A Characterization of Clique Graphs*

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ABSTRACT

In a recent paper [3], Hamelink obtains an interesting sufficient condition for a graph to be a clique graph. In this paper, we give related conditions which are necessary as well as sufficient. As an application of our result we show that Hamelink's condition is also necessary in certain special cases and that here it can be greatly simplified. As another application, we derive certain theorems useful in practice in reducing the question of whether a given graph is a clique graph to whether certain smaller or simpler graphs are.

1. INTRODUCTION

Our graphs will all be finite, non-directed, with no loops or multiple edges. If \( G \) is a graph, \( V(G) \) will denote the set of vertices of \( G \) and \( E(G) \) the set of edges. We denote the adjacency relation by \( I \), i.e., if \( x, y \in V(G) \), then \( xIy \) iff \( (x, y) \in E(G) \). A clique of \( G \) is a maximal complete subgraph. (Some authors use the terminology dominant clique.) Given \( G \), let \( K_1, K_2, \ldots, K_n \) be its cliques. Define \( H \) by \( V(H) = \{K_1, K_2, \ldots, K_n\} \) and \( (K_i, K_j) \in E(H) \) iff \( i \neq j \) and \( K_i \cap K_j \neq \emptyset \). Then we call \( H \) the clique graph of \( G \) and write \( H = K(G) \). The main problem we are concerned with is this: Given a graph \( H \), is it the clique graph of some \( G \)?

2. THE CHARACTERIZATION

Let \( \mathcal{H} \) be a collection of complete subgraphs of a graph \( H \). We shall say \( \mathcal{H} \) has property \( \mathcal{I} \) (for intersection) if whenever \( L_1, L_2, \ldots, L_p \) are in \( \mathcal{H} \)

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and \( L_i \cap L_j \neq \emptyset \) for all \( i, j \) then the total intersection is nonempty, i.e.,

\[
\bigcap_{i=1}^{p} L_i \neq \emptyset.
\]

We say \( \mathcal{K} \) has property \( J_m \) if the above holds whenever \( p \leq m \). Finally, let \( \mathcal{K}(H) \) be the collection of all cliques of the graph \( H \).

**Theorem 1** (Hamelink). If \( \mathcal{K}(H) \) satisfies property \( J \) then \( H \) is a clique graph.

Note how the condition that the points of \( H \) represent cliques is reflected in the cliques of \( H \) itself. The converse of Theorem 1 is not true. To give an example, let \( H \) and \( G \) be the graphs shown in Figure 1. Then \( H = K(G) \), but the set \( \mathcal{K}(H) \) does not satisfy property \( J \). For, take

\[
L_1 = \{A, B, C, D\}, \quad L_2 = \{E, B, F, G\} \quad \text{and} \quad L_3 = \{I, D, G, H\}.
\]

![Figure 1](image_url)

**Theorem 2** (Characterization of Clique Graphs). A graph \( H \) is a clique graph iff there is a collection \( \mathcal{K} \) of complete subgraphs of \( H \) which satisfies the following two properties:

1. \( \mathcal{K} \) covers all the edges of \( H \), i.e., if \( x, y \in H \) and \( xLy \), then \( \{x, y\} \) is contained in some element of \( \mathcal{K} \).
2. \( \mathcal{K} \) satisfies property \( J \).

* We shall describe a simple test for property \( J \) in Sec. 5.
PROOF. The proof of sufficiency is essentially Hamelink's proof of Theorem 1. Let $\mathcal{N} = \{L_1, L_2, \ldots, L_n\}$. Define the graph $G$ as follows:

$$V(G) = V(H) \cup \mathcal{N}.$$  

If $h \in V(H)$, then $hL_i$ iff $h \in L_i$.

$L_iL_j$ iff $i \neq j$ and $L_i \cap L_j \neq \emptyset$.

If $h, h' \in V(H)$, then not $hh'$.

The claim is that $H = K(G)$. To prove this, let $C(h) = \{h\} \cup \{L_i : h \in L_i\}$. It is easy to see that each $C(h)$ is a clique of $G$. Moreover, these are the only cliques of $G$. For, let $C$ be a complete subgraph of $G$. Then, if $C$ contains an element $h$ of $V(H)$, we have $C \subseteq C(h)$. And, otherwise, $C$ is contained in some $C(h)$ by property $\mathcal{J}$.

