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Recurrence relations for the Sheffer sequences

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ABSTRACT

In this paper, using the production matrix of an exponential Riordan array $[g(t), f(t)]$, we give a recurrence relation for the Sheffer sequence for the ordered pair $(g(t), f(t))$. We also develop a new determinant representation for the general term of the Sheffer sequence. As applications, determinant expressions for some classical Sheffer polynomial sequences are derived.

1. Introduction

By a polynomial sequence we shall denote a sequence of polynomials $p_k(x)$, $k = 0, 1, 2, \dots$, where $p_k(x)$ is exactly of degree k for all k. Sequences of polynomials play an important role in mathematics and physics. One of the simplest classes of polynomial sequences is the class of Sheffer sequences that contains relevant sequences such as the Laguerre, the Hermite, the Bernoulli and the Abel polynomials. The systematic study of the class of Sheffer sequences is the object of the modern classical umbral calculus started in the 1970s by Gian-Carlo Rota and his disciples [11].

The notion of Sheffer sequence can be introduced in many ways. In this paper, we will follow the definitions of Rota and Roman, and introduce the concept of Sheffer sequence in terms of formal power series. If the formal power series $g(t)$ has a multiplicative inverse, denoted by $g(t)^{-1}$ or $\frac{1}{g(t)}$, then we

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call $g(t)$ an invertible series. If the series $f(t)$ has a compositional inverse, denoted by $\bar{f}(t)$ and satisfying $\bar{f}(f(t)) = f(\bar{f}(t)) = t$, then we call $f(t)$ a delta series.

Definition 1.1 [11]**.**

(1) Let $g(t)$ be an invertible series and let $f(t)$ be a delta series; we say that the polynomial sequence $s_n(x)$ is the Sheffer sequence for the pair $(g(t), f(t))$ if and only if

$$\sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!} = \frac{1}{g(\bar{f}(t))}e^{x\bar{f}(t)}, \tag{1}$$

where $\bar{f}(t)$ is the compositional inverse of $f(t)$.

(2) The Sheffer sequence for $(1, f(t))$ is the associated sequence for $f(t)$. If $p_n(x)$ is associated to $f(t)$, then $\sum_{n=0}^{\infty} p_n(x)\frac{t^n}{n!} = e^{x\bar{f}(t)}$.

(3) The Sheffer sequence for $(g(t), t)$ is the Appell sequence for $g(t)$. If $a_n(x)$ is Appell for $g(t)$, then $\sum_{n=0}^{\infty} a_n(x)\frac{t^n}{n!} = \frac{e^{xt}}{g(t)}$.

Besides the generating function, there are several other ways to characterize Sheffer sequences. To complement we list the following algebraic ones, the proofs can be found in [11,12].

Theorem 1.2.

(1) A polynomial sequence $p_n(x)$ is the associated sequence for a delta series $f(t)$ if and only if it is of binomial type, that is, if and only if it satisfies the identity

$$p_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}p_k(x)p_{n-k}(y).$$

(2) A polynomial sequence $s_n(x)$ is Sheffer for $(g(t), f(t))$, for some invertible $g(t)$, if and only if

$$s_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}p_k(x)s_{n-k}(y),$$

where $p_n(x)$ is the associated sequence for $f(t)$.

(3) A polynomial sequence $a_n(x)$ is Appell for some invertible $g(t)$, if and only if

$$a_n'(x) = na_{n-1}(x), \quad n = 1, 2, \ldots$$

In a recent work [3], a new definition by means of a determinant form for Appell polynomials is given. In [7], Luzón introduced a new notation $T(f|g)$ to represent the Riordan arrays and gave a recurrence relation for the polynomial sequences associated to Riordan arrays. In this paper, using the production matrix of an exponential Riordan array, we give a recurrence relation for the Sheffer polynomial sequence. A determinant representation for the Sheffer polynomial sequence is obtained. In fact, we show that the general formula for the sequence can be expressed as the characteristic polynomial of the principal submatrix of the production matrix.

