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Circular avoiding sequences with prescribed sum

Dean S. Clark and James T. Lewis

Department of Mathematics, The University of Rhode Island, Kingston, RI 02881, USA

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Abstract

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For given positive integers x, n, and s an x-avoiding circular sequence (of positive integers) of length n and sum s has no set of consecutive terms summing to x, even if wraparound is allowed. A necessary and sufficient condition for the existence of such a sequence is obtained. An effective method to construct avoiding sequences is given. For the cases of most interest the number of avoiding sequences is found.

1. Introduction

In [2–4] we introduced circular avoiding sequences and investigated the minimum sum of such a sequence. Given positive integers x, n, s, the finite sequence $a_1, a_2, \ldots, a_n$ of positive integers is an *x-avoiding circular sequence of length n with sum s* if $a_1 + \cdots + a_n = s$ and if no set of consecutive terms of the periodically extended sequence $a_1, \ldots, a_n, a_1, \ldots, a_n, \ldots$ sums to x; i.e., letting $a_{n+k} = a_k$, $k = 1, 2, \ldots$, we have $a_i + a_{i+1} + \cdots + a_j \neq x$ whenever $1 \leq i \leq j$. If the terms $a_1, \ldots, a_n$ are arranged in a circle, then no arc of terms sums to x, even if wraparound is allowed. For example the sequence 2, 2, 2 avoids every odd integer x but does not avoid any even integer x. For a nontrivial example expressed in the language of graphs, if $a_1, \ldots, a_{89}$ denote the edge weights of a cycle with total weight 187, then the results below ensure the existence of a path with total weight 143.

Correspondence to: Professor J.T. Lewis, Department of Mathematics, University of Rhode Island, Kingston, RI 02881, USA.

Let $\sigma(n,x)=\min\{s\colon a_1,\dots,a_n$ is an x-avoiding circular sequence of length n with sum $s\}$. Some known results (to which we will refer later) for an x-avoiding circular sequence of length n with sum s:

(1) [2, p. 105] We have $s\geq 2n$ (hence $\sigma(n,x)\geq 2n$). Furthermore $\sigma(n,x)=2n$ if and only if $x/\gcd(x,n)$ is odd.

(2) [3, p. 202] If $\gcd(x,2n+1)>1$ then $s\neq 2n+1$.

(3) [3, p. 200] The sequence $a_1,\dots,a_n$ with sum $s>x$ is an x-avoiding circular sequence if and only if $a_i+a_{i+1}+\cdots+a_j\neq x$ and $a_i+a_{i+1}+\cdots+a_j\neq s-x$ whenever $1\leq i\leq j\leq n$.

Here, as usual, $\gcd(x,n)$ denotes the greatest common divisor of x and n. We will later write $d\mid n$ when d divides n. Also $\lfloor k\rfloor$ will denote the greatest integer $\leq k$ and $\lceil k\rceil$ will denote the least integer $\geq k$.

Theorem 2.1 of Section 2 gives a necessary and sufficient condition for the existence of an x-avoiding circular sequence of length n with sum s. Its proof, which uses elementary graph concepts, suggests an efficient method to construct x-avoiding circular sequences. Several examples are given illustrating the proof and using the method. In Section 3 consequences of Theorem 2.1 for $\sigma(n,x)$ are explored. In Section 4 we investigate the *number*, up to rotation, of x-avoiding circular sequences of length n with sum s and give uniqueness results.

2. Existence theorem and examples

This fundamental theorem and its proof (for $s>x$) is due to Hickerson [6] and is included with his permission.

Theorem 2.1. *Let x, n, s be positive integers. There exists an x-avoiding circular sequence of length n with sum s if and only if*

$$n\leq \gcd(x,s)\left\lfloor\frac{s}{2\gcd(x,s)}\right\rfloor. \tag{2.1}$$

Proof. We first consider the main case $s>x$. Let $a_1,\dots,a_n$ be an x-avoiding circular sequence with sum s and set $s_0=0$, $s_k=a_1+\cdots+a_k$, $k=1,\dots,n-1$, $S=\{s_0,\dots,s_{n-1}\}$. Then

(a) $0\in S$,

(b) $S\subseteq\{0,1,\dots,s-1\}$,

(c) S has n elements,

(d) t, u in S implies $(t-u)\not\equiv x \pmod s$.

