PROOF OF GLOBAL CONVERGENCE OF AN EFFICIENT
ALGORITHM FOR PREDICTING TRIP GENERATION,
TRIP DISTRIBUTION, MODAL SPLIT AND
TRAFFIC ASSIGNMENT SIMULTANEOUSLY ON
LARGE-SCALE NETWORKS†

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(Received 3 September 1987)

Communicated by E. Y. Rodin

Abstract—Safwat and Magnanti have developed a combined trip generation, trip distribution, modal split
and traffic assignment model that can predict demand and performance levels on large-scale transportation
networks simultaneously. An efficient algorithm for predicting equilibrium on the model was suggested
by the authors and applied to large-scale systems. The algorithm was found to consistently converge very
rapidly in all cases. A formal proof of convergence was not available at the time of those applications.
This paper provides a formal proof of global convergence of this efficient algorithm under no additional
assumptions on the model. This proof should strengthen the methodology and further encourage its
widespread implementation for transportation planning studies, particularly those of a large-scale nature.

1. INTRODUCTION

Modelling of transportation systems must invariably balance behavioral richness and com-
putational tractability. Towards this end, Safwat and Magnanti [1] developed a combined trip
generation, trip distribution, modal split and traffic assignment model that can predict demand and performance levels on large-scale transportation networks simultaneously, i.e. a Simultaneous Transportation Equilibrium Model (STEM).

The STEM model is formulated as an Equivalent Convex Program (ECP) that can be solved
by a globally convergent feasible direction method. The algorithm finds a feasible descent direction in accordance with the Frank–Wolfe [2] method. Its computational efforts at any given iteration are comparable to the all-or-nothing assignment procedure with fixed demand! The number of iterations required to arrive at a reasonably accurate solution, however, was found to be more than customarily expected [3, 4], because at any given iteration, the direction-finding procedure, for each origin, sets trip generation at a maximum or a minimum value, and assigns this extreme value of trip generation entirely to only one destination.

In order to overcome these computational inefficiencies, particularly for large-scale networks, Brademeyer et al. [5] modified the algorithm, named it the “Stable Performance” (SP) algorithm, and applied it to national intercity multi-user transportation of passengers and freight in Egypt. Safwat and Walton [4] applied a similar modification, called the Logit Distribution of Trips (LDT) algorithm, to urban travel on the large-scale transportation network of Austin, Tex. (i.e. 7096 links, 2137 nodes, 19214 O–D pairs and 520 origins).

The modified procedure (SP or LDT) finds a feasible descent direction essentially by setting trip generation and trip distribution according to their combined logit formulation as postulated in the STEM model. In all cases, SP and LDT were found to consistently converge very rapidly while

†Presented at the 6th Int. Conf. on Mathematical Modelling, Washington University, St Louis, Mo., 4–7 August 1987.
the original algorithm was observed to be relatively slow, though known to be globally convergent (see, for example, Zangwill [6]). A formal proof, however, for the convergence of SP and LDT was not available at the time of their application in the two studies cited above.

The objective of this paper is, therefore, to provide a formal proof of global convergence of the efficient modified procedure, hereafter referred to as the LDT algorithm.

In the next section, a STEM methodology is briefly described. In Section 3, the LDT algorithm is presented. The proof of global convergence of LDT is given in Section 4. Section 5 includes a summary and concluding remarks.

2. A STEM METHODOLOGY

Below is a brief description of a STEM model, an ECP and an algorithm (SPND) for solving the ECP problem in order to predict equilibrium on the STEM model. For a detailed description of the methodology, the reader is referred to Safwat and Magnanti [1].

