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Uniqueness and value distribution of differences of entire functions ☆

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ABSTRACT

We consider the existence of transcendental entire solutions of certain type of non-linear difference equations. As an application, we investigate the value distribution of difference polynomials of entire functions. In particular, we are interested in the existence of zeros of $f^n(z)(\lambda f^m(z+c)+\mu f^m(z))-a$, where f is an entire function, n, m are two integers such that $n \geqslant m > 0$, and λ, μ are non-zero complex numbers. We also obtain a uniqueness result in the case where shifts of two entire functions share a small function.

1. Introduction

A meromorphic function means meromorphic in the whole complex plane. We say that two meromorphic functions f and g share a finite value a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities). We assume that the reader is familiar with the standard symbols and fundamental results of Nevanlinna theory, as found in [7,18]. We use $\sigma(f)$ to denote the order of f and $N_p(r, \frac{1}{f-a})$ to denote the counting function of the zeros of $f-a$, where an m-fold zero is counted m times if $m \leqslant p$ and p times if $m > p$. For a small function a related to f, we define

$$\delta(a, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)}.$$

Recently, Yang and Laine [19] considered the existence of the solutions of a non-linear differential-difference equation of the form

$$f^n + L(z, f) = h, \tag{1}$$

where $L(z, f)$ is a linear differential-difference polynomial in f. They obtained the following result.

Theorem A. *(See [19, Theorem 3.4].) Let P, Q be polynomials. Then a non-linear difference equation*

$$f(z)^2 + P(z)f(z+1) = Q(z)$$

has no transcendental entire solution of finite order.

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Theorem B. *(See [19, Theorem 3.5].) A non-linear difference equation*

$$f(z)^3 + P(z)f(z+1) = c\sin bz \tag{2}$$

where $P(z)$ is a non-constant polynomial and $b, c \in \mathbb{C}$ are non-zero constants, does not admit entire solutions of finite order. If $P(z) = p$ is a non-zero constants, then (2) possesses three distinct entire solution of finite order, provided $b = 3n\pi$ and $p^3 = (-1)^{n+1}\frac{27}{4}c^2$ for a non-zero integer n.

Laine and Yang [9] continued to consider the non-existence of transcendental entire solutions of non-linear differential equations of type

$$f^n + P_d(f) = p_1 e^{a_1 z} + p_2 e^{a_2 z}, \tag{3}$$

and obtained new results which complementing the theorems given by Li and Yang [10,11].

Theorem C. *(See [9, Theorem 3.1].) Let $n \geqslant 3$ be an integer and $P_d(f)$ be a differential polynomial in f of total degree $d \leqslant n-2$ with polynomial coefficients such that $P_d(0) = 0$. Provided that p_1, p_2 are non-vanishing polynomials and a_1, a_2 are distinct non-zero complex constants, then (3) has no transcendental entire solutions.*

Laine and Yang [9] pointed out that a similar conclusion could be proved if the differential polynomial $P_d(f)$ is replaced with a differential-difference polynomial. However, Theorems A, B and C, the degree of the differential-difference polynomial is less than n. Now, we consider the equal-case, we get the following results.

Theorem 1. *Let a, c be non-zero constants, n and m be integers satisfying $n \geqslant m > 0$, $\lambda \neq 0$ be a complex number and let $P(z)$, $Q(z)$ be polynomials. If $n \geqslant 2$, then the difference equation*

$$f(z)^{n+m} + \lambda f(z)^n f(z+c)^m = P(z)e^{Q(z)} + a \tag{4}$$

has no transcendental entire solutions of finite order.

Remark. It seems to us that replacing $f(z)^n f(z+c)^m$ with $f(z)^n \sum_{j=1}^m f(z+c_j)$ and $c_j \neq 0$, or replacing the non-zero value a with $a(z) \not\equiv 0$, where $a(z)$ is a polynomial in z, the same conclusion can be proved.

Let f be a transcendental meromorphic function, and let n be a positive integer. Reminiscent to the value distributions of $f^n f'$, Hayman [5, Corollary to Theorem 9] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geqslant 3$. Mues [15, Satz 3] proved that $f^2 f' - 1$ has infinitely many zeros. Later on, Bergweiler and Eremenko [1, Theorem 2] showed that $ff' - 1$ has infinitely many zeros also. Corresponding to the results above, Laine and Yang [8, Theorem 2] investigated the value distribution of difference products of entire functions.