Property (d) follows from result (3) stated in the introduction. Conversely, a set S satisfying (a)–(d) determines an x-avoiding circular sequence of length n and sum s by ordering the elements of S as $0=s_0<s_1<\cdots<s_{n-1}$ and setting $a_k=s_k-s_{k-1}$, $k=1,\dots,n-1$ and $a_n=s-s_{n-1}$. Hence an x-avoiding circular sequence of length n with sum s exists if and only if a set S satisfying (a)–(d) exists.

Let G be the graph with vertex set $\{0,1,\ldots,s-1\}$ in which vertices t and u are joined by an edge when $(t-u)\equiv \pm x \pmod s$. Then condition (d) means that S is an independent set in G; i.e., a set of vertices no two of which are adjacent. Hence an avoiding sequence of length n exists if and only if some independent set in G, which contains the vertex 0, has cardinality at least n.

Each vertex t in G is adjacent only to $(t+x) \pmod s$ and $(t-x) \pmod s$. Hence G is a disjoint union of cycles (single edges when $s=2x$). Further, t and u are in the same cycle (edge) if and only if $u\equiv(t+kx) \pmod s$ for some integer k, i.e., if and only if $\gcd(x,s) \mid (t-u)$. So the number of cycles (edges) is $\gcd(x,s)$ and each cycle (edge) has $s/\gcd(x,s)$ vertices.

Clearly the largest independent set in a cycle (edge) with k vertices has cardinality $\lfloor k/2 \rfloor$. Hence the largest independent set in G has cardinality $\gcd(x,s)\lfloor s/(2\gcd(x,s))\rfloor$. Adding the restriction that the vertex 0 be in the independent set does not change this. So an x-avoiding circular sequence of length n with sum s exists if and only if (2.1) is satisfied.

Next consider the case $s\le x$ and set $x=qs+r$ with $0\le r<s$. If $r=0$ then no x-avoiding sequence with sum s exists nor is (2.1) satisfied. For $r\ge 1$, $a_1,\ldots,a_n$ is an x-avoiding circular sequence if and only if it is an r-avoiding circular sequence. We can now use the first case of the proof and the fact that $\gcd(r,s)=\gcd(x,s)$ to complete the proof of this case. $\square$

Examples. (1) $x=8$, $n=6$, $s=2n+1=13$. Here $\gcd(x,s)=1$, condition (2.1) is satisfied, and the graph G in the proof of Theorem 2.1 consists of the single cycle 0-8-3-11-6-1-9-4-12-7-2-10-5-0 obtained by starting with 0 and successively adding $8(=x)$, modulo $13(=s)$. One choice of independent set S with cardinality $6(=n)$ containing the vertex 0 is $S=\{0,3,6,9,12,2\}$. Ordering the elements of S gives the sequence of partial sums s_i: 0, 2, 3, 6, 9, 12 and then the x-avoiding circular sequence a_i: 2, 1, 3, 3, 3, 1 as in the proof of Theorem 2.1.

(2) $x=6$, $n=7$, $s=2n+2=16$. Here $\gcd(x,s)=2$ and G consists of the two cycles 0-6-12-2-8-14-4-10-0 and 1-7-13-3-9-15-5-11-1. The choice $S=\{0,12,8,4,1,13,9\}$ gives s_i: 0, 1, 4, 8, 9, 12, 13 and a_i: 1, 3, 4, 1, 3, 1, 3.

(3) $x=16$, $n=4$, $s=9$. Here $s<x$ and so we write (as in the second case of the above proof) $x=qs+r$, or $16=1\cdot 9+7$. We now construct a 7-avoiding circular sequence. In forming the graph G we replace $x(=16)$ by $r(=7)$ and obtain the single cycle (since $\gcd(x,s)=\gcd(r,s)=1$) 0-7-5-3-1-8-6-4-2-0. Choosing $S=\{0,5,1,6\}$ gives s_i: 0, 1, 5, 6 and a_i: 1, 4, 1, 3. A final example where the graph G consists of vertex-disjoint edges is given in Section 4.