2.1. A STEM model

In this subsection, a STEM model that describes users' travel behavior in response to system's performance on a transportation network is presented as follows:

\[ G_i = \alpha_i S_i + E_i, \quad \forall i \in I, \]  
\[ S_i = \max \left\{ 0, \ln \sum_{j \in D_i} \exp(-\theta_j U_{ij} + A_j) \right\}, \quad \forall i \in I, \]  
\[ T_{ij} = G_i \frac{\exp(-\theta_j U_{ij} + A_j)}{\sum_{k \in D_i} \exp(-\theta_k U_{ik} + A_k)}, \quad \forall ij \in R, \]  
\[ C_p \begin{cases} = U_{ij} & \text{if } H_p > 0, \\ \geq U_{ij} & \text{if } H_p = 0, \end{cases}, \quad \forall p \in P, \]  
\[ C_a = \sum_{a \in A} \delta_{ap} C_a, \quad \forall p \in P. \]

In this model, the demand variables are:

- \( G_i \) = the number of trips generated from origin \( i \);
- \( T_{ij} \) = the number of trips distributed from origin \( i \) to destination \( j \);
- \( H_p \) = the number of trips travelling via path \( p \) from any given origin \( i \) to any given destination \( j \);

and

- \( F_a \) = the number of trips using link \( a \).

The performance variables are:

- \( S_i \) = an accessibility variable that measures the expected maximum utility of travel on the transport system as perceived from origin \( i \);
- \( U_{ij} \) = the average minimum "perceived" cost of travel from \( i \) to \( j \);
- \( C_p \) = the average cost of travel via path \( p \) from any given \( i \) to any given \( j \);

and

- \( C_a \) = the average cost of travel on link \( a \) expressed as a function of the number of trips, \( F_a \), on that link.

The remaining quantities are:

- \( E_i \) = a composite measure of the effect that the socio-economic variables, which are exogenous to the transport system, have on trip generation from origin \( i \);
\[ A_j = \text{a composite measure of the effect that the socio-economic variables, which are exogenous to the transport system, have on trip attraction at destination } j; \]
\[ \alpha_i = \text{a parameter that measures the additional number of trips that would be generated from a given origin } i \text{ if the expected maximum utility of travel, as perceived by travellers at } i \text{, increased by unity; } \]
\[ \theta_i = \text{a parameter that measures the sensitivity of travel from a given origin } i \text{ due to changes in the system’s performance between } i \text{ and destinations } j \in D_i; \]

and

\[ \delta_{ap} = \begin{cases} 1 & \text{if link } a \text{ belongs to path } p \\ 0 & \text{otherwise.} \end{cases} \]

The defined sets are:

\[ I = \text{set of origins;} \]
\[ R = \text{set of destinations;} \]
\[ P = \text{set of simple paths in the network;} \]

and

\[ D_i = \text{set of destinations accessible from origin } i. \]

The basic assumptions of this STEM model may be summarized as follows:

1. Trip generation, \( G_i \), is given by any general function as long as it is linearly dependent upon the system’s performance through an accessibility measure, \( S_i \), based on the random utility theory of travel behavior (i.e. the expected maximum utility of travel).
2. Trip distribution, \( T_y \), is given by a logit model whose measured utility functions include the average minimum perceived travel costs, \( U_y \), as a variable with a linear parameter \( \theta_i \).
3. Modal split and trip assignment are simultaneously user optimized. Notice that the STEM framework allows for the modal split to be given by a logit model, or to be (together with trip assignment) system optimized (see Safwat [7]).

### 2.2. An ECP

Consider the following optimization problem (ECP):

Minimize

\[ Z(S, T, H) = J(S) + \psi(T) + \Phi(H) \]

subject to

\[ \sum_{j \in D_i} T_{ij} = \alpha_i S_i + E_i, \quad \forall i \in I, \quad (6) \]

\[ \sum_{p \in P_y} H_p = T_{ij}, \quad \forall ij \in R, \quad (7) \]

\[ S_i \geq 0, \quad \forall i \in I, \]
\[ T_{ij} \geq 0, \quad \forall ij \in R, \]
\[ H_p \geq 0, \quad \forall p \in P, \quad (8) \]
where

\[ J(S) = \sum_{i=1}^{I} \frac{1}{\theta_{i}} \left[ \frac{\alpha_{i} S_{i}^{2} + \alpha_{i} S_{i} - (\alpha_{i} S_{i} + E_{i}) \ln(\alpha_{i} S_{i} + E_{i})}{2} \right]. \]

\[ \psi(T) = \sum_{i=1}^{I} \sum_{j \in b_{i}} (T_{q} \ln T_{q} - A_{q} T_{q} - T_{\theta}), \]

\[ \Phi(H) = \sum_{a \in A} \int_{0}^{F_{a}} C_{\omega}(w) \, dw \]

and

\[ F_{a} = \sum_{p} \Delta_{ap} H_{p}, \quad \forall \, a \in A. \]  \hspace{1cm} (9)