2. Exponential Riordan array

Shapiro et al. [13] introduced the concept of a Riordan array in 1991, then the concept is generalized to the exponential Riordan array by many authors [1,5,6,8,10,14,17]. The connection between the Riordan arrays and the Sheffer sequences has already been pointed out by Shapiro et al. [13] and Sprugnoli [15,16]. The exponential Riordan group is a set of infinite lower-triangular integer matrices, where each matrix is defined by a pair of exponential generating functions $g(t) = 1 + \sum_{n=1}^{\infty} g_n \frac{t^n}{n!}$ and $f(t) = t + \sum_{n=2}^{\infty} f_n \frac{t^n}{n!}$ with $f_0 = 0$ and $g_0 = f_1 = 1$. The associated matrix is the matrix whose jth column has exponential generating function $\frac{1}{j!} g(x) f(x)^j$ (the first column being indexed by 0). The matrix corresponding to the pair $f(t), g(t)$ is denoted by $[g(t), f(t)]$. For example, the exponential Riordan array $[e^t, t]$ corresponds to the Pascal triangle $A = \left(\binom{i}{j}\right)_{i,j \geq 0}$.

The group law is then given by $[g(t), f(t)][h(t), l(t)] = [g(t)h(f(t)), l(f(t))]$. The identity for this law is $I = [1, t]$ and the inverse of $[g(t), f(t)]$ is $[\frac{1}{g(\bar{f}(t))}, \bar{f}(t)]$, where $\bar{f}(t)$ is compositional inverse of $f(t)$. Let $\mathbf{b} = (b_0, b_1, b_2, \ldots)^T$ be a real sequence with exponential generating function $B(t)$, then the sequence $[g(t), f(t)]\mathbf{b}$ has exponential generating function $g(t)B(f(t))$, i.e.

$$[g(t), f(t)]B(t) = g(t)B(f(t)). \tag{2}$$

For example, the exponential Riordan array $[1, e^t - 1]$ corresponds to the Stirling matrix of the second kind, while the exponential Riordan array $[1, \log(1 + t)]$ is the Stirling matrix of the first kind.

Lemma 2.1 [1,5,8]. *Let $A = (a_{n,k})_{n,k \geqslant 0} = [g(t), f(t)]$ be an exponential Riordan array and let*

$$c(t) = c_0 + c_1 t + c_2 t^2 + \cdots, \quad r(t) = r_0 + r_1 t + r_2 t^2 + \cdots \tag{3}$$

be two formal power series such that $r(f(t)) = f'(t)$, $c(f(t)) = g'(t)/g(t)$. Then

$$a_{n+1,k} = \frac{1}{k!} \sum_{i \geqslant k-1} i!(c_{i-k} + k r_{i-k+1}) a_{n,i}, \tag{4}$$

where defining c_{-1}=0. Conversely, starting from the sequences defined by (3), the infinite array $(a_{n,k})_{n,k \geqslant 0}$ defined by (4) is an exponential Riordan array.

For an invertible lower triangular matrix R, its production matrix (also called its Stieltjes matrix, see [4,10]) is the matrix $P = R^{-1}\overline{R}$, where $\overline{R}$ is the matrix R with its first row removed. The production matrix P can be characterized by the matrix equality $RP = DR$, where $D = (\delta_{i+1,j})_{i,j \geqslant 0}$ (δ is the usual Kronecker delta). It is not hard to see that the production matrix $P = (p_{i,j})_{i,j \geqslant 0}$ is Hessenberg, i.e., $p_{i,j} = 0$ when $j > i + 1$.

A consequence of Lemma 2.1 is that the production matrix of $A = [g(t), f(t)]$ is $P = (p_{i,j})_{i,j \geqslant 0}$, where

