \chi_A$ is called a *bistellar k-operation* if $\dim A = k$. Obviously we have $\chi_{(A,B)}^{-1} = \chi_{(B,A)}$. The related equivalence relation is denoted by '$\approx_{\text{bst}}$'. If $\dim B \geqslant 1$, $B = p\cdot B'$, then $\chi_{(A,B)}$ is uniquely determined by p and the facet $F := A\cdot B'$ of $\mathcal{M}$. We then say that F is *visible* from p and we write $\chi_{(A,B)} =: \chi_{p/F}$ (for motivation, see 5).
(5) $\mathcal{M} \xrightarrow{\text{sh,bst}} \mathcal{M}'$, $\mathcal{M} \approx_{\text{sh,bst}} \mathcal{M}'$ is defined in the obvious way.
Note that this notation does not imply any order for the performance of the involved types of operations. An *elementary operation* is an elementary boundary operation or a bistellar operation.

3. Stellar Equivalence

The concept of stellar subdivision is one of the standard tools in the theory of simplicial complexes and has an old and rich tradition. For more information the reader can consult any book about p.l. topology [19, 22]. Later on we need the following fundamental theorem.

(3.1) Theorem. *For arbitrary simplicial complexes the following holds*:

$$|\mathscr{C}'| \cong |\mathscr{C}| \Leftrightarrow \mathscr{C}' \approx_{\mathrm{st}} \mathscr{C}.$$

Remark. From remarks (2) and (3) for (2.2) it follows that the same holds for stellar exchanges.

A complete proof of the above theorem can be found in the book of Glaser [19]. For earlier results see [1, 31]. There exist many theorems of the above type. Ewald and Shephard proved a convex version of (3.1). Indeed, they showed the bistellar equivalence of boundary complexes of simplicial polytopes, but they did not emphasize this.

(3.2) Theorem (Ewald and Shephard [17]). *Boundary complexes of (simplicial) polytopes are stellar (bistellar) equivalent in a geometrical sense. This means that this can be done in such a way that all the spheres appearing in the equivalence are polytopal.*

In the next sections we shall prove some generalizations of this theorem. There are many interesting unsolved problems concerning stellar equivalence. We only mention here the following long-outstanding problem which is not even solved for polytopal spheres (see in [22]).

(3.3) Problem. Let $\mathscr{C}_1$, $\mathscr{C}_2$ be stellar equivalent simplicial complexes. Does there exist a common stellar subdivision

$$\mathscr{C}_1 \xrightarrow{\mathrm{st}} \mathscr{C} \xleftarrow{\mathrm{st}} \mathscr{C}_2?$$

In dimension 2 the answer is known to be yes.

(3.4) Theorem (Ewald [14]). *Let $\mathscr{C}_1$, $\mathscr{C}_2$ be simplicial 2-complexes, $|\mathscr{C}_1| = |\mathscr{C}_2|$. Then there exists a common stellar subdivision $\mathscr{C}$.*

Since about 1970 an interesting connection has developed between the theory of convex bodies and algebraic geometry. Let $P \subseteq \mathbb{Q}^d$ be a full dimensional polytope with $0 \in \operatorname{int} P$ and let Σ be the fan of convex cones spanned by the faces of P. With every cone is associated an affine variety, namely the spectrum of the ring of all Laurent polynomials with support in the dual of the cone. These affine varieties can be glued together in a natural way by using the combinatorial structure of Σ. The resulting variety is a projective toric variety. This far-reaching result leads to a complete characterization of the face numbers of simplicial and simple polytopes [43].

Stellar subdivisions of fans correspond to blow-ups of the associated varieties. Hence (3.2) and (3.4) respectively yield transforms of projective toric varieties and complete toric 3-varieties into projective space by composition of blow-ups and blow-downs [15].

4. Basic Construction Theorems

At the beginning of this section we shall enumerate some basic construction methods which may be of intrinsic interest. The first lemmas deal with permutations of elementary operations.

(4.1) Lemma. *Let $\mathcal{M}_1$, $\mathcal{M}_1'$, $\mathcal{M}_2$ be simplicial n-manifolds such that*
(a) $\mathcal{M}_1' \xrightarrow{\text{sh}} \mathcal{M}_1$;
(b) *$\mathcal{M}_1'$ contains no missing face of $\mathcal{M}_1$;*
(c) $\mathcal{M}_2 = \chi_{(A,B)}\mathcal{M}_1$.
Then the following holds:

$$\mathcal{M}_2' := \chi_{(A,B)}\mathcal{M}_1' \xrightarrow{\text{sh}} \mathcal{M}_2.$$

Proof. $\mathcal{M}_1' \supseteq \mathcal{M}_1$ and (b) imply $B \notin \mathcal{M}_1'$. Hence $\mathcal{M}_2'$ is well defined. On the other hand, every $\rho_{\bar{C}} = \rho_{C \cdot D}$ appearing in the process inverse to (a) is applicable to $\mathcal{M}_2$, because (b) guarantees $D \neq B$. □

(4.2) Lemma. *Let $\mathcal{M}_1$, $\mathcal{M}_1'$, $\mathcal{M}_2$ be simplicial manifolds. Then we have:*

$$\mathcal{M}_2 = \chi_{(A,B)}\mathcal{M}_1, \qquad \mathcal{M}_1' = \rho_{-F}\mathcal{M}_1, \qquad A \nsubseteq F \Rightarrow \mathcal{M}_2', := \rho_{-F}\mathcal{M}_2 = \chi_{(A,B)}\mathcal{M}_1'.$$

Proof. $A \nsubseteq F$ implies $A \cdot \mathcal{B}(B) \cap \mathcal{F}(F) = \varnothing$. Furthermore, the applicability of $\chi_{(A,B)}$ on $\mathcal{M}_1$ implies $B \notin \mathcal{M}_1$ and therefore we have $B \nsubseteq F$. Hence $\mathcal{B}(A) \cdot B \cap \mathcal{F}(F) = \varnothing$ holds too. From this it is easy to verify the applicability of the operations and the validity of the identity on the right-hand side. □

(4.3) Lemma [35]. *Let $\mathcal{M}$ be a simplicial n-manifold and*

$$\kappa_{(A,B)}\mathcal{M} = (\mathcal{M} \backslash A \cdot \mathcal{B}(B) \cdot \mathcal{L}) \cup \mathcal{B}(A) \cdot B \cdot \mathcal{L}$$

and

$$\kappa_{(C,D)}\mathcal{L} = (\mathcal{L} \backslash C \cdot \mathcal{B}(D) \cdot \mathcal{L}') \cup \mathcal{B}(C) \cdot D \cdot \mathcal{L}'.$$

Then the following holds:
(1) $\kappa_{(B \cdot C,D)}\kappa_{(A,B)}\mathcal{M} = \kappa_{(A,B)}\kappa_{(A \cdot C,D)}\mathcal{M}$;
(2) $\operatorname{link}(B \cdot C; \kappa_{(A,B)}\mathcal{M}) = \mathcal{B}(A) \cdot \mathcal{B}(D) \cdot \mathcal{L}'$;
(3) $\operatorname{link}(A \cdot C; \mathcal{M}) = \mathcal{B}(B) \cdot \mathcal{B}(D) \cdot \mathcal{L}'$;
(4) $\operatorname{link}(A; \kappa_{(A \cdot C,D)}\mathcal{M}) = \mathcal{B}(B) \cdot \kappa_{(C,D)}\mathcal{L}$.