3. Implications for $\sigma(n,x)$

The next result gives a convenient formulation of condition (2.1). We use the notation

$$e_2(k) = \max\{e\colon 2^e \mid k\}.$$

Corollary 3.1. *Let x, n, s be positive integers with $s=2n+r$, $r\geq 0$. There exists an x-avoiding circular sequence of length n with sum $s=2n+r$ if and only if either*

$$\gcd(x,2n+r)\leq r \quad or \quad e_2(2n+r)>e_2(x). \tag{3.1}$$

Proof. Since

$$\left\lfloor \frac{s}{2\gcd(x,s)} \right\rfloor = \begin{cases} \dfrac{s}{2\gcd(x,s)}, & \text{if } e_2(s)>e_2(x), \\ \dfrac{s}{2\gcd(x,s)}-1/2, & \text{if } e_2(s)\leq e_2(x), \end{cases}$$

the right-hand side of (2.1) equals

$$\begin{cases} s/2, & \text{if } e_2(s)>e_2(x), \\ \dfrac{s-\gcd(x,s)}{2}, & \text{if } e_2(s)\leq e_2(x). \end{cases}$$

Since $s=2n+r$, the corollary now follows from Theorem 2.1. □

Remark. Using (2.1) and (3.1), result (1) in the introduction is recovered. Also from (3.1) we see that there exists an x-avoiding circular sequence of length n with sum $s=2n+1$ if and only if $\gcd(x,2n+1)=1$. This recovers result (2) of the introduction and proves its converse (conjectured in [3]).

Since the minimum sum $\sigma(n,x)$ satisfies $\sigma(n,x)\geq 2n$, it is natural to consider the "excess" $\sigma(n,x)-2n$. We define $m(x)=\max_{n\geq 1}(\sigma(n,x)-2n)$. It follows easily from Corollary 3.1 that

$$m(x)=\begin{cases} 0, & \text{if } x \text{ is odd}, \\ 1, & \text{if } x=2^k,\ k\geq 1. \end{cases}$$

The next theorem contains a periodicity result for $\sigma(n,x)$ and further information on $m(x)$.

Theorem 3.2. *Let x and n be positive integers. Then*

(1) $\sigma(n+x,x)=\sigma(n,x)+2x$,
(2) $m(x)=2$, *if* $x=4k+2$, $k\geq 1$,
(3) $m(x)\geq 3$, *if* $4\mid x$ *and* $x\neq 2^k$, *all* $k\geq 1$,
(4) $m(x)\leq 2^{e_2(x)+1}-2$.

Proof. Part (1) follows from the fact that n, x, s satisfy (2.1) if and only if $n+x$, x, $s+2x$ do. To prove (2) we have $x=4k+2$, $k\geq 1$. If k is odd take $n=k$ whereas if k is even, $k=2j$ say, take $n=6j+1$. In either case it follows from Corollary 3.1 that $\sigma(n,x)\geq 2n+2$, i.e., $m(x)\geq 2$. But $m(x)\leq 2$ is known [3, p. 205]. To prove (3), let $j \geq 3$ be an odd divisor of x. Since j and 8 are relatively prime, there exists $k \geq 0$ such that $j \mid (8k+3)$. Setting $n = 4k+1$, it follows from Corollary 3.1 that $\sigma(n,x) \geq 2n+3$. To prove (4), note that there is some s in the interval $2n \leq s \leq 2n + 2^{e_2(x)+1} - 2$ which is divisible by $2^{e_2(x)+1}$. For this s, $2\gcd(x,s) \mid s$ and condition (2.1) is satisfied. □

Remarks. (1) The proof of part (4) of Theorem 3.2 is due to Hickerson [6], who also showed that for any integer $e \geq 0$, there exists x such that $e_2(x) = e$ and equality holds in (4).

(2) There does not exist a constant C such that $m(x) \leq C$ for all $x \geq 1$. To see this, if $x = k!$ and $n = 1$ then $s \leq k$ is impossible for an x-avoiding sequence and so $m(x) \geq k-1$.

4. The number of avoiding sequences

Any rotation $a_i, a_{i+1}, \ldots, a_n, a_1, \ldots, a_{i-1}$ of an x-avoiding circular sequence is one also. We are interested in formulas for the *number*, up to rotation, of x-avoiding circular sequences of length n with sum s, i.e., we do not count rotations as different sequences. The two cases of primary interest are $s = 2n$ and $s = 2n+1$. The latter case will be considered first; uniqueness was conjectured in [3] and proved in [6]. In Theorems 4.1 and 4.3 below $\phi(d)$ is Euler's function denoting the number of integers k with $1 \leq k \leq d$ and $\gcd(k,d) = 1$.

