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On approximation properties of a family of linear operators at critical value of parameter

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Abstract

We introduce the family of linear operators

$$\left(A^{\alpha} f\right)(x)=\frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha-1}\left(S_{t} f\right)(x)\, d t, \quad \alpha>0$$

associated to a certain "admissible bunch" of operators S_t, $t>0$, acting on $L_p(\mathbb{R}^n, dm$, and investigate the approximation properties of this family as $\alpha \rightarrow 0^+$. We give some applications to the Riesz and the Bessel potentials generated by the ordinary (Euclidean) and generalized translations.

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[2] In translations from Russian, A.D. Gadjiev, A.D. Gadzhiev, and A.D. Gadžiev all refer to the same person.

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1. Introduction

In this paper, given some "admissible bunch" of linear operators $\{S_t\}_{t>0}$, acting on $L_p(\mathbb{R}^n, dm)$, we introduce the following family of integral operators:

$$(A^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (S_t f)(x)\, dt, \quad \alpha > 0.$$

This family of operators contains (for special choices of "admissible bunch" $\{S_t\}_{t>0}$) the Riesz and the Bessel potentials generated by the ordinary and generalized translation. The classical Riesz potentials, $I^\alpha f$, and the generalized Riesz potentials, $I_\nu^\alpha f$, are defined in terms of Fourier and Fourier–Bessel transforms by the following formulas:

$$F\left(I^\alpha f\right)(x) = |x|^{-\alpha} F(f)(x), \quad x \in \mathbb{R}^n; \tag{1}$$

$$F_\nu\left(I_\nu^\alpha f\right)(x) = |x|^{-\alpha} F_\nu(f)(x), \quad x \in \mathbb{R}_+^n. \tag{2}$$

Similarly, the classical Bessel potentials, $J^\alpha f$, and the generalized Bessel potentials, $J_\nu^\alpha f$, are defined as

$$F\left(J^\alpha f\right)(x) = \left(1 + |x|^2\right)^{-\alpha/2} F(f)(x), \quad x \in \mathbb{R}^n; \tag{3}$$

$$F_\nu\left(J_\nu^\alpha f\right)(x) = \left(1 + |x|^2\right)^{-\alpha/2} F_\nu(f)(x), \quad x \in \mathbb{R}_+^n. \tag{4}$$

These potentials are known as important technical tools in Fourier and Fourier–Bessel harmonic analysis (More information about these potentials can be found in [1–3,5,10–12]).

In this paper we investigate the approximation properties of the family $A^\alpha f$, when the parameter $\alpha > 0$ tends to zero. The paper is organized as follows. Section 2 contains basic notations, definitions and auxiliary lemmas. In particular, the notion of the "admissible bunch" of operators is introduced and some examples are given in the section. The main results of the paper are given in Section 3. This section is devoted to the investigation of approximation properties of the family $(A^\alpha f)(x)$ as $\alpha \to 0^+$. The order of approximation of the Lipschitz functions is also studied. Moreover, some applications to the Riesz and the Bessel potentials generated by the Euclidean and generalized translations are given. It should also be mentioned that the approximation properties of the classical Riesz and Bessel potentials have been studied by Kurokawa [7] before.

2. Preliminaries and auxiliary lemmas

Let $L_p \equiv L_p\left(\mathbb{R}^n, dm\right)$ be the space of m-measurable functions such that

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p\, dm(x)\right)^{1/p} < \infty, \quad 1 \leqslant p < \infty,$$

and let $C_0 \equiv C_0(\mathbb{R}^n)$ be the class of all continuous functions on $\mathbb{R}^n$ vanishing at infinity. We will assume that the set of all compactly supported continuous functions is dense in $L_p(\mathbb{R}^n, dm)$ (e.g., this is the case when m is a Borel measure).

Definition 1. A family $\{S_t\}_{t>0}$ of linear operators on L_p will be called an "admissible bunch" of type $\beta > 0$ if

(a) There exists $c = c(\beta)$ independent from t so that

$$ess\sup_{x\in\mathbb{R}^n} |(S_t f)(x)| \leqslant ct^{-\beta} \|f\|_p ; \tag{5}$$

(b)

$$\sup_{t>0} \|S_t f\|_p \leqslant c \|f\|_p ; \tag{6}$$

(c) the maximal operator

$$\left(S^* f\right)(x) = \sup_{t>0} |(S_t f)(x)|$$

is weak (p, p), i.e.

$$m\left\{x : \left(S^* f\right)(x) > \lambda\right\} \leqslant \left(\frac{c\|f\|_p}{\lambda}\right)^p, \quad \forall \lambda > 0;$$

(d) $\lim_{t\to 0} S_t f = f$ in the L_p-norm. For $f \in L_p \cap C_0$, the convergence is also uniform on $\mathbb{R}^n$.