Theorem D. *(See [8, Theorem 2].) Let f be a transcendental entire function with finite order, and let c be a non-zero complex constant. Then, for $n \geqslant 2$, $f(z)^n f(z+c)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often.*

Some improvements of Theorem D can be found in [12,13]. In the present paper, we consider the value distribution of $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$, where n, m are non-negative integers, and λ, μ are non-zero complex numbers. We obtain the following result which generalize some theorems in [8,12,13].

Theorem 2. *Let f be a transcendental entire function with finite order, c be a non-zero constant, n and m be integers satisfying $n \geqslant m > 0$, and let λ, μ be two complex numbers such that $|\lambda| + |\mu| \neq 0$. If $n \geqslant 2$, then either $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often or $f(z) = e^{\frac{\log t}{c}z}g(z)$, where $t = (-\frac{\mu}{\lambda})^{\frac{1}{m}}$, and $g(z)$ is periodic function with period c.*

Remarks. (1) If $m = 0$ and $\lambda + \mu \neq 0$, then $(\lambda + \mu)f^n$ assumes every non-zero value $a \in \mathbf{C}$ infinitely often provided that $n \geqslant 2$.

(2) It seems to us that replacing the non-zero value $a \in \mathbf{C}$ with $a(z) \not\equiv 0$, where $a(z)$ is a polynomial in z, a similar conclusion can be proved.

(3) When $m > n > 0$. If $\lambda\mu = 0$ and $|\lambda| + |\mu| \neq 0$, $m \geqslant 2$, then Theorem 2 holds. Unfortunately, when $\lambda\mu \neq 0$, $m \geqslant 2$, we do not know whether Theorem 2 holds.

(4) When $\lambda\mu \neq 0$, $m = 1$ and $n = 0$, we can give a counterexample. Namely, let $f(z) = z + e^z$, $\lambda = 1$ and $\mu = -1$. Then $f(z+c) - f(z) = c$, where $c = 2\pi i$. Clearly, $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m)$ cannot assume every non-zero value $a \in \mathbf{C}$ infinitely often.

Corresponding to Theorem 2, we consider the value distribution of $f(z)^n + \mu f(z+c)^m$, where $m \neq n$.

Theorem 3. *Let f be a transcendental entire function with finite order, μ and c be non-zero constants, and let $a(z)$ be a non-zero small function to f. Suppose that n, m are positive integers such that $n > m+1$ (or $m > n+1$). Then the difference polynomial $f(z)^n + \mu f(z+c)^m - a(z)$ has infinitely many zeros.*

Remark. Theorem 3 is not true, if $n = m+1$ (or $m = n+1$). For example, if $m = 1$, $f(z) = e^z + 1$, $c = 2\pi i$ and $\mu = -2$, then $f(z)^2 - 2f(z+c) + 1 = e^{2z}$ has no zeros.

The following result is a partial answer as to what may happen if $n = m$ in Theorem 3.

Theorem 4. *Let f be an entire function with order $1 \leqslant \sigma(f) < \infty$, and suppose that f has infinitely many zeros with the exponent of convergence of zeros $\lambda(f) < 1$. Let μ, a and c be non-zero constants such that $f(z) + \mu f(z+c) \not\equiv 0$. Then the difference polynomial $f(z) + \mu f(z+c) - a$ has infinitely many zeros.*

Set $F(z) = f(z)^n$. Then $F(z+c) = f(z+c)^n$, $\sigma(F) = \sigma(f)$ and $\lambda(F) = \lambda(f)$, so we obtain

Corollary. *Let all conditions of Theorem 4 hold, and let n be a positive integer such that $f(z)^n + \mu f(z+c)^n \not\equiv 0$. Then the difference polynomial $f(z)^n + \mu f(z+c)^n - a$ has infinitely many zeros.*

In 1976, Yang [17] proposed the following problem.

Suppose that f and g are two transcendental entire functions such that f and g share 0 CM and f' and g' share 1 CM. What can be said about the relationship between f and g?

Shibazaki [16] proved the following result.

Theorem E. *Suppose that f and g are entire functions of finite order such that f' and g' share 1 CM. If $\delta(0, f) > 0$ and 0 is a Picard value of g, then either $f \equiv g$ or $f'g' \equiv 1$.*

The following result can be seen as a difference counterpart to Theorem E.