$$p_{i,j} = \frac{i!}{j!}(c_{i-j} + j r_{i-j+1}),$$

$$P = \begin{pmatrix} c_0 & r_0 & 0 & 0 & 0 & 0 & \cdots \\ 1!c_1 & \frac{1!}{1!}(c_0 + r_1) & r_0 & 0 & 0 & 0 & \cdots \\ 2!c_2 & \frac{2!}{1!}(c_1 + r_2) & \frac{2!}{2!}(c_0 + 2r_1) & r_0 & 0 & 0 & \cdots \\ 3!c_3 & \frac{3!}{1!}(c_2 + r_3) & \frac{3!}{2!}(c_1 + 2r_2) & \frac{3!}{3!}(c_0 + 3r_1) & r_0 & 0 & \cdots \\ 4!c_4 & \frac{4!}{1!}(c_3 + r_4) & \frac{4!}{2!}(c_2 + 2r_3) & \frac{4!}{3!}(c_3 + 3r_2) & \frac{4!}{4!}(c_0 + 4r_1) & r_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Furthermore, the bivariate exponential generating function $\phi_P(t, z) = \sum_{n,k} p_{n,k} t^k \frac{z^n}{n!}$ of the matrix P is given by $\phi_P(t, z) = e^{tz}(c(z) + t r(z))$. Note in particular that we have $r(t) = f'(\bar{f}(t))$, $c(t) = \frac{g'(\bar{f}(t))}{g(\bar{f}(t))}$. The sequences $(c_n)_{n \geqslant 0}$ and $(r_n)_{n \geqslant 0}$ are called respectively the c-sequence and the r-sequence of exponential Riordan array A. Since we require $f(t) = t + \sum_{n=2}^{\infty} f_n \frac{t^n}{n!}$ with $f_0 = 0$ and $f_1 = 1$, we have $r_0 = 1$.

Lemma 2.2 [6,17]**.** *If $s_n(x) = \sum_{k=0}^{n} s_{n,k}x^k$ is Sheffer for $(g(t), f(t))$, then the coefficients $s_{n,k}$ are the elements of the exponential Riordan array $\left[\frac{1}{g(\bar{f}(t))}, \bar{f}(t)\right]$. If $x^n = \sum_{k=0}^{n} b_{n,k}s_k(x)$, then $b_{n,k}$ are the elements of the exponential Riordan array $[g(t), f(t)]$.*

Theorem 2.3. *Let B be an invertible lower triangular matrix with production matrix $P = (p_{n,k})_{n,k\geqslant 0}$. Let $B^{-1} = A = (a_{n,k})$ and $a_n(x) = \sum_{k=0}^{n} a_{n,k}x^k$. Then $(a_n(x))_{n\geqslant 0}$ satisfies the recurrence relation of the form:*

$$p_{n,n+1}a_{n+1}(x) = -(p_{n,n} - x)a_n(x) - p_{n,n-1}a_{n-1}(x) - \cdots - p_{n,1}a_1(x) - p_{n,0}a_0(x),$$

with initial condition $a_0(x) = a_{0,0}$ and $p_{0,1}a_1(x) = x - p_{0,0}$.

For $n \geqslant 0$, $a_{n+1}(x)$ is also given by the following determinant formula:

$$\begin{aligned} a_{n+1}(x) &= \frac{(-1)^{n+1}}{p_{0,1}p_{1,2}\cdots p_{n,n+1}} \begin{vmatrix} p_{0,0}-x & p_{0,1} & 0 & \cdots & 0 \\ p_{0,1} & p_{1,1}-x & p_{1,2} & \cdots & 0 \\ p_{0,2} & p_{2,1} & p_{2,2}-x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n,0} & p_{n,1} & p_{n,2} & \cdots & p_{n,n}-x \end{vmatrix} \\ &= \frac{(-1)^{n+1}}{p_{0,1}p_{1,2}\cdots p_{n,n+1}} |P_{n+1} - xI_{n+1}| \\ &= \frac{1}{p_{0,1}p_{1,2}\cdots p_{n,n+1}} |xI_{n+1} - P_{n+1}|. \end{aligned}$$

Proof. Let $E = (1, x, x^2, \ldots)^T$, then $DE = x(1, x, x^2, \ldots)^T$, and $A^{-1}E = (a_0(x), a_1(x), a_2(x), \ldots)^T$, where $D = (\delta_{i+1,j})_{i,j\geqslant 0}$.