If (a) holds for all $\beta > 0$, we call $\{S_t\}_{t>0}$ an "admissible bunch" of infinite type.

Remark 2. The number β in (5) may depend on n and p. The notion "admissible bunch" is close to the notion "admissible semigroup" defined in [3]. The basic difference between these two notions is that, the "admissible bunch" does not require to have semigroup property.

Lemma 3 (*Duoandikoetxea [4, p. 27]*). *Let (X, dm) be a measure space and let $\{T_\varepsilon\}_{\varepsilon>0}$ be a family of linear operators on $L_p(X, dm)$. Denote*

$$\left(T^* f\right)(x) = \sup_{\varepsilon>0} |(T_\varepsilon f)(x)| .$$

If T^* is weak (p, q), i.e.,

$$m\left\{y : \left(T^* f\right)(y) > \lambda\right\} \leqslant \left(\frac{c\|f\|_p}{\lambda}\right)^q, \quad \forall \lambda > 0,$$

then the set

$$\left\{f : f \in L_p(X, dm), \lim_{\varepsilon\to 0} (T_\varepsilon f)(x) = f(x), a.e.\right\}$$

is closed in $L_p(X, dm)$.

Remark 4. Owing to Definition 1(c)–(d), and Lemma 3, it follows that if $f \in L_p$, $1 \leqslant p < \infty$, then $\lim_{t\to 0} (S_t f)(x) = f(x)$, a.e.

Let us give some examples of "admissible bunches" of operators. The most famous examples are the classical Riesz–Bochner, Gauss–Weierstrass, Poisson and Metaharmonic integrals. We consider here the last two examples only, which will be used in the sequel.

(i) *The Poisson integrals* are defined as:

$$(\mathcal{P}_t f)(y) = \int_{\mathbb{R}^n} P(x,t)\, f(y-x)\, dx, \quad \text{where } F[P(\cdot,t)](\xi) = e^{-t|\xi|}. \tag{7}$$

(ii) *The Metaharmonic integrals* are defined as:

$$(\mathcal{M}_t f)(y) = \int_{\mathbb{R}^n} M(x,t)\, f(y-x)\, dx, \tag{8}$$

where $F[M(\cdot,t)](\xi) = e^{-t\sqrt{1+|\xi|^2}}$.

Here F designates the Fourier transform

$$(Fg)(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} g(x)\, dx, \quad \xi\cdot x = \xi_1 x_1 + \cdots + \xi_n x_n. \tag{9}$$

The corresponding kernels in (7) and (8) have the form [12,9] (see also [10,11]):

$$P(x,t) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \frac{t}{\left(|x|^2+t^2\right)^{(n+1)/2}}; \tag{10}$$

$$M(x,t) = \frac{2t}{(2\pi)^{(n+1)/2}} \frac{K_{(n+1)/2}\left(\sqrt{|x|^2+t^2}\right)}{\left(\sqrt{|x|^2+t^2}\right)^{(n+1)/2}}, \tag{11}$$

where $K_{(n+1)/2}(\cdot)$ is the McDonald function. Operators (7) and (8) act on the usual Lebesgue space $L_p(\mathbb{R}^n, dm)$ with $dm(x) = dx = dx_1 \cdots dx_n$, and constitute "admissible bunches" of type $\beta = \frac{n}{p}$ and $\beta = \infty$, respectively (see e.g. [10, p. 217 and p. 257]).

The "admissible bunches" on $L_p(\mathbb{R}^n, dm)$ of different type arise in the Fourier–Bessel harmonic analysis associated with the singular Laplace–Bessel differential operator

$$\Delta_\nu = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2} + \frac{2\nu}{x_n}\frac{\partial}{\partial x_n}, \quad x_n > 0, \quad \nu > 0. \tag{12}$$

Let

$$\mathbb{R}^n_+ = \left\{x : x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n, x_n > 0\right\};$$
$$dm(x) = \mathcal{X}_+(x)\, x_n^{2\nu}\, dx, \quad dx = dx_1 \cdots dx_n. \tag{13}$$