Proof (details are left to the reader). First one must establish (2), which shows that the left-hand side of (1) is well defined. The validity of (3) guarantees the existence of $\kappa_{(A \cdot C,D)}\mathcal{M}$. Then (4) can be proved to show that the right-hand side of (1) is well defined. Finish with the proof of identity (1). □

Next we study how to replace certain constructions by elementary operations.

(4.4) Lemma [37]. *Let $\mathcal{M}$ be a simplicial n-manifold and $\mathcal{K} \subseteq \mathcal{M}$ a shellable n-ball. Then the following holds:*

$$\mathcal{M} \approx_{\text{bst}} (\mathcal{M} \backslash Int(\mathcal{K})) \cup p \cdot Bd(\mathcal{K}).$$

Proof. From our assumption follows the existence of an inverse shelling

$$\mathcal{K} = \rho^+_{A_m} \cdots \rho^+_{A_1}\mathcal{F}(F_0).$$

From this we obtain by induction on m:

$$(\mathcal{M}\backslash Int(\mathcal{K})) \cup p \cdot Bd(\mathcal{K}) = \chi_{A_m} \cdots \chi_{A_1}\chi_{(F_0,p)}\mathcal{M}. \qquad \square$$

(4.5) Lemma [32]. *Let $\mathcal{M}$ be a simplicial n-manifold, $A \in Int(\mathcal{M})$ and $p \in \operatorname{link}(A;\mathcal{M})$ such that:*
(a) $\operatorname{ast}(p; \operatorname{link}(A;\mathcal{M}))$ *is shellable*;
(b) $\operatorname{link}(p;\mathcal{M}) \cap Int(\operatorname{ast}(p;\operatorname{link}(A;\mathcal{M})) = \{\varnothing\}$.
Then we have:

$$\mathcal{M} \approx_{\mathrm{bst}} (\mathcal{M}\backslash st(A;\mathcal{M})) \cup p \cdot \operatorname{ast}(p; \operatorname{link}(A;\mathcal{M}) \cdot \mathcal{B}(A)) =: \mathcal{M}'.$$

Proof. The proof works with induction on the number r of facets of $\operatorname{ast}(p;\operatorname{link}(A;\mathcal{M}))$. In the case of $r=1$ we clearly have $\mathcal{M}' = \chi_A\mathcal{M}$. Otherwise, let $\rho_{+S_r}\cdots\rho_{+S_2}\mathcal{F}(S_1) = \rho^+_{B_r}\cdots\rho^+_{B_2}\mathcal{F}(S_1) = \operatorname{ast}(p;\operatorname{link}(A;\mathcal{M}))$ be an inverse shelling. Then case 1 can be applied on $\mathcal{M}, A\cdot B_r, p$ to obtain $\mathcal{M}'' := \chi_{A\cdot B_r}\mathcal{M} = \chi_{p/A\cdot S_r}\mathcal{M}$. Now the conditions (a), (b) hold for $\mathcal{M}''$, A, p and, furthermore, we have $\rho_{+S_{r-1}}\cdots\rho_{+S_2}\mathcal{F}(S_1) = \rho^+_{B_{r-1}}\cdots\rho^+_{B_2}\mathcal{F}(S_1) = \operatorname{ast}(p;\operatorname{link}(A;\mathcal{M}''))$, which completes the proof.

(4.6) Lemma [37]. *Let $\mathcal{M}$ be a simplicial n-manifold and let $\mathcal{K} \subseteq Bd(\mathcal{M})$ be a shellable $(n-1)$-ball. Then we have*:

$$\mathcal{M} \cup p \cdot \mathcal{K} \xrightarrow{\mathrm{sh}} \mathcal{M}.$$

Proof. Given a shelling $\rho_{-F_1}\cdots\rho_{-F_m}\mathcal{K} = \mathcal{F}(F_0)$ of $\mathcal{K}$. We obtain

$$\mathcal{M} = \rho_{-p\cdot F_0}\rho_{-p\cdot F_1}\cdots\rho_{-p\cdot F_m}(\mathcal{M}\cup p\cdot\mathcal{K}). \qquad \square$$

(4.7) Lemma [37]. *Let M be a simplicial n-manifold, $\mathcal{K} \subseteq Bd(\mathcal{M})$ a shellable $(n-1)$-ball and $\mathcal{K} \subseteq \operatorname{link}(p;\mathcal{M})$ for a vertex $p \in Int(\mathcal{M})$. Then*

$$\mathcal{M} \xrightarrow{\mathrm{sh}} \mathcal{M}\backslash p \cdot Int(\mathcal{K}) =: \mathcal{M}'.$$

Proof. Let $\rho_{-F_1}\cdots\rho_{-F_m}\mathcal{K} = \mathcal{F}(F_1)$ be a shelling of $\mathcal{K}$. By induction on m one obtains:

$$\rho_{-p\cdot F_1}\cdots\rho_{-p\cdot F_m}\mathcal{M} = \mathcal{M}\backslash p\cdot Int(\mathcal{M}) =: \mathcal{M}'$$

and

$$Bd(\mathcal{M}') = (Bd(\mathcal{M})\backslash Int(\mathcal{K})) \cup p \cdot Bd(\mathcal{K}). \qquad \square$$

Now we are able to replace, under certain niceness conditions, stellar subdivisions by elementary operations.

(4.8) Lemma [32]. *Let $\mathcal{M}$ be a simplicial n-manifold and $A \in Int(\mathcal{M})$. If* $\operatorname{link}(A;\mathcal{M})$ *is shellable then*

$$\mathcal{M} \approx_{\mathrm{bst}} \sigma_A\mathcal{M}.$$

Proof. This follows immediately from Lemma (4.4) or (4.5). $\square$

(4.9) Lemma [37]. *Let $\mathcal{M}$ be a simplicial n-manifold and $A \in Bd(\mathcal{M})$. If both* $\operatorname{link}(A;\mathcal{M})$ *and* $\operatorname{link}(A; Bd(\mathcal{M}))$ *are shellable then*

$$\sigma_A\mathcal{M} \xrightarrow{\mathrm{sh,bst}} \mathcal{M}.$$

PROOF. Following Lemma (4.6), the shellability of $\mathrm{clst}(A; Bd(\mathcal{M}))$ implies $\mathcal{M}' := \mathcal{M} \cup p \cdot \mathrm{clst}(A; Bd(\mathcal{M})) \xrightarrow{\mathrm{sh}} \mathcal{M}$. Furthermore, $\mathrm{ast}(p; \mathrm{link}(A; \mathcal{M}')) = \mathrm{link}(A; \mathcal{M})$ is shellable, and it is easy to see that (b) of Lemma (4.5) holds too. Therefore we obtain

$$\begin{aligned}\mathcal{M}' &\approx_{\mathrm{bst}} (\mathcal{M}' \setminus \mathrm{st}(A; \mathcal{M}')) \cup p \cdot \mathcal{B}(A) \cdot \mathrm{link}(A; \mathcal{M}) \\ &= (\mathcal{M} \setminus \mathrm{st}(A; \mathcal{M})) \cup p \cdot \mathcal{B}(A) \cdot \mathrm{link}(A; \mathcal{M}) \\ &\approx \sigma_A \mathcal{M}.\end{aligned}$$