Constraints (6) and (7) are flow conservation equations on the transport network stating: (i) that the number of trips distributed from a given origin to all possible destinations should equal the total number generated from that origin; and (ii) that the number of trips on all paths joining a given origin-destination pair should equal the total number distributed from that origin to that destination. Constraints (8) state, as postulated earlier, that all the decision variables should be non-negative. Expressions (9) define the link-path incidence relationships, stating that the flow on a given link equals the sum of flows on all paths sharing that link.

The objective function \( Z \) has three sets of terms. The last of these, \( \Phi(H) \), corresponds to the familiar transformation introduced by Beckmann et al. [8]. The second set of terms, \( \psi(T) \), is similar to those used by Evans [9] and by Florian and Nguyen [10], as well as in other related models. The first set of terms, \( J(S) \), was introduced by Safwat and Magnanti [1], who proved that under mild monotonicity assumptions on performance functions and non-negativity and inequality assumptions on demand parameters (i.e. \( \theta_{i} > 0, E_{i} > \alpha_{i} > 0 \)), the ECP program has a unique solution that is equivalent to equilibrium on the STEM model.

2.3. An algorithm for predicting equilibrium on the STEM model (SPND)

In this subsection, an algorithm for solving the ECP to predict equilibrium on the STEM model is introduced. The algorithm belongs essentially to the class of feasible-direction methods and is known to be globally convergent (see, for example, Zangwill [6]).

At any given iteration, \( r \), the method involves two main steps. The first step determines a direction for improvement, \( d' \). The second step determines an optimum step size, \( \lambda^{*} \), along that direction. The current solution, \( X' \), is then updated, \( X^{r+1} = X' + \lambda^{*} \cdot d' \), and the process is repeated until a convergence criterion is met.

In accordance with the Frank–Wolfe [2] method, the feasible direction \( d'_L \), in the SPND algorithm, is determined as follows:

\[ d'_L = Y'_L - X', \]

where

\( X' = \) the given current solution \( (S', T', F') \)

and

\( Y'_L = \) the solution of the following linearized subproblem (LP1).

**Subproblem LP1:**

Minimize

\[ Z'_L(Y) = \nabla Z(X') \cdot Y \]

subject to

constraints (6)–(8) and expressions (9).
Below is a description of the steps of the (SPND) algorithm to determine a feasible direction $d^r_r$ at iteration $r$:

Step 1. Uptake link costs by calculating $C^r_r = C^r_r(P^r_r), \forall a \in A$. Set $i = 1$ in an ordered set of origins $I$.

Step 2. Find the shortest path tree from $i$ to all $j \in D_i$. Let $u^r_{ij}$ be the cost of the shortest path from $i$ to $j$.

Step 3. Calculate $w^r_{ij} = \frac{1}{\theta_i} (\ln T^r_{ij} - A_i) + u^r_{ij}, \forall j \in D_i$.

Step 4. Determine $j^*$ satisfying $w^r_{ij} = \min_{j \in D_i} \{w^r_{ij}\}$.

Step 5. Calculate $C^r_i = \frac{1}{\theta_i} [S^r_i - \ln(\alpha_i S^r_i + E_i)]$.

Step 6. If $i < I$, then $i \leftarrow i + 1$ and go to Step 2. Otherwise, continue.

Step 7. Find an optimum solution to LP1 and a feasible direction $d^r_r$ as described below.

The optimum solution to LP1 is the vector $Y^r_i = (\hat{S}^r_i, \hat{T}^r, \hat{H}^r)$, given by

$$
\begin{align*}
\hat{S}^r_i = \\
\begin{cases}
0 & \text{if } C^r_i + w^r_{ij} > 0, \\
S^r_i & \text{if } C^r_i + w^r_{ij} = 0, \quad \forall i \in I, \\
(M_i - E_i) / \alpha & \text{otherwise}.
\end{cases}
\end{align*}
$$

Hence

$$
\hat{G}^r_i = \alpha \hat{S}^r_i + E_i, \quad \forall i \in I,
$$

$$
\hat{T}^r_{ij} = \begin{cases}
\hat{G}^r_i & \text{if } j = j^* \in D_i, \quad \forall ij \in R, \\
0 & \text{otherwise},
\end{cases}
$$

$$
\hat{H}^r_{ip} = \begin{cases}
\hat{G}^r_i & \text{if } p = p^* \in p^*, \quad \forall p \in p^*, \quad \forall ij \in R, \\
0 & \text{otherwise},
\end{cases}
$$

where $M_i$ is the maximum trip generation from $i$ (assuming zero travel cost everywhere in the system).