Theorem 5. *Suppose that f and g are two entire functions of finite order, and let a and b be distinct small functions related to f and g such that $\delta(a) = \delta(a, f) + \delta(a, g) > 1$. If $f(z+c_1)$ and $g(z+c_2)$ share b CM, then exactly one of the following assertions holds.*

(i) *$f(z) \equiv g(z+c)$, where $c = c_2 - c_1$.*
(ii) *$f(z+c_1) = (a-b)e^h + a$, $g(z+c_2) = (a-b)e^{-h} + a$, where $h(z)$ is an entire function.*

2. Some lemmas

The first lemma is a difference analogue of the logarithmic derivative lemma, given by Halburd and Korhonen [4]. Chiang and Feng have obtained similar estimates for the logarithmic difference [2, Corollary 2.5], and this work is independent from [4].

Lemma 1. *(See [4, Theorem 2.1].) Let f be a meromorphic function of finite order, and let $c \in \mathbf{C}$ and $\delta \in (0, 1)$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = o\left(\frac{T(r, f)}{r^\delta}\right) = S(r, f).$$

Lemma 2. *(See [7, Theorem 2.4.2].) Let $f(z)$ be a transcendental meromorphic solution of*

$$f^n A(z, f) = B(z, f),$$

where $A(z, f)$, $B(z, f)$ are differential polynomials in f and its derivatives with small meromorphic coefficients a_λ, in the sense of $m(r, a_\lambda) = S(r, f)$ for all $\lambda \in I$. If $d(B(z, f)) \leqslant n$, then $m(r, A(z, f)) = S(r, f)$.

Lemma 3. *(See [3, Lemma 5.1].) Let f be a finite order meromorphic function, and let c be a non-zero constant. Then for each $\varepsilon > 0$, we have*

$$T\big(r, f(z+c)\big) = T\big(r, f(z)\big) + O\big(r^{\sigma-1+\varepsilon}\big) + O(\log r)$$

and

$$\sigma\big(f(z+c)\big)=\sigma\big(f(z)\big).$$

Lemma 4. *(See [20, Theorem 3.1].) Let $f_j(z)$ $(j=1,2,3)$ be meromorphic functions that satisfy*

$$\sum_{j=1}^{3} f_j(z)\equiv 1.$$

If $f_1(z)$ is not a constant, and

$$\sum_{j=1}^{3} N_2\left(r,\frac{1}{f_j}\right)+\sum_{j=1}^{3}\overline{N}(r,f_j)<\big(\lambda+o(1)\big)T(r),\quad r\in I,$$

where $0\leqslant\lambda<1$, $T(r)=\max_{1\leqslant j\leqslant 3}\{T(r,f_j)\}$, and I has infinite linear measure, then either $f_2(z)\equiv 1$ or $f_3(z)\equiv 1$.

Lemma 5. *(See [18, Theorem 1.51].) Suppose that $f_j(z)$ $(j=1,\dots,n)$ $(n\geqslant 2)$ are meromorphic functions and $g_j(z)$ $(j=1,\dots,n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)}\equiv 0$.
(2) *$1\leqslant j<k\leqslant n$, $g_j(z)-g_k(z)$ are not constants for $1\leqslant j<k\leqslant n$.*
(3) *For $1\leqslant j\leqslant n$, $1\leqslant h<k\leqslant n$,*

$$T(r,f_j)=o\big\{T\big(r,e^{g_h-g_k}\big)\big\},\quad r\to\infty,\ r\notin E,$$

where $E\subset(1,\infty)$ is of finite linear measure.

Then $f_j(z)\equiv 0$.

3. Proof of Theorem 1

Suppose that f is a transcendental entire solution of finite order to Eq. (4). Differentiating (4) and eliminating $e^{Q(z)}$, we get

$$f(z)^{n-m}F(z,f)=-ap^*(z), \tag{5}$$

where

$$F(z,f)=n\lambda P(z)f(z)^{m-1}f'(z)f(z+c)^m+m\lambda P(z)f(z)^m f(z+c)^{m-1}f'(z+c)$$
$$+(m+n)P(z)f(z)^{2m-1}f'(z)-P^*\lambda f(z)^m f(z+c)^m-P^*f(z)^{2m},$$

and $p^*(z)=P'(z)+P(z)Q'(z)$.