Let P be production matrix of A, then $AP = DA$, and $PA^{-1} = A^{-1}D$. Thus $PA^{-1}E = A^{-1}DE$, and $PA^{-1}E = xA^{-1}E$. In matrix form, we have

$$\begin{pmatrix} p_{0,0} & p_{0,1} & 0 & 0 & 0 & \cdots \\ p_{1,0} & p_{1,1} & p_{1,2} & 0 & 0 & \cdots \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} & 0 & \cdots \\ p_{3,0} & p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} a_0(x) \\ a_1(x) \\ a_2(x) \\ a_3(x) \\ \vdots \end{pmatrix} = x \begin{pmatrix} a_0(x) \\ a_1(x) \\ a_2(x) \\ a_3(x) \\ \vdots \end{pmatrix}.$$

$$a_0(x)\begin{pmatrix} p_{0,0} \\ p_{1,0} \\ p_{2,0} \\ p_{3,0} \\ \vdots \end{pmatrix} + \begin{pmatrix} p_{0,1} & 0 & 0 & 0 & \cdots \\ p_{1,1} & p_{1,2} & 0 & 0 & \cdots \\ p_{2,1} & p_{2,2} & p_{2,3} & 0 & \cdots \\ p_{3,1} & p_{3,2} & p_{3,3} & p_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} a_1(x) \\ a_2(x) \\ a_3(x) \\ a_4(x) \\ \vdots \end{pmatrix} = \begin{pmatrix} xa_0(x) \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} + x\begin{pmatrix} 0 \\ a_1(x) \\ a_2(x) \\ a_3(x) \\ \vdots \end{pmatrix}.$$

Since $x\begin{pmatrix} 0 \\ a_1(x) \\ a_2(x) \\ a_3(x) \\ \vdots \end{pmatrix} = x\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} a_1(x) \\ a_2(x) \\ a_3(x) \\ a_4(x) \\ \vdots \end{pmatrix}$, the above matrix equation can be rewritten as

$$\begin{pmatrix} p_{0,1} & 0 & 0 & 0 & \cdots \\ p_{1,1}-x & p_{1,2} & 0 & 0 & \cdots \\ p_{2,1} & p_{2,2}-x & p_{2,3} & 0 & \cdots \\ p_{3,1} & p_{3,2} & p_{3,3}-x & p_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}\begin{pmatrix} a_1(x) \\ a_2(x) \\ a_3(x) \\ a_4(x) \\ \vdots \end{pmatrix} = a_0(x)\begin{pmatrix} x-p_{0,0} \\ -p_{1,0} \\ -p_{2,0} \\ -p_{3,0} \\ \vdots \end{pmatrix}.$$

Therefore, $p_{0,1}a_1(x) = x - p_{0,0}$, and for $n \geqslant 1$, we have

$$p_{n,1}a_1(x) + \cdots + p_{n,n-1}a_{n-1}(x) + (p_{n,n}-x)a_n(x) + p_{n,n+1}a_{n+1}(x) = -p_{n,0}a_0(x),$$

or equivalently

$$p_{n,n+1}a_{n+1}(x) = -(p_{n,n}-x)a_n(x) - p_{n,n-1}a_{n-1}(x) - \cdots - p_{n,1}a_1(x) - p_{n,0}a_0(x).$$

Considering the first $n+1$ equations in $n+1$ variables $a_k(x)$, $1 \leqslant k \leqslant n+1$, the determinant of the coefficients is triangular so that its value is $p_{0,1}p_{1,2}\cdots p_{n,n+1}$. Solving the equations by Cramer's formula, we have

$$a_{n+1}(x) = \frac{1}{p_{0,1}p_{1,2}\cdots p_{n,n+1}}\begin{vmatrix} p_{0,1} & 0 & \cdots & 0 & x-p_{0,0} \\ p_{1,1}-x & p_{1,2} & \cdots & 0 & -p_{0,1} \\ p_{2,1} & p_{2,2}-x & \cdots & 0 & -p_{0,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,n}-x & -p_{n,0} \end{vmatrix}$$

$$= \frac{(-1)^n}{p_{0,1}p_{1,2}\cdots p_{n,n+1}}\begin{vmatrix} x-p_{0,0} & p_{0,1} & 0 & \cdots & 0 \\ -p_{0,1} & p_{1,1}-x & p_{1,2} & \cdots & 0 \\ -p_{0,2} & p_{2,1} & p_{2,2}-x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_{n,0} & p_{n,1} & p_{n,2} & \cdots & p_{n,n}-x \end{vmatrix}.$$

$$= \frac{(-1)^{n+1}}{p_{0,1}p_{1,2}\cdots p_{n,n+1}} \begin{vmatrix} p_{0,0}-x & p_{0,1} & 0 & \cdots & 0 \\ p_{0,1} & p_{1,1}-x & p_{1,2} & \cdots & 0 \\ p_{0,2} & p_{2,1} & p_{2,2}-x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{n,0} & p_{n,1} & p_{n,2} & \cdots & p_{n,n}-x \end{vmatrix}. \quad \square$$

