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A General Left-Definite Theory for Certain Self-Adjoint Operators with Applications to Differential Equations

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We show that any self-adjoint operator A (bounded or unbounded) in a Hilbert space $H = (V, (\cdot,\cdot))$ that is bounded below generates a continuum of Hilbert spaces $\{H_r\}_{r>0}$ and a continuum of self-adjoint operators $\{A_r\}_{r>0}$. For reasons originating in the theory of differential operators, we call each H_r the rth *left-definite space* and each A_r the rth *left-definite operator* associated with (H, A). Each space H_r can be seen as the closure of the domain $\mathscr{D}(A^r)$ of the self-adjoint operator A^r in the topology generated from the inner product $(A^r x, y)$ $(x, y \in \mathscr{D}(A^r))$. Furthermore, each A_r is a unique self-adjoint restriction of A in H_r. We show that the spectrum of each A_r agrees with the spectrum of A and the domain of each A_r is characterized in terms of another left-definite space. The Hilbert space spectral theorem plays a fundamental role in these constructions. We apply these results to two examples, including the classical Laguerre differential expression $\ell[\cdot]$ in which we explicitly find the left-definite spaces and left-definite operators associated with A, the self-adjoint operator generated by $\ell[\cdot]$ in $L^2((0,\infty); t^\alpha e^{-t})$ having the Laguerre polynomials as eigenfunctions.

Key Words: spectral theorem; self-adjoint operator; Hilbert space; Sobolev space; Dirichlet inner product; left-definite Hilbert space; left-definite self-adjoint operator; Laguerre polynomials; Stirling numbers of the second kind.

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1. INTRODUCTION AND MOTIVATION

In this paper, we prove that if A is a self-adjoint operator in a Hilbert space $H = (V, (\cdot,\cdot))$ that is bounded below by a positive constant k, that is, if

$$(Ax, x) \geqslant k(x, x) \qquad (x \in \mathscr{D}(A)),$$

then there are a continuum of unique Hilbert spaces $\{H_r\}_{r>0}$ (which we call *left-definite Hilbert spaces*) and operators $\{A_r\}_{r>0}$ in H_r (called *left-definite operators*), with each A_r being a unique self-adjoint restriction of A in H_r. We explicitly determine these Hilbert spaces H_r, together with their inner products $(\cdot,\cdot)_r$, as specific vector subspaces of H. Moreover, we are able to explicitly specify the domains of each operator A_r as certain left-definite spaces, and we show that the spectrum of each A_r is identical with the spectrum of A. The key result, as we will see, that allows for a determination of these spaces and operators is the classical Hilbert space spectral theorem.

Each of these Hilbert spaces and associated inner products can be viewed as a generalization of a *left-definite* Hilbert space and Dirichlet inner product, respectively, from the theory of self-adjoint differential operators. However, we emphasize that the results developed in this paper apply to *arbitrary* self-adjoint operators in a Hilbert space that are bounded below (see the example in Section 11). It is the case, however, that our original motivation stems from the study of certain differential equations of the form

$$s[y](t) = \lambda w(t) y(t) \qquad (t \in I), \tag{1.1}$$

where $s[\cdot]$ is a Lagrangian symmetric differential expression of order $2n$ given by

$$s[y](t) := \sum_{j=0}^{n} (-1)^j (b_j(t) y^{(j)}(t))^{(j)} \qquad (t \in I). \tag{1.2}$$

Here $I = (a, b)$ is an open interval of the real line $\mathbb{R}$, $w(t) > 0$ for $t \in I$, and each coefficient $b_j(t)$ is positive and infinitely differentiable on I. Such equations arise in the functional analytic study of differential equations having orthogonal polynomial solutions (see [29] for a general discussion of the connections between orthogonal polynomials and differential equations). For further information in this context, see Sections 12 and 13 in this paper for specific examples of differential equations of this type having orthogonal polynomial solutions.

One particular setting for the spectral study of (1.2) is the Hilbert space $L^2(I; w)$, defined by

$$L^2(I; w) = \{f\colon I \to \mathbb{C} \mid f \text{ is Lebesgue measurable and } \int_I |f(t)|^2 w(t)dt < \infty\},$$

with inner product

$$(f, g) = \int_a^b f(t)\bar{g}(t)w(t)dt.$$

Due to the appearance of w on the *right-hand* side of (1.1), it is natural to refer to the $L^2(I; w)$ setting as the classic *right-definite* spectral setting for $w^{-1}s[\cdot]$.

For functions $f, g \in \Delta_{\max}$, the maximal domain of $w^{-1}s[\cdot]$ in $L^2(I; w)$ (see [33, Chap. V] for definitions and notation), we have Green's formula

$$\int_a^b s[f](t)\bar{g}(t)dt = \int_a^b f(t)\overline{s[g]}(t)dt + [f, g](t)|_{t=a}^{t=b} \qquad (f, g \in \Delta_{\max}), \tag{1.3}$$

where $[\cdot, \cdot]$ is the skew-symmetric sesquilinear form for $s[\cdot]$. A related formula – and the central motivating factor for the work that we present in this paper – is *Dirichlet's formula*,

$$\int_a^b s[f](t)\bar{g}(t)dt = \sum_{j=0}^{n} \int_a^b b_j(t)f^{(j)}(t)\bar{g}^{(j)}(t)dt + \{f, g\}(t)|_{t=a}^{t=b} \qquad (f, g \in \Delta_{\max}), \tag{1.4}$$

where $\{\cdot, \cdot\}$ is another bilinear form, closely related to the $[\cdot, \cdot]$ given in (1.3).

There are two well-known operators generated by $w^{-1}s[\cdot]$ in $L^2(I; w)$, the minimal and maximal operators $T_{\min}$ and $T_{\max}$ defined, respectively, by

$$T_{\min}f := w^{-1}s[f] \qquad (f \in \Delta_{\min}),$$

and

$$T_{\max}f := w^{-1}s[f] \qquad (f \in \Delta_{\max}).$$

These operators are adjoints of each other; furthermore, $T_{\min}[\cdot]$ is symmetric in $L^2(I; w)$. The well-established Glazman–Krein–Naimark Theorem (see

[33, Section 18]) of self-adjoint extensions of symmetric differential operators then determines, through appropriate boundary conditions, the various self-adjoint extensions A of $T_{\min}$ (or, equivalently, self-adjoint restrictions of the maximal operator $T_{\max}$).

To continue our motivation for this paper, suppose A: $\mathscr{D}(A) \subset L^2(I; w) \to L^2(I; w)$ is a self-adjoint extension of $T_{\min}$ such that

$$(Af, g) = \int_a^b s[f](t)\bar{g}(t)dt = \sum_{j=0}^{n} \int_a^b b_j(t)f^{(j)}(t)\bar{g}^{(j)}(t)dt \qquad (f, g \in \mathscr{D}(A)); \tag{1.5}$$

That is to say, for all $f, g \in \mathscr{D}(A)$, the evaluation of the Dirichlet form $\{f, g\}(t)|_{t=a}^{t=b}$ in (1.4) is zero (of course, such an A may or may not exist, in general). Furthermore, suppose that $b_0(t) \geqslant k > 0$ for all $t \in I$, where k is a positive constant. Then, from (1.5) and our assumed positivity of the coefficients b_j on (a, b), we find that A satisfies

$$(Af, f) \geqslant k(f, f) \qquad (f \in \mathscr{D}(A)). \tag{1.6}$$

Moreover, we see that $s[\cdot]$ generates, through (1.5), a Sobolev space H_1 with inner product (called the *Dirichlet inner product*)

$$(f, g)_1 := \sum_{j=0}^{n} \int_a^b b_j(t)f^{(j)}(t)\bar{g}^{(j)}(t)dt \qquad (f, g \in H_1); \tag{1.7}$$

for physical reasons, the norm generated from this inner product is also called the *energy norm* (see [32, p. 12]). More specifically, H_1 is defined to be the closure of $\mathscr{D}(A)$ in the topology generated by the norm $\|\cdot\|_1 = (\cdot,\cdot)_1^{1/2}$. Observe that, from (1.5) and (1.7), we have

$$(Af, g) = (f, g)_1 \qquad (f, g \in \mathscr{D}(A)). \tag{1.8}$$

Since the inner product $(\cdot,\cdot)_1$ is generated from the *left-hand* side of (1.1), the literature refers to H_1 as the *left-definite setting* for $w^{-1}s[\cdot]$ and calls H_1 the *left-definite* Hilbert space associated with the expression $w^{-1}s[\cdot]$. Actually, in the notation of this paper, H_1 is the *first* left-definite space associated with A. As this paper shows, there are actually a continuum of left-definite Hilbert spaces associated with such an operator A.

It is possible to extend the identity in (1.8) to obtain

$$(Af, g) = (f, g)_1 \qquad (f \in \mathscr{D}(A), g \in H_1). \tag{1.9}$$

From the inequality (1.6), it follows that $0 \in \rho(A)$, the resolvent set of A. Consequently, we see that $R_0(A) = A^{-1}$ is a bounded operator from H_1 onto $\mathscr{D}(A)$. Furthermore, from the inclusion

$$\mathscr{D}(A) \subset H_1 \subset L^2(I; w)$$

and (1.9), it follows that the operator B: $H_1 \to H_1$, defined by

$$Bf = R_0(A)f \qquad (f \in \mathscr{D}(B) := H_1),$$

is an invertible, self-adjoint operator. The inverse of B, denoted here by A_1, is also a self-adjoint operator. In the literature, A_1 is called the *left-definite operator* associated with A. As we see in this paper, it is more appropriate to name A_1 the *first* left-definite operator associated with A. In fact, we construct a continuum of left-definite self-adjoint operators $\{A_r\}_{r>0}$ associated with the original operator A, with each A_r being a unique self-adjoint restriction of A in H_r.

To emphasize our starting point in this paper, we begin with a self-adjoint operator A that is bounded below in H by a positive constant. In the theory of differential operators, A corresponds to a right-definite operator generated from the differential expression $w^{-1}s[\cdot]$ (as given in (1.1) and (1.2)) in $L^2(I; w)$. However, it is possible that the differential expression $w^{-1}s[\cdot]$ is not right-definite – for example, the function w may be signed on I – and yet $s[\cdot]$ is left-definite (that is, each coefficient $b_j > 0$ on I). This approach is taken by Kong *et al.* in [18] in their left-definite study of the classic, regular Sturm–Liouville equation

$$-(py')' + qy = \lambda wy,$$

on I.

The history of left-definite spectral theory – as it relates to differential operators – can be traced back to the work of Weyl [50] who, in his landmark analysis of second-order Sturm–Liouville differential equations, coined the term *polare-Eigenwertaufgabe* for the study of second-order equations in the left-definite setting. The terminology *left-definite* (actually, the German *Links-definit*) first appeared in the literature in 1965 in a paper by Schäfke and Schneider (see [44]). In his book [17], Kamke uses the term *F-definit* in his study of the differential equation $Fy = \lambda Gy$ (he also uses *G-definit* for his right-definite study of this equation). In [34–36], Niessen and Schneider considered general left-definite singular systems and left-definite s-hermitian problems. Pleijel ([38,39]) provided one of the first concrete examples of such a left-definite setting for a self-adjoint differential operator with his analysis of the classical second-order Legendre equation.

His work was followed soon after by the work of Atkinson *et al.* [3] who examined left-definite square-integrable homogeneous solutions. Later, Everitt [6] gave a complete (first) left-definite analysis of the classical Legendre equation and his student, Onyango-Otieno [37], extended these results by analyzing the appropriate right-definite and first left-definite spectral settings for the differential equations having the classical orthogonal polynomials (Jacobi, Laguerre, Hermite) as solutions. Everitt, in [7], and Bennewitz and Everitt [4] furthered the general theory of left-definite operators associated with second-order differential equations.

During the past 15 years, there have been several additional papers dealing with theory and specific examples of left-definite operators, all within the framework of differential operators. Important results related to second-order equations have been obtained by Krall ([19,20,22,23]), Krall and Littlejohn [21], and Hajmirzaahmad ([16]). Left-definite results for higher-order differential equations have been obtained by Everitt and Littlejohn [11], Everitt *et al.* [9,10,13,14], Loveland [30], and Wellman [49]. More recently, Vonhoff [48] has reconsidered the left-definite analysis of the fourth-order Legendre-type equation based on ideas developed in his thesis [47].

In this paper, we attempt to provide a framework for a general left-definite theory of bounded-below, self-adjoint operators in a Hilbert space. The contents of this paper are as follows. In Section 2 we define the general concept of a left-definite Hilbert space and a left-definite operator associated with a self-adjoint operator that is bounded below. Section 3 contains statements of our main results, with proofs of these theorems given in Sections 6 through 10. In Section 4, we recall the spectral theorem and some of its immediate consequences that we need in our presentation. Our first example of the theory developed in this paper concerns an unbounded self-adjoint operator A in ℓ^2, the classical Hilbert space of square-summable sequences; this example is described in Section 11. In Section 12, we apply the results of this paper to the second-order classical Laguerre differential expression $\ell[\cdot]$. More specifically, for integral values of r, we will explicitly exhibit the left-definite Hilbert space $\{H_r\}$ and the left-definite operators $\{A_r\}$ associated with the self-adjoint operator A in the Hilbert space $H = L^2((0,\infty); t^\alpha e^{-t})$, generated by $\ell[\cdot]$, having the Laguerre polynomials $\{L_m^\alpha(t)\}_{m=0}^\infty$ as eigenfunctions. As we will see, each of the left-definite inner products can be seen as the Dirichlet inner product of the form (1.7) obtained from taking formal integral powers of the differential expression $\ell[\cdot]$. Lastly, in Section 13, we outline a number of other applications (in particular, from [11] and [9]) and open problems resulting from this work.

2. AN ABSTRACT DEFINITION OF A LEFT-DEFINITE SPACE AND A LEFT-DEFINITE OPERATOR

For the remainder of this paper, let V be a vector space (over the complex field $\mathbb{C}$) with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$; the resulting inner product space is denoted $(V,(\cdot,\cdot))$. Suppose V_r (the subscripts will be made clear shortly) is a (vector) subspace (i.e., a linear manifold) of V and let $(\cdot,\cdot)_r$ and $\|\cdot\|_r$ denote, respectively, an inner product (quite possibly different from $(\cdot,\cdot)$) and an associated norm on V_r.

We begin with the following definition of a left-definite Hilbert space.

Definition 2.1. Let $H = (V,(\cdot,\cdot))$ be a Hilbert space. Suppose A: $\mathscr{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$; i.e.,

$$(Ax, x) \geqslant k(x, x) \qquad (x \in \mathscr{D}(A)).$$

Let $H_1 = (V_1,(\cdot,\cdot)_1)$, where V_1 is a subspace of V and $(\cdot,\cdot)_1$ is an inner product on V_1. Then H_1 is said to be a *left-definite* (*Hilbert*) *space* associated with the pair (H, A), if each of the following conditions holds:

(1) H_1 is a Hilbert space,

(2) $\mathscr{D}(A)$ is a subspace of V_1,

(3) $\mathscr{D}(A)$ is dense in H_1,

(4) $(x,x)_1 \geqslant k(x,x)$ $(x \in V_1)$, and

(5) $(x,y)_1 = (Ax,y)$ $(x \in \mathscr{D}(A),\ \ y \in V_1)$.

Given a self-adjoint operator A that is bounded below by a positive constant, it is not clear that a left-definite space H_1 exists for the pair (H, A). In fact, however, we prove the existence and *uniqueness* of this Hilbert space later in this paper; see Theorem 3.1.