□

(4.10) LEMMA [37]. *Let $\mathcal{M}$ be a simplicial n-manifold and $A \in Bd(\mathcal{M})$. If both* $\mathrm{link}(A; \mathcal{M})$ *and* $\mathrm{link}(A; Bd(\mathcal{M}))$ *are shellable then*

$$\mathcal{M} \xrightarrow{\mathrm{sh,bst}} \sigma_A \mathcal{M}.$$

PROOF. Following Lemma (4.5), the shellability of $\mathrm{clst}(A; \mathcal{M})$ implies $\mathcal{M} \approx_{\mathrm{bst}} (\mathcal{M} \setminus st(A; \mathcal{M})) \cup p \cdot Bd(\mathrm{clst}(A; \mathcal{M})) =: \mathcal{M}'$. Now $\mathrm{clst}(A; Bd(\mathcal{M}')) = \mathrm{clst}(A; Bd(\mathcal{M}))$ is a shellable n-ball which is contained in $\mathrm{link}(p; \mathcal{M})$. So we obtain, by Lemma (4.7),

$$\mathcal{M}' \xrightarrow{\mathrm{sh}} \mathcal{M}' \setminus p \cdot \mathrm{st}(A; Bd\mathcal{M}')) \approx \sigma_A \mathcal{M}.$$

□

(4.11) LEMMA. *Let $\mathcal{M}$, $\mathcal{M}'$ be simplicial manifolds. Then*:

$\mathcal{M}' = \chi_{(A,B)} \mathcal{M}$ *and* $\mathrm{clst}(A, \mathcal{M}) \cap Bd(\mathcal{M}) = \mathcal{F}(F)$, *$F$ a facet of $Bd(\mathcal{M}) \Rightarrow \mathcal{M}' \approx_{\mathrm{sh}} \mathcal{M}$.*

PROOF. As bistellar operations do not affect the boundary we clearly have $\mathcal{M}'' := \mathcal{M} \setminus (A \cdot \mathcal{B}(B) \cup \{F\}) = \mathcal{M}' \setminus (\mathcal{B}(A) \cdot B \cup \{F\})$. $F \in \mathrm{clst}(A, \mathcal{M})$ implies $F = A' \cdot B'$, where $A = A' \cdot a$, $B = B' \cdot b$. Let $\dim A = k$ and $A_k, \ldots, A_0$ and $B_{n-k}, \ldots, B_0$ be any ordering of the facets of A and B respectively such that $A_0 = A'$ and $B_0 = B'$. Then the following holds:

$$\mathcal{M}'' = \rho_{-A \cdot B_{n-k}} \cdots \rho_{-A \cdot B_0} \mathcal{M} \quad \text{and} \quad \mathcal{M}'' = \rho_{-B \cdot A_k} \cdots \rho_{-B \cdot A_0} \mathcal{M}'.$$

□

(4.12) LEMMA [35]. *For every simplicial n-ball $\mathcal{K}$, the following holds*:

$$\mathcal{K} \text{ shellable} \Rightarrow \sigma_A \mathcal{K} \text{ shellable}.$$

SKETCH OF THE PROOF. Let be $\sigma_A = \sigma_{(A,a)}$ and let the following inverse shelling of $\mathcal{K}$ be given:

$$(*) \qquad \rho^+_{B_r} \cdots \rho^+_{B_2} \mathcal{F}(F_1) = \rho_{+F_r} \cdots \rho_{+F_2} \mathcal{F}(F_1) = \mathcal{K}.$$

Let us consider one of the facets $F_i \in \mathrm{st}(A; \mathcal{K})$, say $F_i = A \cdot S$, and let $A' := A \cap B_i$ $(B_1 := \varnothing)$. Then we have to replace in $(*)$ ρ_{+F_i} by the sequence $\rho_{+a \cdot A_k \cdot S} \cdots \rho_{+a \cdot A_1 \cdot S}$, where $A_1, \ldots, A_k$ is an order of the facets of A starting with the facets of $\mathrm{st}(A', \mathcal{B}(A))$ if $A' \in \mathcal{B}(A)$ and any arbitrary order otherwise. □

The following decomposition lemma plays and important role for the inductive argument in the proof of the main theorem of this section.

(4.13) LEMMA [34]. *Let $\mathcal{C}$ be a simplicial complex. Then there exists a unique decomposition $\mathcal{C} = \mathcal{B}(P) \cdot \mathcal{C}'$ such that P is a simplexoid (i.e. $\mathcal{B}(P) = \mathcal{B}(T_1), \ldots, \mathcal{B}(T_r)$; $T_1, \ldots, T_r$ simplices) and P is maximal with this property.*

IDEA FOR THE PROOF. Let $\mathcal{D}$ be the simplicial complex which has as facets the missing faces of $\mathcal{C}$. The connected components of $\mathcal{D}$ yield the desired decomposition.

REMARK. Clearly, if P_1, P_2 are simplexoids then $\mathscr{B}(P_1) \cdot \mathscr{B}(P_2)$ is again isomorphic to the boundary complex of a simplexoid.

Now we are able to replace stellar subdivisions by elementary operations without any niceness assumptions.

(4.14) THEOREM [37]. *Let $\mathcal{M}$, $\mathcal{M}'$ be simplicial n-manifolds. Then*

$$|\mathcal{M}'| \cong |\mathcal{M}| \Leftrightarrow \mathcal{M}' \xrightarrow{\text{sh,bst}} \mathcal{M}.$$

In particular, we have $\mathscr{F}(T^n) \xrightarrow{\text{sh,bst}} \mathscr{K}$ and $\mathscr{K} \xrightarrow{\text{sh,bst}} \mathscr{F}(T^n)$ for every simplicial n-ball $\mathscr{K}$.

SKETCH OF THE PROOF. The sufficiency follows at once from remark (2) for (2.1) and remark (1) for (2.2). In order to prove the existence of our transformation we can assume $\mathcal{M}' = \kappa_{(A,B)}\mathcal{M}$ (apply remark for Theorem (3.1)).

Now let $\operatorname{link}(A; \mathcal{M}) = \mathscr{B}(B) \cdot \mathscr{L}$ and let $\mathscr{L} = \mathscr{B}(P) \cdot \mathscr{L}'$ be the unique decomposition of $\mathscr{L}$, according to Lemma (4.13), and let there be given an equivalence $\kappa_r \cdots \kappa_1 \mathscr{L}' = \mathscr{B}(T^{m+1})$ or $\mathscr{F}(T^m)$, according to (3.1).

If $m := \dim \mathscr{L}' \leqslant 2$ or $r \leqslant 2$ then $\mathscr{L}'$ is polytopal and hence shellable (see Steinitz's Theorem in [20]), which implies the shellability of $\mathscr{L}$. From this and $\kappa_{(A,B)} = \sigma_B^{-1}\sigma_A$ we conclude our assertion immediately with the help of Lemmas (4.8), (4.9) and (4.10).

We proceed by induction on m and r. Let $\kappa_1 = \kappa_{(C,D)}$. Then we may apply κ_1 to $\mathscr{B}(P) \cdot \mathscr{L}'$ and obtain $\kappa_{(C,D)}(\mathscr{B}(P) \cdot \mathscr{L}') = \mathscr{B}(P) \cdot \kappa_1 \mathscr{L}'$. Two cases arise.

Case 1: $D \notin \mathcal{M}$. Following (1) of Lemma (4.3), we can construct $\mathcal{M}'$ by steps:

$$\mathcal{M} \xrightarrow{\kappa_{(A \cdot C, D)}} \mathcal{M}_1 \xrightarrow{\kappa_{(A,B)}} \mathcal{M}_2 \xleftarrow{\kappa_{(B \cdot C, D)}} \mathcal{M}'.$$

(2), (3), (3) of Lemma (3.4) then enable us to show that in each step the inductive assumption concerning m (respectively r) is applicable. We remark that $D \notin \mathcal{M}$ if $\dim D = 0$.