Theorem 2.4. *Let $[g(t), f(t)]$ be an exponential Riordan array with the* c*-sequences $(c_i)_{i\geqslant 0}$ and* r*-sequences $(r_i)_{i\geqslant 0}$. Let $(a_n(x))_{n\geqslant 0}$ be the Sheffer polynomial sequence for $(g(t), f(t))$. Then $(a_n(x))_{n\geqslant 0}$ satisfies the recurrence relation of the form:*

$$\begin{aligned} a_{n+1}(x) &= (x - c_0 - nr_1)a_n(x) - \frac{n!}{(n-1)!}(c_1 + (n-1)r_2)a_{n-1}(x) \\ &\quad - \cdots - \frac{n!}{2!}(c_{n-2} + 2r_{n-1})a_2(x) - \frac{n!}{1!}(c_{n-1} + r_n)a_1(x) - n!c_n a_0(x), \end{aligned}$$

with initial condition $a_0(x) = 1$ and $a_1(x) = x - c_0$. For $n \geqslant 0$, $a_{n+1}(x)$ is given by the following determinant formula:

$$\begin{aligned} a_{n+1}(x) &= (-1)^{n+1} \begin{vmatrix} c_0 - x & 1 & 0 & \cdots & 0 \\ 1!c_1 & \frac{1!}{1!}(c_0 + r_1) - x & 1 & \cdots & 0 \\ 2!c_2 & \frac{2!}{1!}(c_1 + r_2) & \frac{2!}{2!}(c_0 + 2r_1) - x & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n!c_n & \frac{n!}{1!}(c_{n-1} + r_n) & \frac{n!}{2!}(c_{n-2} + 2r_{n-1}) & \cdots & \frac{n!}{n!}(c_0 + nr_1) - x \end{vmatrix} \\ &= (-1)^{n+1}|P_{n+1} - xI_{n+1}| = |xI_{n+1} - P_{n+1}|. \end{aligned}$$

Proof. Taking into account $r_0 = 1$ and using Lemma 2.1, the results follow from Theorem 2.3. $\square$

Theorem 2.5. *Let the* r*-sequence of $[1, f(t)]$ be $(r_i)_{i\geqslant 0}$, and let $(a_n(x))_{n\geqslant 0}$ be the associated polynomial sequence to $f(t)$. Then $(a_n(x))_{n\geqslant 0}$ satisfies the recurrence relation of the form:*

$$a_{n+1}(x) = (x - nr_1)a_n(x) - \frac{n!}{(n-2)!}r_2 a_{n-1}(x) - \cdots - \frac{n!}{1!}r_{n-1}a_2(x) - \frac{n!}{0!}r_n a_1(x),$$

with initial condition $a_0(x) = 1$, and $a_1(x) = x$. For $n \geqslant 0$, $a_{n+1}(x)$ is given by the following determinant formula:

$$a_{n+1}(x) = \begin{vmatrix} x & -1 & 0 & \cdots & 0 \\ 0 & x - \frac{1!}{0!}r_1 & -1 & \cdots & 0 \\ 0 & -\frac{2!}{0!}r_2 & x - \frac{2!}{1!}r_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -\frac{n!}{0!}r_n & -\frac{n!}{1!}r_{n-1} & \cdots & x - \frac{n!}{(n-1)!}r_1 \end{vmatrix}.$$

Proof. Since $g(t) = 1$, we have $c(t) = 0$. Let the r-sequence of $[1, f(t)]$ be $(r_i)_{i \geqslant 0}$ with $r_0 = 1$, then

$$p_{i,j} = \begin{cases} \frac{i!}{(j-1)!} \cdot r_{i-j+1}, & \text{if } 1 \leqslant j \leqslant i+1, \\ 0, & \text{otherwise.} \end{cases}$$