If A is a self-adjoint operator in H that is bounded below by a positive number k, then, with assistance from the spectral theorem (see Section 4 and, in particular, Theorem 4.3), we see that A^r is a self-adjoint operator bounded below by $k^r I$ for each $r > 0$. Consequently, we can extend Definition 2.1 to a continuum of left-definite spaces associated with (H, A).

Definition 2.2. Let $H = (V,(\cdot,\cdot))$ be a Hilbert space. Suppose A: $\mathscr{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$; i.e.,

$$(Ax, x) \geqslant k(x, x) \qquad (x \in \mathscr{D}(A)).$$

Let $r > 0$. If there exists a subspace V_r of V and an inner product $(\cdot,\cdot)_r$ on V_r such that $H_r = (V_r, (\cdot,\cdot)_r)$ is a left-definite space associated with the pair (H, A^r), we call H_r an rth *left-definite space associated with the pair* (H, A). Specifically, H_r is an rth left-definite space associated with the pair (H, A) if each of the following conditions hold:

(1) H_r is a Hilbert space,

(2) $\mathscr{D}(A^r)$ is a subspace of V_r,

(3) $\mathscr{D}(A^r)$ is dense in H_r,

(4) $(x,x)_r \geqslant k^r(x,x)$ $(x \in V_r)$, and

(5) $(x,y)_r = (A^r x, y)$ $(x \in \mathscr{D}(A^r),\ \ y \in V_r)$.

From our discussion above, we will see below in Theorem 3.1 that, for each $r > 0$, H_r exists and is unique. At first glance, it appears that the rth left-definite space H_r depends on H, A, and the positive number k satisfying condition (4) in the above definition. In fact, however, each of the left-definite spaces H_r is independent of k; a specific reason will be given in Section 6 after the proof of Theorem 3.1.

We are now in position to define a left-definite operator associated with A.

Definition 2.3. Let $H = (V, (\cdot,\cdot))$ be a Hilbert space. Suppose A: $\mathscr{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by a positive number $k > 0$. Let $r > 0$ and suppose H_r is the rth left-definite space associated with (H, A). If there exists a self-adjoint operator A_r: $H_r \to H_r$ that is a restriction of A; that is to say,

$$A_r x = Ax,$$
$$x \in \mathscr{D}(A_r) \subset \mathscr{D}(A), \tag{2.1}$$

we call such an operator an rth *left-definite operator associated with* (H, A).

In Theorem 3.2 below we prove that if A is a self-adjoint operator that is, is bounded below by a positive number $k > 0$, then for all $r > 0$ there exists a *unique* left-definite operator A_r in H_r associated with (H, A).

3. STATEMENTS OF MAIN RESULTS

There are six main theorems that we prove in this paper concerning left-definite Hilbert spaces and left-definite self-adjoint operators. The Hilbert-space spectral theorem (see [41] or [43]) is essential in establishing most of these results.

Theorem 3.1. *Suppose A is a self-adjoint operator in the Hilbert space $H = (V, (\cdot,\cdot))$ that is bounded below by kI, where $k > 0$. Let $r > 0$. Define $H_r = (V_r, (\cdot,\cdot)_r)$ with*

$$V_r = \mathscr{D}(A^{r/2}) \tag{3.1}$$

and

$$(x, y)_r = (A^{r/2}x, A^{r/2}y) \qquad (x, y \in V_r). \tag{3.2}$$

Then H_r is an rth *left-definite space associated with the pair* (H, A) *in the sense of Definition* 2.2. *Moreover, suppose* $H_r = (V_r, (\cdot,\cdot)_r)$ *and* $H_r' = (V_r', (\cdot,\cdot)_r')$ *are rth left-definite spaces associated with the pair* (H, A). *Then* $V_r = V_r'$ *and* $(x, y)_r = (x, y)_r'$ *for all* $x, y \in V_r = V_r'$; *i.e.,* $H_r = H_r'$. *Consequently* $H_r = (V_r, (\cdot,\cdot)_r)$, *as defined in* (3.1) *and* (3.2), *is* the *unique rth left-definite Hilbert space associated with* (H, A).

Proof. See Section 6. ∎

THEOREM 3.2. *Suppose A is a self-adjoint operator in a Hilbert space H that is bounded below by kI for some $k > 0$. For $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ be the rth left-definite space associated with (H, A). Then there exists a unique left-definite operator A_r in H_r associated with (H, A). More specifically, if there exists a self-adjoint operator $\tilde{A}_r: H_r \to H_r$ such that $\tilde{A}_r x = Ax$ for all $x \in \mathscr{D}(\tilde{A}_r) \subset \mathscr{D}(A)$, then $A_r = \tilde{A}_r$. Furthermore,*

$$\mathscr{D}(A_r) = V_{r+2}. \tag{3.3}$$

and A_r is bounded below by kI in H_r.

Proof. See Section 7. ∎

The following corollary is an immediate consequence of Theorems 3.1 and 3.2. It emphasizes the fact that, set-wise, the domain $\mathscr{D}(A^r)$ of the rth power of A is given by V_{2r} and, in particular, the first and second left-definite spaces associated with A are, respectively, the domain of the positive square root of A and the domain of A. Furthermore, it describes explicitly the domain of the rth left-definite operator in terms of the domain of a certain power of A. Interestingly, we note that the domains of the first and second left-definite operators, A_1 and A_2, are given by $\mathscr{D}(A^{3/2})$ and $\mathscr{D}(A^2)$, respectively.

COROLLARY 3.3. *Suppose A is a self-adjoint operator in the Hilbert space H that is bounded below by kI, where $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ and A_r denote, respectively, the rth left-definite space and the rth left-definite operator associated with (H, A). Then*

(1) $\mathscr{D}(A^r) = V_{2r}$, *in particular,* $\mathscr{D}(A^{1/2}) = V_1$ *and* $\mathscr{D}(A) = V_2$;

(2) $\mathscr{D}(A_r) = \mathscr{D}(A^{(r+2)/2})$, *in particular,* $\mathscr{D}(A_1) = \mathscr{D}(A^{3/2})$ *and* $\mathscr{D}(A_2) = \mathscr{D}(A^2)$.

In the next theorem, we see that when A is a bounded, self-adjoint operator that is bounded below by a positive constant k, then the left-definite theory is trivial. However, the situation is quite different when A is unbounded.

THEOREM 3.4. *Let $H = (V, (\cdot,\cdot))$ be a Hilbert space. Suppose A: $\mathscr{D}(A) \subset H \to H$ is a self-adjoint operator that is bounded below by kI for some $k > 0$. For each $r > 0$, let $H_r = (V_r, (\cdot,\cdot)_r)$ and A_r denote the rth left-definite space and the rth left-definite operator, respectively, associated with (H, A).*

(1) *Suppose A is bounded. Then, for each $r > 0$,*
(i) $V = V_r$;
(ii) *the inner products* $(\cdot,\cdot)$ and $(\cdot,\cdot)_r$ *are equivalent*;
(iii) $A = A_r$.

(2) *Suppose A is unbounded. Then*

(i) *V_r is a proper subspace of V*;
(ii) *V_s is a proper subspace of V_r whenever $0 < r < s$*;
(iii) *the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_s$ are not equivalent for any $s > 0$*;
(iv) *the inner products $(\cdot,\cdot)_r$ and $(\cdot,\cdot)_s$ are not equivalent for any $r, s > 0$, $r \neq s$;*(v)*$\mathscr{D}(A_r)$ is a proper subspace of $\mathscr{D}(A)$ for each $r > 0$*;
(vi) *$\mathscr{D}(A_s)$ is a proper subspace of $\mathscr{D}(A_r)$ whenever $0 < r < s$*;

Proof. See Section 8. ∎

Since, for each $m > 0$, A^m is a self-adjoint operator that is bounded below in H by $k^m I$, we see from Theorems 3.1 and 3.2 that there are a continua of left-definite spaces $\{(H^m)_r\}_{r>0}$ and left-definite operators $\{(A^m)_r\}_{r>0}$ associated with the pair (H, A^m). Furthermore, since A_m is a self-adjoint operator that is bounded below by kI in H_m, there are continua of left-definite spaces $\{(H_m)_r\}_{r>0}$ and left-definite operators $\{(A_m)_r\}_{r>0}$ associated with the pair (H_m, A_m). The following questions naturally arise:

(1) What is the relationship (if any) between the three continua of the left-definite spaces $\{H_r\}_{r>0}$, $\{(H^m)_r\}_{r>0}$, and $\{(H_m)_r\}_{r>0}$?

(2) Since for fixed $m > 0$, $(A_r)^m$ – the mth power of the rth left-definite operator A_r associated with (H, A) – is a self-adjoint restriction of A^m, what is the relationship (if any) between the continuum of left-definite operators $\{(A^m)_r\}_{r>0}$ associated with the pair (H, A^m) and the continuum of self-adjoint operators $\{(A_r)^m)\}_{r>0}$? In particular, is $(A_r)^m$ a left-definite operator associated with (H, A^m); that is to say, is $(A_r)^m \in \{(A^m)_s\}_{s>0}$?

(3) For fixed $m > 0$, what is the relationship (if any) between the continuum of left-definite operators $\{(A_m)_r\}_{r>0}$ associated with the pair (H_m, A_m) and the continuum of left-definite operators $\{A_r\}_{r>0}$ associated with the pair (H, A)?

Each of these questions is answered in the following theorem. In essence, this theorem says that there are no new left-definite spaces or left-definite operators emerging from a consideration of the above questions; that is to say, the original spaces $\{H_r\}_{r>0}$ and operators $\{A_r\}_{r>0}$ encompass all of the left-definite spaces and left-definite operators described above that are associated with the pairs (H, A^m) and (H_m, A_m).

THEOREM 3.5. *Suppose A, H, $\{H_r\}_{r>0}$, and $\{A_r\}_{r>0}$ are as in Theorems* 3.1 *and* 3.2 *above. Fix $m > 0$. For each $r > 0$, let $(H^m)_r = ((V^m)_r, (\cdot,\cdot)_r^m)$ and $(A^m)_r$ denote, respectively, the rth left-definite space and the rth left-definite operator associated with the pair (H, A^m). Then*

(a) $(H^m)_r = H_{mr}$.

(b) $(A_r)^m = (A^m)_{r/m}$ *with* $\mathscr{D}((A_r)^m) = V_{2m+r}$. *Equivalently,* $(A^m)_r = (A_{mr})^m$ *with* $\mathscr{D}((A^m)_r) = V_{2m+mr}$*; that is to say, the* rth *left-definite operator associated with the pair (H, A^m) is the mth power of the (mr)th left-definite operator associated with (H, A).*

Furthermore, let $(H_m)_r = ((V_m)_r, (\cdot,\cdot)_{m,r})$ and $(A_m)_r$ denote the rth left-definite space and the rth left-definite operator, respectively, associated with (H_m, A_m). Then

(c) $(H_m)_r = H_{m+r}$.

(d) $(A_m)_r = A_{m+r}$ *with* $\mathscr{D}((A_m)_r) = V_{m+r+2}$*; in other words, the rth left-definite operator associated with (H_m, A_m) is the $(m+r)$th left-definite operator associated with (H, A).*

Proof. See Section 9. ∎

In addition, we prove the following two theorems concerning the spectra of the left-definite operators $\{A_r\}_{r>0}$.

THEOREM 3.6. *For each $r > 0$, let A_r denote the rth left-definite operator associated with the self-adjoint operator A that is bounded below by kI where $k > 0$. Then*

(a) *The point spectra of A and A_r coincide; i.e., $\sigma_p(A_r) = \sigma_p(A)$.*

(b) *The continuous spectra of A and A_r coincide; i.e., $\sigma_c(A_r) = \sigma_c(A)$.*

(c) *The resolvents of A and A_r coincide; i.e., $\rho(A) = \rho(A_r)$.*

Proof. See Section 10. ∎

Finally, the last general result in this paper is the following theorem.

THEOREM 3.7. *If $\{\varphi_n\}_{n=0}^{\infty}$ is a complete orthogonal set of eigenfunctions of A in H, then for each $r > 0$, $\{\varphi_n\}_{n=0}^{\infty}$ is a complete set of orthogonal* *eigenfunctions of the rth left-definite operator A_r in the rth left-definite space H_r.*

Proof. See Section 10. ∎

4. THE SPECTRAL THEOREM

If A is a self-adjoint operator in a Hilbert space H with inner product $(\cdot,\cdot)$, it is well known (see [43, Chaps. 12 and 13] that there exists a unique operator-valued set functions $E\colon \mathscr{B} \to B(H)$, where $\mathscr{B}$ is the σ-algebra of Borel subsets of $\mathbb{R}$ and $B(H)$ is the Banach algebra of bounded linear operators on H, called the *spectral resolution of the identity*, having the following properties:

$$\begin{aligned}
&(1)\ E(\emptyset) = 0 \text{ and } E(\mathbb{R}) = I.\\
&(2)\ E(\Delta) \text{ is idempotent; that is, } (E(\Delta))^2 = E(\Delta), \text{ for all } \Delta \in \mathscr{B}.\\
&(3)\ E(\Delta) \text{ is self-adjoint in } H \text{ for all } \Delta \in \mathscr{B}.\\
&(4)\ E(\Delta_1 \cap \Delta_2) = E(\Delta_1)E(\Delta_2) = E(\Delta_2)E(\Delta_1) \text{ for all } \Delta_1, \Delta_2 \in \mathscr{B}. \qquad (4.1)\\
&(5)\ E(\Delta_1 \cup \Delta_2) = E(\Delta_1) + E(\Delta_2) \text{ for all } \Delta_1, \Delta_2 \in \mathscr{B} \text{ with } \Delta_1 \cap \Delta_2 = \emptyset.\\
&(6)\ \text{For each } x, y \in H, \text{ the mapping } E_{x,y}\colon \mathscr{B} \to \mathbb{C} \text{ defined by } E_{x,y}(\Delta)\\
&:= (E(\Delta)x, y) \text{ is a complex, regular Borel measure.}
\end{aligned}$$

Since $E(\Delta)$ is a self-adjoint projection for each $\Delta \in \mathscr{B}$, it follows that $\|E(\Delta)\| \leqslant 1$.

A *spectral family* (see [25] or [41]) for a self-adjoint operator A is a one-parameter family $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ of bounded operators in H satisfying:

$$\begin{aligned}
&(1)\quad E_\lambda \text{ is self-adjoint and idempotent for each } \lambda \in \mathbb{R}.\\
&(2)\quad \text{For } \lambda < \mu,\ E_\mu - E_\lambda \text{ is a positive operator.}\\
&(3)\quad \lim_{\lambda\to\infty} E_\lambda x = x \text{ for each } x \in H. \qquad (4.2)\\
&(4)\quad \lim_{\lambda\to-\infty} E_\lambda x = 0 \text{ for each } x \in H.\\
&(5)\quad E_{\lambda+0}x := \lim_{\mu\to\lambda^+} E_\mu x = E_\lambda x \text{ for each } \lambda \in \mathbb{R} \text{ and } x \in H.
\end{aligned}$$

A connection between (4.1) and (4.2) lies in the following lemma; the proof is straightforward.

LEMMA 4.1. *Suppose E is a spectral resolution of the identity in the sense of* (4.1). *For $\lambda \in \mathbb{R}$, define $E_\lambda = E(-\infty, \lambda]$. Then $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a spectral family in the sense of* (4.2).

As mentioned earlier, the Hilbert-space spectral theorem plays a key role in proving the existence and uniqueness of the left-definite spaces $\{H_r\}_{r>0}$ and the left-definite operators $\{A_r\}_{r>0}$ associated with the pair (H, A), where A is a self-adjoint operator in H that is bounded below by kI, for some $k > 0$. In our development of these spaces and operators, we use the spectral resolution of the identity E of A rather than the one-parameter spectral family. However, properties of the spectrum $\sigma(A_r)$ and the resolvent set $\rho(A_r)$ of each left-definite operator A_r are more easily seen through the spectral family rather than the spectral resolution of the identity. Indeed, the following theorem is well known (see [25, Sect. 9.11] and [41, Sect. 132].

