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Solvability of nonlinear variational–hemivariational inequalities

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Abstract

In this paper we study nonlinear elliptic differential equations driven by the p-Laplacian with unilateral constraints produced by the combined effects of a monotone term and of a nonmonotone term (variational–hemivariational inequality). Our approach is variational and uses the subdifferential theory of nonsmooth functions and the theory of accretive and monotone operators. Also using these ideas and a special choice of the monotone term, we prove the existence of a strictly positive smooth solution for a class of nonlinear equations with nonsmooth potential (hemivariational inequality).

Keywords: Generalized subdifferential; Convex subdifferential; p-Laplacian; Principal eigenvalue; m-accretive operator; Maximal monotone operator; Critical point

1. Introduction

In this paper we prove an existence result for variational–hemivariational inequalities driven by the p-Laplacian. Then using the argument of the existence theorem and with a particular choice of the monotone (convex) component of the problem, we prove the

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existence of positive smooth solutions for a class of hemivariational inequalities involving the p-Laplacian differential operator.

So let $Z \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary ∂Z. The problem under consideration is the following:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z, x(z)) - \partial G(x(z)) & \text{a.e. on } Z, \\ x|_{\partial Z} = 0, \quad 2 \leqslant p < \infty. \end{cases} \tag{1.1}$$

Here $j(z,x)$ is a measurable function which is locally Lipschitz in the x-variable and $\partial j(z,x)$ denotes the generalized subdifferential of the locally Lipschitz function $x \to j(z,x)$ (see Section 2). Also $G : X \to \bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is proper, convex, lower semicontinuous and $\partial G(x)$ stands for the subdifferential in the sense of convex analysis of the convex function $x \to G(x)$. So in problem (1.1) we have the combined effects of the unilateral constraints imposed by a monotone (convex) term and by a nonmonotone (nonconvex) term. The presence of the $\partial G(x)$-term (the monotone term), classifies the problem as a variational inequality, while the presence of the $\partial j(z,x)$-term (the nonmonotone term) makes the problem a hemivariational inequality. This explains the nomenclature “variational–hemivariational inequality.”

Hemivariational inequalities (i.e., $G \equiv 0$), have been studied recently by many authors, primarily in the context of semilinear problems (i.e., $p = 2$) and already there is a substantial literature on the subject. For a detailed bibliography, we refer to Gasinski–Papageorgiou [5]. Hemivariational inequalities (as the generalization of variational inequalities, see Showalter [14]), turned out to be a very useful model in describing many problems in mechanics and engineering involving nonconvex and nonsmooth energy functionals. For various applications, we refer to the book of Naniewicz–Panagiotopoulos [13].

In contrast the study of variational–hemivariational inequalities is lagging behind. There are only the works of Goeleven–Motreanu [6] (semilinear problems with G being an indicator function) and Kyritsi–Papageorgiou [8], Marano–Motreanu [12] and Filippakis–Papageorgiou [4] (problems involving the p-Laplacian and with G being an indicator function).

Our approach is variational and combines notions and techniques from nonsmooth analysis and from nonlinear analysis. In the next section, for the convenience of the reader, we review the basic definitions and results from these areas, which we will be using in our analysis. Our main references are the books of Denkowski–Migorski–Papageorgiou [2,3] and of Showalter [14].

2. Mathematical background

Let X be a Banach space. By X^* we denote its topological dual and by $\langle \cdot,\cdot \rangle$ we denote the duality brackets for the pair (X, X^*). A function $\varphi : X \to \mathbb{R}$ is said to be *locally Lipschitz*, if for every $x \in X$ we can find U a neighborhood of x and a constant $k_U > 0$ such that

$$\left|\varphi(y) - \varphi(v)\right| \leqslant k_U \|y - v\| \quad \text{for all } y, v \in U.$$

Recall that if $\psi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a proper (i.e., not identically $+\infty$), convex and lower semicontinuous function, then ψ is locally Lipschitz in the interior of its effective domain $\operatorname{dom}\psi = \{x \in X\colon\ \psi(x) < +\infty\}$. In particular, then a continuous convex function $\psi : X \to \mathbb{R}$ is in fact locally Lipschitz. Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the *generalized directional derivative* of φ at $x \in X$ in the direction $h \in X$ $\varphi^0(x; h)$, is defined by

$$\varphi^0(x;h) \stackrel{\text{df}}{=} \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $\varphi^0(x; \cdot)$ is sublinear continuous. So it is the support function of a nonempty, convex and w^*-compact convex set $\partial\varphi(x)$ defined by

$$\partial\varphi(x) \stackrel{\text{df}}{=} \left\{x^* \in X^*\colon\ \langle x^*, h\rangle \leqslant \varphi^0(x;h) \text{ for all } h \in X\right\}.$$

The multifunction $x \to \partial\varphi(x)$ is known as the *generalized* (or *Clarke*) *subdifferential* of φ. If φ is also convex, then the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, defined by

$$\partial\varphi(x) \stackrel{\text{df}}{=} \left\{x^* \in X^*\colon\ \langle x^*, y - x\rangle \leqslant \varphi(y) - \varphi(x) \text{ for all } y \in X\right\}.$$

Moreover, if $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. If $\varphi, \psi : X \to \mathbb{R}$ are two locally Lipschitz functions and $\lambda \in \mathbb{R}$, then we have

$$\partial(\varphi + \psi) \subseteq \partial\varphi + \partial\psi \quad \text{and} \quad \partial(\lambda\varphi) = \lambda\partial\varphi.$$

Also the generalized subdifferential satisfies a mean value rule. Namely if $\varphi : X \to \mathbb{R}$ is Lipschitz on an open set containing the line segment $[x, y]$, we can find $z = \lambda x + (1 - \lambda)y$ with $\lambda \in (0, 1)$ and $z^* \in \partial\varphi(z)$ such that

$$\varphi(y) - \varphi(x) = \langle z^*, y - x\rangle.$$

In our analysis, we will also use monotone and accretive operators. So let $A : X \to 2^X$. We set $D(A) = \{x \in X\colon\ A(x) \neq \emptyset\}$ (the domain of A) and $\operatorname{Gr} A = \{(x, y) \in X \times X\colon\ y \in A(x)\}$ (the graph of A). We say that A is accretive if for any $(x_i, y_i) \in \operatorname{Gr} A$, $i = 1, 2$, there exists $x^* \in \mathcal{F}(x_1 - x_2)$ such that

$$\langle x^*, x_1 - x_2\rangle \geqslant 0.$$

Here $\mathcal{F} : X \to 2^{X^*}$ is the duality map of X, i.e., $\mathcal{F}(x) = \{x^* \in X^*\colon\ \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}$ for each $x \in X$. An accretive operator $A : X \to 2^X$ is said to be m-accretive, if $R(I + A) = X$.

Given an accretive operator $A : X \to 2^X$ and $\lambda > 0$, we define the following two well-known operators:

$$J_\lambda = (I + \lambda A)^{-1}, \quad \text{the resolvent of } A \quad \text{and}$$

$$A_\lambda = \frac{1}{\lambda}(I - J_\lambda), \quad \text{the Yosida approximation of } A.$$

Note that $D(J_\lambda) = D(A_\lambda) = R(I + \lambda A)$. Also J_λ is single-valued and nonexpansive, i.e.,

$$\left\|J_\lambda(x) - J_\lambda(y)\right\| \leqslant \|x - y\| \quad \text{for every } x, y \in R(I + \lambda A),$$

while A_λ is single-valued accretive and Lipschitz continuous with Lipschitz constant $\frac{2}{\lambda}$. Moreover, we have $J_\lambda(x) \to x$ as $\lambda \downarrow 0$ for each $x \in D(A) \cap [\bigcap_{\lambda>0} R(I+\lambda A)]$ and $A_\lambda(x) \in A(J_\lambda(x))$ for every $x \in R(I+\lambda A)$. Finally, an accretive operator $A : X \to 2^X$ is m-accretive if and only if $R(I+\lambda A) = X$ for all $\lambda > 0$.