We get that $F(z,f)$ cannot vanish identically by repeating the reasoning as [8, Theorem 2]. Set

$$F^*(z,f)=n\frac{\lambda P(z)f(z)^{m-1}f'(z)f(z+c)^m}{f(z)^{2m}}+m\frac{\lambda P(z)f(z)^m f(z+c)^{m-1}f'(z+c)}{f(z)^{2m}}$$
$$+(m+n)\frac{P(z)f(z)^{2m-1}f'(z)}{f(z)^{2m}}-\frac{P^*\lambda f(z)^m f(z+c)^m+P^*f(z)^{2m}}{f(z)^{2m}}. \tag{6}$$

Then from (5), we have

$$f^{n+m}F^*(z,f)=-ap^*(z). \tag{7}$$

Applying Lemmas 1 and 2, we obtain

$$m\big(r,F^*(z,f)\big)=S(r,f)$$

and

$$m\big(r,fF^*(z,f)\big)=S(r,f).$$

From (6) and (7) we know that the poles of $F^*(z,f)$ may be located only at the zeros of $f(z)$. If $F^*(z,f)$ has infinitely many poles, then from that a zero of $f(z)$ with multiplicity t should be a pole of multiplicity $mt+1$ of $F^*(z,f)$. Since $n\geqslant 2$, we know that the left side of (7) must have infinitely many zeros, which is a contradiction to $p^*(z)$ being a polynomial. So we obtain

$$N\big(r, F^*(z, f)\big) = O(\log r)$$

and

$$N\big(r, fF^*(z, f)\big) = O(\log r).$$

Hence

$$T\big(r, F^*(z, f)\big) = S(r, f)$$

and

$$T\big(r, fF^*(z, f)\big) = S(r, f).$$

Therefore

$$T\big(r, f(z)\big) = S(r, f),$$

which is a contradiction.

4. Proof of Theorem 2

Let $a \in \mathbf{C} \setminus \{0\}$ be arbitrary. If $\lambda f(z+c)^m + \mu f(z)^m \equiv 0$, by [14, Satz 19], p. 98, we know that $f(z)$ can be written in the form $f(z) = e^{\frac{\log t}{c} z} g(z)$, where $t = (-\frac{\mu}{\lambda})^{\frac{1}{m}}$, $g(z)$ is a periodic function with period c. Now suppose $\lambda f(z+c)^m + \mu f(z)^m \not\equiv 0$. We consider the following two cases.

Case 1. Suppose $\lambda = 0$. Then $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m) = \mu f(z)^{m+n}$. Let $F(z) = \mu f(z)^{m+n} - a$. By the second main theorem, we get

$$\begin{aligned}(m+n)T(r, f) = T(r, F) + S(r, f) &\leqslant \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F+a}\right) + S(r, F)\\ &\leqslant \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f)\\ &\leqslant T(r, f) + \overline{N}\left(r, \frac{1}{F}\right) + S(r, f).\end{aligned}$$

Since $n \geqslant 2$, we get $m + n > 2$, and F must have infinitely many zeros.

Case 2. Suppose $\lambda \neq 0$. Assume on the contrary to the assertion that $f(z)^n(\lambda f(z+c)^m + \mu f(z)^m) - a$ has finitely many zeros. Then

$$f(z)^n\big(\lambda f(z+c)^m + \mu f(z)^m\big) - a = H(z)e^{Q(z)},$$

where $H(z)$, $Q(z)$ are polynomials. When $\mu \neq 0$, Theorem 2 holds by Theorem 1. When $\mu = 0$, a simple modification of the proof of Theorem 1 yields Theorem 2.

5. Proof of Theorem 3

Suppose on the contrary to the assertion that $f(z)^n + \mu f(z+c)^m - a(z)$ has finitely many zeros. Then by Hadamard factorization theorem, there exist two polynomials $P(z)$ and $Q(z)$ such that

$$f(z)^n + \mu f(z+c)^m - a(z) = P(z)e^{Q(z)}. \tag{8}$$

Case 1. If $n > m + 1$, then differentiating (8) and eliminating $e^{Q(z)}$, we have

$$\begin{aligned}f(z)^{n-1}\left(nf'(z) - \left(Q'(z) + \frac{P'(z)}{P(z)}\right)f(z)\right) &= a'(z) + m\mu f(z+c)^{m-1} f'(z+c)\\ &\quad + \left(Q'(z) + \frac{P'(z)}{P(z)}\right)\big(\mu f(z+c)^m - a(z)\big).\end{aligned} \tag{9}$$