Theorem 4.2. *Suppose $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ is a spectral family, satisfying the conditions of* (4.2), *of a self-adjoint operator A. For $\lambda_0 \in \mathbb{R}$, we have*:

(a) $\lambda_0 \in \sigma_p(A)$ (*the point spectrum*) *if and only if* $E_{\lambda_0} \neq E_{\lambda_0 - 0}$.

(b) $\lambda_0 \in \sigma_c(A)$ (*the continuous spectrum*) *if and only if* $E_{\lambda_0} = E_{\lambda_0 - 0}$ *and* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *is not constant on any neighborhood of* λ_0 *in* $\mathbb{R}$.

(c) $\lambda_0 \in \rho(A)$ (*the resolvent set*) *if and only if there exists* $\varepsilon > 0$ *such that* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ *is constant on* $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$.

We are now in position to state the spectral theorem in a Hilbert space (see [43, Theorems 13.24 and 13.30]).

Theorem 4.3. (The Spectral Theorem). *Let A be a self-adjoint operator* (*bounded or unbounded*) *in a Hilbert space $H = (V, (\cdot,\cdot))$. Let E be the spectral resolution of the identity associated with A. Then, for each $r > 0$, the self-adjoint operator A^r has a* (*densely defined*) *domain $\mathscr{D}(A^r)$ given by*

$$\mathscr{D}(A^r) = \left\{ x \in H \,\middle|\, \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} < \infty \right\}, \tag{4.3}$$

and is characterized by the identities

$$(A^r x, y) = \int_{\mathbb{R}} \lambda^r dE_{x,y} \qquad (x \in \mathscr{D}(A^r),\ \ y \in H) \tag{4.4}$$

and

$$\|A^r x\|^2 = \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} \qquad (x \in \mathscr{D}(A^r)). \tag{4.5}$$

Conversely, suppose F: $\mathscr{B} \to B(H)$ is a spectral resolution of the identity. Then there exists a unique self-adjoint operator $\tilde{A}$ in H with (*densely defined*) *domain*

$$\mathscr{D}(\tilde{A}) = \left\{ x \in H \,\middle|\, \int_{\mathbb{R}} \lambda^2 dF_{x,x} < \infty \right\}$$

that is characterized by

$$(\tilde{A}x, y) = \int_{\mathbb{R}} \lambda dF_{x,y} \qquad (x \in \mathscr{D}(\tilde{A}),\ \ y \in H),$$

and

$$\|\tilde{A}x\|^2 = \int_{\mathbb{R}} \lambda^2 dF_{x,x} \qquad (x \in \mathscr{D}(\tilde{A})).$$

Moreover, in this theorem, we can replace the interval $\mathbb{R}$ of integration in each of the above integrals with the spectrum of the self-adjoint operator. In particular, for a self-adjoint operator A that is bounded below by kI for $k > 0$, we can replace the interval of integration $\mathbb{R}$ with $[k, \infty)$ since, in this case, the spectrum $\sigma(A) \subset [k, \infty)$ (see [43, Theorem 12.32]).

5. TECHNICAL LEMMAS

The following results will be used extensively in Sections 6 through 10.

LEMMA 5.1. *Suppose A is a self-adjoint operator in a Hilbert space $H = (V, (\cdot, \cdot))$ and suppose E is the spectral resolution of the identity for A. Then*

$$\begin{aligned} E_{E(\Delta_1)x,y}(\Delta_2) &= E_{x,y}(\Delta_1 \cap \Delta_2) \\ &= E_{x,E(\Delta_2)y}(\Delta_1) = E_{x,E(\Delta_1)y}(\Delta_2) \qquad (\Delta_1, \Delta_2 \in \mathscr{B}), \end{aligned} \tag{5.1}$$

$$E_{x,x}(\Delta) = \|E(\Delta)x\|^2 \qquad (\Delta \in \mathscr{B}), \tag{5.2}$$

and

$$\int_{\mathbb{R}} dE_{x,y} = (E(\mathbb{R})x, y) = (x, y) \qquad (x, y \in H). \tag{5.3}$$

Proof. These properties follow directly from the definition of E so the proof is omitted. ∎

LEMMA 5.2. *Suppose A is a self-adjoint operator in a Hilbert space $H = (V, (\cdot, \cdot))$ that is bounded below by kI for some $k > 0$. Suppose E is the spectral resolution of the identity for A. Then, for each $s > 0$, A^s and $E(\Delta)$ commute for* *all* $\Delta \in \mathscr{B}$; *that is to say*,

$$E(\Delta)A^s x = A^s E(\Delta)x \qquad (\Delta \in \mathscr{B};\ x \in \mathscr{D}(A^s)). \tag{5.4}$$

Proof. Let $\Delta \in \mathscr{B}$ and $x \in \mathscr{D}(A^s)$. Then, for any $B \in \mathscr{B}$,

$$\begin{aligned} E_{E(\Delta)x,E(\Delta)x}(B) &= (E(B)E(\Delta)x, E(\Delta)x) \\ &= (E(\Delta \cap B)x, x) \qquad \text{by (2), (3), and (4) of (4.1)} \\ &= ((E(\Delta \cap B))^2 x, x) \\ &= (E(\Delta \cap B)x, E(\Delta \cap B)x) \qquad \text{by (2) and (3) of (4.1)} \\ &= \|E(\Delta \cap B)x\|^2 = E_{x,x}(\Delta \cap B) \qquad \text{by (5.2)} \\ &\leqslant E_{x,x}(B). \end{aligned}$$

Hence

$$\begin{aligned} \int_{\mathbb{R}} \lambda^{2s} dE_{E(\Delta)x,E(\Delta)x} &= \int_{[k,\infty)} \lambda^{2s} dE_{E(\Delta)x,E(\Delta)x} \\ &\leqslant \int_{[k,\infty)} \lambda^{2s} dE_{x,x} = \int_{\mathbb{R}} \lambda^{2s} dE_{x,x} < \infty. \end{aligned}$$

Thus, from (4.3), we see that

$$E(\Delta)x \in \mathscr{D}(A^s). \tag{5.5}$$

Moreover, for $y \in H$, we have from (5.1) that

$$E_{E(\Delta)x,y}(B) = E_{x,E(\Delta)y}(B);$$

hence, from (4.4) and the self-adjointness of $E(\Delta)$, we see that

$$\begin{aligned} (A^s E(\Delta)x, y) &= \int_{\mathbb{R}} \lambda^s dE_{E(\Delta)x,y} = \int_{\mathbb{R}} \lambda^s dE_{x,E(\Delta)y} \\ &= (A^s x, E(\Delta)y) = (E(\Delta)A^s x, y), \end{aligned}$$

that is to say,

$$(A^s E(\Delta)x - E(\Delta)A^s x, y) = 0 \qquad (y \in H),$$

from which it follows that $A^s E(\Delta) = E(\Delta)A^s$. ∎

LEMMA 5.3. *Suppose A is a self-adjoint operator in the Hilbert space $H = (V, (\cdot,\cdot))$ that is bounded below by kI for some $K > 0$. Let E be the spectral* *resolution of the identity of* A. *Then*, *for each* $s > 0$ *and* $\Delta \in \mathscr{B}$, *we have*

$$E_{A^s x,y}(\Delta) = \int_\Delta \lambda^s dE_{x,y} \qquad (x \in \mathscr{D}(A^s),\ \ y \in H), \tag{5.6}$$

and

$$E_{x,A^s y}(\Delta) = \int_\Delta \lambda^s dE_{x,y} \qquad (x \in H,\ \ y \in \mathscr{D}(A^s)). \tag{5.7}$$

That is to say,

$$dE_{A^s x,y} = \lambda^s dE_{x,y} \qquad (x \in \mathscr{D}(A^s),\ \ y \in H) \tag{5.8}$$

and

$$dE_{x,A^s y} = \lambda^s dE_{x,y} \qquad (x \in H,\ \ y \in \mathscr{D}(A^s)). \tag{5.9}$$

REMARK 5.1. *The identities in* (5.8) *and* (5.9) *are understood to mean*, *in the sense of the Radon–Nikodym theorem*,

$$\int_{\mathbb{R}} f(\lambda) dE_{A^s x,y} = \int_{\mathbb{R}} f(\lambda)\lambda^s dE_{x,y} \tag{5.10}$$

and

$$\int_{\mathbb{R}} f(\lambda) dE_{x,A^s y} = \int_{\mathbb{R}} f(\lambda)\lambda^s dE_{x,y}, \tag{5.11}$$

respectively, *for each nonnegative Borel measurable function* $f\colon \mathbb{R} \to [0, \infty]$; *see* [42, pp. 121–126].

Proof of Lemma 5.3. Let $s > 0$; for $x \in \mathscr{D}(A^s)$ and $y \in H$ we see that

$$\begin{aligned} E_{A^s x,y}(\Delta) &= (E(\Delta)A^s x, y) \\ &= (A^s E(\Delta)x, y) && \text{by Lemma 5.2} \\ &= \int_{\mathbb{R}} \lambda^s dE_{E(\Delta)x,y} && \text{by (4.4)} \\ &= \int_\Delta \lambda^s dE_{x,y} && \text{from (5.1)} \end{aligned} \tag{5.12}$$

The identity in (5.7) follows in a similar fashion. ∎

6. EXISTENCE AND UNIQUENESS OF THE LEFT-DEFINITE SPACES: PROOF OF THEOREM 3.1

Proof. Existence of the Left-Definite Spaces. Let $r > 0$. To show that $H_r = (V_r, (\cdot,\cdot)_r)$, defined in (3.1) and (3.2), is a left-definite space for the pair (H, A) we need to establish the five properties listed in Definition 2.2.

(i) H_r is a Hilbert space.

Suppose $\{x_n\} \subset H_r$ is Cauchy. From (3.2), we see that

$$\|x_n - x_m\|_r = \|A^{r/2}(x_n - x_m)\|,$$

where $\|\cdot\|_r$ and $\|\cdot\|$ are the norms generated, respectively, from the inner products $(\cdot,\cdot)_r$ and $(\cdot,\cdot)$. Hence $\{A^{r/2}x_n\}$ is Cauchy in H so there exists $y \in H$ such that

$$A^{r/2}x_n \to y \text{ in } H \qquad \text{as } n \to \infty. \tag{6.1}$$

Moreover, from (5.3) and Theorem 4.3, we have

$$\begin{aligned} k^r\|x_n - x_m\|^2 &= k^r \int_{\mathbb{R}} dE_{x_n - x_m,\, x_n - x_m} = k^r \int_{[k,\infty)} dE_{x_n - x_m,\, x_n - x_m} \\ &\leqslant \int_{[k,\infty)} \lambda^r dE_{x_n - x_m,\, x_n - x_m} \\ &= \|A^{r/2}(x_n - x_m)\|^2. \end{aligned}$$

Therefore, $\{x_n\}$ is Cauchy in H. From the completeness of H, there exists $x \in H$ such that

$$x_n \to x \text{ in } H \qquad \text{as } n \to \infty. \tag{6.2}$$

From (6.1) and (6.2) and the fact that $A^{r/2}$ is closed (being self-adjoint from Theorem 4.3), we see that $x \in \mathscr{D}(A^{r/2}) = H_r$ and $A^{r/2}x = y$. In particular, H_r is complete.

(ii) $\mathscr{D}(A^r) \subset V_r \subset H$.

Let $x \in \mathscr{D}(A^r)$. If $k \leqslant 1$, then

$$\begin{aligned}\int_{\mathbb{R}} \lambda^r dE_{x,x} &= \int_{[k,\infty)} \lambda^r dE_{x,x} \\ &\leqslant \int_{[k,1]} \lambda^r dE_{x,x} + \int_{(1,\infty)} \lambda^r dE_{x,x} \\ &\leqslant \int_{[k,1]} dE_{x,x} + \int_{(1,\infty)} \lambda^{2r} dE_{x,x} \\ &\leqslant \int_{\mathbb{R}} dE_{x,x} + \int_{\mathbb{R}} \lambda^{2r} dE_{x,x} \\ &= \|x\|^2 + \|A^r x\|^2 < \infty \qquad \text{by (4.5),}\end{aligned}$$

so that $x \in \mathscr{D}(A^{r/2}) = V_r$. A similar calculation shows that if $k > 1$, then

$$\int_{\mathbb{R}} \lambda^r dE_{x,x} \leqslant \|A^r x\|^2 < \infty,$$

so $x \in V_r$.

(iii) $\mathscr{D}(A^r)$ is dense in H_r.

Let $x \in H_r = \mathscr{D}(A^{r/2})$. Define, for each $n \in \mathbb{N}$, $x_n = E(-\infty, n]x$. From (2), (3), and (4) of (4.1), we see that for $\Delta \in \mathscr{B}$,

$$\begin{aligned}E_{x_n,x_n}(\Delta) &= (E(\Delta)x_n, x_n) \\ &= (E(\Delta)E(-\infty,n]x, E(-\infty,n]x) \\ &= (E(\Delta \cap (-\infty,n])x, x) \\ &= E_{x,x}(\Delta \cap (-\infty,n]).\end{aligned}$$

Consequently, for $n \geqslant k$,

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{2r} dE_{x_n,x_n} &= \int_{(-\infty,n]\cap[k,\infty)} \lambda^{2r} dE_{x,x} \\ &= \int_{[k,n]} \lambda^{2r} dE_{x,x} \\ &\leqslant n^{2r} \int_{\mathbb{R}} dE_{x,x} = n^{2r}\|x\|^2 < \infty,\end{aligned}$$

from which it follows that $x_n \in \mathscr{D}(A^r)$ for $n \geqslant k$. Moreover, from Properties (1) and (5) of (4.1), we see that

$$E(n,\infty) = I - E(-\infty, n]$$

and hence

$$x - x_n = E(n,\infty)x \qquad (n \geqslant 1).$$

Thus

$$
\begin{aligned}
\|x - x_n\|_r^2 &= (A^{r/2}(x - x_n), A^{r/2}(x - x_n)) \\
&= (A^{r/2}E(n,\infty)x, A^{r/2}E(n,\infty)x) \\
&= (A^{r/2}E(n,\infty)x, A^{r/2}x) \qquad \text{by Lemma 5.2 and (4) of (4.1)} \\
&= \int_{\mathbb{R}} \lambda^{r/2} dE_{E(n,\infty)x, A^{r/2}x} \qquad \text{by (4.4)} \\
&= \int_{(n,\infty)} \lambda^{r/2} dE_{x, A^{r/2}x} \qquad \text{by (5.1)} \\
&= \int_{(n,\infty)} \lambda^{r} dE_{x,x} \qquad \text{by (5.9)} \qquad (6.3)
\end{aligned}
$$

Define μ: $\mathscr{B} \to \mathbb{R}$ by $\mu(\Delta) = \int_\Delta \lambda^r dE_{x,x}$; then μ is a finite, positive measure on $\mathscr{B}$. Let $\Delta_n = (n, \infty)$ for each $n \in \mathbb{N}$; since $\Delta_n \searrow \varnothing$, we have $\mu(\Delta_n) \to 0$ as $n \to \infty$ (see [42, Theorem 1.19, Part (e)]). Consequently, from (6.3), we see that $\|x - x_n\|_r \to 0$ as $n \to \infty$. It follows that $\mathscr{D}(A^r)$ is dense in H_r.

(iv) $(x,x)_r \geqslant k^r(x,x)$ $(x \in V_r)$.