Case 2: $D \in \mathcal{M}$. This case can be reduced to Case 1. As mentioned above, we may assume $\dim D \geqslant 1$. Let $D = p \cdot E$, p a vertex of D. Then we subdivide $\mathscr{L}'$ in the 0-face p (which clearly yields an isomorphic complex), $\kappa_{(p,q)}\mathscr{L}' = (\mathscr{L}' \setminus p \cdot \operatorname{link}(p; \mathscr{L}')) \cup q \cdot \operatorname{link}(p; \mathscr{L}')$, where q is a new vertex not contained in $\mathcal{M}$. From Lemma (4.3), we then derive

$$\mathcal{M} \xrightarrow{\kappa_{(B \cdot p, q)}} \mathcal{M}_1 \xrightarrow{\kappa_{(A,B)}} \mathcal{M}_2 \xleftarrow{\kappa_{(A \cdot p, q)}} \mathcal{M}',$$

from which we obtain our assertion by the inductive argument or by applying Case 1 in the second step respectively. □

5. TRANSFORMATIONS OF CLOSED MANIFOLDS AND SPHERES

In 1978, Ewald realized the connection between bistellar operations and shellings. Moving along a suitable ray starting from a vertex of a simplicial polytope P 'one can see' (having realized P in a suitable manner) all the facets of P in a certain ordering. This implies both a shelling of the boundary complex of P and a bistellar equivalence between the boundary complex of P and that of a simplex.

(5.1) THEOREM (Ewald [13]). *Let P be a simplicial d-polytope and let p be a vertex of P. Then there exists a (geometrical) bistellar equivalence*

$$\chi_{p/F_r} \cdots \chi_{p/F_1} \mathscr{B}(P) = \mathscr{B}(T^d).$$

REMARKS. (1) An alternative proof works with the help of Gale-diagrams (see [26]).
(2) Kleinschmidt [24] has generalized this process to non-simplicial polytopes.
(3) So-called regular bistellar operations of fans were used to prove that a complete smooth toric 3-variety can be transformed into a projective one by blow-ups with non-singular centers [11, 16].

(5.2) THEOREM (Bruggesser and Mani [8]). *Boundary complexes of polytopes are shellable (this can be done starting with the facets of the star of an arbitrary vertex of the polytope).*

This deep result has produced many applications. As already mentioned, (5.2) was the basis for McMullen's proof of the upper bound conjecture. With the help of shellings of simplicial polytopes, Blind and Mani [7] proved in 1986 the conjecture of Perles that simple polytopes have isomorphic boundary complexes provided that their 1-skeletons are isomorphic.

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring with the natural grading by degree, where the variables are interpreted as the vertices of a $(d-1)$-dimensional simplicial complex $\mathscr{C}$. Then let I be the ideal generated by the missing faces of $\mathscr{C}$. Factoring out I from R yields the so-called *Stanley–Reisner ring A* of $\mathscr{C}$ (see [38, 42]). A result of Reisner [38] in 1976 states that the Stanley–Reisner rings of homological spheres are Cohen–Macaulay rings. One consequence of this result was Stanley's new proof of the upper bound theorem for convex polytopes and its extension to homological spheres [41]. Much effort has been spent to obtain combinatorial proofs of the results of Reisner and Stanley. In 1979 Kind and Kleinschmidt proved that shellable simplicial complexes are Cohen–Macaulay [23]. Another method was used by Stanley [40].

Combining the global construction in (5.1) with similar local processes enabled us to prove:

(5.3) THEOREM (Pachner [33]). *Let P, P' be simplicial d-polytopes with the same number of vertices. Then there exists a (geometrical) bistellar equivalence*

$$\chi_r \cdots \chi_1 \mathscr{B}(P) = \mathscr{B}(P')$$

such that all the polytopes appearing in the equivalence have the same number of vertices (in particular, one can choose P' to be a stacked polytope (see [4, 20]).

It has turned out that shellability is not a property which holds for general spheres.

(5.4) THEOREM (Edwards [12]). *There exist non-shellable triangulated (topological!) 5-spheres.*

For this reason it was surprising that Theorem (5.1) could be generalized to simplicial spheres.

(5.5) THEOREM (Pachner [35]). *Let $\mathscr{M}$, $\mathscr{M}'$ be closed simplicial manifolds. Then we have*

$$|\mathscr{M}'| \cong |\mathscr{M}| \Leftrightarrow \mathscr{M}' \approx_{\text{bst}} \mathscr{M}.$$

PROOF. Replace the corresponding keywords in the proof of Theorem (4.14). □

(5.6) COROLLARY. *Every simplicial n-sphere is bistellar equivalent to the boundary complex of the* $(n+1)$*-simplex.*

For simplicial 3-spheres with up to 9 vertices this was proved by computationally constructing [3] all these spheres. Using the ideas of Kind and Kleinschmidt and based on (5.6), Lee has recently found a new proof of the Cohen–Macaulay property for simplicial spheres [27].
A statement of Mandel [28] seems to imply that Theorem (5.3) cannot be generalized to simplicial spheres. But the following is known:

(5.7) THEOREM (Pachner [32]). *Let* $\mathcal{S}$ *be a simplicial n-sphere. If* $\mathcal{S}$ *can be transformed into* $\mathcal{B}(T^{n+1})$ *by bistellar operations without bistellar n-operations then* $\mathcal{S}$ *can be transformed into the boundary complex of a stacked polytope by bistellar operations without changing the number of vertices during the process.*

For the construction of special collars in Section 6 we need the following strengthening of a theorem in [35].

(5.8) THEOREM. *Every simplicial n-sphere* $\mathcal{S}$ *is the boundary complex of a shellable simplicial* $(n+1)$*-ball* $\mathcal{K}$. $\mathcal{K}$ *can be chosen such that* $\mathcal{S}$ *is full in* $\mathcal{K}$.

PROOF. Following Theorem (5.6), it is sufficient to prove our assertion for $\mathcal{S}' = \chi_{(A,B)}\mathcal{S}$, assuming that the assertion holds for $\mathcal{S}$.

Case 1: $B \notin \mathcal{K}$. Then $\mathcal{K}' := \rho'_A\, \mathcal{K}$ is a shellable ball with boundary complex $\mathcal{S}'$.

Case 2: $B \in \mathcal{K}$. Indeed we then have $B \in Int(\mathcal{K})$ and, following Lemma (4.12), $\sigma_B\mathcal{K}$ is again a shellable ball with boundary complex $\mathcal{S}$. As $B \notin \sigma_B\mathcal{K}$ we can then apply Case 1 to obtain a shellable ball $\mathcal{K}'$ with boundary complex $\mathcal{S}'$.

Stellar subdivisions applied in all faces $C \in Int(\mathcal{K}')$ which are missing faces of $\mathcal{S}'$ then yield the desired ball (Lemma (4.12)). □

For further information about bistellar equivalence and related problems the reader can consult [13, 25, 33, 25].

6. TRANSFORMATIONS OF MANIFOLDS WITH BOUNDARY AND BALLS

A survey of shelling can be found in [10]. In addition to the facts already presented we mention the following important result.