Theorem 4.1. *Let x, n, s be positive integers with* $\gcd(x,s) = 1$.

(1) *The number, up to rotation, of x-avoiding circular sequences of length n with sum s is equal to the number of solutions, up to rotation, in positive integers $g_1, \ldots, g_n$ of the equation $g_1 + \cdots + g_n = s - n$. The number of these is*

$$(1/n) \sum_{d \mid \gcd(n,s)} \phi(d) \binom{(s-n)/d - 1}{n/d - 1}.$$

(2) *When* $\gcd(x, 2n+1) = 1$, *there is a unique, up to rotation, x-avoiding circular sequence of length n with sum $s = 2n+1$.*

Proof. To prove (1), when $\gcd(x,s) = 1$ the graph G in the proof of Theorem 2.1 consists of a single cycle of length s. We temporarily revoke our agreement about rotations not being counted as different sequences. Then a solution of $g_1 + \cdots + g_n = s - n$, all $g_i \geq 1$, defines an independent set S containing the vertex 0 as follows. Proceeding from vertex 0 on the cycle, omit g_1 vertices and select the next one for S. Then omit g_2 vertices and select the next one for S. Continuing, the independent set S thus obtained can be ordered to give a sequence s_i of partial sums and then a circular avoiding sequence a_i. (In Example (1) of Section 2 the solution $g_1 = \cdots = g_5 = 1$, $g_6 = 2$ of $g_1 + \cdots + g_6 = 7$ yields the S, s_i, a_i given there.) This process gives a one-to-one correspondence between solutions of $g_1+\cdots+g_n=s-n$, all $g_i\geq 1$, and circular avoiding sequences. Now note that a rotation of $g_1,\ldots,g_n$ gives a translation modulo s of the corresponding independent set S, which in turn leads to a rotation of the corresponding circular avoiding sequence a_i. Also, a rotation of a circular avoiding sequence a_i can arise only from a rotation of the corresponding g_i sequence. This establishes the first sentence of (1).

To prove the second sentence of (1), we use aspects of the Polya theory of counting; specifically, Burnside's lemma [1, p. 310]. Let C_n denote the cyclic group of order n generated by the permutation $\Pi=(1\,2\cdots n)$. If d is a divisor of n, each of the $\phi(d)$ permutations $\Pi^{kn/d}$ with $\gcd(k,d)=1$ fixes the solutions of $g_1+\cdots+g_n=s-n$ which consist of a block of length n/d repeated d times. The number of solutions of this type is the number of solutions in positive integers of $g_1+\cdots+g_{n/d}=(s-n)/d$. As is well known [5, p. 3] this number is

$$\binom{(s-n)/d-1}{n/d-1}$$

if $d\mid(s-n)$ and 0 otherwise. Hence by Burnside's lemma the number of orbits (solutions of $g_1+\cdots+g_n=s-n$ which are different under rotation) is

$$(1/n)\sum_{\Pi^j\in C_n}\{\text{number of solutions fixed by } \Pi^j\}$$
$$=(1/n)\sum_{d\mid\gcd(n,s-n)}\phi(d)\binom{(s-n)/d-1}{n/d-1}.$$

Since $\gcd(n,s-n)=\gcd(n,s)$, the formula in (1) is obtained.

Part (2) follows immediately from (1), since $g_1+\cdots+g_n=n+1$, all $g_i\geq 1$, has the unique, up to rotation, solution $g_1=\cdots=g_{n-1}=1$, $g_n=2$. □

Example. $x=9$, $n=10$, $s=2n+2=22$. Here $\gcd(x,s)=1$ and so the number, up to rotation, of 9-avoiding circular sequences of length 10 with sum 22 equals the number, up to rotation, of solutions of $g_1+\cdots+g_{10}=12$, all $g_i\geq 1$. It is easy to see there are six such solutions. This is also the result of the formula in (1).

We now turn to the interesting case $s=2n$, $n=x$. As we will see (Corollary 4.4) this will enable us to handle the case $s=2n$, $n\neq x$ also.