Here $\nu > 0$ is a fixed parameter and $\mathcal{X}_+(x)$ is the characteristic function of $\mathbb{R}^n_+$, i.e. $\mathcal{X}_+(x) = 1$ if $x_n > 0$ and $\mathcal{X}_+(x) = 0$ if $x_n \leqslant 0$. The relevant Fourier–Bessel transform F_ν associated to the measure (13) and the operator (12) is defined by

$$(F_\nu f)(\xi) = \int_{\mathbb{R}^n_+} f(x) e^{-i\xi'\cdot x'} j_{\nu-\frac{1}{2}}(\xi_n x_n)\, x_n^{2\nu}\, dx, \tag{14}$$

where $\xi'\cdot x' = \xi_1 x_1 + \cdots + \xi_{n-1}x_{n-1}$, $j_\lambda(\tau) = 2^\lambda \Gamma(\lambda+1)\, \mathcal{J}_\lambda(\tau) \big/ \tau^\lambda$, $\mathcal{J}_\lambda(\tau)$ is the Bessel function of the first kind. The Fourier–Bessel harmonic analysis is adopted to the generalized convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_+} f(y) \left(T^y g\right)(x)\, y_n^{2\nu}\, dy, \quad x \in \mathbb{R}^n_+ \tag{15}$$

generated by the generalized (Bessel) translation

$$\left(T^{y} f\right)(x)=\frac{\Gamma(v+1/2)}{\Gamma(v)\,\Gamma(1/2)}\int_0^{\pi} f\left(x'-y',\sqrt{x_n^2-2x_n y_n\cos\alpha+y_n^2}\right)\sin^{2v-1}\alpha d\alpha \tag{16}$$

(see e.g. [1,2,5,6,8,14]). Actually we deal with the usual (Euclidean) translation in $x'=(x_1,\ldots,x_{n-1})$ and the generalized translation with respect to x_n-variable.

The corresponding generalized Poisson and Metaharmonic integrals, $\{\mathcal{P}_t^{(v)} f\}_{t>0}$ and $\{\mathcal{M}_t^{(v)} f\}_{t>0}$, which also fall into the scope of Definition 1, are defined as:

(i′)
$$\left(\mathcal{P}_t^{(v)} f\right)(y)=\int_{\mathbb{R}_+^n} P^{(v)}(x,t)\left(T^x f\right)(y)\,x_n^{2v}\,dx,\qquad y\in\mathbb{R}_+^n;$$
$$F_v\left[P^{v}(\cdot,t)\right](\xi)=e^{-t|\xi|},\quad t>0,\ \ \xi\in\mathbb{R}_+^n; \tag{17}$$

(ii′)
$$\left(\mathcal{M}_t^{(v)} f\right)(y)=\int_{\mathbb{R}_+^n} M^{(v)}(x,t)\left(T^x f\right)(y)\,x_n^{2v}\,dx,$$
$$F_v\left[M^{(v)}(\cdot,t)\right](\xi)=e^{-t\sqrt{1+|\xi|^2}},\quad t>0,\ \ \xi\in\mathbb{R}_+^n, \tag{18}$$

where F_v is the Fourier–Bessel transform defined by (14).

Operators (17) and (18) represent “admissible bunches” of type $\beta=\frac{n+2v}{p}$ and $\beta=\infty$, respectively. The corresponding kernels have the form:

$$P^{(v)}(x,t)=\frac{2\Gamma((n+2v+1)/2)}{\pi^{n/2}\Gamma(v+1/2)}\frac{t}{\left(|x|^2+t^2\right)^{(n+2v+1)/2}}, \tag{19}$$

$$M^{(v)}(x,t)=\frac{2^{-v+3/2}t}{(2\pi)^{n/2}\,\Gamma(v+1/2)}\frac{K_{(n+2v+1)/2}\left(\sqrt{|x|^2+t^2}\right)}{\left(\sqrt{|x|^2+t^2}\right)^{(n+2v+1)/2}}. \tag{20}$$

More information about these integral operators can be found in [1,2,5].

From now on the letters $c, c_0, c_1, c_2, \ldots$ will be used for constants. As usual, we will write “$\varphi(\alpha)=\mathrm{O}(1)$ as $\alpha\to 0$” if the function $\varphi(\alpha)$ is bounded as $\alpha\to 0$.

Lemma 5. *Let the kernels $P(x,t)$, $M(x,t)$, $P^{(v)}(x,t)$ and $M^{(v)}(x,t)$ be defined as in* (10), (11), (19) *and* (20), *respectively. Then, there exists $c>0$ such that*

(a) $M(x,t)\leqslant cP(x,t),\quad \forall x\in\mathbb{R}^n, t>0;$
(b) $M^{(v)}(x,t)\leqslant cP^{(v)}(x,t),\quad \forall x\in\mathbb{R}_+^n, t>0.$

Proof. Taking into account the following well known estimation for the McDonald function [10, p. 257]

$$\frac{K_\gamma(r)}{r^\gamma}\leqslant\left\{\begin{array}{ll} c_0\frac{e^{-r}}{r^\gamma\sqrt{r}} & \text{if } r>0\\ c_0 r^{-2\gamma} & \text{if } 0<r\leqslant 1\end{array}\right\}\leqslant c_1 r^{-2\gamma},\qquad (0<r<\infty,\gamma\geqslant 1/2),$$

we have for $\gamma = (n+1)/2$

$$M(x,t) = c_2 t \frac{K_\gamma\left(\sqrt{|x|^2+t^2}\right)}{\left(\sqrt{|x|^2+t^2}\right)^\gamma} \leqslant \frac{c_3 t}{\left(|x|^2+t^2\right)^{(n+1)/2}} \overset{(10)}{=} cP(x,t).$$