To prove the necessity of the conditions, suppose $H = K(G)$. Let $V(G) = \{g_1, g_2, \ldots, g_n\}$, let $V(H) = \{h_1, h_2, \ldots, h_m\}$, and let $K_1, K_2, \ldots, K_m$ denote the cliques of $G$, labeled in such a way that $h_iL_j$ iff $K_i \cap K_j \neq \emptyset$. For $i = 1, 2, \ldots, n$, define $L_i = \{h_j : g_i \in K_j\}$. Each $L_i$ is complete because, if $g_i$ and $g_k$ are in $L_i$, then $g_i \in K_j \cap K_k$ and so $h_iL_k$. The claim is that $\mathcal{N} = \{L_1, L_2, \ldots, L_n\}$ satisfies properties (1) and (2). Property (1) is satisfied because if $h_iL_j$ then $K_j \cap K_i \neq \emptyset$. Finally, $\mathcal{N}$ satisfies property $\mathcal{J}$. For, suppose $L_i \cap L_j \neq \emptyset$, pairwise intersect. Then, for all $j, k$, there is a point $h_{jk}$ in $L_{ij} \cap L_{ik}$. Thus $g_{ij} \in K_{jk}$ and therefore we have $g_{ij}L_k$. It follows that $\{g_{i_1}, g_{i_2}, \ldots, g_{i_r}\}$ is contained in some clique $K_s$ of $G$ and thus $h_s \in \bigcap_{i=1}^{r} L_{i}$. Q.E.D.


3. THE CASE OF CLIQUE NUMBER $\leq 3$

There are certain situations in which the conditions of Theorem 2 may be simplified, i.e., where the conditions of Hamelink become necessary as well as sufficient. This fact will follow by a simple application of Theorem 2. We first require one lemma.

LEMMA 1. Suppose $\mathcal{N}$ is a collection of complete subgraphs of a graph $H$, $\mathcal{N}$ satisfies (1) and (2) of Theorem 2, and suppose no member of $\mathcal{N}$ is contained in any other. Then $\mathcal{N}$ contains a 2-element set iff this set is a clique of $H$. 
Proof. Every 2-element clique is contained in $\mathcal{K}$ by property (1). Conversely, suppose $L_1 = \{h, h'\} \in \mathcal{K}$ and there is a point $h'' \neq h, h'$ which is adjacent to both $h$ and $h'$. Then there are sets $L_2$ and $L_3$ in $\mathcal{K}$ such that $\{h, h''\} \subseteq L_2$ and $\{h', h''\} \subseteq L_3$. It follows that $L_1, L_2, L_3$ pairwise intersect but have no point in common, violating property $\mathcal{J}$. Q.E.D.

Definition. $\omega(H) = \text{clique number of } H = \max\{|L| : L \text{ is a clique of } H\}$.

Theorem 3. If $\omega(H) \leq 3$, then $H$ is a clique graph iff $\mathcal{K}(H)$ satisfies property $\mathcal{J}$.

Proof. If $H$ is a clique graph then there is some collection $\mathcal{K}$ of complete subgraphs satisfying properties (1) and (2) of Theorem 2. Let $\mathcal{K}'$ be the collection of all (setwise) maximal elements of $\mathcal{K}$ together with all one-element cliques of $H$. This collection still satisfies properties (1) and (2). We shall show that $\mathcal{K}' = \mathcal{K}(H)$. $\mathcal{K}' \subseteq \mathcal{K}(H)$ follows directly by Lemma 1 since $\omega(H) \leq 3$. To show $\mathcal{K}(H) \subseteq \mathcal{K}'$, suppose $L \in \mathcal{K}(H)$. That $L \in \mathcal{K}'$ follows easily if $|L| < 3$. Thus, let $L = \{h_1, h_2, h_3\}$. By property (1), $\mathcal{K}'$ has elements $L_1, L_2, L_3$ containing $\{h_1, h_3\}, \{h_1, h_3\}$ and $\{h_2, h_3\}$, respectively. Since $\mathcal{K}'$ satisfies property (2), there is a point $h$ in $L_1 \cap L_2 \cap L_3$. Since $h$ is in each $L_i$, it is adjacent to or equal to each point $h_i$. Thus $\{h_1, h_2, h_3, h\}$ is complete in $H$ and $\omega(H) \leq 3$ implies that $h = h_i$, some $i$. If $i = 1$, say, then $L_3 = \{h_1, h_2, h_3\}$ and so $L \in \mathcal{K}'$.

The converse follows by Theorem 2. Q.E.D.