Example. $n=x=6$, $s=12$. The graph G in the proof of Theorem 2.1 consists of six edges 0–6, 1–7, 2–8, 3–9, 4–10, 5–11. There are $2^5=32$ independent sets S which include the vertex 0. These lead to the 6-avoiding sequences 111117, 111252, 112143, 113412, 122322, 131313. Each of the first five of these sequences appears as six different rotations, whereas 131313 has minimum period 2 and only two different rotations.

In general, if a sequence has minimum period d, then $d\mid n$ and the sequence consists of a block of length d repeated n/d times. The sequence has d different rotations.

Recall the classical Möbius function μ defined, for positive integer $m=p_1^{\alpha_1}\cdots p_t^{\alpha_t}$ (prime factorization) by

$$\mu(m)=\begin{cases}1, & \text{if } m=1,\\ 0, & \text{if any } \alpha_i>1,\\ (-1)^t, & \text{if } \alpha_1=\cdots=\alpha_t=1.\end{cases}$$

A known property [5, p. 10] needed below is

$$\sum_{d\mid m}\mu(d)=\begin{cases}1, & \text{if } m=1,\\ 0, & \text{if } m>1.\end{cases}$$

The next lemma, a modification of the classical Möbius Inversion Formula [5, p. 11], will be used in proving part (6) of Theorem 4.3 below.

Lemma 4.2. *Let f, g be real functions defined on the set of positive integers. If*

$$f(n)=\sum_{\substack{d\mid n\\ n/d\text{ odd}}} g(d)$$

then

$$g(n)=\sum_{\substack{d\mid n\\ d\text{ odd}}}\mu(d)f(n/d).$$

Proof.

$$\sum_{\substack{d\mid n\\ d\text{ odd}}}\mu(d)f(n/d)$$

$$=\sum_{\substack{d\mid n\\ d\text{ odd}}}\mu(d)\sum_{\substack{d'\mid(n/d)\\ (n/d)/d'\text{ odd}}} g(d')$$

$$=\sum\sum_{(d,d')\in D}\mu(d)g(d') \quad \text{where } D=\{(d,d'):\ d\mid n,\ d\text{ odd},\ d'\mid(n/d),\ (n/d)/d'\text{ odd}\}$$

$$=\sum\sum_{(\delta,e)\in E}\mu(\delta)g(e) \quad \text{where } E=\{(\delta,e):\ e\mid n,\ n/e\text{ odd},\ \delta\mid(n/e)\}$$

$$=\sum_{\substack{e\mid n\\ n/e\text{ odd}}} g(e)\sum_{\delta\mid(n/e)}\mu(\delta)=g(n).$$

The third equation follows from the easily verified fact that $D=E$. The last equation follows from

$$\sum_{\delta\mid(n/e)}\mu(\delta)=\begin{cases}1, & \text{if } n/e=1,\\ 0, & \text{if } n/e>1.\end{cases} \qquad \square$$

Theorem 4.3. *For positive integer n, let $T(n)$ be the number, up to rotation, of n-avoiding circular sequences of length n with sum $s=2n$. Let $N(n)$ be the number, up to rotation, of such sequences with minimum period n. Then*

(1) $T(n) \geq \lceil 2^{n-1}/n \rceil$.

(2) $T(n) = \sum_{\substack{d \mid n \\ n/d \text{ odd}}} N(d)$.

(3) $\sum_{\substack{d \mid n \\ n/d \text{ odd}}} dN(d) = 2^{n-1}$.

(4) *For odd prime p, $N(p)=(2^{p-1}-1)/p$ and $T(p)=N(p)+1$.*

(5) $T(2^k)=N(2^k)=2^{2^k-k-1}$, $k \geq 1$.

(6) $N(n)=(1/n) \sum_{\substack{d \mid n \\ d \text{ odd}}} \mu(d) 2^{n/d-1}$.

(7) $T(n) = (1/n) \sum_{\substack{d \mid n \\ d \text{ odd}}} \phi(d) 2^{n/d-1}$.