The results follow from Lemma 2.1 and Theorem 2.4. □

If $f(t) = t$, then $r(t) = 1$. Let the c-sequence of $[g(t), t]$ be $(c_i)_{i \geqslant 0}$, then

$$p_{i,j} = \begin{cases} \frac{i!}{j!} \cdot c_{i-j}, & \text{if } 0 \leqslant j \leqslant i, \\ 1, & \text{if } j = i+1, \\ 0, & \text{otherwise,} \end{cases}$$

$$P = \begin{pmatrix} c_0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1!c_1 & \frac{1!}{1!}c_0 & 1 & 0 & 0 & 0 & \cdots \\ 2!c_2 & \frac{2!}{1!}c_1 & \frac{2!}{2!}c_0 & 1 & 0 & 0 & \cdots \\ 3!c_3 & \frac{3!}{1!}c_2 & \frac{3!}{2!}c_1 & \frac{3!}{3!}c_0 & 1 & 0 & \cdots \\ 4!c_4 & \frac{4!}{1!}c_3 & \frac{4!}{2!}c_2 & \frac{4!}{3!}c_3 & \frac{4!}{4!}c_0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence, we have following result.

Theorem 2.6. *Let the c-sequence of $[g(t), t]$ be $(c_i)_{i \geqslant 0}$, and let $(a_n(x))_{n \geqslant 0}$ be the Appell polynomial sequence for $g(t)$. Then $(a_n(x))_{n \geqslant 0}$ satisfies the recurrence relation of the form:*

$$a_{n+1}(x) = (x - c_0)a_n(x) - \frac{n!}{(n-1)!}c_1 a_{n-1}(x) - \cdots - \frac{n!}{2!}c_{n-2}a_2(x) - \frac{n!}{1!}c_{n-1}a_1(x) - n!c_n a_0(x),$$

with initial condition $a_0(x) = 1$, and $a_1(x) = x - c_0$. For $n \geqslant 0$, $a_{n+1}(x)$ is given by the following determinant formula:

$$a_{n+1}(x) = \begin{vmatrix} x - c_0 & -1 & 0 & \cdots & 0 \\ -1!c_1 & x - c_0 & -1 & \cdots & 0 \\ -2!c_2 & -\frac{2!}{1!}c_1 & x - c_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -n!c_n & -\frac{n!}{1!}c_{n-1} & -\frac{n!}{2!}c_{n-2} & \cdots & x - c_0 \end{vmatrix}.$$

3. Examples

Example 1. The exponential Riordan array $A = [e^{\frac{vt^2}{2}}, t]$ has general term

$$a_{n,k} = \begin{cases} \frac{n!}{k!} \frac{(\frac{v}{2})^{\frac{n-k}{2}}}{(\frac{n-k}{2})!}, & \text{if } n \geqslant k \text{ and } n-k \text{ even,} \\ 0, & \text{otherwise,} \end{cases}$$

and the generating functions for its c-sequence and r-sequence are $c(t) = vt$ and $r(t) = 1$ respectively.

Let $(a_n(x))_{n\geqslant 0}$ be the Appell sequence for $g(t) = e^{\frac{vt^2}{2}}$. Then $a_0(x) = 1$, $a_1(x) = x$, and for $n \geqslant 1$

$$a_{n+1}(x) = xa_n(x) - nva_{n-1}(x),$$

with the general formula

$$a_{n+1}(x) = (-1)^{n+1}\begin{vmatrix} -x & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ v & -x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2v & -x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3v & -x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & nv & -x \end{vmatrix}.$$

For $n = 2$, we have $a_3(x) = (-1)^3\begin{vmatrix} -x & 1 & 0 \\ v & -x & 1 \\ 0 & 2v & -x \end{vmatrix} = -3vx+x^3$. When $v = 2$, we have $a_n(2x) = H_n(x)$ are the Hermite polynomials (see [2]).

Example 2. The exponential Riordan array $A = [1, \frac{t}{1-t}]$ has general term

$$a_{n,k} = \begin{cases} \frac{n!}{k!}\binom{n-1}{k-1}, & \text{if } n \geqslant k \geqslant 0, \\ 0, & \text{otherwise,} \end{cases}$$

and the generating functions for its c-sequence and r-sequence are $c(t) = 0$ and $r(t) = (1+t)^2$ respectively.