Actually it turns out that, if $\omega(H) \leq 3$, property $\mathcal{J}$ is equivalent to the much weaker property $\mathcal{J}_3$. This will follow from the next lemma, and will give us a very simple criterion for clique graphs if $\omega(H) \leq 3$.

Lemma 2. Suppose $\omega(H) \leq m$ and $\mathcal{K}$ is a collection of complete subgraphs of $H$. Then $\mathcal{K}$ satisfies property $\mathcal{J}$ iff it satisfies property $\mathcal{J}_m$.

Proof. The case $m = 1$ is trivial. Suppose $m > 1$ and suppose $\mathcal{K}$ satisfies $\mathcal{J}_m$ but not $\mathcal{J}$. Then there are distinct $L_1, L_2, \ldots, L_p$ in $\mathcal{K}$ which pairwise intersect but have no point in common. We may assume that $L_i \not\subseteq L_j$ for $i \neq j$. For the collection of all minimal elements among $L_1, L_2, \ldots, L_p$ has this property. Note first that $|L_i \cap L_j| = m - 1$, all $i \neq j$. For, $|L_i \cap L_j| < |L_i| \leq m$. Suppose $|L_i \cap L_j| = r < m - 1$. Let $L_i \cap L_j = \{k_1, k_2, \ldots, k_t\}$. Then for each $u$ there is $L_u$ such that $k_u \notin L_u$. Hence $\{L_i, L_j, L_{i_2}, L_{i_3}, \ldots, L_{i_t}\}$ consists of $\leq m$ elements of $\mathcal{K}$ which pairwise intersect but have no point in common, violating property $\mathcal{J}_m$. 
Since $|L_i \cap L_j| = m - 1$, all $i \neq j$, since $|L_i| \leq m$, all $i$, and since $\bigcap_{i=1}^p L_i = \emptyset$, it follows that there are distinct points $h_1, h_2, \ldots, h_{m+1}$ in $H$ so that

$$L_i = \{h_1, h_2, \ldots, h_{i-1}, h_i, h_{i+1}, \ldots, h_{m+1}\},$$

where the symbol $h_i$ means $h_i$ is omitted. But now the points $h_j$ and $h_k$ are adjacent in $H$ for all $j \neq k$, because $h_j, h_k$ are in the complete subgraph $L_i$ for $i \neq j, k$. Thus $\{h_1, h_2, \ldots, h_{m+1}\}$ is a complete subgraph of $H$, and this violates $\omega(H) \leq m$. Q.E.D.

**Theorem 4.** If $\omega(H) \leq 3$, then $H$ is a clique graph iff $\mathcal{K}(H)$ satisfies property $\mathcal{J}_3$.

**Proof.** Theorem 3 and Lemma 2. Q.E.D.

**Definition.** A graph $H_1$ is a partial subgraph of a graph $H_2$ if $V(H_1) \subseteq V(H_2)$ and $E(H_1) \subseteq E(H_2)$.

**Corollary.** If $\omega(H) \leq 3$, then $H$ is a clique graph iff it has no partial subgraph isomorphic to the graph of Figure 2.

![Figure 2](image)

**Proof.** Suppose $H$ has such a partial subgraph. Since $\omega(H) \leq 3$, the three outer triangles are cliques. These pairwise intersect but have no point in common, violating property $\mathcal{J}_3$ for $\mathcal{K}(H)$. Conversely, suppose $\mathcal{K}(H)$ does not satisfy property $\mathcal{J}_3$. Let $K_1, K_2, K_3$ be three cliques which pairwise intersect but have no point in common. Using $\omega(H) \leq 3$, it is easy to prove that each $K_i$ is a triangle. Moreover, $|K_i \cap K_j| = 1$, $i \neq j$. For suppose, for example, $|K_1 \cap K_2| = 2$. Let $K_1 = \{h_1, h_2, h_3\}$ and let $K_2 = \{h_1, h_2, h_4\}$. Then, since $K_1 \cap K_3 \neq \emptyset$, $K_2 \cap K_3 \neq \emptyset$, and $K_1 \cap K_2 \cap K_3 = \emptyset$, we conclude $K_3 = \{h_3, h_4, h_5\}$, some $h_5$. It follows that $\{h_1, h_2, h_3, h_4\}$ is complete, violating $\omega(H) \leq 3$. Thus, $K_1, K_2, K_3$ are triangles with no common point, each pair of which has exactly one point in common. This implies that the vertices of $K_1, K_2, K_3$ are the vertices of a partial subgraph isomorphic to the graph of Figure 2. Q.E.D.
4. REDUCTION THEOREMS

As a further application of Theorem 2, we present some results which might be useful as tools in reducing the question of whether a given graph is a clique graph to whether certain smaller or simpler graphs are clique graphs. The proofs are straightforward using the characterization.