Let $x \in V_r$. Then, from (4.5),

$$
\begin{aligned}
(x,x)_r &= \int_{\mathbb{R}} \lambda^r dE_{x,x} \\
&= \int_{[k,\infty)} \lambda^r dE_{x,x} \\
&\geqslant k^r \int_{[k,\infty)} dE_{x,x} \\
&= k^r \int_{\mathbb{R}} dE_{x,x} \\
&= k^r(x,x).
\end{aligned}
$$

(v) $(x, y)_r = (A^r x, y)$ $(x \in \mathscr{D}(A^r),\ \ y \in V_r)$.

Let $x \in \mathscr{D}(A^r)$ and $y \in H_r = \mathscr{D}(A^{r/2})$. By part (ii) of this proof, we see that $x \in \mathscr{D}(A^{r/2})$. From (4.4), we have

$$
\begin{aligned}
(x, y)_r = (A^{r/2}x, A^{r/2}y) &= \int_{\mathbb{R}} \lambda^{\frac{r}{2}} dE_{x, A^{r/2}y} \\
&= \int_{\mathbb{R}} \lambda^r dE_{x,y} \qquad \text{by (5.9)} \\
&= (A^r x, y), \qquad (6.4)
\end{aligned}
$$

as required.

Properties (i)–(v) show that, for each $r > 0$, $H_r = (V_r, (\cdot,\cdot)_r)$ is an rth left-definite space associated with the pair (H, A).

Uniqueness of the Left-Definite Space. Suppose $H_r = (V_r, (\cdot,\cdot)_r)$ and $H_r' = (V_r', (\cdot,\cdot)_r')$ are left-definite spaces associated with the pair (H, A) for some $r > 0$. Fix $x \in V_r'$. By Property 3 of Definition 2.2, there exists $\{x_n\} \subset \mathscr{D}(A^r)$ such that $x_n \to x$ in H_r' as $n \to \infty$; that is,

$$\|x_n - x\|_r' \to 0 \qquad (n \to \infty).$$

On account of Property 5 of Definition 2.2, we see that

$$\|x_n - x_m\|_r' = \|x_n - x_m\|_r,$$

and hence that $\{x_n\}$ is Cauchy in H_r. Consequently, there exists $\hat{x} \in V_r$ such that $\|x_n - \hat{x}\|_r \to 0$ as $n \to \infty$. From Property 4 of 2.2, we see that

$$\|x_n - x\| \leqslant \frac{1}{k^{r/2}} \|x_n - x\|_r'$$

and

$$\|x_n - \hat{x}\| \leqslant \frac{1}{k^{r/2}} \|x_n - \hat{x}\|_r.$$

Hence $x = \hat{x} \in V_r$. By symmetry, it follows that $V_r = V_r'$. Moreover, for $x, y \in V_r = V_r'$, we have

$$(x, y)_r' = \lim_{n\to\infty} (x_n, y)_r' = \lim_{n\to\infty} (A^r x_n, y) = \lim_{n\to\infty} (x_n, y)_r = (x, y)_r.$$

This completes the proof of Theorem 3.1. ∎

In Section 2, we remarked that each of the left-definite spaces $\{H_r\}_{r>0}$ associated with (H, A) is independent of $k > 0$, where A is self-adjoint and bounded below by kI. Indeed, this follows from the above theorem. For, suppose $H_r(k) = (V_r(k), (\cdot,\cdot)_{r,k})$ (respectively, $H_r(k') = (V_r(k'), (\cdot,\cdot)_{r,k'})$) is the rth left-definite space associated with the pair (H, A), where A is a self-adjoint operator that is bounded below by kI (respectively, $k'I$). By the above theorem,

$$V_r(k) = \mathscr{D}(A^{r/2}) = V_r(k')$$

and

$$(x, y)_{r,k} = (A^{r/2}x, A^{r/2}y) = (x, y)_{r,k'} \qquad (x, y \in V_r(k) = V_r(k')).$$

That is to say, $H_r(k) = H_r(k')$.

7. PROOF OF THEOREM 3.2

Proof. Let $r > 0$. Define $E(r)$ to be the operator-valued mapping, defined on the Borel sets of $\mathbb{R}$, by

$$E(r)(\Delta) = E(\Delta) \qquad (\Delta \in \mathscr{B}), \tag{7.1}$$

where E is the spectral resolution of the identity associated with A. We first show that $E(r)$ is a spectral resolution of the identity in H_r. For $x \in H_r$ we have, from the definition of the inner product $(\cdot, \cdot)_r$,

$$\begin{aligned}
\|E(r)(\Delta)x\|_r^2 &= (A^{r/2}E(\Delta)x, A^{r/2}E(\Delta)x) \\
&= (A^{r/2}E(\Delta)x, A^{r/2}x) \\
&= \int_{\mathbb{R}} \lambda^{r/2} dE_{E(\Delta)x, A^{r/2}x} \qquad \text{by (4.4)} \\
&= \int_{\Delta} \lambda^{r/2} dE_{x, A^{r/2}x} \qquad \text{by (5.1)} \\
&= \int_{\Delta} \lambda^{r} dE_{x,x} \qquad \text{by (5.9)} \\
&\leqslant \int_{[k,\infty)} \lambda^{r} dE_{x,x} \\
&= \|x\|_r^2 \qquad \text{by (6.4),}
\end{aligned}$$

that is to say, $E(r)(\Delta) \in B(H_r)$ for all $\Delta \in \mathscr{B}$. By the definition of $E(r)$, it is clear that Properties (1), (2), (4), and (5) of (4.1) are satisfied. Moreover, for $x, y \in H_r$,

$$\begin{aligned}
(E(r)(\Delta)x, y)_r &= (A^{r/2}E(\Delta)x, A^{r/2}y) \\
&= (E(\Delta)A^{r/2}x, A^{r/2}y) \qquad \text{by Lemma 5.2} \\
&= (A^{r/2}x, E(\Delta)A^{r/2}y) \qquad \text{since } E(\Delta) \text{ is self-adjoint} \\
&= (A^{r/2}x, A^{r/2}E(\Delta)y) \\
&= (x, E(r)(\Delta)y)_r.
\end{aligned}$$

Hence $E(r)(\Delta)$ is self-adjoint for each $\Delta \in \mathscr{B}$. It remains to show that Property (6) of (4.1) holds for $E(r)$. For $\Delta \in \mathscr{B}$ and $x, y \in H_r$,

$$\begin{aligned} E(r)_{x,y}(\Delta) &= (E(r)(\Delta)x, y)_r \\ &= (A^{r/2}E(\Delta)x, A^{r/2}y) \\ &= \int_{\mathbb{R}} \lambda^{r/2} dE_{E(\Delta)x, A^{r/2}y} \\ &= \int_{\Delta} \lambda^{r/2} dE_{x, A^{r/2}y} \\ &= \int_{\Delta} \lambda^{r} dE_{x,y} \qquad \text{by (5.9).} \end{aligned}$$

Thus, $E(r)_{x,y}$ is a complex, regular Borel measure on $\mathscr{B}$; moreover, we have the formal measure identity

$$dE(r)_{x,y} = \lambda^r dE_{x,y}. \tag{7.2}$$

It follows from the spectral theorem (see Theorem 4.3) that, for each $r > 0$, there exists a unique self-adjoint operator A_r: $\mathscr{D}(A_r) \subset H_r \to H_r$ with domain

$$\mathscr{D}(A_r) = \left\{ x \in H_r \,\middle|\, \int_{\mathbb{R}} \lambda^2 dE(r)_{x,x} < \infty \right\}. \tag{7.3}$$

Furthermore, we have the identities

$$(A_r x, y)_r = \int_{\mathbb{R}} \lambda dE(r)_{x,y} \qquad (x \in \mathscr{D}(A_r),\ y \in H_r), \tag{7.4}$$

and

$$\|A_r x\|^2 = \int_{\mathbb{R}} \lambda^2 dE(r)_{x,x} \qquad (x \in \mathscr{D}(A_r)). \tag{7.5}$$

From (7.2), we see that

$$\int_{\mathbb{R}} \lambda^2 dE(r)_{x,x} = \int_{\mathbb{R}} \lambda^{r+2} dE_{x,x};$$

it follows from (3.1), (4.3), and (7.3) that $\mathscr{D}(A_r) = V_{r+2}$.

Note that, for $x \in \mathscr{D}(A_r)$,

$$\begin{aligned} k^r \int_{\mathbb{R}} \lambda^2 dE_{x,x} &= k^r \int_{[k,\infty)} \lambda^2 dE_{x,x} \\ &\leqslant \int_{[k,\infty)} \lambda^{r+2} dE_{x,x} = \int_{\mathbb{R}} \lambda^2 dE(r)_{x,x} \qquad \text{by (7.2)} \\ &< \infty, \end{aligned}$$

and hence that $\mathscr{D}(A_r) \subset \mathscr{D}(A)$. We now show that $A_r x = Ax$ for $x \in \mathscr{D}(A_r)$. To this end, fix $x \in \mathscr{D}(A_r)$ and let $y \in H_r$. Then, from (7.2) and (7.4),

$$\begin{aligned} (A_r x, y)_r &= \int_{\mathbb{R}} \lambda dE(r)_{x,y} \\ &= \int_{\mathbb{R}} \lambda^{r+1} dE_{x,y}. \end{aligned} \tag{7.6}$$

On the other hand, from (5.8) and (5.9),

$$\begin{aligned} (Ax, y)_r &= (A^{r/2} Ax, A^{r/2} y) \\ &= \int_{\mathbb{R}} \lambda^{r/2} dE_{Ax, A^{r/2} y} \\ &= \int_{\mathbb{R}} \lambda^{r+1} dE_{x,y}. \end{aligned} \tag{7.7}$$

Comparing (7.6) and (7.7), we conclude that

$$A_r x = Ax \qquad (x \in \mathscr{D}(A_r)). \tag{7.8}$$

To show that A_r is bounded below by kI in H_r, let $x \in \mathscr{D}(A_r) \subset V_r = \mathscr{D}(A^{r/2})$. Then, from (7.7),

$$\begin{aligned} (A_r x, x)_r &= \int_{\mathbb{R}} \lambda^{r+1} dE_{x,x} \\ &= \int_{\mathbb{R}} \lambda dE_{A^{r/2}x, A^{r/2}x} \qquad \text{from (5.8) and (5.9)} \\ &= (A(A^{r/2}x), A^{r/2}x) \qquad \text{from (4.4)} \\ &\geqslant k(A^{r/2}x, A^{r/2}x) \\ &= k(x,x)_r. \end{aligned}$$

To establish uniqueness, suppose $\tilde{A}_r\colon H_r \to H_r$ is a self-adjoint operator such that $\tilde{A}_r x = Ax$ for all $x \in \mathscr{D}(\tilde{A}_r) \subset \mathscr{D}(A)$. Then for $x \in \mathscr{D}(\tilde{A}_r)$,

$$\begin{aligned}(\tilde{A}_r x, \tilde{A}_r x)_r &= (A^{r/2}Ax, A^{r/2}Ax)\\ &= \int_{\mathbb{R}} \lambda^r dE_{Ax,Ax} \qquad \text{by (4.5)}\\ &= \int_{\mathbb{R}} \lambda^{r+2} dE_{x,x} \qquad \text{by (5.8) and (5.9)}\\ &= \int_{\mathbb{R}} \lambda^2 dE(r)_{x,x} \qquad \text{by (7.2).}\end{aligned} \tag{7.9}$$

However, by (7.3), we see that $x \in \mathscr{D}(A_r)$ and hence, from (7.8), we have

$$\tilde{A}_r x = Ax = A_r x.$$

In particular, A_r is a self-adjoint extension of the self-adjoint operator $\tilde{A}_r$ which forces $A_r = \tilde{A}_r$. ∎

8. PROOF OF THEOREM 3.4

Proof. (a) If A is bounded, then so is A^r for any $r > 0$; consequently, we may take $\mathscr{D}(A^r) = V$ for all $r > 0$. Hence, from Property 2 of Definition 2.2, we see that $V_r = V$ for all $r > 0$. Moreover, from Properties 4 and 5 of Definition 2.2,

$$k^r(x,x) \leqslant (x,x)_r = (A^r x, x) \leqslant \|A^r\|(x,x) \qquad (x \in V),$$

so the inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_r$ are equivalent. This completes the proof of Part (a).

(b) If A is unbounded, so is A^r for each $r > 0$. Consequently, since $V_r = \mathscr{D}(A^{r/2})$, it is impossible for $V_r = V$; this proves (i).

To show (ii), let $0 < r < s$ and suppose $x \in V_s$ so $\int_{\mathbb{R}} \lambda^s dE_{x,x} < \infty$. If $k > 1$, then

$$\int_{\mathbb{R}} \lambda^r dE_{x,x} = \int_{[k,\infty)} \lambda^r dE_{x,x} \leqslant \int_{[k,\infty)} \lambda^s dE_{x,x} = \int_{\mathbb{R}} \lambda^s dE_{x,x} < \infty.$$

If $k \leqslant 1$, then

$$\begin{aligned}\int_{\mathbb{R}} \lambda^r dE_{x,x} &= \int_{[k,\infty)} \lambda^r dE_{x,x} \\ &= \int_{[k,1]} \lambda^r dE_{x,x} + \int_{(1,\infty)} \lambda^r dE_{x,x} \\ &\leqslant \int_{\mathbb{R}} dE_{x,x} + \int_{\mathbb{R}} \lambda^s dE_{x,x} \\ &= \|x\|^2 + \int_{\mathbb{R}} \lambda^s dE_{x,x} < \infty.\end{aligned}$$

In either case, we see that $x \in V_r$. Suppose, for some $0 < r < s$, $V_s = V_r$; that is to say, $\mathscr{D}(A^{s/2}) = \mathscr{D}(A^{r/2})$. Write $s = r + \varepsilon$; then, from the identity $A^{s/2}x = A^{\varepsilon/2}(A^{r/2}x)$, we see that

$$\mathscr{R}(A^{r/2}) \subset \mathscr{D}(A^{\varepsilon/2}),$$

where $\mathscr{R}(A^{r/2})$ denotes the range of $A^{r/2}$. However, since $A^{r/2}$ is bounded below by $k^{r/2}I$, we have $0 \in \rho(A^{r/2})$. Hence, from well-known results, we have $\mathscr{R}(A^{r/2}) = H$, forcing $\mathscr{D}(A^{\varepsilon/2}) = V$. This implies, of course, that $A^{\varepsilon/2}$ is bounded, contradicting our hypothesis. Hence, for $0 < r < s$, V_s is a proper subspace of V_r.

To prove (iii), let $x \in V \backslash V_s$ so

$$\|x\|^2 = \int_{[k,\infty)} dE_{x,x} < \infty \qquad \text{but} \qquad \int_{[k,\infty)} \lambda^s dE_{x,x} = \infty.$$

For $n \in \mathbb{N}$, $n > k$, let

$$x_n = E[k,n)x. \tag{8.1}$$

Clearly each $x_n \in V$; moreover, since $E_{x_n,x_n}(\Delta) = E_{x,x}(\Delta \cap [k,n))$, we have

$$\begin{aligned}(x_n, x_n)_s &= \int_{[k,\infty)} \lambda^s dE_{x_n,x_n} \\ &= \int_{[k,n)} \lambda^s dE_{x,x} \\ &\leqslant n^s \|x\|^2 < \infty,\end{aligned}$$

so $x_n \in V_s$ for each $n > k$. On the other hand, for $n > k$,

$$\begin{aligned}(x_n, x_n)_s &= \int_{[k,n)} \lambda^s dE_{x,x} \\ &\to \int_{[k,\infty)} \lambda^s dE_{x,x} = \infty,\end{aligned}$$

while

$$\begin{aligned}(x_n, x_n) &= \int_{[k,\infty)} dE_{x_n,x_n} \\ &= \int_{[k,n)} dE_{x,x} \\ &\leqslant \int_{[k,\infty)} dE_{x,x} \\ &= \|x\|^2.\end{aligned}$$

Consequently, it is impossible for a positive constant c to exist such that

$$(x, x)_s \leqslant c(x, x) \qquad (x \in V_s).$$

The proof of (iv) is identical to (iii) with V_r replacing V where $r < s$.