The path flows can be decomposed into link flows using the link–path incidence relationship as follows:

$$
\hat{F}^r_a = \begin{cases}
\sum_{ij \in R} \delta_{ap} \hat{G}^r_i & \text{if link "a" belongs to path } p^* \text{ between } ij^*, \\
0 & \text{otherwise}.
\end{cases}
$$

Hence, the feasible direction at iteration $r$ is the vector $d^r_r$ with the following components:

$$
d^r_i = \hat{S}^r_i - S^r_i, \quad \forall i \in I, \\
d^r_{ij} = \hat{T}^r_{ij} - T^r_{ij}, \quad \forall ij \in R, \\
d^r_a = \hat{F}^r_a - F^r_a, \quad \forall a \in A.
$$

The main computational effort in this direction-finding algorithm is finding the set of shortest paths from all origins to all destinations in Step 2, which is identical with that of the traffic assignment problem with fixed demand. The additional calculations in Steps 3–5 are insignificant compared to those in Step 2. Step 7 just loads the shortest paths with the total demand to the most "needy" destinations, which involves even less effort than the all-or-nothing loading procedure. This procedure is referred to as the Shortest Path to the most Needy Destination "SPND" algorithm as dictated by its direction-finding step.
3. AN EFFICIENT ALGORITHM FOR PREDICTING EQUILIBRIUM ON THE STEM MODEL (LDT)

The SPND algorithm, summarized in the preceding section, was found to require more iterations than customarily expected [3, 4], because at any given iteration its direction-finding procedure, for each origin \( i \), involves:

1. setting trip generation at its maximum value \( M_i \) or at its minimum value \( E_i \)
2. assigning this extreme value of trip generation entirely to only one (i.e. most needy) destination.

In order to overcome these inefficiencies, the LDT algorithm determines a feasible direction essentially by solving the logit formulation of the combined trip generation and trip distribution demand model.

That is, the feasible direction \( \mathbf{d}' \) at iteration \( r \) involves a Logit Distribution of Trips (i.e. LDT) and is determined as follows:

Steps 1 & 2. Identical with the SPND algorithm.

Step 3. Find \( \mathbf{d}' = \mathbf{Y}' - \mathbf{X}' \), where the components of the vector \( \mathbf{Y}' = (L', Q', V') \) are given by

\[
L'_i = \max \left\{ 0, \ln \sum_{j \in D_i} \exp(-\theta_i U_{ij} + A_{ij}) \right\}, \quad \forall i \in I,
\]

\[
Q'_{ij} = \left( \alpha_i L'_i + E_i \right) \frac{\exp(-\theta_i U_{ij} + A_{ij})}{\sum_{k \in D_i} \exp(-\theta_i U_{ik} + A_{ik})}, \quad \forall ij \in R,
\]

\[
B'_p = \begin{cases} Q_{ij}' & \text{if } p = p^* \in P_y, \\ 0 & \text{otherwise}, \end{cases} \quad \forall p \in P_y, \ ij \in R,
\]

\[
V'_a = \sum_{ij \in R} \sum_{p \in P_y} \delta_{ap} B'_p, \quad \forall a \in A;
\]

and, hence the components of the vector \( \mathbf{d}' \) are

\[
d'_i = L'_i - S'_i, \quad \forall i \in I,
\]

\[
d'_{ij} = Q'_{ij} - T'_{ij}, \quad \forall ij \in R,
\]

\[
d'_a = V'_a - F'_a, \quad \forall a \in A.
\]

As indicated in the Introduction, this algorithm was applied to large-scale systems by Brademeyer et al. [5] to intercity travel in Egypt and in a slightly different form by Safwat and Walton [4] to urban travel in Austin, Tex. The algorithm was found in all cases to converge very rapidly to an equilibrium solution. A formal proof, however, for its global convergence was not available at that time.