(6.1) THEOREM (Rudin [39], Grünbaum [21]). *There exist non-shellable simplicial balls.*

As we have seen in Theorem (4.14), one succeeds with additional bistellar operations. It would certainly be more convenient to deal with boundary operations alone. In order to replace bistellar operations by shellings and inverse shellings we need special partial collars:

(6.2) LEMMA. *Let $\mathcal{M}$ be a simplicial manifold and F be a facet of $Bd(\mathcal{M})$. Then there exists a simplicial manifold $\mathcal{M}'$ such that:*

(1) $\mathcal{M}' \xrightarrow{\text{sh}} \mathcal{M}$;
(2) $\mathcal{M}$ *is full in* $\mathcal{M}'$;
(3) $Bd(\mathcal{M}') \cap Bd(\mathcal{M}) = \mathcal{F}(F)$.

PROOF. Let $A \in Bd(\mathcal{M})$ be. Then link $(A; Bd(\mathcal{M}))$ is a simplicial sphere and Theorem (5.5) asserts the existence of a ball $\mathcal{K}$ such that:
(a) $Bd(\mathcal{K}) = \text{link}(A; Bd(\mathcal{M}))$;
(b) $Bd(\mathcal{K})$ is full in $\mathcal{K}$;
(c) $\mathcal{K}$ is shellable.
From (a) and (b) it follows that $\mathcal{M}' := \mathcal{M} \cup A \cdot \mathcal{K}$ is again a simplicial manifold with $Bd(\mathcal{M}') = (\mathcal{M} \backslash \text{st}(A; Bd(\mathcal{M})) \cup \mathcal{B}(A) \cdot \mathcal{K}$. Furthermore, (c) implies the shellability of $\mathcal{B}(A) \cdot \mathcal{K}$, and $\mathcal{B}(A) \cdot \mathcal{K}$ is contained in link$(a; \mathcal{M}')$ for every vertex a of A. Hence we obtain $\mathcal{M}' \xrightarrow{\text{sh}} \mathcal{M}$ directly from Lemma (4.7). Further, (b) implies that $\mathcal{M}'$ contains no missing face of $\mathcal{M}$.
Application of the above process to all the faces of $\mathcal{M} \backslash \mathcal{F}(F)$ after having them ordered by decreasing dimension yields the desired manifold. □

REMARK. Generalizations of (6.2) are obvious.

We are now able to prove the main theorem of this paper.

(6.3) THEOREM. *Let $\mathcal{M}'$, $\mathcal{M}$ be simplicial manifolds with boundary. Then the following holds:*

$$|\mathcal{M}'| \cong |\mathcal{M}| \Leftrightarrow \mathcal{M}' \approx_{\text{sh}} \mathcal{M}.$$

PROOF. Following Theorem (4.14), it is sufficient to prove $\mathcal{M}' := \chi_{(A,B)}\mathcal{M} \approx_{\text{sh}} \mathcal{M}$. We may assume that $\mathcal{M}$ is connected. From this it follows that $\mathcal{M}$ is strongly connected (see [2]) and hence that there exists a sequence $F_0, \ldots, F_r$ of facets of $\mathcal{M}$ such that $F_0 \in \text{st}(A, \mathcal{K})$, $F := F_r$ is a facet having one of its facets in the boundary of $\mathcal{M}$ and F_{i-1}, F_i have a facet in common. We choose such a sequence with minimum r and then prove by induction on r, as follows.
There exists a simplicial n-manifold $\mathcal{M}_1$ such that

$$\mathcal{M}_1 \xrightarrow{\text{sh}} \mathcal{M}, \qquad \mathcal{M}_1' := \chi_{(A,B)}\mathcal{M}_1 \xrightarrow{\text{sh}} \mathcal{M}' \qquad \text{and} \qquad \mathcal{M}_1' \approx_{\text{sh}} \mathcal{M}_1.$$

Let S be the facet of F contained in $Bd(\mathcal{M})$ and then let $\mathcal{M}_2$ be the simplicial n-manifold constructed in Lemma (6.2) with respect to $\mathcal{M}$, S. From (2) of Lemma (6.2) it follows that we can apply $\chi_{(A,B)}$ on $\mathcal{M}_2$ and Lemma (4.1) yields $\mathcal{M}_2' := \chi_{(A,B)}\mathcal{M}_2 \xrightarrow{\text{sh}} \mathcal{M}'$.
In case $r = 0$, (3) of Lemma (6.2) enables us to apply Lemma (4.11), which yields $\mathcal{M}_2' \xrightarrow{\text{sh}} \mathcal{M}_2$. That means that our assertion holds for $\mathcal{M}_1 := \mathcal{M}_2$.
Otherwise (1) and (3) of Lemma (6.2) allow us to apply ρ_{-F} on $\mathcal{M}_2$. The minimality of r implies $F \notin \text{st}(A; \mathcal{M}_2)$. Hence $\chi_{(A,B)}$ can be applied on $\mathcal{M}_3 := \rho_{-F}\mathcal{M}_2$, and Lemma (4.2) yields $\rho_{-F}\mathcal{M}_3' = \mathcal{M}_2'$, where $\mathcal{M}_3' := \chi_{(A,B)}\mathcal{M}_3$. The inductive assumption applied to $\mathcal{M}_3$ now proves our assertion. □

(6.4) Corollary (Pachner [37]). *For every simplicial n-ball* $\mathcal{K}$,

$$\mathcal{K} \approx_{\text{sh}} \mathcal{F}(T^n).$$

(6.5) Corollary (Pachner [37]). *Let* $\mathcal{S}$ *be a simplicial n-sphere and* $p \in \operatorname{vert}(\mathcal{S}_1)$. *Then there exists a transformation*

$$\chi^{\pm}_{p/F_r} \cdots \chi^{\pm}_{p/F_1} \mathcal{S} = \mathcal{B}(T^{n+1}).$$

Proof. Apply Theorem (6.3) on $\operatorname{ast}(p; \mathcal{S})$.

7. Pseudoshellings and Face Numbers

Let $f_i(\mathcal{C})$ be the number of i-dimensional faces of a simplicial $(d-1)$-complex. The vector $f(\mathcal{C}) := (f_0, \ldots, f_{d-1})$ will be called the *f-vector* of $\mathcal{C}$. For many purposes the *h-vector* $h(\mathcal{C})$ defined by

$$(7.1)\quad h_i(\mathcal{C}) := \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\mathcal{C}), \qquad i = 0, \ldots, d,$$

is easier to deal with. For the algebraic interpretation of $h(\mathcal{C})$ in Stanley–Reisner rings the reader is referred to [42].

Unlike stellar subdivisions, elementary operations allow an explicit computation of the numbers of faces. Following Corollary (6.4), $\mathcal{K} \approx_{\text{sh}} \mathcal{F}(T^{d-1})$ holds for every simplicial $(d-1)$-ball $\mathcal{K}$. Let λ_i^- and λ_i^+ respectively denote the numbers of elementary k-shellings and inverse elementary k-shellings if $\mathcal{K}$ is constructed from $\mathcal{F}(T^{d-1})$ in this way. It is well known and easy to calculate with the help of (7.1) that $h(\rho_A^- \mathcal{K}) = h(\mathcal{K}) - e_k$ holds, where e_k denotes the k-th unit vector $(k = 0, \ldots, d)$ and $\dim A = k - 1$. Hence we obtain:

$$(7.2)\quad h_i(\mathcal{K}) = \lambda^+_{d-1-i} - \lambda^+_{i-1}, \qquad i = 1, \ldots, d-1.$$

These equations obviously remain true if we use generalized shellings which change the f-vector in the same way. Many authors have already considered simplicial complexes which can be constructed from a single simplex by inverse generalized shellings (compare [9, 23]). These complexes are in general not manifolds, but they are of the following type:

(7.3) Definition. A *totally strongly connected* complex $\mathcal{C}$ is a pure simplicial complex with the property that $\operatorname{link}(A, \mathcal{C})$ is strongly connected for every $A \in \mathcal{C}$ ($A = \varnothing$ included!).