To show (v), we remark that, by definition, $\mathscr{D}(A_r) \subset \mathscr{D}(A)$. Suppose, in fact, $\mathscr{D}(A_r) = \mathscr{D}(A)$ for some $r > 0$. Then, from Theorem 3.2 and Corollary 3.3, we have $V_{r+2} = V_2$. However, from part (ii), this implies $r = 0$, which is impossible.

The proof of part (vi) is similar. ∎

9. PROOF OF THEOREM 3.5

Proof. From (3.1), we see that

$$(V^m)_r = \mathscr{D}((A^m)^{r/2}) = \mathscr{D}(A^{mr/2}) = V_{mr}$$

and

$$(x, y)^m_r = (A^{mr/2}x, A^{mr/2}y) = (x, y)_{mr} \qquad (x, y \in (V^m)_r = V_{mr}),$$

and we see that

$$(H^m)_r = H_{mr},$$

establishing (a) of Theorem 3.5. To show (b), first observe from (4.3) of Theorem 4.3 that

$$\begin{aligned}
\mathscr{D}((A_r)^m) &= \left\{x \in H \,\middle|\, \int_{\mathbb{R}} \lambda^{2m} dE(r)_{x,x} < \infty\right\} \\
&= \left\{x \in H \,\middle|\, \int_{\mathbb{R}} \lambda^{2m+r} dE_{x,x} < \infty\right\} \qquad \text{by (7.2)} \\
&= \mathscr{D}(A^{(2m+r)/2}) \qquad \text{by (4.3)} \\
&= V_{2m+r} \qquad \text{by (3.1)}, \qquad (9.1)
\end{aligned}$$

where $E(r)$ is the spectral resolution of the identity of A_r in H_r. Consequently, the operator $(A_r)^m$: $H_r \to H_r$ given by

$$\begin{aligned}
(A_r)^m x &= A^m x \\
x \in \mathscr{D}((A_r)^m) &= V_{2m+r}
\end{aligned} \qquad (9.2)$$

is a self-adjoint restriction of A^m in the rth left-definite space H_r.

On the other hand, since

$$(H^m)_{r/m} = H_r$$

and

$$\mathscr{D}((A^m)_{r/m}) = (V^m)_{r/m+2} = V_{2m+r},$$

we see that the (r/m)th left-definite operator $(A^m)_{r/m}$: $H_r \to H_r$ associated with the pair (H, A^m) is given by

$$\begin{aligned}
(A^m)_{r/m} x &= A^m x \\
x \in \mathscr{D}((A^m)_{r/m}) &= V_{2m+r}.
\end{aligned} \qquad (9.3)$$

From the uniqueness part of Theorem 3.2, we conclude from (9.2) and (9.3) that

$$(A^m)_{r/m} = (A_r)^m,$$

proving the first statement in (b). The second part of (b) follows in a similar manner.

Regarding Part (c) of the theorem, note from (3.1) and (9.1) that

$$(V_m)_r = \mathscr{D}((A_m)^{r/2}) = V_{m+r}. \qquad (9.4)$$

Moreover, for $x, y \in (V_m)_r = V_{m+r}$,

$$\begin{aligned}(x, y)_{m,r} &= ((A_m)^{r/2}x, (A_m)^{r/2}y)_m \\ &= (A^{r/2}x, A^{r/2}y)_m \qquad \text{since } A_m \text{ is a restriction of } A \\ &= (A^{m/2}(A^{r/2}x), A^{m/2}(A^{r/2}y)) \qquad \text{by (3.2)} \\ &= (A^{(m+r)/2}x, A^{(m+r)/2}y) \\ &= (x, y)_{m+r} \qquad \text{by (3.2)}.\end{aligned}$$

Consequently, we see that $(H_m)_r = H_{m+r}$.
From (3.3) and (9.4), we see that

$$\mathscr{D}((A_m)_r) = (V_m)_{r+2} = V_{m+r+2}.$$

Therefore, the left-definite operator $(A_m)_r$: $H_{m+r} \to H_{m+r}$ is given by

$$\begin{aligned}&(A_m)_r x = A_m x = Ax \\ &x \in \mathscr{D}((A_m)_r) = V_{m+r+2}.\end{aligned}$$

On the other hand, from (3.3), the left-definite operator A_{m+r} is a self-adjoint restriction of A in H_{m+r} with domain $\mathscr{D}(A_{m+r}) = V_{m+r+2}$. Thus, from the uniqueness condition given in Theorem 3.2, we conclude that

$$(A_m)_r = A_{m+r}.$$

The proof of Theorem 3.5 is now complete. ∎

COROLLARY 9.1. *With the same conditions and notation as in Theorem* 3.5, *we have*

$$(A^m)_1 = (A_m)^m. \tag{9.5}$$

That is to say, the first left-definite operator associated with (H, A^m) *is the mth power of the mth left-definite operator associated with* (H, A).

We remark on an interesting application of this corollary in the last section of this paper (see Remark 13.3).

10. PROOFS OF THEOREM 3.6 AND THEOREM 3.7

Proof of Theorem 3.6. For each $r > 0$, we denote the associated spectral family (see (4.2)) of $E(r)$, the spectral resolution of the identity for A_r

(see 7.1), to be $\{E_\lambda(r)\}_{\lambda\in\mathbb{R}}$, where each $E_\lambda(r)$ is defined by

$$E_\lambda(r) := E(r)(-\infty, \lambda].$$

From (7.1), we see that

$$E_\lambda(r) = E_\lambda \qquad (r > 0,\ \lambda \in \mathbb{R}).$$

Consequently, from Theorem 4.2, we have that $\sigma_p(A) = \sigma_p(A_r)$, $\sigma_c(A) = \sigma_c(A_r)$, and $\rho(A) = \rho(A_r)$, for each $r > 0$. However, we include a separate proof that the point spectra of A and each A_r are equal; this proof is important in that it shows that the eigenfunctions of A are the same as the eigenfunctions of each A_r.

Let $r > 0$. Suppose $\mu \in \sigma_p(A)$; hence there exists a nonzero $x \in \mathscr{D}(A)$ such that $Ax = \mu x$. Clearly, $x \in \mathscr{D}(A^n)$ so that from Theorem 4.3,

$$\int_{\mathbb{R}} \lambda^{2n} dE_{x,x} < \infty \qquad (n \in \mathbb{N})$$

and $A^n x = \mu^n x$. Choose $n \in \mathbb{N}$ such that $r + 2 < 2n$. Then, if $k \leqslant 1$,

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{r+2} dE_{x,x} &= \int_{[k,\infty)} \lambda^{r+2} dE_{x,x}\\ &= \int_{[k,1]} \lambda^{r+2} dE_{x,x} + \int_{(1,\infty)} \lambda^{r+2} dE_{x,x}\\ &\leqslant \int_{[k,1]} dE_{x,x} + \int_{(1,\infty)} \lambda^{2n} dE_{x,x}\\ &\leqslant \|x\|^2 + \|A^n x\|^2 < \infty \qquad \text{by (4.5) and (5.3).}\end{aligned}$$

If $k > 1$, then

$$\begin{aligned}\int_{\mathbb{R}} \lambda^{r+2} dE_{x,x} &= \int_{[k,\infty)} \lambda^{r+2} dE_{x,x}\\ &\leqslant \int_{[k,\infty)} \lambda^{2n} dE_{x,x}\\ &= \|A^n x\|^2 < \infty.\end{aligned}$$

Consequently, $x \in \mathscr{D}(A_r) \subset \mathscr{D}(A)$, $A_r x = Ax = \mu x$, and $\sigma_p(A) \subset \sigma_p(A_r)$. Since A_r is a restriction of A, the inclusion $\sigma_p(A_r) \subset \sigma_p(A)$ is clear. ∎

To prove Theorem 3.7, we begin by first proving the following lemma.

LEMMA 10.1. *Suppose A is a self-adjoint operator in H. If $Ax = \mu x$ then, for each $s > 0$,*

$$A^s x = \mu^s x.$$

Proof. Let $y \in H$ and $\Delta \in \mathscr{B}$. Then, from (5.12),

$$\begin{aligned}\mu E_{x,y}(\Delta) &= (E(\Delta)\mu x, y) = (E(\Delta)Ax, y)\\ &= \int_\Delta \lambda dE_{x,y}. \end{aligned} \tag{10.1}$$

On the other hand,

$$\mu E_{x,y}(\Delta) = \mu \int_\Delta dE_{x,y}. \tag{10.2}$$

Define σ: $\mathscr{B} \to \mathbb{C}$ by

$$\sigma(\Delta) = \int_\Delta (\lambda - \mu) dE_{x,y}.$$

From (10.1) and (10.2) we see that σ is the zero measure; that is to say, if $f(\lambda)$ is any Borel measurable function then

$$\int_\Delta f(\lambda) d\sigma = 0 \qquad (\Delta \in \mathscr{B}).$$

Choose $\Delta \in \mathscr{B}$ such that $\mu \notin \Delta$ and let $f(\lambda) = 1/(\lambda - \mu)$. Then

$$0 = \int_\Delta f(\lambda) d\sigma = \int_\Delta f(\lambda)(\lambda - \mu) dE_{x,y} = \int_\Delta dE_{x,y}.$$

That is to say, if $\mu \notin \Delta$ then $E_{x,y}(\Delta) = 0$. Hence, for $y \in H$,

$$\begin{aligned}(A^s x, y) &= \int_{\mathbb{R}} \lambda^s dE_{x,y} = \int_{\{\mu\}} \lambda^s dE_{x,y}\\ &= \mu^s \int_{\{\mu\}} dE_{x,y} = \mu^s \int_{\mathbb{R}} dE_{x,y}\\ &= (\mu^s x, y).\end{aligned}$$

It follows that $A^s x = \mu^s x$. ∎

We are now in position to prove Theorem 3.7.

Proof. Suppose that $\{\varphi_n\}$ is a complete set of eigenfunctions of A with $A\varphi_n = \lambda_n \varphi_n$ $(n \in \mathbb{N}_0)$. From Theorem 3.6, we see that $\{\varphi_n\} \subset \mathscr{D}(A_r)$. To show that $\{\varphi_n\}$ is complete in H_r, it suffices to show that if $f \in H_r$ satisfies

$$(f, \varphi_n)_r = 0 \qquad (n \in \mathbb{N}_0),$$

then $f = 0$ in H_r. Now

$$0 = (f, \varphi_n)_r = (A^{r/2} f, A^{r/2} \varphi_n) = \lambda_n^{r/2} (A^{r/2} f, \varphi_n)$$

by Lemma 10.1. Since $\lambda_n > 0$, we see that $(A^{r/2} f, \varphi_n) = 0$ $(n \in \mathbb{N}_0)$ and hence, from the completeness of $\{\varphi_n\}$ in H, we have that $A^{r/2} f = 0$. Consequently,

$$\|f\|_r^2 = (A^{r/2} f, A^{r/2} f) = 0,$$

and hence $f = 0$ in H_r. ∎

11. EXAMPLE: A SELF-ADJOINT OPERATOR IN ℓ^2

Let ℓ^2 denote the usual Hilbert space of square-summable sequences of complex numbers with inner product

$$(x, y) = \sum_{n=1}^{\infty} x_n \overline{y_n}$$

for $x = (x_n)_{n=1}^{\infty} = (x_1, x_2, \ldots, x_n, \ldots)$ and $y = (y_n)_{n=1}^{\infty} = (y_1, y_2, \ldots, y_n, \ldots,) \in \ell^2$.

Define A: $\ell^2 \to \ell^2$ by

$$Ax = (x_1, 2x_2, \ldots, nx_n, \ldots),$$

for

$$x \in \mathscr{D}(A) = \left\{ x = (x_n)_{n=1}^{\infty} \in \ell^2 \,\middle|\, \sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty \right\}.$$

It is not difficult to show that A is an unbounded, self-adjoint operator with spectrum $\sigma(A) = \mathbb{N}$. Moreover,

$$(Ax, x) = \sum_{n=1}^{\infty} n|x_n|^2 \geqslant \sum_{n=1}^{\infty} |x_n|^2 = (x, x),$$

so A is bounded below by $1I$ in ℓ^2.

The spectral resolution of the identity $E\colon \mathscr{B} \to B(H)$ associated with A is given by

$$E(B)x = \sum_{n \in \mathbb{N} \cap B} x_n e_n \quad (B \in \mathscr{B};\ x = (x_n)_{n=1}^{\infty} \in \ell^2),$$

where

$$e_n = (\delta_{n,m})_{m=1}^{\infty} \qquad (n \in \mathbb{N}) \tag{11.1}$$

and, for each $n, m \in \mathbb{N}$, $\delta_{n,m}$ is the Kronecker delta function. Moreover,

$$E_{x,y}(B) = (E(B)x, y) = \sum_{n \in \mathbb{N} \cap B} x_n \overline{y_n} \quad (B \in \mathscr{B};\ x = (x_n)_{n=1}^{\infty},\ y = (y_n)_{n=1}^{\infty} \in \ell^2).$$

In particular,

$$E_{x,y}(\{n\}) = x_n \overline{y_n} \qquad (n \in \mathbb{N}) \tag{11.2}$$

and

$$\int_{\mathbb{R}} \lambda^{2r} dE_{x,x} = \sum_{n=1}^{\infty} \int_{\{n\}} \lambda^{2r} dE_{x,x} = \sum_{n=1}^{\infty} n^{2r} |x_n|^2 \qquad (x = (x_n)_{n=1}^{\infty} \in \ell^2).$$

Hence, for each $r > 0$, we see from (4.3) that

$$\mathscr{D}(A^r) = \left\{ x = (x_n)_{n=1}^{\infty} \,\middle|\, \sum_{n=1}^{\infty} n^{2r} |x_n|^2 < \infty \right\}. \tag{11.3}$$

For each $r > 0$, define

$$V_r = \left\{ x = (x_n)_{n=1}^{\infty} \in \ell^2 \,\middle|\, \sum_{n=1}^{\infty} n^{r} |x_n|^2 < \infty \right\} \tag{11.4}$$

and define $(\cdot, \cdot)_r\colon V_r \times V_r \to \mathbb{C}$ by

$$(x, y)_r = \sum_{n=1}^{\infty} n^r x_n \overline{y_n} \qquad (x = (x_n)_{n=1}^{\infty},\ y = (y_n)_{n=1}^{\infty} \in V_r). \tag{11.5}$$

Let

$$H_r = (V_r, (\cdot, \cdot)_r). \tag{11.6}$$

Our first result concerning the left-definite theory associated with (ℓ^2, A) is given in

THEOREM 11.1. *For each* $r > 0$, *the inner product space* H_r, *defined in* (11.4), (11.5), *and* (11.6), *is the* rth *left-definite Hilbert space associated with* (ℓ^2, A).

Proof. We must show that, for each $r > 0$, H_r satisfies the five properties listed in Definition 2.2.

(i) H_r is a Hilbert space.