The part (b) is proved analogously. $\square$

We need the following Lipschitz classes

$$\Lambda_\lambda = \left\{ f : f \in L_\infty\left(\mathbb{R}^n\right) \|f(x-y) - f(x)\|_\infty \leqslant c|y|^\lambda \right\}; \tag{21}$$

$$\tilde{\Lambda}_\lambda = \left\{ f : f \in L_\infty\left(\mathbb{R}^n_+\right), \left\|\left(T^y f\right)(x) - f(x)\right\|_\infty \leqslant c|y|^\lambda \right\}, \tag{22}$$

where $0 < \lambda \leqslant 1$; T^y is the generalized translation (16); $\|g\|_\infty = \sup|g(x)|$; supremum is taken over $\mathbb{R}^n$ in (21), and over $\mathbb{R}^n_+$ in (22), respectively.

Lemma 6. (a) *Let $S_t f$ be one of the operators $\mathcal{P}_t f$ and $\mathcal{M}_t f$. If $f \in \Lambda_\lambda$, then*

$$\|S_t f - f\|_\infty = \mathrm{O}(1)\, t^\lambda \quad \text{as } t \to 0. \tag{23}$$

(b) *Let $S_t f$ be one of the operators $\mathcal{P}_t^{(v)} f$ and $\mathcal{M}_t^{(v)} f$. If $f \in \tilde{\Lambda}_\lambda$, then*

$$\|S_t f - f\|_\infty = \mathrm{O}(1)\, t^\lambda \quad \text{as } t \to 0. \tag{24}$$

Proof. We will prove only the case $S_t f = \mathcal{M}_t^{(v)} f$, with $f \in \tilde{\Lambda}_\lambda$. (The other cases are proved analogously.)

Since $\int_{\mathbb{R}^n_+} M^{(v)}(y,t)\, y_n^{2v}\, dy = e^{-t}$, we have

$$\begin{aligned}(\mathcal{M}_t^{(v)} f)(x) - f(x) &= \int_{\mathbb{R}^n_+} M^{(v)}(y,t) \Big(\left(T^y f\right)(x) - e^t f(x) \Big) y_n^{2v}\, dy \\ &= \int_{\mathbb{R}^n_+} M^{(v)}(y,t) \Big(\left(T^y f\right)(x) - f(x) \Big) y_n^{2v}\, dy + \left(e^t - 1\right) f(x).\end{aligned}$$

This yields

$$\left\|\mathcal{M}_t^{(v)} f - f\right\|_\infty \leqslant \int_{\mathbb{R}^n_+} M^{(v)}(y,t) \left\|T^y f - f\right\|_\infty y_n^{2v}\, dy + \left(e^t - 1\right) \|f\|_\infty = i_1 + i_2.$$

Further, by Lemma 5(b) and (22) we have

$$\begin{aligned} i_1 &\leqslant c_1 \int_{\mathbb{R}^n_+} P^{(v)}(y,t) \left\|T^y f - f\right\|_\infty y_n^{2v}\, dy \\ &\leqslant c_2 \int_{\mathbb{R}^n_+} \frac{t}{\left(|y|^2+t^2\right)^{(n+2v+1)/2}} |y|^\lambda\, y_n^{2v}\, dy = c_3 t^\lambda. \end{aligned}$$

Since $e^t - 1 = t + \mathrm{O}(1)\, t^2$ as $t \to 0$, it follows that $i_2 = \mathrm{O}(1)\, t$ as $t \to 0$.

Finally, for $0 < \lambda \leqslant 1$ we get

$$\left\|\mathcal{M}_t^{(v)} f - f\right\|_\infty = \mathrm{O}(1)\, t^\lambda + \mathrm{O}(1)\, t = \mathrm{O}(1)\, t^\lambda \quad \text{as } t \to 0. \qquad \square$$

Now we introduce a class of integral operators generated by an "admissible bunch" $\{S_t\}_{t>0}$ (see Definition 1). Given an "admissible bunch" $\{S_t\}_{t>0}$ of type $\beta > 0$ and a complex number α with $\operatorname{Re}\alpha > 0$, we define the following family of integral operators:

$$\left(A^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} (S_t f)(x)\, dt. \tag{25}$$

For $f \in L_p(\mathbb{R}^n, dm)$, $1 \leqslant p < \infty$, the expression (25) is well defined a.e. on $\mathbb{R}^n$ provided $0 < \operatorname{Re}\alpha < \beta$. Indeed,

$$\left(A^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \left(\int_0^1 + \int_0^{\infty} \right) t^{\alpha-1} (S_t f)(x)\, dt = i_1 + i_2.$$

Denote $a = \operatorname{Re}\alpha$. By Definition 1(a)–(b)

$$\|i_1\|_p \leqslant c\, \|f\|_p \int_0^1 t^{a-1}\, dt < \infty \quad \text{and} \quad \|i_2\|_p \leqslant c\, \|f\|_p \int_1^{\infty} t^{a-\beta-1}\, dt < \infty.$$

Therefore $(A^{\alpha} f)(x) = i_1 + i_2$ is finite a.e.