If the operator takes values in X^*, then the corresponding notion is that of monotonicity. So let $A : X \to 2^{X^*}$. As before $D(A) = \{x \in X\colon A(x) \neq \emptyset\}$ (the domain of A) and $\operatorname{Gr} A = \{(x,x^*) \in X \times X^*\colon x^* \in A(x)\}$ (the graph of A). We say that A is monotone if for any $(x_i, x_i^*) \in \operatorname{Gr} A$, $i = 1,2$, we have

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geqslant 0. \tag{2.1}$$

We say that A is strictly monotone, if equality in (2.1) implies that $x_1 = x_2$. Moreover, A is maximal monotone, if

$$\langle x^* - y^*, x - y \rangle \geqslant 0 \quad \text{for all } (x,x^*) \in \operatorname{Gr} A$$

implies that $(y,y^*) \in \operatorname{Gr} A$. In other words, $\operatorname{Gr} A$ is not properly included in the graph of another monotone operator, i.e., $\operatorname{Gr} A$ is maximal with respect to inclusion among the graphs of all monotone operators. An operator $A : X \to 2^{X^*}$ is said to be coercive if either $D(A)$ is bounded or $D(A)$ is unbounded and $\inf[\|x^*\|\colon x^* \in A(x)] \to +\infty$ as $\|x\| \to \infty$, $x \in D(A)$. A single-valued operator $A : X \to X^*$ with $D(A) = X$, is said to be demicontinuous, if $x_n \to x$ in X implies $A(x_n) \xrightarrow{w} A(x)$ in X^* (i.e., A is strong-to-weak sequentially continuous). An operator $A : X \to X^*$ which is monotone demicontinuous, it is maximal monotone. In addition a maximal monotone, coercive operator $A : X \to 2^{X^*}$ is surjective (i.e., $R(A) = X^*$).

If $X = H$ is a Hilbert space identified with its dual, then the duality map $\mathcal{F}$ of H is the identity operator. So the notions of accretivity and monotonicity coincide. Moreover, $A : H \to 2^H$ is maximal monotone if and only if $R(I + A) = H$. Therefore the notions of maximal monotonicity and m-accretivity coincide.

Our analysis also involves the principal eigenvalue of $(-\Delta_p, W_0^{1,p}(Z))$. So briefly let us recall what is known about it. Consider the following nonlinear elliptic eigenvalue problem:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \lambda |x(z)|^{p-2}x(z) & \text{a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{2.2}$$

The least real number λ for which (2.2) has a nontrivial solution, is called the first *eigenvalue* of $(-\Delta_p, W_0^{1,p}(Z))$. We know (see Gasinski–Papageorgiou [5] and the references therein) that λ_1 is positive, isolated and simple (i.e., the associated eigenspace is one-dimensional). Moreover, there is a variational characterization of λ_1, via the Rayleigh quotient, i.e.,

$$\lambda_1 = \min\left[\frac{\|Dx\|_p^p}{\|x\|_p^p}\colon x \neq 0,\ x \in W_0^{1,p}(Z)\right]. \tag{2.3}$$

The minimum in (2.3) is realized at the normalized eigenfunction u_1. Note that if u_1 minimizes the Rayleigh quotient, then so does $|u_1|$ and so we infer that u_1 does not change sign on Z. Hence we may assume that $u_1(z) \geqslant 0$ a.e. on Z. In fact, using the nonlinear regularity theory and the nonlinear strong maximum principle (see Gasinski–Papageorgiou [5, Section I.5.3]), we can say that $u_1 \in C^{1,\beta}(\bar{Z})$ with $0 < \beta < 1$ and $u_1(z) > 0$ for all $z \in Z$.

Finally by $\Gamma_0(X)$ we denote the cone of all functions $\varphi : X \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ which are proper (i.e., not identically $+\infty$), convex and lower semicontinuous.

3. Existence theorem

In this section we prove the existence of a solution for problem (1.1). For this purpose our hypotheses on the data of (1.1) are the following:

$\underline{H(j)_1}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function, such that $j(\cdot, 0) \in L^1(Z)$ and
(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$ we have

$$|u| \leqslant a(z) + c|x|^{p-1} \quad \text{with } a \in L^\infty(Z)_+,\ c > 0;$$

(iv) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \leqslant \lambda_1$ a.e. on Z with strict inequality on a set of positive measure and

$$\limsup_{|x| \to +\infty} \frac{pj(z,x)}{|x|^p} \leqslant \theta(z) \quad \text{and} \quad \limsup_{|x| \to +\infty} \frac{u}{|x|^{p-2}x} \leqslant \theta(z)$$

uniformly for almost all $z \in Z$.

Remark 3.1. The following nonsmooth locally Lipschitz integrands satisfy hypotheses $H(j)_1$:

$$j_1(z,x) = \max\left\{\frac{\theta(z)}{p}|x|^p, \frac{\lambda_1}{2p}\sqrt{|x|}\right\} - x^2 \ln|x|$$

with $\theta \in L^\infty(z)_+$ as in $H(j_1)$(iv) and $\dfrac{\lambda_1}{2} \leqslant \theta(z)$ a.e. on Z and

$$j_2(z,x) = \begin{cases} \frac{\ln|x|}{|x|} - 1 & \text{if } x < -1, \\ \sin(\frac{\pi}{2}x) & \text{if } |x| \leqslant 1, \\ \frac{\theta(z)}{p}x^p - \ln x + 1 - \frac{\theta(z)}{p} & \text{if } x > 1, \end{cases}$$

with $\theta \in L^\infty(Z)_+$ as in $H(j)_1$(iv).

$\underline{H(G)}$: $G : \mathbb{R} \to \bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ is a proper (i.e., not identically $+\infty$), convex and lower semicontinuous function (i.e., $G \in \Gamma_0(\mathbb{R})$) such that $G(0) = 0$ and there exists $y_0 \in L^q(Z)$ ($\frac{1}{p} + \frac{1}{q} = 1$) such that $\int_Z G^*(y_0(z))\,dz < +\infty$ (here by $G^*(\cdot)$ we denote the conjugate (Fenchel transform) of $G(\cdot)$, i.e., $G^*(y) = \sup[yx - G(x)\colon x \in \mathbb{R}]$, see Denkowski–Migorski–Papageorgiou [2, p. 536]).

We start with a simple lemma which clarifies the nonuniform nonresonance conditions at $\pm\infty$ in hypothesis $H(j)_1$(iv).

Lemma 3.2. If $\theta \in L^\infty(Z)_+$, $\theta(z) \leqslant \lambda_1$ *a.e. on* Z *and the inequality is strict on a set of positive measure,* **then** *there exists* $\xi > 0$ *such that*

$$\psi(x) = \|Dx\|_p^p - \int_Z \theta(z)\big|x(z)\big|^p\,dz \geqslant \xi\|Dx\|_p^p \quad \textit{for all } x \in W_0^{1,p}(Z).$$

Proof. From (2.3) and the hypothesis on θ, we have that $\psi \geqslant 0$. We argue indirectly. Suppose the lemma is not true. Then exploiting the p-homogeneity of ψ, we can find a sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ such that

$$\|Dx_n\|_p = 1 \quad \text{for all } n \geqslant 1 \quad \text{and} \quad \psi(x_n) \downarrow 0.$$

By virtue of the Poincaré inequality, the sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. So by passing to a suitable subsequence if necessary, we may assume that $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$ and $x_n \to x$ in L^p(Z) (recall that $W_0^{1,p}(Z)$ is embedded compactly into $L^p(Z)$). Exploiting the weak lower semicontinuity of the norm functional in a Banach space, in the limit as $n \to \infty$, we have

$$\begin{aligned} &\psi(x) = \|Dx\|_p^p - \int_Z \theta(z)\big|x(z)\big|^p\,dz \leqslant 0 \\ \Rightarrow\quad &\|Dx\|_p^p \leqslant \int_Z \theta(z)\big|x(z)\big|^p\,dz \leqslant \lambda_1\|x\|_p^p \\ \Rightarrow\quad &\|Dx\|_p^p = \lambda_1\|x\|_p^p \quad \big(\text{see (2.3)}\big). \end{aligned} \tag{3.1}$$

It follows that $x = 0$ or $x = \pm u_1$. If $x = 0$, then $x_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|Dx_n\|_p = 1$ for all $n \geqslant 1$. Hence $x = \pm u_1$, and so $|x(z)| = |u_1(z)| > 0$ for all $z \in Z$ (see Section 2). Then from the first inequality in (3.1) and the hypothesis on $\theta \in L^\infty(Z)_+$, we obtain

$$\|Dx\|_p^p < \lambda_1\|x\|_p^p,$$

a contradiction to (2.3). This proves the lemma. □

Using this lemma and a variational method, we can prove the following existence theorem for problem (1.1).