For each $n \in \mathbb{N}$, let $\mathbf{x}_n = (x_{n,1}, x_{n,2}, \ldots)$ and suppose that $\{\mathbf{x}_n\}_{n=1}^{\infty}$ is Cauchy in H_r. Let $\varepsilon > 0$; then there exists $N = N(\varepsilon) \in \mathbb{N}$ such that for $m, n \geqslant N$ we have

$$\|\mathbf{x}_m - \mathbf{x}_n\|_r^2 < \varepsilon^2.$$

In particular,

$$\varepsilon^2 > \sum_{j=1}^{\infty} j^r |x_{m,j} - x_{n,j}|^2 \geqslant |x_{m,j} - x_{n,j}|^2 \qquad (j \in \mathbb{N};\ m, n \geqslant N). \tag{11.7}$$

Hence, for each $j \in \mathbb{N}$, $\{x_{n,j}\}_{n=1}^{\infty}$ is Cauchy in $\mathbb{C}$ so there exists $\alpha_j \in \mathbb{C}$ such that

$$x_{n,j} \to \alpha_j \qquad (n \to \infty).$$

Let

$$\mathbf{x} = (\alpha_1, \alpha_2, \ldots, \alpha_j, \ldots).$$

From (11.7), we see that, for each $p \in \mathbb{N}$,

$$\sum_{j=1}^{p} j^r |x_{m,j} - x_{n,j}|^2 < \varepsilon^2 \qquad (m, n \geqslant N);$$

letting $n \to \infty$ in this equality yields

$$\sum_{j=1}^{p} j^r |x_{m,j} - \alpha_j|^2 \leqslant \varepsilon^2 \qquad (m \geqslant N).$$

If we now let $p \to \infty$, we see that

$$\|\mathbf{x}_m - \mathbf{x}\|_r^2 = \sum_{j=1}^{\infty} j^r |x_{m,j} - \alpha_j|^2 \leqslant \varepsilon^2 \qquad (m \geqslant N).$$

That is to say, $\mathbf{x}_m \to \mathbf{x}$ in H_r. Moreover,

$$\|\mathbf{x}\|_r \leqslant \|\mathbf{x} - \mathbf{x}_N\|_r + \|\mathbf{x}_N\|_r \leqslant \varepsilon + \|\mathbf{x}_N\|_r < \infty,$$

so $\mathbf{x} \in H_r$. Hence H_r is complete.

(ii) $\mathscr{D}(A^r) \subset V_r \subset \ell^2$.

Let $x = (x_1, x_2, \ldots, x_n, \ldots) \in \mathscr{D}(A^r)$. From (11.3) and the inequality $n^{2r}|x_n|^2 \geqslant n^r|x_n|^2$ $(n \in \mathbb{N})$, we have from the Comparison Test for Infinite Series that $x \in V_r$.

(iii) $\mathscr{D}(A^r)$ is dense in H_r.

Define, for each $n \in \mathbb{N}$,

$$e_{n,r} = e_n / n^{r/2},$$

where e_n is given in (11.1). It is well known that $\{e_n\}_{n=1}^{\infty}$ is a complete orthonormal set in ℓ^2. Furthermore, it is easy to see that $\{e_{n,r}\}_{n=1}^{\infty}$ is an orthonormal set in H_r. Moreover, if $x = (x_1, x_2, \ldots, x_n, \ldots) \in H_r$ is such that

$$0 = (x, e_{n,r})_r = n^{r/2} x_n \qquad (n \in \mathbb{N}),$$

then $x = 0$. Hence $\{e_{n,r}\}_{n=1}^{\infty}$ is a complete orthonormal set in H_r. From well-known Hilbert space results (see [42, Theorem 4.18]), we see that the set E of all finite linear combinations of elements from $\{e_{n,r}\}_{n=1}^{\infty}$ is dense in H_r. But since $e_{n,r} \in \mathscr{D}(A^r)$ for each $n \in \mathbb{N}$ and $\mathscr{D}(A^r)$ is a subspace of V_r, we have $E \subset \mathscr{D}(A^r)$; consequently, $\mathscr{D}(A^r)$ is dense in H_r.

(iv) $(x, x)_r \geqslant (x, x)$ $(x \in V_r)$.

Let $x \in V_r$. Then

$$(x, x)_r = \sum_{n=1}^{\infty} n^r |x_n|^2 \geqslant \sum_{n=1}^{\infty} |x_n|^2 = (x, x).$$

(v) $(x, y)_r = (A^r x, y)$ $(x \in \mathscr{D}(A^r),\ y \in V_r)$.

Let $x = (x_1, x_2, \ldots) \in \mathscr{D}(A^r)$ and $y = (y_1, y_2, \ldots) \in V_r$. From (4.4) and (11.2), we see that

$$(A^r x, y) = \sum_{n=1}^{\infty} \int_{\{n\}} \lambda^r dE_{x,y} = \sum_{n=1}^{\infty} n^r x_n \overline{y_n} = (x, y)_r.$$

This completes the proof of the theorem. ∎

From Theorems 3.2 and 3.6, we have the following result concerning the rth left-definite operator A_r associated with (ℓ^2, A).

THEOREM 11.2. *For each $r > 0$, let A_r: $H_r \to H_r$ be defined by*

$$A_r x = (x_1, 2x_2, \ldots, nx_n, \ldots) \qquad (x = (x_n)_{n=1}^{\infty} \in \mathscr{D}(A_r)),$$

where

$$\mathscr{D}(A_r) = \left\{ x = (x_n)_{n=1}^{\infty} \in \ell^2 \,\middle|\, \sum_{n=1}^{\infty} n^{r+2}|x_n|^2 < \infty \right\}.$$

Then A_r is the rth *left-definite operator associated with the pair (ℓ^2, A). In particular, A_r is an unbounded, self-adjoint operator in H_r with $\sigma(A_r) = \mathbb{N}$.*

12. EXAMPLE: THE LAGUERRE DIFFERENTIAL EQUATION AND LAGUERRE POLYNOMIALS

In this section, we determine explicitly:

(a) the sequence $\{H_n\}_{n=1}^{\infty}$ of left-definite spaces associated with the self-adjoint differential operator A in $L^2((0,\infty); t^{\alpha}e^{-t})$, generated by the classical second-order Laguerre differential expression $\ell[\cdot]$ defined by

$$\ell[y](t) := \frac{1}{t^{\alpha}e^{-t}}(-(t^{\alpha+1}e^{-t}y'(t))' + kt^{\alpha}e^{-t}y(t)) \qquad (t \in (0,\infty)), \tag{12.1}$$

having the Laguerre polynomials $\{L_m^{\alpha}(t)\}_{m=0}^{\infty}$ as eigenfunctions;

(b) the sequence of left-definite self-adjoint operators $\{A_n\}_{n=1}^{\infty}$ associated with $(L^2((0,\infty); t^{\alpha}e^{-t}), A)$, and their domains $\{\mathscr{D}(A_n)\}_{n=1}^{\infty}$; and

(c) the domains $\mathscr{D}(A^n)$ of each integral power A^n of A. In particular, we give a new characterization of the domain $\mathscr{D}(A)$ of A that is independent of $\alpha > -1$ (see Corollary 12.9).

Even though the theory developed to this point guarantees the existence of a continuum of left-definite spaces $\{H_r\}_{r>0}$ and left-definite operators $\{A_r\}_{r>0}$ (they are all *differential* operators), we can only explicitly determine the left-definite spaces, their inner products, and the domains of the left-definite operators when r is a positive integer. A careful explanation for why this is the case will be given later in this section.

For the rest of this section, we fix $\alpha > -1$; moreover, unless otherwise specified, we shall assume that k is a fixed, positive constant. To simplify the notation, we refer to certain self-adjoint operators as A, A^n, A_n, etc., instead of $A_{\alpha,k}, A^n_{\alpha,k}, A_{n,\alpha,k}$, etc., respectively; likewise, we suppress the dependence on α and k when we refer to the various left-definite spaces and the Laguerre differential expressions.

In most textbooks on special functions it is customary to set $k = 0$ in the Laguerre equation. However, for spectral reasons, it is necessary that $k > 0$; a specific reason for this will be given shortly.

When $\lambda = m \in \mathbb{N}_0$, the equation $\ell[y](t) = (\lambda + k)y(t)$, which in nonsymmetric form can be rewritten as

$$ty'' + (1 + \alpha - t)y' + my = 0,$$

has a polynomial solution $L_m^\alpha(t)$ of degree m; the sequence of polynomials $\{L_m^\alpha(t)\}_{n=0}^\infty$ is called the *generalized Laguerre* or *Laguerre–Sonine* polynomials. These polynomials form a complete orthogonal set in the Hilbert space

$$L_\alpha^2(0,\infty) := L^2((0,\infty); t^\alpha e^{-t}) \tag{12.2}$$

of Lebesgue measurable functions $f\colon (0,\infty) \to \mathbb{C}$ satisfying $\|f\| < \infty$, where $\|\cdot\|$ is the norm generated from the inner product $(\cdot,\cdot)$, defined by

$$(f,g) := \int_0^\infty f(t)\bar{g}(t)t^\alpha e^{-t} dt \qquad (f,g \in L_\alpha^2(0,\infty)). \tag{12.3}$$

In fact, with the mth Laguerre polynomial defined by

$$L_m^\alpha(t) = \left(\frac{1}{\Gamma(\alpha+1)\binom{m+\alpha}{m}}\right)^{1/2} \sum_{j=0}^m \frac{(-1)^j}{j!}\binom{m+\alpha}{m-j} t^j \qquad (m \in \mathbb{N}_0),$$

it is the case that $\{L_m^\alpha(t)\}_{m=0}^\infty$ is orthonormal in $L_\alpha^2(0,\infty)$; that is,

$$(L_m^\alpha, L_r^\alpha) = \delta_{m,r} \qquad (m, r \in \mathbb{N}_0), \tag{12.4}$$

where $\delta_{m,r}$ is the Kronecker delta function. We refer the reader to [40, Chap. 12] or [45, Chap. V] for various properties of the Laguerre polynomials. One particular property that we will repeatedly use throughout this section is the derivative formula

$$\frac{d^j(L_m^\alpha(t))}{dt^j} = C_m(\alpha, j)L_{m-j}^{\alpha+j}(t) \qquad (m, j \in \mathbb{N}_0), \tag{12.5}$$

where

$$C_m(\alpha, j) = (-1)^j (P(m,j))^{1/2} \tag{12.6}$$

and

$$P(m,j) = m(m-1)\cdots(m-j+1) \qquad (m, j \in \mathbb{N}_0; j \leqslant m). \tag{12.7}$$

From (12.5) and the orthonormality of the Laguerre polynomials, we see that

$$\int_0^\infty \frac{d^j(L_m^\alpha(t))}{dt^j}\frac{d^j(L_r^\alpha(t))}{dt^j} t^{\alpha+j}e^{-t}dt = P(m,j)\delta_{m,r} \qquad (m,r,j \in \mathbb{N}_0). \tag{12.8}$$

The maximal domain Δ of $t^{-\alpha}e^t\ell[\cdot]$ in $L_\alpha^2(0,\infty)$ is defined to be

$$\Delta = \{f \in L_\alpha^2(0,\infty) \mid f, f' \in AC_{\text{loc}}(0,\infty); t^{-\alpha}e^t\ell[f] \in L_\alpha^2(0,\infty)\}. \tag{12.9}$$

Define the operator A: $L_\alpha^2(0,\infty) \to L_\alpha^2(0,\infty)$ by

$$Af(t) = \ell[f](t) \qquad (f \in \mathscr{D}(A),\ \text{a.e. } t > 0), \tag{12.10}$$

where the domain of A is given by

$$\mathscr{D}(A) = \left\{ f \in \Delta \,\middle|\, \lim_{t \to 0^+} t^{\alpha+1}e^{-t}f'(t) = 0 \right\} \tag{12.11}$$

when $-1 < \alpha < 1$ and

$$\mathscr{D}(A) = \Delta \tag{12.12}$$

in the case that $\alpha \geqslant 1$. Then, as can be seen by the Glazman–Krein–Naimark theory [33, Theorem 4, Section 18.1], A is a self-adjoint operator and has the Laguerre polynomials $\{L_m^\alpha(t)\}_{m=0}^\infty$ as a complete set of eigenfunctions; moreover, the spectrum of A is given by

$$\sigma(A) = \{m + k \mid m \in \mathbb{N}_0\}. \tag{12.13}$$

For further details on the spectral theory of the Laguerre equation and other second-order classical differential equations, the reader is referred to [2, Appendix II, Sect. 9]; [46, Chap. IV], and the account in [37].

It is also well-known (for example, see [37]) that

$$(Af, f) = \int_0^\infty [t^{\alpha+1}e^{-t}|f'(t)|^2 + kt^\alpha e^{-t}|f(t)|^2]dt \geqslant k(f,f)$$
$$(f \in \mathscr{D}(A)). \tag{12.14}$$

That is, A is bounded below in $L_\alpha^2(0,\infty)$ by kI. It is this inequality that explains the importance of the positivity of k in (12.1). Consequently, we can apply Theorems 3.1, 3.2, and 3.6. Note that $(\cdot,\cdot)_1$, defined by

$$(f,g)_1 = \int_0^\infty [t^{\alpha+1}e^{-t}f'(t)\bar{g}'(t) + kt^\alpha e^{-t}f(t)\bar{g}(t)]dt \qquad (f, g \in \mathscr{D}(A)),$$

is an inner product; in fact, it is the inner product for the first left-definite space associated with the pair $(L^2_\alpha(0,\infty), A)$. Moreover, the closure of $\mathscr{D}(A)$ in the topology generated from this inner product is the first left-definite space H_1 associated with $(L^2_\alpha(0,\infty), A)$.

We now turn our attention to the explicit construction of the sequence of left-definite inner products $(\cdot,\cdot)_n$ $(n \in \mathbb{N})$ associated with $(L^2_\alpha(0,\infty), A)$. As we will see, these are generated from the integral powers $\ell^n[\cdot]$ $(n \in \mathbb{N})$ of the Laguerre expression $\ell[\cdot]$, given inductively by

$$\ell^1[y] = \ell[y],\ \ell^2[y] = \ell(\ell[y]), \ldots, \ell^n[y] = \ell(\ell^{n-1}[y]) \qquad (n \in \mathbb{N}).$$

A key to the explicit determination of these powers is certain numbers $\{b_j(n,k)\}_{j=0}^n$ which we now define.

Definition 12.1. For $n \in \mathbb{N}$ and $j \in \{0, 1, \ldots, n\}$, define

$$b_j(n,k) := \sum_{i=0}^{j} \frac{(-1)^{i+j}}{j!} \binom{j}{i} (k+i)^n. \tag{12.15}$$

If we expand the term $(k+i)^n$ in (12.15) and switch the order of summation, we find that

$$\begin{aligned} b_j(n,k) &= \sum_{m=0}^{n} \left(\sum_{i=0}^{j} \frac{(-1)^{i+j}}{j!} \binom{j}{i} i^{n-m} \right) \binom{n}{m} k^m \\ &= \sum_{m=0}^{n} \binom{n}{m} S_{n-m}^{(j)} k^m, \end{aligned} \tag{12.16}$$

where

$$S_n^{(j)} := \sum_{i=0}^{j} \frac{(-1)^{i+j}}{j!} \binom{j}{i} i^n \qquad (n, j \in \mathbb{N}_0) \tag{12.17}$$

is the Stirling number of the second kind. By definition, $S_n^{(j)}$ is the number of ways of partitioning n elements into j nonempty subsets (in particular, $S_0^j = 0$ for any $j \in \mathbb{N}$); we refer the reader to [1, pp. 824–825] for various properties of these numbers. Consequently, we see that

$$b_0(n,k) = \begin{cases} 0 & \text{if } k = 0 \\ k^n & \text{if } k > 0, \end{cases} \tag{12.18}$$

and, for $j \in \{1, 2, \ldots, n\}$,

$$b_j(n,k) = \begin{cases} S_n^j & \text{if } k = 0 \\ \sum_{m=0}^{n-1} \binom{n}{m} S_{n-m}^{(j)} k^m & \text{if } k > 0. \end{cases} \tag{12.19}$$

In order to develop more properties of these numbers, we first prove the following lemma.