4. PROOF OF GLOBAL CONVERGENCE OF THE LDT ALGORITHM

Since the LDT algorithm is a feasible direction method used to solve a convex program, its proof of convergence essentially involves showing that:

(i) the direction determined at any given iteration in the process is a descent direction
(ii) the movement along that direction does not "jam" into a point that is not a solution.

In this section we prove that under the same mild assumptions of the STEM model, the LDT algorithm is indeed globally convergent.
Proof of global convergence of the LDT algorithm

Theorem

Suppose that \( \theta_i > 0 \) and \( E_i > \alpha_i > 0, \forall i \in I \), and that \( \forall a \in A \), the performance function \( C_a(F_a') \) is real-valued, non-negative, continuous and strictly increasing. Suppose also that there is an initial feasible solution to the STEM model \( X^0 \) with \( T^0_y > 0, \forall ij \in R \). Then the sequence of solutions \( \{X'\}_r \) generated by the LDT algorithm will converge in the limit to the unique equilibrium that exists on the STEM model.

Proof. Safwat and Magnanti [1] proved that under the assumptions of this theorem the ECP problem has a unique solution that is equivalent to the STEM model. Also notice that the initial feasible solution \( X^0 \) with \( T^0_y > 0, ij \in R \), always exists since

\[
\sum_{j \in D_i} T^0_y \geq E_i > \alpha_i > 0.
\]

It remains to show that the direction \( d' \) determined by the LDT procedure is a feasible descent direction, and that the algorithm does not "jam" at a point that is not the solution.

That \( d' \) is a feasible descent direction is proven by showing that \( \nabla Z(X')d' < 0 \) for any further \( r \) in the procedure. This inner product may be expressed as follows:

\[
\nabla Z(X')d' = \sum_{i \in I} \frac{\alpha_i}{\theta_i} [S_i' - \ln(\alpha_i S_i' + E_i)] (L_i' - S_i')
\]

\[
+ \sum_{ij \in R} \left( \frac{1}{\theta_i} \ln T^0_y - A_j \right) (Q^0_y - T^0_y)
\]

\[
+ \sum_{a \in A} C_a(F_a') \cdot (V^0_a - F^0_a).
\]

Using the equality

\[
\alpha_i (L_i' - S_i') = \sum_{j \in D_i} (Q^0_y - T^0_y),
\]

the first two summations in equation (10) reduce to

\[
\sum_{ij \in R} \left( \frac{1}{\theta_i} [S_i' - \ln(\alpha_i S_i' + E_i)] + \ln T^0_y - A_j \right) (Q^0_y - T^0_y).
\]

Let \( W_i(T'_i) = \{w_i(T'_i): j \in D_i\} \) denote the vector of the inverse demand function from origin \( i \) as a function of the minimum travel costs, \( w_i^r: j \in D_i \), that generated the solution demand vector \( T'_i \) at iteration \( r \).

Safwat and Magnanti [1] showed that this inverse exists and that under the assumptions of the theorem, i.e.

\[
\sum_{j \in D_i} T^0_y \geq E_i, \quad \forall i \in I; \quad T^0_y > 0, \quad \forall ij \in R,
\]

the Jacobian matrix of the vector of the inverse is negative definite (i.e. the inverse is a strictly decreasing function). The components of the inverse are given by

\[
w_i^r(T'_i) = \frac{1}{\theta_i} \left[ A_j - \ln T^0_y + \sum_{k \in D_i} T^0_k - \frac{1}{\alpha_i} \left( \sum_{k \in D_i} T^0_k - E_i \right) \right].
\]

It can be easily verified that the quantity \( 1/\theta_i \) in each term of expression (11) is equal to \( -w_i^r(T'_i) \) equation (12) and, hence, expression (11) reduces to

\[
- \sum_{ij \in R} w_i^r(Q^0_y - T^0_y).
\]