Theorem 3.3. If *hypotheses* $H(j)_1$ *and* $H(G)$ *hold,* **then** *problem* (1.1) *has a solution* $x \in W_0^{1,p}(Z)$.

Proof. Let $\varphi_1 : W_0^{1,p}(Z) \to \mathbb{R}$ and $\varphi_2 : W_0^{1,p}(Z) \to \bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ be defined by

$$\varphi_1(x) = -\int_Z j\big(z, x(z)\big)\,dz \quad \text{and}$$

$$\varphi_2(x) = \begin{cases} \frac{1}{p}\|D(x)\|_p^p + \int_Z G(x(z))\,dz & \text{if } G(x(\cdot)) \in L^1(Z), \\ +\infty & \text{otherwise.} \end{cases}$$

We know that φ_1 is locally Lipschitz and φ_2 is proper, convex and lower semicontinuous, i.e., $\varphi_2 \in \Gamma_0(W_0^{1,p}(Z))$ (see Denkowski–Migorski–Papageorgiou [2, pp. 617 and 589, respectively]). We set $\varphi(x) = \varphi_1(x) + \varphi_2(x)$ for all $x \in W_0^{1,p}(Z)$.

Because of hypothesis $H(j)_1$ and using the mean value theorem for locally Lipschitz functions (see Section 2) and the fact that $j(\cdot,0) \in L^1(Z)$, we see that for almost all $z \in Z$ and all $x \in \mathbb{R}$,

$$\big|j(z,x)\big| \leqslant \hat{\alpha}(z) + \hat{c}|x|^p \quad \text{with } \hat{\alpha} \in L^1(Z)_+,\ \hat{c} > 0. \tag{3.2}$$

Using (3.2) and the first inequality in hypothesis $H(j)_1$(iv), given $\varepsilon > 0$ we can find $\alpha_\varepsilon \in L^1(Z)$ such that for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have

$$j(z,x) \leqslant \frac{1}{p}\big(\theta(z) + \varepsilon\big)|x|^p + \alpha_\varepsilon(z). \tag{3.3}$$

Then for all $x \in W_0^{1,p}(Z)$,

$$\begin{aligned}
\varphi(x) &= \varphi_1(x) + \varphi_2(x) \\
&\geqslant \frac{1}{p}\|Dx\|_p^p - \int_Z j\big(z, x(z)\big)\,dz \quad \big(\text{since } G \geqslant 0, \text{ see hypothesis } H(G)\big) \\
&\geqslant \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta(z)\big|x(z)\big|^p\,dz - \frac{\varepsilon}{p}\|x\|_p^p - \|\alpha_\varepsilon\|_1 \quad \big(\text{see (3.3)}\big) \\
&\geqslant \frac{\xi}{p}\|Dx\|_p^p - \frac{\varepsilon}{p\lambda_1}\|Dx\|_p^p - \|\alpha_\varepsilon\|_1 \quad \big(\text{see Lemma 3.2 and (2.3)}\big).
\end{aligned}$$

Choose $\varepsilon < \lambda_1\xi$. We obtain

$$\varphi(x) \geqslant \beta_1\|Dx\|_p^p - \beta_2 \quad \text{for some } \beta_1, \beta_2 > 0 \text{ and all } x \in W_0^{1,p}(Z).$$

From this inequality we infer that φ is coercive. Note that the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$ implies that φ_1 is completely continuous. Hence φ is weakly lower semicontinuous and so we can apply the Weierstrass theorem and generate $x \in W_0^{1,p}(Z)$ such that

$$\varphi(x) = \min \varphi.$$

Invoking the Ekeland variational principle (see Denkowski–Migorski–Papageorgiou [3, p. 93]), we obtain a sequence $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ such that

$$\varphi(x_n) \downarrow \varphi(x) \quad \big(\text{i.e., } \{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z) \text{ is a minimizing sequence}\big) \quad \text{and}$$
$$\varphi(x_n) \leqslant \varphi(v) + \frac{1}{n}\|v - x_n\| \quad \text{for all } v \in W_0^{1,p}(Z).$$

Given $\lambda \in [0,1]$ and $h \in W_0^{1,p}(Z)$, we set $v = (1-\lambda)x_n + \lambda h = x_n + \lambda(h - x_n)$. Also let $I_G : W_0^{1,p}(Z) \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by

$$I_G(x) = \begin{cases} \int_Z G(x(z))\,dz & \text{if } G(x(\cdot)) \in L^1(Z), \\ +\infty & \text{otherwise.} \end{cases}$$

We know that $I_G \in \Gamma_0(W_0^{1,p}(Z))$. We have

$$\begin{aligned}
-\frac{1}{n}\|h-x_n\| &\leqslant \frac{\varphi_1(x_n+\lambda(h-x_n))-\varphi_1(x_n)}{\lambda} \\
&\quad + \frac{1}{p\lambda}\bigl[\bigl\|D\bigl(x_n+\lambda(h-x_n)\bigr)\bigr\|_p^p - \|Dx_n\|_p^p\bigr] + I_G(h) - I_G(x_n) \\
&\quad \text{(since } I_G \text{ is convex)}, \\
\Rightarrow \quad -\frac{1}{n}\|h-x_n\| &\leqslant \varphi_1^0(x_n; h-x_n) + \bigl\langle A(x_n), h-x_n\bigr\rangle + I_G(h) - I_G(x_n) \\
&\quad \text{for all } h \in W_0^{1,p}(Z). \qquad (3.4)
\end{aligned}$$

Here $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ is the nonlinear operator defined by

$$\bigl\langle A(x), y\bigr\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N}\, dz \quad \text{for all } x, y \in W_0^{1,p}(Z),$$

with $\langle\cdot,\cdot\rangle$ being the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$, $\frac{1}{p}+\frac{1}{q}=1$. It is easy to see that A is monotone, demicontinuous hence it is maximal monotone (see Section 2). In (3.4) we put $h=0$. We obtain

$$\begin{aligned}
&-\frac{1}{n}\|x_n\| \leqslant \varphi_1^0(x_n; -x_n) + \bigl\langle A(x_n), -x_n\bigr\rangle - I_G(x_n) \\
&\bigl(\text{since } I_G(0)=0, \text{ see hypothesis } H(G)\bigr). \qquad (3.5)
\end{aligned}$$

Recalling that $\varphi_1^0(x_n;\cdot)$ is the support function of the set $\partial\varphi_1(x_n)$ which is w-compact in $W^{-1,q}(Z)$, we can find $-u_n \in \partial\varphi_1(x_n)$ such that

$$\varphi_1^0(x_n; -x_n) = \langle u_n, x_n\rangle.$$

From Proposition 3.5.36, p. 614, and Theorem 5.5.39, p. 617, of Denkowski–Migorski–Papageorgiou [2], we have that $u_n \in S^q_{\partial j(\cdot, x_n(\cdot))} = \{u \in L^q(Z)\colon\ u(z) \in \partial j(z, x_n(z)) \text{ a.e. on } Z\}$ ($\frac{1}{p}+\frac{1}{q}=1$) for all $n \geqslant 1$. So

$$\varphi_1^0(x_n; -x_n) = \int_Z u_n x_n\, dz.$$

Using this in (3.5), we obtain

$$\bigl\langle A(x_n), x_n\bigr\rangle - \int_Z u_n x_n\, dz = \|Dx_n\|_p^p - \int_Z u_n x_n\, dz \leqslant \frac{1}{n}\|x_n\|, \quad n \geqslant 1. \qquad (3.6)$$

We claim that $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded. Suppose that this is not true. By passing to a suitable subsequence if necessary, we may assume that $\|x_n\| \to +\infty$ as $n \to \infty$. Set $y_n = \frac{x_n}{\|x_n\|}$, $n \geqslant 1$. We may assume that

$$y_n \xrightarrow{w} y \quad \text{in } W_0^{1,p}(Z), \qquad y_n \to y \quad \text{in } L^p(Z), \qquad y_n(z) \to y(z) \quad \text{a.e. on } Z \quad \text{and}$$
$$\bigl|y_n(z)\bigr| \leqslant k(z) \quad \text{a.e. on } Z \quad \text{for all } n \geqslant 1 \text{ with } k \in L^p(Z)$$