LEMMA 12.1. *Let $m, n \in \mathbb{N}$.*

(1) *If $m \leqslant n$,* $\sum_{j=m}^{n} (-1)^j \binom{n}{j} \binom{j}{m} = (-1)^n \delta_{n,m}$.

(2) *If $m < n$,* $\sum_{j=m}^{n-1} (-1)^j \binom{n}{j} \binom{j}{m} = (-1)^{n-1}\binom{n}{m}$.

Proof. The proof of (1) follows from the identity

$$\begin{aligned} \binom{n}{m} t^m (1+t)^{n-m} &= \binom{n}{m} \sum_{r=0}^{n-m} \binom{n-m}{r} t^{r+m} \\ &= \binom{n}{m} \sum_{j=m}^{n} \binom{n-m}{j-m} t^j \\ &= \sum_{j=m}^{n} \binom{n}{j} \binom{j}{m} t^j. \end{aligned}$$

Consequently, for $m < n$,

$$0 = \sum_{j=m}^{n} (-1)^j \binom{n}{j} \binom{j}{m} = \sum_{j=m}^{n-1} (-1)^j \binom{n}{j} \binom{j}{m} + (-1)^n \binom{n}{m};$$

this proves (2). ∎

LEMMA 12.2. *For each $n \in \mathbb{N}$, the numbers $b_j = b_j(n,k)$, defined in* (12.15), *satisfy the following properties.*

(1) *For $k > 0$, the numbers $\{b_j(n,k)\}_{j=0}^{n}$ are positive.*

(2) *They are the unique solutions to the equations*

$$(m+k)^n = \sum_{j=0}^{n} P(m,j)b_j \qquad (m \in \mathbb{N}_0), \tag{12.20}$$

where $P(m,j)$ *is defined in* (12.7).

Proof. Property (1) follows immediately from the positivity of the Stirling numbers of the second kind and the formulas listed in (12.18) and (12.19).

Since both sides of (12.20) are polynomials in m of degree n and since $P(m,j)$ is a polynomial in m of degree j, it is clear that the numbers $\{b_j\}_{j=0}^{n}$ exist and are unique. Furthermore, it is clear that, for fixed $n \in \mathbb{N}$ and $k > 0$, each b_j is independent of $m \in \mathbb{N}_0$ in (12.20). By setting $m = 0$ in (12.20), we obtain

$$b_0 = k^n,$$

which agrees with (12.15) when $j = 0$.

Suppose the number b_j, given in (12.15), satisfy (12.20) for $j = 0, 1, \ldots, r-1 < n$. Then, with $j = r \leqslant n$ and $m = r$, we see that

$$(r+k)^n = \sum_{j=0}^{n} P(r,j)b_j = \sum_{j=0}^{r} P(r,j)b_j \qquad \text{since } P(r,j) = 0 \text{ if } j > r,$$

so that

$$\begin{aligned} r!b_r &= (r+k)^n - \sum_{j=0}^{r-1} P(r,j)b_j \\ &= (r+k)^n - \sum_{j=0}^{r-1} \sum_{i=0}^{j} \frac{(-1)^{i+j}}{j!} P(r,j) \binom{j}{i} (k+i)^n. \end{aligned}$$

Switching the order of summation yields

$$\begin{aligned} b_r &= \frac{(r+k)^n}{r!} - \sum_{i=0}^{r-1} \frac{(-1)^i (k+i)^n}{r!} \sum_{j=i}^{r-1} (-1)^j \binom{r}{j} \binom{j}{i} \\ &= \frac{(r+k)^n}{r!} + \sum_{i=0}^{r-1} \frac{(-1)^{i+r}}{r!} \binom{r}{i} (k+i)^n \qquad \text{by Lemma 12.1, Part (2)} \\ &= \sum_{i=0}^{r} \frac{(-1)^{i+r}}{r!} \binom{r}{i} (k+i)^n. \end{aligned}$$

This completes the proof. ∎

With $\mathscr{P}$ denoting the space of all (possibly complex-valued) polynomials, we are now in a position to prove the following theorem.

THEOREM 12.3. *Let $n \in \mathbb{N}$ and let $\ell[\cdot]$ denote the Laguerre differential expression defined in* (12.1). *Then*

$$\int_0^\infty \ell^n[p](t)\bar{q}(t)t^\alpha e^{-t}dt$$
$$= \sum_{j=0}^n b_j(n,k) \int_0^\infty p^{(j)}(t)\bar{q}^{(j)}(t)t^{\alpha+j}e^{-t}dt \qquad (p, q \in \mathscr{P}). \qquad (12.21)$$

Furthermore, $\ell^n[\cdot]$ is Lagrangian symmetrizable with the symmetry factor $w(t) = t^\alpha e^{-t}$, and the Lagrangian symmetric form of $t^\alpha e^{-t}\ell^n[\cdot]$ is given by

$$t^\alpha e^{-t}\ell^n[y](t) = \sum_{j=0}^n (-1)^j (b_j(n,k)t^{\alpha+j}e^{-t}y^{(j)}(t))^{(j)}, \qquad (12.22)$$

where $\{b_j(n,k)\}_{j=0}^n$ are the numbers defined in (12.15) *or* (12.18) *and* (12.19).

Proof. Since the Laguerre polynomials $\{L_m^\alpha(t)\}_{m=0}^\infty$ form a basis for $\mathscr{P}$, it suffices to show that (12.21) is valid for $p = L_m^\alpha(t)$ and $q = L_r^\alpha(t)$, where $m, r \in \mathbb{N}_0$ are arbitrary. From the identity

$$\ell^n[L_m^\alpha](t) = (m+k)^n L_m^\alpha(t) \qquad (m \in \mathbb{N}_0) \qquad (12.23)$$

it follows, with this particular choice of p and q, that the left-hand side of (12.21) reduces to $(m+k)^n\delta_{m,r}$. On the other hand, from (12.8), the right-hand side of (12.21) becomes

$$\sum_{j=0}^n P(m,j)b_j(n,k)\delta_{m,r}.$$

From Lemma 12.2, Part (2), we conclude that (12.21) is true for our choice of polynomials p and q.

To prove (12.22), define the differential expression

$$m[y](t) := \frac{1}{t^\alpha e^{-t}} \sum_{j=0}^n (-1)^j (b_j(n,k)t^{\alpha+j}e^{-t}y^{(j)}(t))^{(j)}. \qquad (12.24)$$

For $p, q \in \mathscr{P}$ and $[a,b] \subset (0,\infty)$, we apply integration by parts to obtain

$$\begin{aligned}&\int_a^b m[p](t)\bar{q}(t)t^{\alpha}e^{-t}dt\\ &= \sum_{j=0}^{n} (-1)^j b_j(n,k) \sum_{r=1}^{j} (-1)^{r+1} (p^{(j)}(t)t^{\alpha+j}e^{-t})^{(j-r)} \bar{q}^{(r-1)}(t)|_a^b\\ &\quad + \sum_{j=0}^{n} b_j(n,k) \int_a^b p^{(j)}(t)\bar{q}^{(j)}(t)t^{\alpha+j}e^{-t}dt.\end{aligned}$$

Now, for any $p \in \mathscr{P}$, $(p^{(j)}(t)t^{\alpha+j}e^{-t})^{(j-r)} = tp_{j,r}(t)t^{\alpha}e^{-t}$ for some $p_{j,r} \in \mathscr{P}$. Consequently, as $a \to 0^+$ and $b \to \infty$, we see that

$$\begin{aligned}&\int_0^\infty m[p](t)\bar{q}(t)t^{\alpha}e^{-t}dt\\ &= \sum_{j=0}^{n} b_j(n,k) \int_0^\infty p^{(j)}(t)\bar{q}^{(j)}(t)t^{\alpha+j}e^{-t}dt \qquad (p,q \in \mathscr{P}).\end{aligned} \tag{12.25}$$

Hence, from (12.21) and (12.25), we see that for all polynomials p and q, we have

$$(\ell^n[p] - m[p], q) = 0.$$

From the density of polynomials in $L^2_\alpha(0,\infty)$, it follows that

$$\ell^n[p](t) = m[p](t) \qquad (t > 0) \tag{12.26}$$

for all polynomials p. This latter identity implies that the expression $\ell^n[\cdot]$ has the form given in (12.22). ∎

For example, we see from this theorem that

$$t^{\alpha}e^{-t}\ell^2[y](t) = (t^{\alpha+2}e^{-t}y'')'' - ((2k+1)t^{\alpha+1}e^{-t}y')' + k^2t^{\alpha}e^{-t}y$$

and

$$\begin{aligned}t^{\alpha}e^{-t}\ell^3[y](t) = &-(t^{\alpha+3}e^{-t}y''')''' + ((3k+3)t^{\alpha+2}e^{-t}y'')''\\ &-((3k^2+3k+1)t^{\alpha+1}e^{-t}y')' + k^3t^{\alpha}e^{-t}y.\end{aligned}$$

The following corollary lists some additional properties of $\ell^n[\cdot]$.

COROLLARY 12.4. *Let $n \in \mathbb{N}$. Then*

(1) *the* nth *power of the classical Laguerre differential expression*,

$$\mathscr{L}[y](t) := -ty''(t) + (t - 1 - \alpha)y'(t),$$

is symmetrizable with symmetry factor $w(t) = t^{\alpha}e^{-t}$ and has the Lagrangian symmetric form

$$t^{\alpha}e^{-t}\mathscr{L}^{n}[y](t) := \sum_{j=1}^{n} (-1)^{j}(S_{n}^{(j)}t^{\alpha+j}e^{-t}y^{(j)}(t))^{(j)},$$

where $S_n^{(j)}$ is the Stirling number of the second kind defined in (12.17);

(2) *the bilinear form $(\cdot,\cdot)_n$ defined on $\mathscr{P} \times \mathscr{P}$ by*

$$(p,q)_n := \sum_{j=0}^{n} b_j(n,k) \int_0^{\infty} p^{(j)}(t)\bar{q}^{(j)}(t)t^{\alpha+j}e^{-t}dt \qquad (p,q \in \mathscr{P}) \tag{12.27}$$

is an inner product when $k > 0$ and satisfies

$$(\ell^{n}[p],q) = (p,q)_n \qquad (p,q \in \mathscr{P}); \tag{12.28}$$

(3) *the Laguerre polynomials $\{L_m^{\alpha}(t)\}_{m=0}^{\infty}$ are orthogonal with respect to the inner product $(\cdot,\cdot)_n$, and in fact*,

$$(L_m^{\alpha}, L_r^{\alpha})_n = \sum_{j=0}^{n} b_j(n,k) \int_0^{\infty} \frac{d^j L_m^{\alpha}(t)}{dt^j} \frac{d^j L_r^{\alpha}(t)}{dt^j} t^{\alpha+j}e^{-t}dt = (m+k)^n \delta_{m,r}. \tag{12.29}$$

Proof. The proof of (1) follows immediately from Theorem 12.3 and the identities (12.18) and (12.19). The proof of (2) is clear since the numbers $\{b_j(n,k)\}_{j=0}^{n}$ are positive when $k > 0$. The identity in (12.28) follows from (12.25) and (12.26). Lastly, (12.29) is a special case of (12.28), using (12.4) and (12.23). ∎

For results that follow in this section, it is convenient to introduce the following notation. For $n \in \mathbb{N}$, let

$$AC_{\text{loc}}^{(n-1)}(0,\infty) := \{f\colon (0,\infty) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\text{loc}}(0,\infty)\}.$$

Also, for $\alpha > -1$ and $j \in \mathbb{N}_0$, let $L^2_{\alpha+j}(0,\infty)$ be the Hilbert space defined by

$$L^2_{\alpha+j}(0,\infty) := \{f\colon (0,\infty) \to \mathbb{C} \mid f \text{ is Lebesgue measurable and} \\ \left.\int_0^\infty |f|^2 t^{\alpha+j} e^{-t} dt < \infty \right\}, \tag{12.30}$$

with inner product $\int_0^\infty f(t)\bar{g}(t)t^{\alpha+j}e^{-t}dt$ $(f, g \in L^2_{\alpha+j}(0,\infty))$.

Definition 12.2. *For each $n \in \mathbb{N}$, define*

$$V_n := \{f\colon (0,\infty) \to \mathbb{C} \mid f \in AC^{(n-1)}_{\mathrm{loc}}(0,\infty); \\ f^{(j)} \in L^2_{\alpha+j}(0,\infty)\ (j = 0, 1, \ldots, n)\}, \tag{12.31}$$

where each $L^2_{\alpha+j}(0,\infty)$ is defined in (12.30), *and let $(\cdot,\cdot)_n$ and $\|\cdot\|_n$ denote, respectively, the inner product*

$$(f, g)_n = \sum_{j=0}^n b_j(n,k) \int_0^\infty f^{(j)}(t)\bar{g}^{(j)}(t)t^{\alpha+j}e^{-t}dt \qquad (f, g \in V_n), \tag{12.32}$$

(*see* (12.27) *and* (12.28)) *and the norm $\|f\|_n = (f, f)_n^{1/2}$, where the numbers $b_j(n,k)$ are defined in* (12.15).

The inner product $(\cdot,\cdot)_n$, defined in (12.32), is a Sobolev inner product and is more commonly called the *Dirichlet inner product* associated with the symmetric differential expression (12.22). We remark that, for each $r > 0$, the spectral theorem abstractly gives the rth left-definite inner product to be

$$(f, g)_r = \int_{\mathbb{R}} \lambda^r dE_{f,g} \qquad (f, g \in V_r),$$

where E is the spectral resolution of the identity for A. However, unlike in the previous example, we are able to determine this inner product in terms of the differential expression $\ell^r[\cdot]$ only when $r \in \mathbb{N}$.

We aim to show (see Theorem 12.8) that

$$H_n = (V_n, (\cdot,\cdot)_n)$$

is the nth left-definite space associated with the pair $(L^2_\alpha(0,\infty), A)$, where A is defined in (12.10), (12.11), and (12.12). We begin by showing that H_n is a complete inner product space.