If $nf'(z) - (Q'(z) + \frac{P'(z)}{P(z)})f \equiv 0$, then we have $f(z)^n = AP(z)e^{Q(z)}$, where A is a non-zero constant. Writing $f = he^{\frac{Q}{n}}$, where h is a polynomial, and substituting f into (8), we get

$$(A-1)P(z)e^{Q(z)} + \mu h(z+c)e^{\frac{mQ(z+c)}{n}} - a(z) \equiv 0. \tag{10}$$

Clearly, $A \neq 1$. Let $g = e^{\frac{Q}{n}}$. From (10) and Lemma 3, we get

$$nT(r,g) \leqslant mT(r,g) + O\left(r^{\sigma(g)-1+\varepsilon}\right) + O(\log r),$$

which is a contradiction. Therefore, we obtain $nf'(z) - (Q'(z) + \frac{P'(z)}{P(z)})f \neq 0$. Combining (9) and Lemma 2, we have

$$T\left(r, nf'(z) - \left(Q'(z) + \frac{P'(z)}{P(z)}\right)f\right) = S(r,f)$$

and

$$T\left(r, f\left(nf'(z) - \left(Q'(z) + \frac{P'(z)}{P(z)}\right)f\right)\right) = S(r,f).$$

Hence

$$T(r,f) = S(r,f),$$

which is a contradiction.

Case 2. If $n < m-1$, then set $F(z) = f(z+c)$, $F(z-c) = f(z)$ follows. We obtain

$$F(z)^m + \frac{1}{\mu}F(z-c)^n - \frac{1}{\mu}a(z-c) = P^*(z)e^{Q^*(z)}.$$

Similarly as in Case 1, we get the conclusion, completing the proof of Theorem 3.

6. Proof of Theorem 4

Since $f(z)$ is an entire function of order $1 \leqslant \sigma(f) < \infty$ and has infinitely many zeros with $\lambda(f) < 1$, we can write $f(z) = h(z)e^{P(z)}$ by the Hadamard factorization theorem, where $h(z)$ is the product of the zeros of $f(z)$, is also an entire function and $\lambda(f) = \lambda(h) = \sigma(h) < 1$, and $P(z)$ is a non-constant polynomial. If $f(z) + \mu f(z+c) - a$ has finitely many zeros, we obtain

$$h(z)e^{P(z)} + \mu h(z+c)e^{P(z+c)} - a = Q(z)e^{Q^*(z)}, \tag{11}$$

where $Q(z)$, $Q^*(z)$ are polynomials. Clearly

$$h(z)e^{P(z)} + \mu h(z+c)e^{P(z+c)} - a - Q(z)e^{Q^*(z)} \equiv 0. \tag{12}$$

Case 1. If $P(z+c) - P(z) \equiv a_1$, where a_1 is a constant, then we obtain $P(z) = Az + B$, where $A \neq 0$. Substituting $P(z) = Az + B$ into (12), we have

$$e^{Az+B}\left(h(z) + \mu h(z+c)e^{a_1}\right) - a - Q(z)e^{Q^*(z)} \equiv 0. \tag{13}$$

If $Q^*(z) - Az - B \equiv a_2$, where a_2 is a constant, then $Q^*(z) = Az + C$. By (13), we get

$$e^{Az+B}\left(h(z) + \mu h(z+c)e^{a_1} - Q(z)e^{C-B}\right) \equiv a, \tag{14}$$

$a \neq 0$, which is a contradiction.

If $Q^*(z) - Az - B \not\equiv a_2$, then we get $a \equiv 0$, from (13) and Lemma 5, which also is a contradiction. Hence $P(z+c) - P(z) \not\equiv a_1$.

Case 2. If $P(z+c) - Q^*(z) \equiv b_1$, where b_1 is a constant, then we get

$$\left(\mu h(z+c)e^{b_1} - Q(z)\right)e^{Q^*(z)} + h(z)e^{P(z)} - a \equiv 0. \tag{15}$$

Clearly, $Q^*(z) - P(z) \not\equiv b_2$, otherwise, we get $P(z+c) - P(z) \equiv b_1 + b_2$, which is a contradiction. When $Q^*(z) - P(z) \not\equiv b_2$, applying Lemma 5 to (15), we get $h(z) \equiv 0$, which is a contradiction. So $P(z+c) - Q^*(z) \not\equiv b_1$.

Case 3. Similarly as in Case 2, we get $P(z) - Q^*(z) \not\equiv c_1$, where c_1 is a constant.