Proof. To prove (1), note that the graph G in the proof of Theorem 2.1 consists of n vertex-disjoint edges; hence there are 2^{n-1} independent sets S containing the vertex 0. There is a one-to-one correspondence between these and avoiding sequences (with rotations considered as different sequences). But each sequence has at most n different rotations. To prove (2), note that each avoiding sequence counted in $T(n)$ having minimum period d, where $d \mid n$, consists of a block of length d repeated n/d times. This block must sum to $2d$ and be a d-avoiding circular sequence (since $n=d(n/d)$ is avoided), i.e., the block is a sequence counted in $N(d)$. Also n/d must be odd since n is avoided. Conversely, each d-avoiding sequence counted in $N(d)$, where $d \mid n$ and n/d is odd, can be periodically extended to an n-avoiding sequence of length n counted in $T(n)$. Part (3) is proved similarly, noting that a sequence with minimum period d has d different rotations. The first equation of (4) is immediate, since (3) gives $N(1)+pN(p)=2^{p-1}$ and we have $N(1)=1$. The second equation of (4) follows from (2). The first equation of (5) also follows from (2), since $d \mid 2^k$, $2^k/d$ odd implies $d=2^k$. For the second equation in (5), we have only $d=2^k$ in (3) and so $2^k N(2^k)=2^{2^k-1}$. Using (3) and the lemma (with $f(n)=2^{n-1}$, $g(n)=nN(n)$) we obtain (6). To prove (7), we substitute (6) into (2), collect terms with the same power of 2 displayed, and use the well-known formula [1, p. 77]

$$\sum_{d' \mid \delta} \mu(d')/d' = \phi(\delta)/\delta$$

as follows:

$$T(n) = \sum_{\substack{d \mid n \\ n/d \text{ odd}}} (1/d) \left[\sum_{\substack{d' \mid d \\ d' \text{ odd}}} \mu(d') 2^{d/d'-1} \right]$$

$$= \sum_{\substack{\delta \mid n \\ \delta \text{ odd}}} \left[\sum_{d' \mid \delta} \frac{\mu(d')}{(n/\delta)d'} \right] 2^{n/\delta - 1}$$

$$= (1/n) \sum_{\substack{\delta \mid n \\ \delta \text{ odd}}} \phi(\delta) 2^{n/\delta - 1}. \qquad \square$$

Remark. $T(2^k) = N(2^k) = 2^{2^k - k - 1}$ is also the number of DeBruijn sequences of length 2^{k+1}, cf. [5, p. 110]. It would be interesting to exhibit a one-to-one correspondence.

Corollary 4.4. *Let x and n be positive integers with $d = \gcd(x, n)$. Then the number, up to rotation, of x-avoiding circular sequences with sum $s = 2n$ equals*

$$\begin{cases} T(d), & \text{if } x/d \text{ is odd,} \\ 0, & \text{if } x/d \text{ is even} \end{cases}$$

where $T(d)$ is defined in Theorem 4.3.

Proof. By [3, p. 200] such a sequence consists of a block of length d, summing to $2d$ and avoiding d, repeated n/d times. (The *minimum* period of the sequence need not be d.) There are, up to rotation, $T(d)$ such sequences when x/d is odd. When x/d is even there are no such sequences (result (1) in the introduction). $\square$

Example. $x = 100$, $n = 140$. Using Corollary 4.4 and Theorem 4.3, the number, up

Table 1
Number of n-avoiding circular sequences of length n with sum $2n$

n	$N(n)$	$T(n)$
1	1	1
2	1	1
3	1	2
4	2	2
5	3	4
6	5	6
7	9	10
8	16	16
9	28	30
10	51	52
16	2,048	2,048
20	26,214	26,216

to rotation, of 100-avoiding circular sequences of length $n = 140$ with sum $s = 2n = 280$ is (since $d = \gcd(100, 140) = 20$)

$$T(20) = (1/20) \sum_{\substack{d \mid n \\ d \text{ odd}}} \phi(d) 2^{n/d-1} = (1/20)[2^{19} + 4 \cdot 2^3] = 26{,}216.$$

Using Theorem 4.3 we can easily compute the entries in Table 1. We note that the smallest value of n for which the inequality (1) of Theorem 4.3 is strict is $n = 9$.

Remark. We have uniqueness, up to rotation, of x-avoiding circular sequences of length n with sum s in these cases:

(1) $s = 2n + 1$, $\gcd(x, 2n + 1) = 1$ (Theorem 4.1),

(2) $s = 2n$, $x/\gcd(x, n)$ odd, $\gcd(x, n) = 1$ or 2 (Corollary 4.4 and Table 1).

It can be shown that, when $n > 1$, these are the only instances of uniqueness.

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