(recall that $W_0^{1,p}(Z)$ is embedded compactly in $L^p(Z)$ and see Denkowski–Migorski–Papageorgiou [2, p. 147]). From hypothesis $H(j)_1$(iii) we have

$$\frac{|u_n(z)|}{\|x_n\|^{p-1}} \leqslant \frac{a(z)}{\|x_n\|^{p-1}} + c\left|y_n(z)\right|^{p-1} \quad \text{a.e. on } Z,$$
$$\Rightarrow \quad \left\{\frac{u_n(\cdot)}{\|x_n\|^{p-1}}\right\}_{n\geqslant 1} \subseteq L^q(Z) \quad \text{is bounded.} \tag{3.7}$$

We may assume that $\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} f$ in $L^q(Z)$. Given $\varepsilon > 0$ and $n \geqslant 1$, we introduce the following two sets:

$$Z^+_{\varepsilon,n} = \left\{z \in Z\colon\ x_n(z) > 0,\ \frac{u_n(z)}{x_n(z)^{p-1}} \leqslant \theta(z) + \varepsilon\right\} \quad \text{and}$$
$$Z^-_{\varepsilon,n} = \left\{z \in Z\colon\ x_n(z) < 0,\ \frac{u_n(z)}{|x_n(z)|^{p-2}x_n(z)} \leqslant \theta(z) + \varepsilon\right\}.$$

Remark that $x_n(z) \to +\infty$ a.e. on $\{y > 0\}$ and $x_n(z) \to -\infty$ a.e. on $\{y < 0\}$. So by virtue of the second inequality in hypothesis $H(j)_1$(iv), we have

$$\hat{\chi}^+_{\varepsilon,n}(z) = \chi_{Z^+_{\varepsilon,n}}(z) \to 1 \quad \text{a.e. on } \{y > 0\} \quad \text{and}$$
$$\hat{\chi}^-_{\varepsilon,n}(z) = \chi_{Z^-_{\varepsilon,n}}(z) \to 1 \quad \text{a.e. on } \{y < 0\}.$$

Also we have

$$\hat{\chi}^+_{\varepsilon,n}(z)\frac{u_n(z)}{\|x_n\|^{p-1}} = \hat{\chi}^+_{\varepsilon,n}(z)\frac{u_n(z)}{x_n(z)^{p-1}}y_n(z)^{p-1} \leqslant \hat{\chi}^+_{\varepsilon,n}(z)\big(\theta(z)+\varepsilon\big)y_n(z)^{p-1}.$$

Taking weak limits in $L^q(\{y > 0\})$, we obtain

$$f(z) \leqslant \big(\theta(z)+\varepsilon\big)y(z)^{p-1} \quad \text{a.e. on } \{y > 0\}.$$

Letting $\varepsilon \downarrow 0$, we finally have that

$$f(z) \leqslant \theta(z)y(z)^{p-1} \quad \text{a.e. on } \{y > 0\}. \tag{3.8}$$

Arguing similarly, using $\hat{\chi}^-_{\varepsilon,n}$ this time, we infer that

$$f(z) \geqslant \theta(z)\left|y(z)\right|^{p-2}y(z) \quad \text{a.e. on } \{y < 0\}. \tag{3.9}$$

Moreover, from (3.7) it is clear that

$$f(z) = 0 \quad \text{a.e. on } \{y = 0\}. \tag{3.10}$$

Because of (3.8)–(3.10), we can say that

$$f(z)y(z) \leqslant \theta(z)\left|y(z)\right|^p \quad \text{a.e. on } Z. \tag{3.11}$$

We return to (3.6) and we divide with $\|x_n\|^p$. We have

$$\|Dy_n\|_p^p - \int\limits_Z \frac{u_n}{\|x_n\|^{p-1}}y_n\,dz \leqslant \frac{1}{n\|x_n\|^{p-1}},$$

$$\Rightarrow \quad \|Dy\|_p^p \leqslant \int_Z fy\,dz,$$

$$\Rightarrow \quad \|Dy\|_p^p \leqslant \int_Z \theta|y|^p\,dz \quad \big(\text{see (3.11)}\big), \tag{3.12}$$

$$\Rightarrow \quad \|Dy\|_p^p \leqslant \lambda_1\|y\|_p^p \quad \big(\text{see hypothesis } H(j)_1\text{(iv)}\big). \tag{3.13}$$

From (2.3) and (3.13) it follows that

$$\|Dy\|_p^p = \lambda_1\|y\|_p^p \quad \Rightarrow \quad y = 0 \text{ or } y = \pm u_1.$$

If $y = 0$, then $y_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|y_n\| = 1, n \geqslant 1$. So $y = \pm u_1$, hence $|y(z)| > 0$ for all $z \in Z$ (see Section 2). From this, (3.12) and the hypothesis on $\theta \in L^\infty(Z)_+$, we infer that

$$\|Dy\|_p^p < \lambda_1\|y\|_p^p,$$

which contradicts (2.3). So $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$ is bounded and we may assume that

$$x_n \xrightarrow{w} x \quad \text{in } W_0^{1,p}(Z) \quad \text{and} \quad x_n \to x \quad \text{in } L^p(Z).$$

Recall that

$$-\frac{1}{n}\|h - x_n\| \leqslant \varphi_1^0(x_n; h - x_n) + \big\langle A(x_n), h - x_n\big\rangle + I_G(h) - I_G(x_n)$$
$$\text{for all } h \in W_0^{1,p}(Z),\ n \geqslant 1. \tag{3.14}$$

Set $h = x$ and as earlier choose $\hat{u}_n \in S^q_{\partial j(\cdot, x_n(\cdot))} = \{u \in L^q(Z)\colon\ u(z) \in \partial j(z, x_n(z))$ a.e. on $Z\}$ such that

$$\varphi_1^0(x_n; x - x_n) = -\int_Z \hat{u}_n(x - x_n)\,dz, \quad n \geqslant 1.$$

So we can write that

$$-\frac{1}{n}\|x - x_n\| \leqslant \big\langle A(x_n), x - x_n\big\rangle - \int_Z \hat{u}_n(x - x_n)\,dz + I_G(x) - I_G(x_n). \tag{3.15}$$

Note that

$$\int_Z \hat{u}_n(x - x_n)\,dz \to 0 \quad \text{and} \quad I_G(x) \leqslant \liminf_{n\to\infty} I_G(x_n) \quad \big(\text{since } I_G \in \Gamma_0\big(W_0^{1,p}(Z)\big)\big).$$

So if we pass to the pass to the limit as $n \to \infty$ in (3.15), we obtain

$$0 \leqslant \liminf_{n\to\infty}\big\langle A(x_n), x - x_n\big\rangle \quad \Rightarrow \quad \limsup_{n\to\infty}\big\langle A(x_n), x_n - x\big\rangle \leqslant 0.$$

Because A is maximal monotone, it is generalized pseudomonotone (see Denkowski–Migorski–Papageorgiou [3, p. 58] and Showalter [14, p. 41]). So we have

$$\big\langle A(x_n), x_n\big\rangle \to \big\langle A(x), x\big\rangle \quad \Rightarrow \quad \|Dx_n\|_p \to \|Dx\|_p.$$

Also $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^{\mathbb{N}})$. Because $L^p(Z, \mathbb{R}^{\mathbb{N}})$ is uniformly convex, it has the Kadec–Klee property (see Denkowski–Migorski–Papageorgiou [2, p. 309]) and so $Dx_n \to D(x)$ in $L^p(Z, \mathbb{R}^{\mathbb{N}})$. Therefore $x_n \to x$ in $W_0^{1,p}(Z)$.

Recall that from the choice of $\{x_n\}_{n\geqslant 1} \subseteq W_0^{1,p}(Z)$, the choice $v = x_n + \lambda(h - x_n)$ made earlier in the proof and from the convexity of φ_2, we have

$$\begin{aligned}
-\frac{1}{n}\|h - x_n\| &\leqslant \frac{\varphi_1(x_n + \lambda(h - x_n)) - \varphi_1(x_n)}{\lambda} \\
&\quad + \frac{1}{\lambda}\big[\varphi_2\big((1-\lambda)x_n + \lambda h\big) - \varphi_2(x_n)\big] \\
&\leqslant \frac{\varphi_1(x_n + \lambda(h - x_n)) - \varphi_1(x_n)}{\lambda} + \varphi_2(h) - \varphi_2(x_n), \\
\Rightarrow \quad -\frac{1}{n}\|h - x_n\| &\leqslant \varphi_1^0(x_n; h - x_n) + \varphi_2(h) - \varphi_2(x_n), \\
\Rightarrow \quad 0 &\leqslant \varphi_1^0(x; h - x) + \varphi_2(h) - \varphi_2(x) \quad \text{for all } h \in W_0^{1,p}(Z). \qquad (3.16)
\end{aligned}$$

To obtain (3.16) we have used the upper semicontinuity of $\varphi_1^0(\cdot;\cdot)$ (see Denkowski–Migorski–Papageorgiou [2, p. 602]), the lower semicontinuity of φ_2 and the fact that $x_n \to x$ in $W_0^{1,p}(Z)$ as $n \to \infty$.