From Cases 1, 2 and 3 and applying Lemma 5 to (12), we get $h(z) \equiv 0$, which is a contradiction. Therefore, $f(z) + \mu f(z+c) - a$ has infinitely many zeros.

7. Proof of Theorem 5

The former part of Theorem 5 follows by using the same reasoning as in [6] with apparent modification. For the convenience of the reader, we give a complete proof.

From $\delta(a) > 1$, we can easily get $\delta(a,f) > 0$ and $\delta(a,g) > 0$. Now we take a positive number ε such that $(2 + 2\varepsilon - \delta(a)) < 1$, $\delta(a,f) - \varepsilon > 0$ and $\delta(a,g) - \varepsilon > 0$. Then we have

$$\left(\delta(a,f) - \varepsilon\right)T(r,f) \leqslant m\left(r, \frac{1}{f-a}\right) \tag{16}$$

and

$$\big(\delta(a,g)-\varepsilon\big)T(r,g)\leqslant m\left(r,\frac{1}{g-a}\right) \tag{17}$$

as $r\to\infty$. By Lemma 1, we deduce that

$$m\big(r,f(z+c_1)\big)\leqslant m(r,f)+S(r,f), \tag{18}$$

and

$$m\left(r,\frac{1}{f-a}\right)\leqslant m\left(r,\frac{1}{f(z+c_1)-a}\right)+S(r,f). \tag{19}$$

Set

$$F(z)=\frac{f(z+c_1)-a}{b-a},\qquad G(z)=\frac{g(z+c_2)-a}{b-a}. \tag{20}$$

From (16), (18)–(20), we get

$$\begin{aligned}\big(\delta(a,f)-\varepsilon\big)T(r,f)&\leqslant m\left(r,\frac{1}{f(z+c_1)-a}\right)+S(r,f)\\&\leqslant m\big(r,f(z+c_1)\big)+S(r,f)\\&\leqslant T(r,F)+S(r,f)\leqslant T(r,f)+S(r,f).\end{aligned} \tag{21}$$

Similarly,

$$\big(\delta(a,g)-\varepsilon\big)T(r,g)\leqslant T(r,G)+S(r,g)\leqslant T(r,g)+S(r,g). \tag{22}$$

Hence

$$S(r,F)=S(r,f),\qquad S(r,G)=S(r,g).$$

Again from (16) and (19), we obtain that

$$\begin{aligned}\big(\delta(a,f)-\varepsilon\big)T(r,F)&\leqslant\big(\delta(a,f)-\varepsilon\big)T(r,f)+S(r,f)\\&\leqslant m\left(r,\frac{1}{f(z+c_1)-a}\right)+S(r,f)\\&\leqslant T(r,F)-N\left(r,\frac{1}{F}\right)+S(r,f).\end{aligned} \tag{23}$$

So we have

$$N\left(r,\frac{1}{F}\right)\leqslant\big(1-\delta(a,f)+\varepsilon\big)T(r,F). \tag{24}$$

By the same reasoning, we get

$$N\left(r,\frac{1}{G}\right)\leqslant\big(1-\delta(a,g)+\varepsilon\big)T(r,G). \tag{25}$$

Since $f(z+c_1)$ and $g(z+c_2)$ share b CM, we obtain that,

$$\frac{f(z+c_1)-b}{g(z+c_2)-b}=e^{h(z)}, \tag{26}$$

where $h(z)$ is a polynomial. From (26) we have

$$F(z)-G(z)e^{h(z)}+e^{h(z)}\equiv 1.$$

Set $F_1(z)=F(z)$, $F_2(z)=G(z)e^{h(z)}$, $F_3(z)=e^{h(z)}$. Then

$$F_1+F_2+F_3=1,$$

and

$$T(r)=\max_{1\leqslant j\leqslant 3}\big\{T(r,F_j)\big\},\qquad S(r)=o\big(T(r)\big).$$

From (24) and (25), we get

$$\sum_{j=1}^{3} N_2\left(r, \frac{1}{F_j}\right) + \sum_{j=1}^{3} \overline{N}(r, F_j) \leqslant N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) + S(r)$$
$$\leqslant \big(2 + 2\varepsilon - \delta(a)\big)T(r) + S(r).$$

By Lemma 4, we get that $F_2 = 1$ or $F_3 = 1$. If $F_2 = 1$, the conclusion (ii) holds, while if $F_3 = 1$, the conclusion (i) holds.

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