The family of operators (25) contains the Riesz and the Bessel potentials generated by the ordinary and generalized translation.

The classical Riesz potentials, $I^{\alpha} f$, and the Bessel potentials, $J^{\alpha} f$, initially defined in terms of Fourier transform by (1) and (3), have the following integral representations via the Poisson and Metaharmonic integrals, respectively:

$$\left(I^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} (\mathcal{P}_t f)(x)\, dt \quad \text{(Stein, Weiss [13])}. \tag{26}$$

$$\left(J^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} (\mathcal{M}_t f)(x)\, dt \quad \text{(Lizorkin [9])}. \tag{27}$$

The analogous representations of the generalized Riesz and Bessel potentials, initially defined by (2) and (4), respectively, have exactly the same form with the superscript (v) in notation of the corresponding semigroups (17) and (18):

$$\left(I_v^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} \left(\mathcal{P}_t^{(v)} f\right)(x)\, dt, \tag{28}$$

$$\left(J_v^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} t^{\alpha-1} \left(\mathcal{M}_t^{(v)} f\right)(x)\, dt, \tag{29}$$

(see [1,2,4]).

Remark 7. It is clear that all these formulas (26)–(29) have the form (25).

3. The approximation properties of the family $A^{\alpha} f$ as $\alpha \to 0^+$

Theorem 8. *Let* $f \in L_p\left(\mathbb{R}^n, dm\right)$, $1 \leqslant p < \infty$, *and the family of operators* $\{A^{\alpha}\}_{\alpha>0}$ *be defined by* (25). *Then*

(a) $\lim_{\alpha\to 0^+} (A^{\alpha} f)(x) = f(x)$ *for almost all* $x \in \mathbb{R}^n$;
(b) *If* $f \in L_p \cap C_0$, *then convergence is uniform on* $\mathbb{R}^n$.

Proof. (a) Let $x \in \mathbb{R}^n$ be such a point that $\lim_{t\to 0} (S_t f)(x) = f(x)$ (see Remark 4). Then given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that $|(S_t f)(x) - f(x)| < \varepsilon$ for all $0 < t < \delta_1$ and there exists $\delta_2 > 0$ such that $\left(1 - e^{-t}\right) < \varepsilon$ for all $0 < t < \delta_2$. Taking the number δ as $0 < \delta < \min\{\delta_1, \delta_2, 1\}$, we have

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_0^\delta t^{\alpha-1}\left[(S_t f)(x) - e^{-t} f(x)\right] dt\right| \leqslant \frac{1}{\Gamma(\alpha)}\int_0^\delta t^{\alpha-1} |(S_t f)(x) - f(x)|\, dt \\
&\quad + \frac{|f(x)|}{\Gamma(\alpha)}\int_0^\delta t^{\alpha-1}\left(1 - e^{-t}\right) dt \leqslant \frac{\varepsilon}{\Gamma(\alpha)}\int_0^\delta t^{\alpha-1}\, dt + \frac{\varepsilon |f(x)|}{\Gamma(\alpha)}\int_0^\delta t^{\alpha-1}\, dt \\
&\quad = \frac{\varepsilon\delta^\alpha}{\alpha\Gamma(\alpha)}(1 + |f(x)|) = \frac{\varepsilon\delta^\alpha}{\Gamma(\alpha+1)}(1 + |f(x)|) = \mathrm{O}(1)\,\varepsilon \quad \text{as } \alpha \to 0^+. \qquad (30)
\end{aligned}$$

Further,

$$\begin{aligned}
&\left|\frac{1}{\Gamma(\alpha)}\int_\delta^\infty t^{\alpha-1}\left[(S_t f)(x) - e^{-t} f(x)\right] dt\right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)}\int_\delta^\infty t^{\alpha-1}\left(|(S_t f)(x)| + e^{-t}|f(x)|\right) dt \\
&\quad \overset{(5)}{\leqslant} \frac{c_1}{\Gamma(\alpha)}\int_\delta^\infty t^{\alpha-1}\left[t^{-\beta}\|f\|_p + e^{-t}|f(x)|\right] dt \\
&\quad \leqslant \frac{c_1}{\Gamma(\alpha)}\left(\|f\|_p\int_\delta^\infty t^{\alpha-\beta-1}\, dt + |f(x)|\,\delta^{\alpha-1}\int_\delta^\infty e^{-t}\, dt\right) \\
&\quad = \frac{c_1}{\Gamma(\alpha)}\left(\frac{\|f\|_p}{\beta-\alpha}\delta^{\alpha-\beta} + |f(x)|\,\delta^{\alpha-1}e^{-\delta}\right) = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+. \qquad (31)
\end{aligned}$$