We want to preserve this property when allowing additional generalized shellings.

(7.4) Definition. Let $\mathcal{C}, \mathcal{C}'$ be totally strongly connected $(d-1)$-complexes, $A \in \mathcal{C}$, $\dim A = k$. Then we call $\hat{\rho}_{+F}\mathcal{C} := \hat{\rho}^+_{(A,B)}\mathcal{C} := \mathcal{C}'$ an *inverse* (*elementary*) *k-pshelling* (*= pseudoshelling*) provided that $\mathcal{C}' = \mathcal{C} \cup B \cdot \mathcal{F}(A)$, where $F = A \cdot B$ is a $(d-1)$-simplex such that $\mathcal{F}(F) \cap \mathcal{C} = A \cdot \mathcal{B}(B)$ holds. The inverse operation is called an *elementary* $(d-1-k)$*-pshelling* and is denoted by $\hat{\rho}_{-F} = \hat{\rho}^-_{(B,A)}$.

Remarks and Examples. (1) Note that a pseudoshelling is not determined by A or B alone. The applicability of $\hat{\rho}^-_{(B,A)}\mathcal{C}'$ implies $B \cdot \mathcal{B}(A) \subseteq Bd(\mathcal{C}')$. The notations '$\xrightarrow{\text{psh}}$', '$\approx_{\text{psh}}$' are defined as usual.

(2) We allow inverse $(d-1)$-$p$shellings $\rho_{(F,\varnothing)}\mathscr{C}$. F a $(d-1)$-simplex. This implies $\mathscr{F}(F)\cap\mathscr{C}=\{F\}\cdot\varnothing=\varnothing$. From this follows $\varnothing\notin\mathscr{C}$, which implies $\mathscr{C}=\varnothing$ and $\varnothing\xrightarrow{\text{psh}}\mathscr{F}(T^{d-1})$. Thus, given an equivalence $\varnothing\approx_{\text{psh}}\mathscr{C}$, we always may assume that there appears precisely one inverse $(d-1)$-$p$seudoshelling and no (-1)-$p$shelling (which is the inverse operation).
(3) Let $\mathscr{S}$ be a simplicial $(d-1)$-sphere and F a facet of $\mathscr{S}$. Then $\hat{\rho}_{-F}\mathscr{S}$ is a simplicial ball.

From remark (3) of Definition (7.4) and Corollary (6.4) we obtain:

(7.5) $\mathscr{C}$ simplicial ball or sphere $\Rightarrow\varnothing\approx_{\text{psh}}\mathscr{C}$.

Let be $\varnothing\approx_{\text{psh}}\mathscr{C}$. As for elementary operations, then let $\lambda_k^-(\mathscr{C})$ [respectively $\lambda_k^+(\mathscr{C})$] denote the number of k-$p$shellings and inverse k-$p$shellings in this process (starting with the empty complex $\varnothing$). Clearly these numbers do not depend only on $\mathscr{C}$, but also on the present equivalence. (7.2) generalizes to:

(7.6) $(\lambda_{d-1-i}^+-\lambda_{i-1}^-)$ $(\mathscr{C})$ depends only on $\mathscr{C}$ as $h_i(\mathscr{C})=(\lambda_{d-1-i}^+-\lambda_{i-1}^-)$ $(\mathscr{C})$ holds for $i=0,\ldots,d$.

In particular, $h_0=1$.

Examples. $h(\mathscr{F}(T^{d-1}))=(1,0,\ldots,0)$, $h(\mathscr{B}(T^d))=(1,\ldots,1)$.

The number of facets increases by one or decreases by one respectively if an inverse elementary $p$shelling or an elementary $p$shelling is applied. Obviously the number of vertices is calculated as follows (compare remark (2) for (7.4)):

$$f_0=d(\lambda_{d-1}^+-\lambda_{-1}^-)(\mathscr{C})+(\lambda_{d-2}^+-\lambda_0^-)(\mathscr{C})=d+(\lambda_{d-2}^+-\lambda_0^-)(\mathscr{C})$$

Hence we obtain from (7.6):

(7.7) $f_0(\mathscr{C})=d+h_1(\mathscr{C})$ and $f_{d-1}(\mathscr{C})=\sum_{i=0}^{d}h_i(\mathscr{C})$.

Obviously, every inverse k-$p$shelling of $\mathscr{C}$ induces one inverse j-$p$shelling in $\text{skel}_{d-2}(\mathscr{C})$ for $j=k-1,\ldots,-1$. Thus we obtain:

$$\lambda_{k-1}^+(\text{skel}_{d-2}(\mathscr{C}))=\sum_{j=k}^{d-1}\lambda_j^+(\mathscr{C})$$

and, analogously,

$$\lambda_k^-(\text{skel}_{d-2}(\mathscr{C}))=\sum_{j=0}^{k}\lambda_j^-(\mathscr{C}).$$

Consequently, we obtain:

(7.8) Let $\varnothing\approx_{\text{psh}}\mathscr{C}$, $\dim\mathscr{C}=d-1$. Then:

(1) $\varnothing\approx_{\text{psh}}\text{skel}_{d-2}(\mathscr{C})$;
(2) $(h_1-h_{i-1})(\text{skel}_{d-2}(\mathscr{C}))=h_i(\mathscr{C})$, $i=0,\ldots,d-1$, and $h_i(\text{skel}_{d-2}(\mathscr{C}))=\sum_{j=0}^{i}h_j(\mathscr{C})$.

Together with (7.7) this leads to:

(7.9) $f_j(\mathscr{C}):=\sum_{i=0}^{j+1}\binom{d-i}{d-j-1}h_i(\mathscr{C})$, $\qquad j=-1,\ldots,d-1$.

Let us now consider bistellar operations. Let $\mu_i(\mathcal{M}, \mathcal{M}')$ denote the number of bistellar k-operations in an equivalence $\mathcal{M}' \approx_{\text{bst}} \mathcal{M}$ (transforming $\mathcal{M}$ into $\mathcal{M}'$). For a simplicial $(d-1)$-sphere we write $\mu_i(\mathcal{S}) := \mu_i(\mathcal{B}(T^d), \mathcal{S})$. From (7.1) it follows easily that a bistellar k-operation decreases h_i for $k+1 \leqslant i \leqslant d-1-k$ by one if $2k \leqslant d-1$, and increases h_i for $d-k \leqslant i \leqslant k$ by one if $2k \geqslant d-1$. Hence we obtain:

(7.10) Let $\mathcal{M} \approx_{\text{bst}} \mathcal{M}'$. Then:

(1) $(\mu_{d-1-i} - \mu_i)(\mathcal{M}, \mathcal{M}') = (h_{i+1} - h_i)(\mathcal{M}') - (h_{i+1} - h_i)(\mathcal{M})$, $\quad 0 \leqslant i \leqslant d-1$. That means $(\mu_{d-1-i} - \mu_i)(\mathcal{M}, \mathcal{M}')$ depends only on $\mathcal{M}$, $\mathcal{M}'$. In particular, for spheres we have $(\mu_{d-1-i} - \mu_i)(\mathcal{S}) = (h_{i+1} - h_i)(\mathcal{S})$.
(2) $h_i(\mathcal{M}') - h_i(\mathcal{M}) = \sum_{j=0}^{i-1} (\mu_{d-1-j} - \mu_j)(\mathcal{M}, \mathcal{M}')$.