Let $(a_n(x))_{n\geqslant 0}$ be the associated sequence for $f(t) = \frac{t}{1-t}$. Then $a_0(x) = 1$, $a_1(x) = x$, and for $n \geqslant 1$

$$a_{n+1}(x) = (x-2n)a_n(x) - n(n-1)a_{n-1}(x).$$

In fact, for $n \geqslant 1$, we have $a_n(x) = \sum_{k=1}^{n}(-1)^{n-k}\frac{n!}{k!}\binom{n-1}{k-1}x^k$.

The determinant formula is

$$a_{n+1}(x) = (-1)^n\begin{vmatrix} x & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2-x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 4-x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 6 & 6-x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & n(n-1) & 2n-x \end{vmatrix}.$$

For $n = 2$, we have $a_3(x) = (-1)^2\begin{vmatrix} x & 1 & 0 \\ 0 & 2-x & 1 \\ 0 & 2 & 4-x \end{vmatrix} = 6x - 6x^2 + x^3$.

Example 3. Considering the exponential Riordan array $A = [(\cosh t)^y, \tanh t]$, we have $A^{-1} = \left[(1-t^2)^{\frac{y}{2}}, \ln\sqrt{\frac{1+t}{1-t}}\,\right]$. From Lemma 2.1 we obtain $r(t) = 1 - t^2$ and $c(t) = yt$. It follows at once that the general entry of production matrix of P

$$p_{i,j} = \begin{cases} i(y-i+1), & \text{if } j = i-1, \\ 0, & \text{if } j = i, \\ 1, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

The first rows of P are

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ y & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 2(y-1) & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 3(y-2) & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 4(y-3) & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let $(a_n(x))_{n\geqslant 0}$ be the Sheffer sequence for $((\cosh t)^y, \tanh t)$. Then $a_0(x) = 1$, $a_1(x) = x$, and for $n \geqslant 1$

$$a_{n+1}(x) = xa_n(x) - n(y-n+1)a_{n-1}(x).$$

The determinant formula is

$$a_n(x) = (-1)^n \begin{vmatrix} -x & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ y & -x & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2(y-1) & -x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 3(y-2) & -x & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -x & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & (n-1)(y-n+2) & -x \end{vmatrix}.$$

Note that in this example $a_n(x)$ are the Cayley continuants of order n (see [9]).

Example 4. Considering the exponential Riordan array $A = [1 - t, t - \frac{t^2}{2}]$. Its inverse is $A^{-1} = [\frac{1}{\sqrt{1-2t}}, 1 - \sqrt{1-2t}\,]$, which corresponds to the Bessel matrix of the first kind

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 3 & 3 & 1 & 0 & 0 & \cdots \\ 15 & 15 & 6 & 1 & 0 & \cdots \\ 105 & 105 & 45 & 10 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The generating function of the r-sequence of exponential Riordan array $A = [1 - t, t - \frac{t^2}{2}]$ is $r(t) = \sqrt{1-2t}$, and the generating function of its c-sequence is $c(t) = -\frac{1}{\sqrt{1-2t}}$. The product matrix of A begins

$$P = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ -1 & -2 & 1 & 0 & 0 & \cdots \\ -3 & -3 & -3 & 1 & 0 & \cdots \\ -15 & -12 & -6 & -4 & 1 & \cdots \\ -105 & -75 & -30 & -10 & -5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

whose general entry is

$$p_{n,k} = \begin{cases} \frac{(n+1)!}{k!(n-k+1)2^{n-k}}\binom{2n-2k}{n-k}, & \text{if } 0 \leqslant k \leqslant n, \\ 1, & \text{if } k = n+1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $(a_n(x))_{n\geqslant 0}$ be the Sheffer sequence for $(1 - t, t - \frac{t^2}{2})$. Then $a_0(x) = 1$, and for $n \geqslant 1$, $a_n(x) = |xI_n - P_n|$, where P_n is the nth principal submatrix of the production matrix P. For $n = 3$, we have

$$a_3(x) = |xI_3 - P_3| = \begin{vmatrix} x+1 & -1 & 0 \\ 1 & x+2 & -1 \\ 3 & 3 & x+3 \end{vmatrix} = 15 + 15x + 6x^2 + x^3.$$

Note that in this example $a_n(x)$ are the Bessel polynomials with exponents in decreasing order (see [5]).

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