We set $\psi_1(h) = \varphi_1^0(x; h - x)$ and $\psi_2(h) = \varphi_2(h) - \varphi_2(x)$ for all $h \in W_0^{1,p}(Z)$. Then we have:

(a) ψ_1 is continuous convex and $\partial\psi_1(x) = \partial\varphi_1(x)$, where the first subdifferential is in the sense of convex analysis and the second is a generalized subdifferential.
(b) $\psi_2 \in \Gamma_0(W_0^{1,p}(Z))$ and $\partial\psi_2(x) = \partial\varphi_2(x)$, where both subdifferentials are in the sense of convex analysis.

Since ψ_1 is continuous, the calculus rules for the convex subdifferential (see Denkowski–Migorski–Papageorgiou [2, p. 549]), imply that

$$\begin{aligned}
\partial(\psi_1 + \psi_2)(x) &= \partial\psi_1(x) + \partial\psi_2(x) \\
&= \partial\varphi_1(x) + \partial\varphi_2(x) \quad \big(\text{see (a) and (b) above}\big). \qquad (3.17)
\end{aligned}$$

From the definition of the convex subdifferential (see Section 2), we have

$$\begin{aligned}
\partial(\psi_1 + \psi_2)(x) = \big\{x^* \in W^{-1,q}(Z)\colon\ \langle x^*, h - x\rangle &\leqslant \psi_1(h) + \psi_2(h) - \psi_2(x) \\
&= \varphi_1^0(x; h - x) + \varphi_2(h) - \varphi_2(x) \\
&\quad \text{for all } h \in W_0^{1,p}(Z)\big\}. \qquad (3.18)
\end{aligned}$$

So we obtain

$$\begin{aligned}
&0 \in \partial(\psi_1 + \psi_2)(x) \quad \big(\text{see (3.16), (3.18)}\big) \\
&\quad \Rightarrow \quad 0 \in \partial\varphi_1(x) + \partial\varphi_2(x) \quad \big(\text{see (3.17)}\big). \qquad (3.19)
\end{aligned}$$

Let $\hat{A} : \hat{D} \subseteq L^q(Z) \to L^q(Z)$ be the nonlinear operator defined by

$$\hat{A}(x) = A(x) \quad \text{for all } x \in \hat{D} = \{x \in W_0^{1,p}(Z):\ A(x) \in L^q(Z)\}.$$

Evidently $C_c^\infty(Z) \subseteq \hat{D}$. From Calvert [1, Lemma 3.1] (see also Li–Zhen [11, Proposition 2.1]), we have that $\hat{A}$ is m-accretive.

Also let $S : D(S) \subseteq L^q(Z) \to 2^{L^q(Z)}$ be defined by

$$S(x) = S^q_{\partial G(x(\cdot))} = \{v \in L^q(Z):\ v(z) \in \partial G(x(z)) \text{ a.e. on } Z\}$$
$$\text{for all } x \in D(S) = \{x \in L^q(Z):\ S^q_{\partial G(x(\cdot))} \neq \emptyset\}.$$

We claim that S is m-accretive. First we show that S is accretive. To this end let $(x_1, v_1), (x_2, v_2) \in \operatorname{Gr} S$. We set

$$y_1(z) = x_1(z) + v_1(z) \quad \text{and} \quad y_2(z) = x_2(z) + v_2(z).$$

We have

$$x_1(z) = (I + \partial G(\cdot))^{-1}(y_1(z)) \quad \text{and} \quad x_2(z) = (I + \partial G(\cdot))^{-1}(y_2(z)).$$

Exploiting the nonexpansiveness of the resolvent operator corresponding to $\partial G(\cdot)$, we obtain

$$\begin{aligned} &|x_1(z) - x_2(z)| \leqslant |y_1(z) - y_2(z)| \quad \text{a.e. on } Z,\\ &\Rightarrow \quad \|x_1 - x_2\|_q \leqslant \|y_1 - y_2\|_q = \|x_1 + v_1 - (x_2 + v_2)\|_q,\\ &\Rightarrow \quad S \quad \text{is accretive.} \end{aligned}$$

To show the m-accretivity of S, we need to show that $R(I + S) = L^q(Z)$. To this end let $h \in L^q(Z)$ and consider the multifunction

$$z \to L(z) = \{x \in \mathbb{R}:\ (I + \partial G(\cdot))^{-1}(h(z)) = x\}.$$

Because $G \in \Gamma_0(\mathbb{R})$ (see hypothesis $H(G)$), we have that $\partial G(\cdot)$ is maximal monotone. From the maximal monotonicity of the operator $\partial G(\cdot)$ we have that $L(z) \neq \emptyset$ for all $z \in Z$. Also from Hu–Papageorgiou [7, p. 362], we know that the function $z \to (I + \partial G(\cdot))^{-1}(h(z))$ is Lebesgue measurable. Therefore the function

$$(z, x) \to \xi(z, x) = (I + \partial G(\cdot))^{-1}(h(z)) - x$$

is a Caratheodory function, i.e., it is measurable in $z \in Z$ and continuous in $x \in \mathbb{R}$, hence it is jointly measurable. So we have

$$\operatorname{Gr} L = \{(z, x) \in Z \times \mathbb{R}:\ \xi(z, x) = 0\} \in \mathcal{L}_Z \times B(\mathbb{R}),$$

with $\mathcal{L}_Z$ being the Lebesgue σ-field of Z and $B(\mathbb{R})$ the Borel σ-field of $\mathbb{R}$. We can apply the Yankon–von Neumann–Aumann selection theorem (see Hu–Papageorgiou [7, p. 158]) and obtain a Lebesgue measurable function $x : Z \to \mathbb{R}$ such that

$$x(z) \in L(z) \quad \text{for all } z \in Z.$$

Since $(I + \partial G(\cdot))^{-1}(0) = 0$ a.e. on Z (recall that $0 \in \partial G(0)$, see hypothesis $H(G)$), from the nonexpansiveness of the resolvent operator, we have

$$|x(z)| \leqslant |h(z)| \quad \text{a.e. on } Z \quad \Rightarrow \quad x \in L^q(Z).$$

Clearly $h \in (I+S)(x)$ and because $h \in L^q(Z)$ was arbitrary, we conclude that $R(I+S) = L^q(Z)$. This proves that S is m-accretive.

Next let $\eta : L^p(Z) \to L^q(Z)$ be defined by

$$\eta(x)(\cdot) = \left|x(\cdot)\right|^{p-2} x(\cdot).$$

We know that $\eta(x)(\cdot) = \|x\|_p^{p-2} \mathcal{F}(x)(\cdot)$, where $\mathcal{F}$ is the duality map of $L^p(Z)$ (see Hu–Papageorgiou [7, p. 317] and Showalter [14, p. 93]). If by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(Z), L^q(Z))$, for every $x \in \hat{D}$ and every $\lambda > 0$, we have

$$\begin{aligned}\left(\hat{A}(x), \eta\left(S_\lambda(x)\right)\right)_{pq} &= \left\langle A(x), \eta\left(S_\lambda(x)\right)\right\rangle \\ &= \int_Z \|Dx\|^{p-2}\left(Dx, D\eta\left(\partial G_\lambda(x)\right)\right)_{\mathbb{R}^{\mathbb{N}}} dz. \end{aligned} \tag{3.20}$$