This leads to an easy proof of the *Dehn–Sommerville equations.*

(7.11) For closed simplicial $(d-1)$-manifolds $(h_{d-i} - h_1)(\mathcal{M})$ are topological invariants, $0 \leqslant i \leqslant d$. In particular, for a sphere $\mathcal{S}$ it holds that $(h_{d-i} - h_i)(\mathcal{S}) = 0$.

Proof. $|\mathcal{M}'| \cong |\mathcal{M}|$ implies $\mathcal{M}' \approx_{\text{bst}} \mathcal{M}$ (Theorem (5.5)) and, following (7.10), we then obtain:

$$(h_{d-i} - h_i)(\mathcal{M}') - (h_{d-i} - h_i)(\mathcal{M}) = \sum_{j=0}^{d-i-1} (\mu_{d-1-j} - \mu_j)(\mathcal{M}, \mathcal{M}') - \sum_{j=0}^{i-1} (\mu_{d-1-j} - \mu_j)(\mathcal{M}, \mathcal{M}')$$
$$= \sum_{j=0}^{d-1} \mu_{d-1-j}(\mathcal{M}, \mathcal{M}') - \sum_{j=0}^{d-1} \mu_j(\mathcal{M}, \mathcal{M}') = 0.$$

In particular, for spheres we obtain

$$(h_{d-1} - h_i)(\mathcal{S}) = (h_{d-i} - h_i)(\mathcal{B}(T^d)) = 0.$$

Remark. For $i = 0$ this yields the Euler equation (use (7.8)).

Pseudoshellings make it possible to construct manifolds other than spheres or balls.

(7.12) Theorem. *For every orientable closed simplicial 2-manifold $\mathcal{M}$, $\varnothing \approx_{\text{psh}} \mathcal{M}$. Also $(h_2 - h_1)(\mathcal{M}) = -6g(\mathcal{M})$, $g(\mathcal{M})$ the genus of $\mathcal{M}$.*

Proof. Let $F = x \cdot y \cdot z$ and $F' = x' \cdot y' \cdot z'$ be two disjoint triangles of $\mathcal{M}$. $\mathrm{Skel}_1(\mathcal{M})$ contains a path $x = x_0, x_1, \ldots, x_r = x'$. Using stellar subdivisions and Theorem (6.3) we may assume $x_0 \cdot x_i \notin \mathcal{M}$ for $i = 1, \ldots, r$. Now we are able to construct a surface $\mathcal{M}'$ of genus $g(\mathcal{M}') = g(\mathcal{M}) + 1$ by elementary $p$shellings, and its inverses by the following steps:

$$\mathcal{C}_1 := \hat{\rho}^-_{(F', \varnothing)} \hat{\rho}^-_{(F, \varnothing)} \mathcal{M},$$
$$\mathcal{C}_2 := \hat{\rho}^+_{(x_{r-1}, x_0 \cdot x_r)} \hat{\rho}^+_{(x_{r-2}, x_0 \cdot x_{r-1})} \cdots \hat{\rho}^+_{(x_1, x_0 \cdot x_2)} \mathcal{C}_1,$$
$$\mathcal{C}_3 := \hat{\rho}^+_{(\varnothing, x \cdot x' \cdot z')} \hat{\rho}^+_{(z, x \cdot z')} \hat{\rho}^+_{(y', z \cdot z')} \hat{\rho}^+_{(y, z \cdot y')} \hat{\rho}^+_{(x', y \cdot y')} \hat{\rho}'_{(x, y \cdot x')} \mathcal{C}_2,$$
$$\mathcal{M}' := \hat{\rho}^-_{(x_0 \cdot x_2, x_1)} \cdots \hat{\rho}^-_{(x_0 \cdot x_{r-1}, x_{r-2})} \hat{\rho}^-_{(x_0 \cdot x_{r-1} \cdot x_r, \varnothing)} \mathcal{C}_3.$$

From this follows

$$h(\mathcal{M}') - h(\mathcal{M}) = -2e_2 + re_1 + (5e_1 + e_2) - (e_2 + (r-1)e_1) = 4e_1 - 2e_2.$$

Together with (7.5) and (7.11) this implies our assertion. □

(7.13) Conjecture. $\varnothing \approx_{\text{psh}} \mathcal{M}$ holds for simplicial manifolds.

REMARK. We believe that a general proof is possible with the help of *handle-body theorems* (see [36]).

Finally, we shall present a combinatorial interpretation for some known consequences of the Dehn–Sommerville equations (see [30]). Let $\mathscr{S}$ be a simplicial $(d-1)$-sphere, $p \in \mathrm{vert}(\mathscr{S})$ and $\mathscr{K} := \mathrm{ast}(p; \mathscr{S})$, $\mathscr{L} := \mathrm{link}(p; \mathscr{S})$. From Corollary (6.4) we obtain an equivalence $\mathscr{F}(T^{d-1}) \approx_{\mathrm{sh}} \mathscr{K}$ which yields bistellar equivalences $\mathscr{B}(T^d) \approx_{\mathrm{bst}} \mathscr{S}$ and $\mathscr{B}(T^{d-1}) \approx_{\mathrm{bst}} \mathscr{L}$ (compare Corollary (6.5)). Obviously we have that every elementary k-shelling of $\mathscr{K}$ induces a bistellar k-operation of $\mathscr{S}$ and $\mathscr{L}$. Also every inverse elementary k-shelling of $\mathscr{K}$ induces a bistellar k-operation of $\mathscr{L}$ and a bistellar $(k+1)$-operation of $\mathscr{S}$ respectively. Hence we obtain:

$$\mu_j(\mathscr{S}) = \lambda_{j-1}^+(\mathscr{K}) + \lambda_j^-(\mathscr{K}), \qquad j = 0, \ldots, d,$$

and

$$\mu_j(\mathscr{L}) = \lambda_j^+(\mathscr{K}) + \lambda_j^-(\mathscr{K}), \qquad j = 0, \ldots, d-1.$$

From (7.6) and (7.10) it then easily follows that:

(7.14) (1) $(h_{i+1} - h_i)(\mathscr{S}) = (h_{i+1} - h_{d-1})(\mathscr{K}), \qquad i = 0, \ldots, d-1,$

$$h_i(\mathscr{S}) = (h_0 + \cdots + h_i)(\mathscr{K}) - (h_{d+1-i} + \cdots + h_d)(\mathscr{K}), \qquad i = 0, \ldots, d;$$

(2) $(h_i - h_{i-1})(\mathscr{L}) = (h_i - h_{d-i})(\mathscr{K}), \qquad i = 0, \ldots, d-2,$

$$h_i(\mathscr{L}) = (h_0 + \cdots + h_i)(\mathscr{K}) - (h_{d-i} + \cdots + h_d)(\mathscr{K}), \qquad i = 0, \ldots, d-1.$$

This certainly implies:

(7.15) $h_i(\mathscr{S}) = h_i(\mathscr{K}) + h_{i-1}(\mathscr{L}) = h_{d-i}(\mathscr{K}) + h_i(\mathscr{L}), \qquad i = 0, \ldots, d.$

There are similar equations in the paper by McMullen and Walkup [30]. We hope that additional ideas will solve the following problem:

(7.16) PROBLEM. Find combinatorial proofs of the following. For every, simplicial ball $\mathscr{K}$, it holds that:
(1) $h(\mathscr{K}) \geqslant 0$;
(2) $(h_{i+1} - h_{d-i})(\mathscr{K}) \geqslant 0$, $2i \leqslant d-1$.