Now by making use of (30) and (31) we have

$$\begin{aligned}
&\left|\left(A^\alpha f\right)(x) - f(x)\right| \\
&\quad = \left|\frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(S_t f)(x)\, dt - \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{-t} f(x)\, dt\right| \\
&\quad \leqslant \frac{1}{\Gamma(\alpha)}\int_0^\delta t^{\alpha-1}\left|(S_t f)(x) - e^{-t} f(x)\right| dt + \frac{1}{\Gamma(\alpha)}\int_\delta^\infty t^{\alpha-1}\left|(S_t f)(x) - e^{-t} f(x)\right| dt \\
&\quad = \mathrm{O}(1)\,\varepsilon + \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+.
\end{aligned}$$

The last estimate yields

$$\limsup_{\alpha\to 0}\left|\left(A^\alpha f\right)(x) - f(x)\right| \leqslant c\varepsilon, \qquad c = c(x).$$

Since $\varepsilon > 0$ is arbitrary we have

$$\lim_{\alpha\to 0}\left|\left(A^\alpha f\right)(x) - f(x)\right| = 0.$$

(b) Let now $f \in L_p \cap C_0$. Using the notation $\|g\|_\infty = \sup|g(x)|$, we have from (30) and (31)

$$\|A^\alpha f - f\|_\infty \leqslant \varepsilon\frac{\delta^\alpha}{\Gamma(\alpha+1)}\left(1 + \|f\|_\infty\right) + \alpha\frac{c_1}{\Gamma(\alpha+1)}\left(\frac{\|f\|_p}{\beta-\alpha}\delta^{\alpha-\beta} + \|f\|_\infty\,\delta^{\alpha-1}e^{-\delta}\right).$$

The last expression leads to $\limsup_{\alpha\to 0} \|(A^\alpha f) - f\|_\infty \leqslant \varepsilon(1 + \|f\|_\infty)$, $\forall\varepsilon > 0$, and therefore, $\lim_{\alpha\to 0} \|A^\alpha f - f\|_\infty = 0$.

The proof of the theorem is completed. □

Corollary 9. *Owing to the formulas* (26)–(29), *the statement of the Theorem* 8 *is valid*, *in particular*, *for the operators* I^α, J^α, I_ν^α *and* J_ν^α.

Remark 10. The approximation properties of the families $I^\alpha f$ and $J^\alpha f$ as $\alpha \to 0^+$ have been studied by Kurokawa [7] before.

The next theorem gives an estimation for the order of approximation of the Lipschitz functions (see (21) and (22)). Below the notation $L_{p,\nu}$ stands for $L_p(\mathbb{R}^n, dm)$ with $dm\,(x) = \mathcal{X}_+\,(x)\,x_n^{2\nu}\,dx$ (see (13)).

Theorem 11. (a) *Let* $f \in L_p(\mathbb{R}^n,\ dx) \cap \Lambda_\lambda$, $1 \leqslant p < \infty$, $0 < \lambda \leqslant 1$. *Let further* A^α *be any of the potentials* I^α *and* J^α, $\alpha > 0$. *Then*

$$\left\|A^\alpha f - f\right\|_\infty = \mathrm{O}\,(1)\,\alpha \quad \textit{as } \alpha \to 0^+.$$

(b) *Let* $f \in L_{p,\nu} \cap \tilde{\Lambda}_\lambda$, $1 \leqslant p < \infty$, $0 < \lambda \leqslant 1$. *Let further* A^α *be any of the generalized potentials* I_ν^α *and* J_ν^α, $\alpha > 0$. *Then*

$$\left\|A^\alpha f - f\right\|_\infty = \mathrm{O}\,(1)\,\alpha \quad \textit{as } \alpha \to 0^+.$$