Here G_λ is the Moreau–Yosida regularization of G. We know that G_λ is differentiable (with the derivative denoted by ∂G_λ) and $S_\lambda(x)(\cdot) = \partial G_\lambda(x(\cdot)) \in L^q(Z)$ for all $x \in L^q(Z)$ (see Hu–Papageorgiou [7, pp. 349–350] and Showalter [14, p. 162]). Because $\partial G_\lambda = (\partial G)_\lambda$, it is Lipschitz continuous and monotone. So using the chain rule for Sobolev functions (see Denkowski–Migorski–Papageorgiou [2, p. 348]), we have

$$\begin{aligned}&\left\|Dx(z)\right\|^{p-2}\left(Dx(z), D\left(\eta\left(\partial G_\lambda\left(x(z)\right)\right)\right)\right)_{\mathbb{R}^{\mathbb{N}}} \\ &\quad = (p-1)\left|\partial G_\lambda\left(x(z)\right)\right|^{p-1}\left(\frac{d}{dx}\partial G_\lambda\right)\left(x(z)\right)\left\|Dx(z)\right\|^p \quad \text{a.e. on } Z. \end{aligned} \tag{3.21}$$

Since $(\frac{d}{dx}\partial G_\lambda)(x) \geqslant 0$ for all $x \in \mathbb{R}$ (due to the monotonicity of $\partial G_\lambda(\cdot)$), using (3.21) in (3.20), we obtain

$$\left(\hat{A}(x), \eta\left(S_\lambda(x)\right)\right)_{pq} \geqslant 0 \quad \text{for all } x \in \hat{D}.$$

Applying Theorem 7.44, p. 394, of Hu–Papageorgiou [7], we conclude that

$$x \to \hat{A}(x) + S(x) \quad \text{is } m\text{-accretive}. \tag{3.22}$$

It is immediate from the definitions of $\hat{A}$, S and φ_2 that

$$\hat{A} + S \subseteq \partial\varphi_2 \cap \left(W_0^{1,p}(Z) \times L^q(Z)\right). \tag{3.23}$$

Clearly $\partial\varphi_2 \cap (W_0^{1,p}(Z) \times L^q(Z))$ is accretive in $L^q(Z) \times L^q(Z)$ (recall that $2 \leqslant p < \infty$ which implies that $W_0^{1,p}(Z) \subseteq L^q(Z)$). Combining this with (3.22) and (3.23), we conclude that

$$\hat{A} + S = \partial\varphi_2 \cap \left(W_0^{1,p}(Z) \times L^q(Z)\right). \tag{3.24}$$

Because of (3.19), we can find $u \in \partial\varphi_1(x)$ and $w \in \partial\varphi_2(x)$ such that

$$0 = u + w \quad \Rightarrow \quad w = -u. \tag{3.25}$$

Recall that $-u \in S^q_{\partial j(\cdot,x(\cdot))}$. Therefore $w \in \partial\varphi_2(x) \cap L^q(Z)$ and so from (3.24) we have that

$$w = \hat{A}(x) + v \quad \text{with } v \in S(x) = S^q_{\partial G(x(\cdot))}.$$

Thus finally from (3.25), we have

$$\hat{A}(x) + v + u = 0.$$

Let $\zeta \in C_c^\infty(Z)$. Taking duality brackets with ζ, we obtain

$$\begin{aligned}
&\big(\hat{A}(x), \zeta\big)_{pq} = (-u, \zeta)_{pq} + (-v, \zeta)_{pq},\\
&\Rightarrow \quad \big\langle A(x), \zeta\big\rangle = \int_Z (-u)\zeta\, dz + \int_Z (-v)\zeta\, dz,\\
&\Rightarrow \quad \int_Z \|Dx\|^{p-2}(Dx, D\zeta)_{\mathbb{R}^N}\, dz = \int_Z (-u)\zeta\, dz + \int_Z (-v)\zeta\, dz.
\end{aligned}$$

Note that $\operatorname{div}(\|Dx\|^{p-2}Dx) \in W^{-1,q}(Z)$ (see Denkowski–Migorski–Papageorgiou [2, p. 362]). Also the adjoint of the gradient operator $D \in \mathcal{L}(W_0^{1,p}(Z), L^p(Z))$ is the operator $-\operatorname{div} \in \mathcal{L}(L^q(Z), W^{-1,q}(Z))$. So we have

$$\big\langle -\operatorname{div}\big(\|Dx\|^{p-2}Dx\big), \zeta\big\rangle = \int_Z (-u)\zeta\, dz + \int_z (-v)\zeta\, dz.$$

Because $C_c^\infty(Z)$ is dense in $W_0^{1,p}(Z)$, we conclude that

$$\begin{aligned}
&-\operatorname{div}\big(\big\|Dx(z)\big\|^{p-2}Dx(z)\big) = -u(z) - v(z)\\
&\quad \in \partial j\big(z, x(z)\big) - \partial G\big(x(z)\big) \quad \text{a.e. on } Z \text{ and } x|_{\partial Z} = 0,\\
&\Rightarrow \quad x \in W_0^{1,p}(Z) \text{ is a solution of problem (1.1).} \qquad \square
\end{aligned}$$

Remark 3.4. If $\int_Z j(z,0)\, dz \leqslant 0$ and there exists $x_0 \in \mathbb{R}, x_0 \neq 0$ such that $\int_Z j(z, x_0)\, dz > 0$, then we can guarantee that the solution $x \in W_0^{1,p}(Z)$ obtained in Theorem 3.3 is nontrivial.

A case of special interest is when $G(x) = i_C(x)$ with $C \subseteq \mathbb{R}$ being a closed, convex subset, $0 \in C$. Then Theorem 3.3, implies that there exists $x \in W_0^{1,p}(Z)$ and $u \in S^q_{\partial j(\cdot, x(\cdot))}$ such that

$$\int_Z \|Dx\|^{p-2}(Dx, Dy - Dx)_{\mathbb{R}^N}\, dz \geqslant \int_Z u(y - x)\, dz$$

for all $y \in \hat{C} = \{y \in W_0^{1,p}(Z)\colon\ y(z) \in C \text{a.e. on } Z\}$. For example, we can have $C = \mathbb{R}_+$ in which case $\hat{C} = W_0^{1,p}(Z)_+ =$ the positive cone of the Sobolev space $W_0^{1,p}(Z)$.

4. Positive solutions

In this section using the method of the proof of Theorem 3.3 and some additional hypotheses on the nonsmooth potential $j(z,x)$, we prove the existence of a strictly positive, smooth solution for the following hemivariational inequality:

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z,x(z)) & \text{a.e. on } Z, \\ x|_{\partial Z} = 0. \end{cases} \tag{4.1}$$

Now the hypotheses on $j(z,x)$ are the following:

$\underline{H(j)_2}$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function, such that $\int_Z j(z,0)\,dz \leqslant 0$ $\partial j(z,0) \subseteq \mathbb{R}_+$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$ we have
$$|u| \leqslant a(z) + c|x|^{p-1} \quad \text{with } a \in L^\infty(Z)_+,\ c > 0;$$
(iv) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \leqslant \lambda_1$ a.e. on Z with strict inequality on a set of positive measure and
$$\limsup_{x\to+\infty} \frac{pj(z,x)}{x^p} \leqslant \theta(z) \quad \text{and} \quad \limsup_{x\to+\infty} \frac{u}{x^{p-1}} \leqslant \theta(z)$$
uniformly for almost all $z \in Z$;
(v) for almost all $z \in Z$, all $x \geqslant 0$ and all $u \in \partial j(z,x)$ we have
$$-c_0 x^{p-1} \leqslant u;$$
(vi) there exists $M > 0$ such that for almost all $z \in Z$, all $x \geqslant M$ and all $u \in \partial j(z,x)$, we have
$$u \geqslant 0 \quad \text{or} \quad u \leqslant 0$$
and there exists $x_0 > 0$ such that $\int_Z j(z,x_0)\,dz > 0$.

Remark 4.1. Let $\theta \in L^\infty(Z)_+$ be as in hypothesis $H(j)_2$(iv). The following nonsmooth locally Lipschitz integrands satisfy hypotheses $H(j)_2$:

$$j_1(z,x) = \begin{cases} x - e^x + 1 & \text{if } x \leqslant 0, \\ \frac{\theta(z)}{p}x^p - x^r \ln x & \text{if } x \geqslant 0 \text{ with } 1 < r < p, \end{cases} \quad \text{and}$$
$$j_2(z,x) = \begin{cases} \sin x & \text{if } x < 0, \\ \tan^{-1} x & \text{if } 0 \leqslant x \leqslant 1, \\ \frac{\theta(z)}{p}x^p - \frac{\theta(z)}{p} + \frac{\pi}{4} & \text{if } x > 1. \end{cases}$$

In the next theorem $C_0^1(\bar{Z}) = \{x \in C^1(\bar{Z}):\ x|_{\partial Z} = 0\}$.