(1) is known to be true (although not by strictly combinatorial methods), while the truth of (2) has yet to be determined by any means, although (2) is true if $Bd(\mathscr{K})$ is polytopal.

ACKNOWLEDGEMENT

I would like to thank Andreas Krecht, who has proved the details of a part of the theorems.

REFERENCES

1. J. W. Alexander, The combinatorial theory of complexes, *Ann. Math.*, (2) **31** (1930), 292–320.
2. P. S. Alexandroff and H. Hopf, *Topologie I*, Springer-Verlag, Berlin, 1935.

3. A. Altshuler, J. Bokowski and L. Steinberg, The classification of simplicial 3-spheres with nine vertices into polytopes and nonpolytopes, *Discr. Math.*, **31** (1980), 115–124.
4. D. Barnette, A proof of the lower bound conjecture for convex polytopes, *Pac. J. Math.*, **46** (1973), 349–354.
5. L. J. Billera and C. W. Lee, A proof of the sufficiency of McMullen's conditions for f-vectors of simplicial convex polytopes, *J. Combin. Theory, Ser. A*, **31** (1981), 237–255.
6. A. Björner, Shellable and Cohen–Macaulay partially ordered sets, *Trans. Am. Math. Soc.*, **260** (1980), 159–183.
7. R. Blind and P. Mani-Levitska, Puzzles and polytope isomorphisms, *Aequationes Math.*, **34** (1987), 287–297.
8. H. Bruggesser and P. Mani, Shellable decompositions of cells and spheres, *Math. Scand.*, **29** (1972), 197–205.
9. G. Danaraj and V. Klee, Shellings of spheres and polytopes, *Duke Math. J.*, **41** (1974), 443–451.
10. G. Danaraj and V. Klee, Which spheres are shellable? *Ann. Discr. Math.*, **2** (1978), 33–52.
11. V. I. Danilov, Birational geometry of toric 3-folds, *Math. USSR Izv.* **21** (2) (1983), 269–280.
12. R. D. Edwards, The double suspension of a certain homology 3-sphere is S^5, *Am. Math. Soc. Notices*, **22** (1975), A-334.
13. G. Ewald, Über die stellare Äquivalenz konvexer Polytope, *Result. Math.*, **1** (1978), 54–64.
14. G. Ewald, Über stellare Unterteilung von Simplizialkomplexen, *Archiv Math.*, **46** (1986), 153–158.
15. G. Ewald, Torische Varietäten und konvexe Polytope, *Stud. Scient. Math. Hung.*, **22** (1987), 11–18.
16. G. Ewald, Blow-ups of smooth toric 3-varieties. *Abh. Math. Sem. Hamb.*, **57** (1987), 193–201.
17. G. Ewald and G. C. Shephard, Stellar subdivisions of boundary complexes of convex polytopes, *Math. Ann.*, **210** (1974), 7–16.
18. G. Ewald, P. Kleinschmidt, U. Pachner and Chr. Schulz, Neuere Entwicklungen in der kombinatorischen Konvexgeometrie, in: *Contributions to Geometry: Proceedings of the Geometry Symposium in Siegen, 1978*, Birkhäuser-Verlag, Basel, 1979.
19. L. C. Glaser, *Geometrical Combinatorial Topology, Vol. 1*, Van Nostrand Reinhold, New York 1970.
20. B. Grünbaum, *Convex Polytopes*, Interscience Publishers, New York, 1967.
21. B. Grünbaum, Two non-shellable triangulations of the 3-cell, manuscript, 1972.
22. J. F. P. Hudson, *Piecewise Linear Topology*, Math. Lect. Note Ser. New York: Benjamin, 1969.
23. B. Kind and P. Kleinschmidt, Schälbare Cohen–Macauley Komplexe und ihre Parametrisierung, *Math. Z.*, **167** (1979), 173–179.
24. P. Kleinschmidt, Stellare Abänderungen und Schälbarkeit von Komplexen und Polytopen, *J. Geom.*, **11**(2) (1978), 161–176.
25. C. W. Lee, Two combinatorial properties of a class of simplicial polytopes, *Isr. J. Math.*, **47**(4) (1984), 261–269.
26. C. W. Lee, Regular triangulations of convex polytopes, to appear.
27. C. W. Lee, PL-spheres and convex polytopes, in preparation.
28. A. Mandel, Topology of oriented matroids, Ph.D. thesis, University of Waterloo, Ontario, Canada, 1982.
29. P. McMullen, The maximum numbers of faces of a convex polytope. *Mathematika*, **17** (1970), 179–184.
30. P. McMullen and D. W. Walkup, A generalized lower-bound conjecture for simplicial polytopes, *Mathematika*, **18** (1971), 264–273.
31. M. H. A. Newman, Combinatorial topology of convex regions, *Proc. Natl. Acad. Sci.*, **16**(3) (1930), 240–242.
32. U. Pachner, Bistellare Äquivalenz kombinatorischer Mannigfaltigkeiten, *Archiv Math.*, **30** (1978), 89–98.
33. U. Pachner, Über die bistellare Äquivalenz simplizialer Sphären und Polytope, *Math. Z.*, **176** (1981), 565–576.
34. U. Pachner, Diagonalen in Simplizialkomplexen, *Geom. Ded.*, **24** (1987) 1–28.
35. U. Pachner, Konstruktionsmethoden und das kombinatorische Homöomorphieproblem für Triangulationen kompakter semilinearer Mannigfaltigkeiten, *Abh. Math. Sem. Hamb.*, **57** (1987), 69–86.
36. U. Pachner, Ein Henkeltheorem für geschlossene semilineare Mannigfaltigkeiten, *Result. Math.*, **12** (1987), 386–394.
37. U. Pachner, Shellings of simplicial balls and p.l. manifolds with boundary, *Discr. Math.*, **81** (1990), 37–47.
38. G. Reisner, Cohen–Macaulay quotients of polynomial rings, *Adv. Math.*, **21** (1976), 30–49.
39. M. E. Rudin, An unshellable triangulation of a tetrahedron, *Bull. Am. Math. Soc.*, **64** (1958), 90–91.
40. R. P. Stanley, Cohen–Macaulay rings and constructible polytopes, *Bull. Am. Math. Soc.*, **81** (1975), 133–135.

41. R. P. Stanley, The upper bound conjecture and Cohen–Macaulay rings, *Stud. Appl. Math.*, **54** (1975), 135–142.
42. R. P. Stanley, Cohen–Macaulay complexes, in: *Higher Combinatorics*, NATO Adv. Study Inst., Ser. C: Math. Phys. Sci. 31, Reidel, Dordrecht, 1977, pp. 51–62.
43. R. P. Stanley, The number of faces of a simplicial convex polytope, *Adv. Math.*, **35** (1980), 236–238.

Received 25 January 1990 and accepted 14 September 1990

UDO PACHNER
Abteilung für Mathematik,
Ruhr-Universität Bochum,
Universitätsstraße 150,
D-4630 Bochum, F.R.G.