Proof. We will prove only the statement $\left\|I_\nu^\alpha f - f\right\|_\infty = \mathrm{O}\,(1)\,\alpha$ as $\alpha \to 0^+$. (The other statements of the theorem are proved analogously, using Lemma 6 and the inequality (5).) We have

$$\begin{aligned}
\left(I_\nu^\alpha f\right)(x) - f\,(x) &\overset{(28)}{=} \frac{1}{\Gamma\,(\alpha)} \int_0^\infty t^{\alpha-1} (\mathcal{P}_t^{(\nu)} f)\,(x)\;dt - f\,(x) \\
&= \frac{1}{\Gamma\,(\alpha)} \int_0^\infty t^{\alpha-1} \left((\mathcal{P}_t^{(\nu)} f)\,(x) - e^{-t} f\,(x) \right) dt \\
&= \frac{1}{\Gamma\,(\alpha)} \int_0^1 t^{\alpha-1} \left((\mathcal{P}_t^{(\nu)} f)\,(x) - f\,(x) \right) dt \\
&\quad + \frac{f\,(x)}{\Gamma\,(\alpha)} \int_0^1 t^{\alpha-1} \left(1 - e^{-t}\right) dt \\
&\quad + \frac{1}{\Gamma\,(\alpha)} \int_0^\infty t^{\alpha-1} \left((\mathcal{P}_t^{(\nu)} f)\,(x) - e^{-t} f\,(x) \right) dt.
\end{aligned} \tag{32}$$

Further,

$$\begin{aligned}
\left\|I_\nu^\alpha f - f\right\|_\infty \leqslant{}& \frac{1}{\Gamma\,(\alpha)} \int_0^1 t^{\alpha-1} \left\|\mathcal{P}_t^{(\nu)} f - f\right\|_\infty dt + \frac{\|f\|_\infty}{\Gamma\,(\alpha)} \int_0^1 t^{\alpha-1} \left(1 - e^{-t}\right) dt \\
&+ \frac{1}{\Gamma\,(\alpha)} \int_0^\infty t^{\alpha-1} \left(\left\|\mathcal{P}_t^{(\nu)} f\right\|_\infty + e^{-t}\,\|f\|_\infty \right) dt = i_1 + i_2 + i_3.
\end{aligned}$$

The relation (24) with $S_t f = \mathcal{P}_t^{(\nu)} f$ leads to

$$i_1 \leqslant \frac{c}{\Gamma\,(\alpha)} \int_0^1 t^{\alpha+\lambda-1}\,dt = \alpha \frac{c}{\Gamma\,(\alpha+1)} \frac{1}{\alpha+\lambda} = \mathrm{O}\,(1)\,\alpha \quad \text{as } \alpha \to 0^+.$$

Since $(1-e^{-t}) = t + \mathrm{O}(1)\,t^2$ as $t \to 0$, we have

$$i_2 = \frac{\mathrm{O}(1)}{\Gamma(\alpha)} \int_0^1 t^\alpha\, dt = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+.$$

Now by making use of the inequality (5) with $\beta = \frac{n+2\nu}{p}$, we have

$$\begin{aligned} i_3 &\leqslant \frac{1}{\Gamma(\alpha)} \left(c\,\|f\|_p \int_0^\infty t^{\alpha-\beta-1}\, dt + \|f\|_\infty \int_0^\infty e^{-t}\, dt \right) \\ &= \frac{\mathrm{O}(1)}{\Gamma(\alpha)} = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+. \end{aligned}$$

Finally,

$$\left\| I_\nu^\alpha f - f \right\|_\infty \leqslant i_1 + i_2 + i_3 = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+. \qquad \square$$

Remark 12. It is interesting to observe that the order of approximation does not depend on the "Lipschitz degree" λ of the function *f*.

The following theorem constitutes a local behavior of the family $(A^\alpha f)(x)$ as $\alpha \to 0^+$ at a "Lipschitz point" x_0 of f. Given a λ, $0 < \lambda \leqslant 1$ and $x_0 \in \mathbb{R}^n$, we define (compare with (21))

$$\Lambda_\lambda(x_0) = \left\{ f : |f(x_0 - y) - f(x_0)| \leqslant c_f\, |y|^\lambda,\ \forall\, |y| \leqslant 1 \right\}.$$

Similarly, for $0 < \lambda \leqslant 1$ and $x_0 \in \mathbb{R}^n_+$ we define (compare with (22))

$$\tilde{\Lambda}_\lambda(x_0) = \left\{ f : \left| \left(T^y f \right)(x_0) - f(x_0) \right| \leqslant c_f\, |y|^\lambda,\ \forall\, |y| \leqslant 1, \left(y \in \mathbb{R}^n_+ \right) \right\},$$

where T^y is the generalized translation given by (16).