Theorem 4.2. **If** *hypotheses* $H(j)_2$ *hold,* **then** *problem* (4.1) *has a solution* $x \in C_0^1(\bar{Z})$ *such that* $x(z) > 0$ *for all* $z \in Z$ *and* $\frac{\partial x}{\partial n}(z) < 0$ *for all* $z \in \partial Z$.

Proof. Let

$$G(x)=i_{\mathbb{R}_+}(x)=\begin{cases}0 & \text{if } x\geqslant 0,\\ +\infty & \text{if } x<0.\end{cases}$$

Evidently $G\geqslant 0$ and $G\in \Gamma_0(\mathbb{R})$. In this case it is more convenient to make the following choices of φ_1 and φ_2:

$$\varphi_1(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z j\big(z,x(z)\big)\,dz,\quad x\in W_0^{1,p}(Z)\quad\text{and}$$

$$\varphi_2(x)=\begin{cases}\int_Z G(x(z))\,dz & \text{if } G(x(\cdot))\in L^1(Z),\\ +\infty & \text{otherwise},\end{cases}\qquad x\in W_0^{1,p}(Z).$$

Let $C=\{x\in W_0^{1,p}(Z)\colon\ x(z)\geqslant 0 \text{ a.e. on } Z\}$ (i.e., $C=W_0^{1,p}(Z)_+$ the positive cone of the Sobolev space $W_0^{1,p}(Z)$). Evidently

$$\varphi_2(x)=i_C(x)=\begin{cases}0 & \text{if } x\in C,\\ +\infty & \text{otherwise}.\end{cases}$$

From the first part of the proof of Theorem 3.3 and because $\operatorname{dom}\varphi_2=C$, we obtain an $x\in C$, $x\neq 0$ (see the remark after the proof of Theorem 3.3), such that

$$0\in\partial\varphi_1(x)+\partial\varphi_2(x). \tag{4.2}$$

We should point out that since $\operatorname{dom}\varphi_2=C$, in hypothesis $H(j)_2$(iv) we can assume that the limits in the two inequalities are taken only in the positive direction (i.e., as $x\to+\infty$, compare with hypothesis $H(j)$(iv)). From (4.2), we infer that there exists $u\in S^q_{\partial j(\cdot,x(\cdot))}$ such that

$$A(x)-u\in-\partial\varphi_2(x)=-N_C(x),$$

where $N_C(x)$ is the normal cone to the closed convex set C at $x\in C$, i.e., $N_C(x)=\{x^*\in W^{-1,q}(Z)\colon\ \langle x^*,y-x\rangle\leqslant 0 \text{ for all } y\in C\}$ (see Denkowski–Migorski–Papageorgiou [2, p. 622]). So

$$\big\langle A(x)-u,y-x\big\rangle\geqslant 0\quad\text{for all } y\in C.$$

Let $h\in W_0^{1,p}(Z)$ and $\varepsilon>0$ be arbitrary and set $y=(x+\varepsilon h)^+=x+\varepsilon h+(x+\varepsilon h)^-\in C$. We have

$$\begin{aligned}&0\leqslant\big\langle A(x)-u,\varepsilon h\big\rangle+\big\langle A(x)-u,(x+\varepsilon h)^-\big\rangle,\\ &\quad\Rightarrow\quad -\big\langle A(x)-u,(x+\varepsilon h)^-\big\rangle\leqslant\big\langle A(x)-u,\varepsilon h\big\rangle.\end{aligned} \tag{4.3}$$

We estimate the left-hand side of (4.3). Then

$$-\big\langle A(x)-u,(x+\varepsilon h)^-\big\rangle=-\big\langle A(x),(x+\varepsilon h)^-\big\rangle+\int_Z u(x+\varepsilon h)^-\,dz. \tag{4.4}$$

Assume that the first option in hypothesis $H(j)_2$(vi) holds, namely for almost all $z\in Z$, all $x\geqslant M$ and all $u\in\partial j(z,x)$, we have $u\geqslant 0$. Set

$$Z_-^\varepsilon=\big\{z\in Z\colon\ (x+\varepsilon h)(z)<0\big\}\quad\text{and}\quad \hat{Z}_-^\varepsilon=\big\{z\in Z_-^\varepsilon\colon\ x(z)>0\big\}.$$

We know that

$$D(x+\varepsilon h)^{-}(z)=\begin{cases}-D(x+\varepsilon h)(z) & \text{a.e. on } Z_{-}^{\varepsilon},\\ 0 & \text{otherwise}\end{cases}$$

(see Denkowski–Migorski–Papageorgiou [2, p. 348]). So we have

$$\begin{aligned}-\big\langle A(x),(x+\varepsilon h)^{-}\big\rangle &= -\int_{Z}\|Dx\|^{p-2}\big(Dx, D(x+\varepsilon h)^{-}\big)_{\mathbb{R}^{\mathbb{N}}}\,dz\\ &= \int_{Z_{-}^{\varepsilon}}\|Dx\|^{p-2}\big(Dx, D(x+\varepsilon h)\big)_{\mathbb{R}^{\mathbb{N}}}\,dz\\ &\geqslant \varepsilon\int_{Z_{-}^{\varepsilon}}\|Dx\|^{p-2}(Dx, Dh)_{\mathbb{R}^{\mathbb{N}}}\,dz\\ &= \varepsilon\int_{\hat{Z}_{-}^{\varepsilon}}\|Dx\|^{p-2}(Dx, Dh)_{\mathbb{R}^{\mathbb{N}}}\,dz. \end{aligned} \tag{4.5}$$

The last equality follows from the fact that $Dx(z)=0$ a.e. on $\{x=0\}$ (by Stampacchia's Theorem, see Denkowski–Migorski–Papageorgiou [2, p. 349]).

Also we have

$$\begin{aligned}\int_{Z}u(x+\varepsilon h)^{-}\,dz &= -\int_{Z_{-}^{\varepsilon}}u(x+\varepsilon h)\,dz\\ &= -\int_{Z_{-}^{\varepsilon}\cap\{x<M\}}u(x+\varepsilon h)\,dz-\int_{Z_{-}^{\varepsilon}\cap\{x\geqslant M\}}u(x+\varepsilon h)\,dz.\end{aligned}$$

We estimate each summand of the right-hand side separately. So

$$-\int_{Z_{-}^{\varepsilon}\cap\{x<M\}}u(x+\varepsilon h)\,dz=-\int_{Z_{-}^{\varepsilon}\cap\{x=0\}}u(x+\varepsilon h)\,dz-\int_{Z_{-}^{\varepsilon}\cap\{0<x<M\}}u(x+\varepsilon h)\,dz.$$

By hypothesis $\partial j(z,0)\subseteq\mathbb{R}_{+}$ a.e. on Z. So $u(z)\geqslant 0$ a.e. on $Z_{-}^{\varepsilon}\cap\{x=0\}$. Also since $x(z)\geqslant 0$ a.e. on Z, we have that $h(z)<0$ a.e. on Z_{-}^{ε}. So we obtain

$$-\int_{Z_{-}^{\varepsilon}\cap\{x=0\}}u(x+\varepsilon h)\,dz=-\int_{Z_{-}^{\varepsilon}\cap\{x=0\}}\varepsilon uh\,dz\geqslant 0.$$

Therefore

$$\begin{aligned}-\int_{Z_{-}^{\varepsilon}\cap\{x<M\}}u(x+\varepsilon h)\,dz &\geqslant -\int_{Z_{-}^{\varepsilon}\cap\{0<x<M\}}u(x+\varepsilon h)\,dz\\ &\geqslant \beta_1\int_{Z_{-}^{\varepsilon}\cap\{0<x<M\}}(x+\varepsilon h)\,dz\end{aligned}$$

$$\text{for some } \beta_1 > 0 \quad \big(\text{see hypothesis } H(j)_2\text{(iii)}\big)$$
$$\geqslant \varepsilon\beta_1 \int_{\hat{Z}^\varepsilon_- \cap \{x<M\}} h\, dz \quad \big(\text{since } x \geqslant 0\big). \tag{4.6}$$