The last summation in equation (10) can be expressed as

\[
\sum_{ij \in R} U_i^r \cdot Q^0_y - \sum_{a \in A} C_a(F_a') \cdot F^0_a
\]

since the link flow variables, \( V^0_a, \forall a \in A \), are obtained by assigning \( Q^0_y \) to the minimum cost paths given the link costs \( C_a' = C_a(F_a') \) at iteration \( r \).
Adding and subtracting the quantity
\[ \sum_{ij \in R} U'^*_i T'^*_j \]
to and from expression (13) or (14) and then combining expressions (13) and (14) yields the following expression for equation (10):
\[ \nabla Z(X') \delta' = \sum_{ij \in R} (U'^*_i - w'^*_i) (Q'^*_i - T'^*_j) + \sum_{ij \in R} U'^*_i T'^*_j - \sum_{a \in A} C'^*_a F'^*_a. \]  
(15)
The last two summations in equation (15) cancel each other since they represent two equal quantities with opposite sign.

Each term in the first summation in equation (15) involves multiplying the difference between two values of a component of the inverse demand function, times the difference between the two corresponding values of the demand function. The inverse is a strictly decreasing function, as is the function itself, implying that each term in equation (15) is strictly negative, unless of course, \( Q'^*_i = T'^*_j \) and hence, \( U'_i = w'_i \), for some \( ij \in R \), in which case the term(s) vanish(es). Other terms in equation (15), however, will still be strictly negative implying that \( \delta' \) is indeed a feasible descent direction. All terms in equation (15) will vanish only at equilibrium, i.e. in the limit as \( r \to \infty \).

It remains to show that the sequence of solutions generated by the algorithm does not "jam" at a non-optimal solution. This is ensured by observing that there is sufficient room to move within the feasible region along \( \delta' \) even in the limit, i.e.
\[ \text{a constant } \delta > 0 \text{ exists such that for any } \tau, \quad 0 \leq \tau \leq \delta, \quad X' + \tau \delta' \text{ is feasible for } r \to \infty. \]  
(16)
This ensures the closeness of the map that optimizes the objective function \( Z \) along the direction \( \delta' \) [6, 11]. In our case it is clear that condition (16) is fulfilled. To see this we refer to Safwat and Magnanti [1] who showed that at equilibrium all \( T'^*_a \) must be strictly positive. The fact that our initial feasible solution has \( T'^*_a = 0, \forall ij \in R \), and that the performance functions are real-valued, ensures that \( T'^*_a > 0, \forall ij \in R \), at all iterations \( r = 1, 2, \ldots, \infty \). Furthermore, we argue that \( S'^*_a > 0, \forall i \in I \). This implies that at equilibrium there must be at least one destination which is "attractive" for any given origin \( i \), i.e.
\[ -\theta_i U'^*_i + A_j > 0 \quad \text{for at least one } j \in D_i, \forall i \in I. \]

It should be clear, however, that even if it so happened that \( S'^*_a = 0 \) for some origin \( i \), condition (16) still holds and the procedure will indeed converge to the solution point.

5. SUMMARY AND CONCLUSIONS

A combined trip generation, trip distribution, modal split and traffic assignment (i.e. a STEM) model was developed by Safwat and Magnanti [1]. The STEM model attempts to achieve a balance between behavioral richness and computational attractiveness. The model is formulated as an ECP which can be solved by several globally convergent algorithms (e.g. SPND). A more efficient algorithm (i.e. LDT) was recently suggested by the authors and applied to large-scale systems. The LDT algorithm was found to consistently converge very rapidly compared with the relatively slower SPND procedure.

In this paper the authors have provided a formal proof of global convergence of the LDT algorithm under the same assumptions as postulated for the STEM model. This proof has strengthened the approach and should encourage its further implementation for large-scale transportation planning studies. A "similar" procedure for the combined trip distribution/traffic assignment model was developed by Evans [9]. Her proof used results from Rockafeller's theory [12] and was certainly full of insight, although cumbersome. Ours used the theory of global convergence of feasible direction methods by Zangwill [6] and is greatly simplified. Needless to say, the STEM model combines trip generation in addition to the components of Evans' model. Of course, the attractive properties of the combined trip generation/distribution demand model in STEM have helped greatly in the development of such an efficient and convergent LDT algorithm.
REFERENCES