Theorem 13. (a) *Let* $f \in L_p(\mathbb{R}^n, dx) \cap \Lambda_\lambda(x_0)$, $1 \leqslant p < \infty$, $0 < \lambda \leqslant 1$. *Let further* A^α *be any of the potentials* I^α *and* J^α, $\alpha > 0$. *Then*

$$\left(A^\alpha f \right)(x_0) - f(x_0) = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+.$$

(b) *Let* $f \in L_{p,\nu} \cap \tilde{\Lambda}_\lambda(x_0)$, $1 \leqslant p < \infty$, $0 < \lambda \leqslant 1$. *Let further* A^α *be any of the generalized potentials* I_ν^α *and* J_ν^α, $\alpha > 0$. *Then*

$$\left(A^\alpha f \right)(x_0) - f(x_0) = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+.$$

Proof. As in Theorem 11, we will prove only the case of $A^\alpha = I_\nu^\alpha$. The other statements of the theorem are proved analogously by making use of Lemma 6. Using (32), we have

$$\begin{aligned} \left(I_\nu^\alpha f \right)(x_0) - f(x_0) &= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} \left((\mathcal{P}_t^{(\nu)} f)(x_0) - f(x_0) \right) dt \\ &\quad + \frac{f(x_0)}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} \left(1 - e^{-t} \right) dt \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^\infty t^{\alpha-1} \left((\mathcal{P}_t^{(\nu)} f)(x_0) - e^{-t} f(x_0) \right) dt \\ &= i_1 + i_2 + i_3. \end{aligned}$$

Further,

$$\begin{aligned}\left|\left(\mathcal{P}_t^{(v)} f\right)(x_0) - f(x_0)\right| &= \left|\int_{\mathbb{R}_+^n} P^{(v)}(y,t)\left(\left(T^y f\right)(x_0) - f(x_0)\right) y_n^{2v}\,dy\right| \\ &\leqslant \int_{|y|<1} P^{(v)}(y,t)\left|\left(T^y f\right)(x_0) - f(x_0)\right| y_n^{2v}\,dy \\ &\quad + \int_{|y|>1} P^{(v)}(y,t)\left|\left(T^y f\right)(x_0)\right| y_n^{2v}\,dy \\ &\quad + \left|f(x_0)\right| \int_{|y|>1} P^{(v)}(y,t)\, y_n^{2v}\,dy = j_1 + j_2 + j_3.\end{aligned}$$

Since $f \in \tilde{\Lambda}_\lambda(x_0)$, we have

$$j_1 \overset{(19)}{\leqslant} c_1 \int_{|y|<1} \frac{t}{\left(|y|^2 + t^2\right)^{(n+2v+1)/2}} |y|^\lambda y_n^{2v}\,dy \leqslant c_2 t^\lambda.$$

By the Hölder inequality,

$$\begin{aligned} j_2 &\leqslant \|f\|_p \left(\int_{|y|>1} \left(P^{(v)}(y,t)\right)^{p'} y_n^{2v}\,dy\right)^{1/p'} \\ &= c_3 t \left(\int_{|y|>1} \left(|y|^2 + t^2\right)^{-p'(n+2v+1)/2} y_n^{2v}\,dy\right)^{1/p'} \\ &\leqslant c_4 t \left(\int_{|y|>1} |y|^{-p'(n+2v+1)} y_n^{2v}\,dy\right)^{1/p'} \leqslant c_5 t.\end{aligned}$$

Similarly,

$$j_3 = c_6 t \int_{|y|>1} \left(|y|^2 + t^2\right)^{-(n+2v+1)/2} y_n^{2v}\,dy \leqslant c_7 t.$$

Therefore

$$\left|(\mathcal{P}_t^{(v)} f)(x_0) - f(x_0)\right| = \mathrm{O}(1)\, t^\lambda \quad \text{as } t \to 0. \tag{33}$$

Using (33), we get

$$|i_1| \leqslant \frac{c}{\Gamma(\alpha)} \int_0^1 t^{\alpha+\lambda-1}\,dt = \frac{c}{\Gamma(\alpha)} \frac{1}{\alpha+\lambda} = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+. \tag{34}$$

Further, since $\left(1 - e^{-t}\right) = t + \mathrm{O}(1)\, t^2$ as $t \to 0$, we get

$$|i_2| \leqslant \frac{c}{\Gamma(\alpha)} \int_0^1 t^\alpha\,dt = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+. \tag{35}$$

By making use of (5) with $\beta = (n+2v)/p$, we have

$$\begin{aligned}|i_3| &\leqslant \frac{1}{\Gamma(\alpha)} \left(c\,\|f\|_p \int_0^\infty t^{\alpha-\beta-1}\,dt + |f(x_0)| \int_0^\infty e^{-t}\,dt\right) \\ &= \frac{\mathrm{O}(1)}{\Gamma(\alpha)} = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+. \end{aligned} \tag{36}$$

Finally, by (34)–(36), it follows that

$$\left(I_\nu^\alpha f\right)(x_0) - f(x_0) = \mathrm{O}(1)\,\alpha \quad \text{as } \alpha \to 0^+.$$

The proof is completed. □

Remark 14. As in Theorem 11, the order of approximation does not depend on the "Lipschitz degree" λ of the function *f*.

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