Also since $u(z) \geqslant 0$ a.e. on $\{x \geqslant M\}$, we have

$$-\int_{Z^\varepsilon_- \cap \{x \geqslant M\}} u(x+\varepsilon h)\, dz \geqslant 0. \tag{4.7}$$

Therefore from (4.6) and (4.7), we infer that

$$\int_Z u(x+\varepsilon h)^-\, dz \geqslant \varepsilon\beta_1 \int_{\hat{Z}^\varepsilon_- \cap \{x<M\}} h\, dz. \tag{4.8}$$

Using (4.5) and (4.8) in (4.4), we obtain

$$-\big\langle A(x) - u, (x+\varepsilon h)^-\big\rangle \geqslant \varepsilon \int_{\hat{Z}^\varepsilon_-} \|Dx\|^{p-2}(Dx, Dh)_{\mathbb{R}^\mathbb{N}}\, dz + \varepsilon\beta_1 \int_{\hat{Z}^\varepsilon_- \cap \{x<M\}} h\, dz. \tag{4.9}$$

Returning to (4.3), using (4.9) and then dividing with $\varepsilon > 0$, we obtain

$$\int_{\hat{Z}^\varepsilon_-} \|Dx\|^{p-2}(Dx, Dh)_{\mathbb{R}^\mathbb{N}}\, dz + \beta_1 \int_{\hat{Z}^\varepsilon_- \cap \{x<M\}} h\, dz \leqslant \big\langle A(x) - u, h\big\rangle.$$

Note that $|\hat{Z}^\varepsilon_-|_N \to 0$ as $\varepsilon \downarrow 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^\mathbb{N}$). So in the limit as $\varepsilon \downarrow 0$, we obtain

$$0 \leqslant \big\langle A(x) - u, h\big\rangle \quad \text{for all } h \in W^{1,p}_0(Z) \quad \Rightarrow \quad A(x) = u.$$

Next suppose that the second option in hypothesis $H(j)_2$(vi) is valid, namely for almost all $z \in Z$, all $x \geqslant M$ and all $u \in \partial j(z,x)$, we have $u \leqslant 0$. In this case we have

$$-\int_{Z^\varepsilon_-} u(x+\varepsilon)\, dz = -\int_{Z^\varepsilon_- \cap \{x<M\}} u(x+\varepsilon h)\, dz - \int_{Z^\varepsilon_- \cap \{x \geqslant M\}} u(x+\varepsilon h)\, dz$$
$$\geqslant -\int_{\hat{Z}^\varepsilon_- \cap \{x<M\}} u(x+\varepsilon h)\, dz - \int_{Z^\varepsilon_- \cap \{x \geqslant M\}} u(x+\varepsilon h)\, dz$$
$$\big(\text{since } \partial j(z,0) \subseteq \mathbb{R}_+ \text{ a.e. on } Z\big)$$
$$\geqslant \varepsilon\beta_2 \int_{\hat{Z}^\varepsilon_- \cap \{x<M\}} h\, dz - \varepsilon \int_{Z^\varepsilon_- \cap \{x \geqslant M\}} uh\, dz$$
$$\text{for some } \beta_2 > 0 \ \big(\text{recall } x \geqslant 0 \text{ and see } H(j)_2\text{(iii)}\big). \tag{4.10}$$

Using (4.5) and (4.10) in (4.4), we have

$$-\langle A(x)-u,(x+\varepsilon h)^-\rangle \geqslant \varepsilon \int_{\hat{Z}_-^\varepsilon} \|Dx\|^{p-2}(Dx,Dh)_{\mathbb{R}^N}\,dz + \varepsilon\beta_2 \int_{\hat{Z}_-^\varepsilon\cap\{x<M\}} h\,dz - \varepsilon \int_{\hat{Z}_-^\varepsilon\cap\{x\geqslant M\}} uh\,dz. \tag{4.11}$$

Combining (4.3) and (4.11) and dividing with $\varepsilon > 0$, we obtain

$$\int_{\hat{Z}_-^\varepsilon} \|Dx\|^{p-2}(Dx,Dh)_{\mathbb{R}^N}\,dz + \beta_2 \int_{\hat{Z}_-^\varepsilon\cap\{x<M\}} h\,dz - \int_{\hat{Z}_-^\varepsilon\cap\{x\geqslant M\}} uh\,dz \leqslant \langle A(x)-u,h\rangle.$$

As before $|\hat{Z}_-^\varepsilon|_N \to 0$ as $\varepsilon \downarrow 0$. So in the limit we have

$$0 \leqslant \langle A(x)-u,h\rangle \quad \text{for all } h\in W_0^{1,p}(Z) \quad \Rightarrow \quad A(x)=u.$$

So in both cases we have

$$A(x)=u \quad \text{for some } u\in S^q_{\partial j(\cdot,x(\cdot))}.$$

From this it follows that

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z,x(z)) & \text{a.e. on Z},\\ x|_{\partial Z}=0, \end{cases}$$

i.e., $x\in W_0^{1,p}(Z)$, $x\geqslant 0$, $x\neq 0$ solves problem (4.1).

By virtue of Theorem 7.1 of Ladyzhenskaya–Uraltseva [9] (see also Gasinski–Papageorgiou [5, p. 115]), we have $x\in L^\infty(Z)$. Then using Theorem 1 of Lieberman [10] (see also Gasinski–Papageorgiou [5, p. 116]), we have that $x\in C_0^1(\bar{Z})$. Because of hypothesis $H(j)_2$(v), we have

$$\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) \leqslant c_0|x(z)|^{p-1} \quad \text{a.e. on } Z.$$

Invoking Theorem 5 of Vazquez [15], we conclude that

$$x(z)>0 \quad \text{for all } z\in Z \quad \text{and} \quad \frac{\partial x}{\partial n}(z)<0 \quad \text{for all } z\in\partial Z. \qquad \square$$

Remark 4.3. If $C_0^1(\bar{Z})_+ = \{x\in C_0^1(\bar{Z})\colon x(z)\geqslant 0\}$ (the positive cone in $C_0^1(\bar{Z})$), then from the properties of $x\in C_0^1(\bar{Z})$ obtained in Theorem 4.2, we have that $x\in \operatorname{int} C_0^1(\bar{Z})_+$.

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References

[1] B. Calvert, Perturbation by Nemytskii operators of $m - T$ accretive operators in L^q, $q \in (1, +\infty)$, Rev. Roumaine Math. Pures Appl. XXII (1977) 883–906.
[2] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Theory, Kluwer Academic/Plenum, New York, 2003.
[3] Z. Denkowski, S. Migórski, N.S. Papageorgiou, An Introduction to Nonlinear Analysis: Applications, Kluwer Academic/Plenum, New York, 2003.
[4] M.E. Filippakis, N.S. Papageorgiou, Solutions for nonlinear variational inequalities with a nonsmooth potential, Abstr. Appl. Anal. 2004 (2004) 635–649.
[5] L. Gasiński, N.S. Papageorgiou, Nonsmooth Critical Point Theory and Nonlinear Boundary Value Problems, CRC Press, Boca Raton, FL, 2005.
[6] D. Goeleven, D. Motreanu, A degree theoretic approach for the study of eigenvalue problems in variational–hemivariational inequalities, Differential Integral Equations 10 (1997) 893–904.
[7] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis. Volume I: Theory, Kluwer Academic, Dordrecht, 1997.
[8] S. Kyritsi, N.S. Papageorgiou, An obstacle problem for nonlinear hemivariational inequalities at resonance, J. Math. Appl. 276 (2002) 292–313.
[9] O. Ladyzhenskaya, N. Uraltseva, Linear and Quasilinear Elliptic Equations, Academic Press, New York, 1968.
[10] G. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 (1988) 1203–1219.
[11] W. Li, H. Zhen, The applications of sums of ranges of accretive operators to nonlinear equations involving the p-Laplacian operator, Nonlinear Anal. 24 (1995) 185–193.
[12] S. Marano, D. Motreanu, Infinitely many critical points for nondifferentiable functions and applications to a Neumann-type problem involving the p-Laplacian, J. Differential Equations 182 (2002) 108–120.
[13] Z. Naniewicz, P. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications, Dekker, New York, 1995.
[14] R.E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, Math. Surveys Monogr., vol. 49, Amer. Math. Soc., Providence, RI, 1997.
[15] J. Vazquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191–202.