**Theorem 5.** Suppose $H$ is disconnected and $H_1, H_2, \ldots, H_p$ are its components. Then $H$ is a clique graph iff each $H_i$ is.

**Proof.** Trivial (even without the characterization).

**Theorem 6.** Suppose $H$ is a connected graph with a cut-point $h$. Let $H - h = H_1' + H_2'$, $H_1' \cap H_2' = \emptyset$, and suppose there is no edge from $H_1'$ to $H_2'$. If $H_i = H_i' + h$, then $H$ is a clique graph iff $H_1$ and $H_2$ are.

**Proof.** If $\mathcal{K}_i$ is a collection of complete subgraphs of $H_i$ satisfying (1) and (2), then $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ satisfies (1) and (2) for $H$. Conversely, if $\mathcal{K}_i$ is a collection of complete subgraphs of $H$ satisfying (1) and (2), then $\mathcal{K}_i = \{L \in \mathcal{K} : L \subseteq V(H_i)\}$ is a collection of complete subgraphs satisfying (1) and (2) for $H_i$. Q.E.D.

**Corollary 6.1.** Suppose $H$ is a connected graph with a cut-point $h$. Let $H_1', H_2', \ldots, H_n'$ be the components of $H - h$ and let $H_i$ be the subgraph generated by $H_i'$ plus the vertex $h$. Then $H$ is a clique graph iff each $H_i$ is.

**Proof.** The argument is similar.

**Definition.** Suppose $H$ is a graph and $S$ is a subset of $V(H)$ so that $h, h' \in S$ implies not $hIh'$. Then $S$ is a **independent set**. If in addition $S$ is a cut set, $S$ will be called an **independent cut-set**.

**Corollary 6.2.** Suppose $H$ is a connected graph and $S$ is an independent cut-set of $H$. Let $H - S = H_1' + H_2'$, $H_1' \cap H_2' = \emptyset$ and suppose that there is no edge from $H_1'$ to $H_2'$. If $H_i$ is the subgraph of $H$ generated by $H_i'$ plus $S$, then $H$ is a clique graph iff $H_1$ and $H_2$ are.

**Proof.** The argument is again similar.

**Corollary 6.3.** Suppose $H$ is a connected graph and for some $h$, $\{h' : hIh'\}$ is an independent set. Then $H$ is a clique graph iff $H - h$ is.
5. A Test for Property $\mathcal{J}$

One of the weaknesses of the clique graph criterion given in Theorem 2 is that property $\mathcal{J}$ is not easy to verify. To verify that a collection of sets has this property, one needs to look at all subsets.

To improve on this, suppose $\mathcal{F} = \{E_1, E_2, \ldots, E_n\}$ is a family of sets and for all points $x, y, z$ in $E = \bigcup E_i$, define $\mathcal{F}(x, y, z)$ to be the subfamily of all sets containing at least two of the points $x, y, z$. Then we note that the family $\mathcal{F}$ has property $\mathcal{J}$ iff for all $x, y, z$ in $E$,

$$\bigcap \{E_i: E_i \in \mathcal{F}(x, y, z)\} \neq \emptyset.$$ 

This intersection property follows directly from property $\mathcal{J}$. The converse is proved by induction. If the sets $E_1, E_2, \ldots, E_r$ pairwise intersect, we find $x_1, x_2, x_3$ so that for $j \leq 3$, $x_j \in \bigcap_{k \neq j} E_{ik}$. Then each $E_{ik}$ is in $\mathcal{F}(x_1, x_2, x_3)$.

Another way of stating this result is that the family $\mathcal{F}$ has property $\mathcal{J}$ iff for all points $x, y, z$ in $E$, there is a point $w$ in $E$ with the property that each set $E_i$ containing two of the points $x, y, z$ also contains $w$. This result was pointed out to us by Claude Berge (personal communication). He proved it by noting that a family $\mathcal{F}$ has property $\mathcal{J}$ iff its dual family $\mathcal{F}^*$ has a so-called faithful graph representation (see Berge [1] for definitions) and by using a criterion due to Gilmore [2] for a family of sets to have a faithful graph representation.

References