Theorem 12.5. *For each $n \in \mathbb{N}$, H_n is a Hilbert space.*

Proof. Suppose $\{f_m\}_{m=1}^{\infty}$ is Cauchy in H_n. Since each of the numbers $b_j(n,k)$ is positive, we have in particular that $\{f_m^{(n)}\}_{m=1}^{\infty}$ is Cauchy in $L^2_{\alpha+n}(0,\infty)$ and hence there exists $g_{n+1} \in L^2_{\alpha+n}(0,\infty)$ such that

$$f_m^{(n)} \to g_{n+1} \qquad \text{in } L^2_{\alpha+n}(0,\infty).$$

Fix t, $t_0 > 0$ (t_0 will be chosen shortly) and assume $t_0 \leqslant t$. From Hölder's inequality,

$$\begin{aligned}
&\int_{t_0}^{t} |f_m^{(n)}(u) - g_{n+1}(u)|du \\
&= \int_{t_0}^{t} |f_m^{(n)}(u) - g_{n+1}(u)|u^{\frac{\alpha+n}{2}} e^{-u/2} u^{\frac{-\alpha-n}{2}} e^{u/2} du \\
&\leqslant \left(\int_{t_0}^{t} |f_m^{(n)}(u) - g_{n+1}(u)|^2 u^{\alpha+n} e^{-u} du\right)^{1/2} \cdot \left(\int_{t_0}^{t} u^{-\alpha-n} e^{u} du\right)^{1/2} \\
&= M(t_0,t)\left(\int_{t_0}^{t} |f_m^{(n)}(u) - g_{n+1}(u)|^2 u^{\alpha+n} e^{-u} du\right)^{1/2} \to 0 \qquad \text{as } m \to \infty.
\end{aligned}$$

Consequently, since $f_m^{(n-1)} \in AC_{\mathrm{loc}}(0,\infty)$, we see that

$$f_m^{(n-1)}(t) - f_m^{(n-1)}(t_0) = \int_{t_0}^{t} f_m^{(n)}(u)du \to \int_{t_0}^{t} g_{n+1}(u)du, \tag{12.33}$$

and, in particular, $g_{n+1} \in L^1_{\mathrm{loc}}(0,\infty)$. Furthermore, from the definition of $(\cdot,\cdot)_n$, we have see that the sequence $\{f_m^{(n-1)}\}_{m=0}^{\infty}$ is also Cauchy in $L^2_{\alpha+n-1}(0,\infty)$; hence there exists a function $g_n \in L^2_{\alpha+n-1}(0,\infty)$ such that

$$f_m^{(n-1)} \to g_n \qquad \text{in } L^2_{\alpha+n-1}(0,\infty).$$

Repeating the above argument, we see that $g_n \in L^1_{\mathrm{loc}}(0,\infty)$ and, for any $t, t_1 > 0$,

$$f_m^{(n-2)}(t) - f_m^{(n-2)}(t_1) = \int_{t_1}^{t} f_m^{(n-1)}(u)du \to \int_{t_1}^{t} g_n(u)du. \tag{12.34}$$

Moreover, there exists a subsequence $\{f_{m_{k,n-1}}^{(n-1)}\}$ of $\{f_m^{(n-1)}\}_{m=1}^{\infty}$ such that

$$f_{m_{k,n-1}}^{(n-1)}(t) \to g_n(t) \qquad \text{a.e. } t > 0.$$

Choose $t_0 > 0$ in (12.33) such that $f_{m_{k,n-1}}^{(n-1)}(t_0) \to g_n(t_0)$ and then pass through this subsequence in (12.33) to obtain

$$g_n(t) - g_n(t_0) = \int_{t_0}^{t} g_{n+1}(u)du \qquad (\text{a.e. } t > 0).$$

That is to say,

$$g_n \in AC_{\text{loc}}(0,\infty) \qquad \text{and} \qquad g_n'(t) = g_{n+1}(t) \text{ a.e. } t > 0. \tag{12.35}$$

Next, we see that $\{f_m^{(n-2)}\}_{m=1}^{\infty}$ is Cauchy in $L^2_{\alpha+n-2}(0,\infty)$ so there exists $g_{n-1} \in L^2_{\alpha+n-2}(0,\infty)$ such that

$$f_m^{(n-2)} \to g_{n-1} \qquad \text{in } L^2_{\alpha+n-2}(0,\infty).$$

As above, we find that $g_{n-1} \in L^1_{\text{loc}}(0,\infty)$; moreover, for any $t, t_2 > 0$

$$f_m^{(n-3)}(t) - f_m^{(n-3)}(t_2) = \int_{t_2}^{t} f_m^{(n-2)}(u)du \to \int_{t_2}^{t} g_{n-1}(u)du,$$

and there exists a subsequence $\{f_{m_{k,n-2}}^{(n-2)}\}$ of $\{f_m^{(n-2)}\}$ such that

$$f_{m_{k,n-2}}^{(n-2)}(t) \to g_{n-1}(t) \qquad \text{a.e. } t > 0.$$

In (12.34), choose $t_1 > 0$ such that $f_{m_{k,n-2}}^{(n-2)}(t_1) \to g_{n-1}(t_1)$ and pass through the subsequence $\{f_{m_{k,n-2}}^{(n-2)}\}$ in (12.34) to obtain

$$g_{n-1}(t) - g_{n-1}(t_1) = \int_{t_1}^{t} g_n(u)du \qquad (\text{a.e. } t > 0).$$

Consequently, $g_{n-1} \in AC^{(1)}_{\text{loc}}(0,\infty)$ and $g''_{n-1}(t) = g'_n(t) = g_{n+1}(t)$ a.e. $t > 0$. Continuing in this fashion, we obtain $n+1$ functions $g_{n+1-j} \in L^2_{\alpha+n-j}(0,\infty) \cap L^1_{\text{loc}}(0,\infty)$ $(j = 0, 1, \ldots, n)$ such that

(i) $f_m^{(n-k)} \to g_{n-k+1}$ in $L^2_{\alpha+n-k}(0,\infty)$ $(k = 0, 1, \ldots, n)$,

(ii) $g_1 \in AC^{(n-1)}_{\text{loc}}(0,\infty)$; $g_2 \in AC^{(n-2)}_{\text{loc}}(0,\infty), \ldots, g_n \in AC_{\text{loc}}(0,\infty)$,

(iii) $g'_{n-k}(t) = g_{n-k+1}(t)$ a.e. $t > 0$ $(k = 0, 1, \ldots, n-1)$,

(iv) $g_1^{(j)} = g_{j+1}$ $(j = 0, 1, \ldots, n)$.

In particular, we see that $f_m^{(j)} \to g_1^{(j)}$ in $L^2_{\alpha+j}(0,\infty)$ for $j = 0, 1, \ldots, n$ and $g_1 \in V_n$. Hence, we see that

$$\begin{aligned} \|f_m - g_1\|_n^2 &= \sum_{j=0}^{n} b_j(n,k) \int_0^{\infty} |f_m^{(j)}(u) - g_1^{(j)}(u)|^2 u^{\alpha+j} e^{-u} du \\ &\to 0 \qquad \text{as } m \to \infty. \end{aligned}$$

Hence H_n is complete.

We now show that $\mathscr{P}$ is dense in H_n; consequently, $\{L_m^{\alpha}(t)\}_{m=0}^{\infty}$ is a complete orthogonal set in H_n. We remark that we cannot appeal to Theorem 3.7 to conclude that the Laguerre polynomials are complete in H_n.

Indeed, we do not know at this point that H_n is the nth left-definite space associated with $(L^2_\alpha(0,\infty), A)$.

THEOREM 12.6. *The Laguerre polynomials* $\{L^\alpha_m(t)\}_{m=0}^\infty$ *form a complete orthogonal set in the space* H_n. *In particular, the space* $\mathcal{P}$ *of polynomials is dense in* H_n.

Proof. Let $f \in H_n$; in particular, $f^{(n)} \in L^2_{\alpha+n}(0,\infty)$. Consequently, from the completeness and orthonormality of $\{L^{\alpha+n}_m(t)\}_{m=0}^\infty$ in $L^2_{\alpha+n}(0,\infty)$, it follows that

$$\sum_{m=0}^{r} c_{m,n} L_m^{\alpha+n} \to f^{(n)} \qquad \text{as } r \to \infty \text{ in } L^2_{\alpha+n}(0,\infty),$$

where the numbers $\{c_{m,n}\}_{m=0}^\infty \subset \ell^2$ are the Fourier coefficients of $f^{(n)}$ defined by

$$c_{m,n} = \int_0^\infty f^{(n)}(t) L_m^{\alpha+n}(t) t^{\alpha+n} e^{-t} dt \qquad (m \in \mathbb{N}_0).$$

For $r \geqslant n$, define the polynomials

$$p_r(t) = \sum_{m=n}^{r} \frac{c_{m-n,n}}{C_m(\alpha,n)} L_m^\alpha(t),$$

where the numbers $\{C_m(\alpha,n)\}_{m\geqslant n}$ are defined in (12.6). Then, using the derivative formula (12.5) for the Laguerre polynomials, we see that

$$p_r^{(j)}(t) = \sum_{m=n}^{r} \frac{c_{m-n,n} C_m(\alpha,j)}{C_m(\alpha,n)} L_{m-j}^{\alpha+j}(t) \qquad (j = 1,2,\ldots), \tag{12.36}$$

and, in particular, as $r \to \infty$,

$$p_r^{(n)} = \sum_{m=n}^{r} c_{m-n,n} L_{m-n}^{\alpha+n} \to f^{(n)} \qquad \text{in } L^2_{\alpha+n}(0,\infty).$$

Furthermore, from [42, Theorem 3.12], there exists a subsequence $\{p_{r_j}^{(n)}\}$ of $\{p_r^{(n)}\}$ such that

$$p_{r_j}^{(n)}(t) \to f^{(n)}(t) \qquad \text{as } j \to \infty \text{ (a.e. } t > 0). \tag{12.37}$$

Returning to (12.36), observe that since $C_m(\alpha,j)/C_m(\alpha,n) \to 0$ as $m \to \infty$ for $j = 0,1,\ldots,n-1$, we see that

$$\left\{\frac{c_{m-n,n} C_m(\alpha,j)}{C_m(\alpha,n)}\right\}_{m=n}^\infty$$

is a square-summable sequence. Consequently, from the completeness of $\{L_m^{\alpha+j}(t)\}_{m=0}^\infty$ in $L^2_{\alpha+j}(0,\infty)$ and the Riesz–Fischer theorem (see [42, Chap. 4, Theorem 4.17], there exists $g_j \in L^2_{\alpha+j}(0,\infty)$ such that

$$p_r^{(j)} \to g_j \quad \text{in } L^2_{\alpha+j}(0,\infty) \text{ as } r \to \infty \ (j = 0, 1, \dots, n-1). \tag{12.38}$$

Since, for a.e. $a, t > 0$,

$$\begin{aligned} p_{r_j}^{(n-1)}(t) - p_{r_j}^{(n-1)}(a) &= \int_a^t p_{r_j}^{(n)}(u)du \to \int_a^t f^{(n)}(u)du \\ &= f^{(n-1)}(t) - f^{(n-1)}(a) \qquad (j \to \infty), \end{aligned}$$

we see that, as $j \to \infty$,

$$p_{r_j}^{(n-1)}(t) \to f^{(n-1)}(t) + c_1 \text{ a.e. } t > 0, \tag{12.39}$$

where c_1 is some constant. From (12.38), with $j = n-1$, we deduce that

$$g_{n-1}(t) = f^{(n-1)}(t) + c_1 \text{ a.e. } t > 0.$$

Next, from (12.39) and one integration, we obtain

$$p_{r_j}^{(n-2)}(t) \to f^{(n-2)}(t) + c_1 t + c_2 \qquad (j \to \infty),$$

for some constant c_2 and hence, from (12.38),

$$g_{n-2}(t) = f^{(n-2)}(t) + c_1 t + c_2.$$

We continue this process to see that, as $r \to \infty$,

$$p_r^{(j)} \to f^{(j)} + q_{n-j-1} \qquad \text{in } L^2_{\alpha+j}(0,\infty) \ (j = 1, 2, \dots, n),$$

where q_{n-j-1} is a polynomial of degree $\leqslant n-j-1$ $(q_{-1} = 0)$ satisfying

$$q'_{n-j-1}(t) = q_{n-j-2}(t).$$

For each $r \geqslant n$, define the polynomials

$$\pi_r(t) := p_r(t) - q_{n-1}(t)$$

and observe that

$$\begin{aligned} \pi_r^{(j)} &= p_r^{(j)} - q_{n-1}^{(j)} \\ &= p_r^{(j)} - q_{n-j-1} \\ &\to f^{(j)} \qquad \text{in } L^2_{\alpha+j}(0,\infty). \end{aligned}$$

Hence,

$$\begin{aligned}\|f-\pi_r\|_n^2 &= \sum_{j=0}^{n} b_j(n,k)\int_0^\infty |f^{(j)}(u)-\pi_r^{(j)}|^2 u^{\alpha+j}e^{-u}du\\ &\to 0 \qquad \text{as } r\to\infty. \quad \blacksquare\end{aligned}$$

The next result, which gives a simpler characterization of the function space V_n, follows from ideas in the above proof of Theorem 12.6.

THEOREM 12.7. *For each* $n \in \mathbb{N}$,

$$V_n = \{f\colon (0,\infty)\to\mathbb{C} \mid f\in AC_{\mathrm{loc}}^{(n-1)}(0,\infty);\, f^{(n)}\in L^2_{\alpha+n}(0,\infty)\}, \tag{12.40}$$

where $L^2_{\alpha+n}(0,\infty)$ *is defined in* (12.30).

Proof. Define

$$V_n' = \{f\colon (0,\infty)\to\mathbb{C} \mid f\in AC_{\mathrm{loc}}^{(n-1)}(0,\infty);\, f^{(n)}\in L^2_{\alpha+n}(0,\infty)\};$$

it is clear, from the definition of V_n in (12.31), that $V_n \subset V_n'$. Conversely, suppose $f\in V_n'$ so $f^{(n)}\in L^2_{\alpha+n}(0,\infty)$ and $f\in AC_{\mathrm{loc}}^{(n-1)}(0,\infty)$. From the proof of Theorem 12.6, we see that there are polynomials $\{\pi_r\}\subset L^2_{\alpha+j}(0,\infty)$ such that

$$\pi_r^{(j)}\to f^{(j)} \text{ in } L^2_{\alpha+j}(0,\infty)\ (j=0,1,\ldots,n-1).$$

That is, $f^{(j)}\in L^2_{\alpha+j}(0,\infty)$ for $j=0,1,\ldots,n-1$, so $f\in V_n$. $\blacksquare$

We are now in position to prove the main result of this section.

THEOREM 12.8. (a) *For* $\alpha>-1$ *and* $k>0$, *let* $A\colon \mathscr{D}(A)\subset L^2_\alpha(0,\infty)\to L^2_\alpha\times(0,\infty)$ *denote the self-adjoint operator, defined in* (12.10), (12.11), *and* (12.12), *having the Laguerre polynomials* $\{L_m^\alpha(t)\}_{m=0}^\infty$ *as a complete set of eigenfunctions. For each* $n\in N$, *let* V_n *be given as in* (12.31) *or* (12.40) *and let* $(\cdot,\cdot)_n$ *denote the inner product defined in* (12.27). *Then* $H_n=(V_n,(\cdot,\cdot)_n)$ *is the nth left-definite space associated with the pair* $(L^2_\alpha(0,\infty),A)$. *Moreover, the Laguerre polynomials* $\{L_m^\alpha(t)\}_{m=0}^\infty$ *form a complete orthogonal set in* H_n, *satisfying the orthogonality relation* (12.29).

(b) *Define*

$$A_n\colon \mathscr{D}(A_n)\subset H_n\to H_n$$

by

$$A_n f(t) = \ell[f](t) \qquad (\text{a.e. } t \in (0,\infty))$$

for

$$f \in \mathscr{D}(A_n) := \{f\colon (0,\infty) \to \mathbb{C} \mid f \in AC_{\text{loc}}^{(n+1)}(0,\infty);\, f^{(n+2)} \in L^2_{\alpha+n+2}(0,\infty)\}, \tag{12.41}$$

where $\ell[\cdot]$ *is the Laguerre differential expression defined in* (12.1). *Then* A_n *is the nth left-definite self-adjoint operator associated with the pair* $(L^2_\alpha(0,\infty), A)$. *Furthermore, the Laguerre polynomials* $\{L^\alpha_m(t)\}_{m=0}^\infty$ *are eigenfunctions of* A_n *and the spectrum of* A_n *is explicitly given by*

$$\sigma(A_n) = \{m + k \mid m \in \mathbb{N}_0\}.$$

Proof. To show that H_n is the nth left-definite space for the pair $(L^2_\alpha(0,\infty), A)$, we must show that the five conditions in Definition 2.2 are satisfied.

(i) H_n is complete.

The proof of this condition is given in Theorems 12.5 and 12.7.

(ii) $\mathscr{D}(A^n) \subset H_n \subset L^2_\alpha(0,\infty)$.

Let $f \in \mathscr{D}(A^n)$. Since the Laguerre polynomials $\{L^\alpha_m(t)\}_{m=0}^\infty$ form a complete orthonormal set in $L^2_\alpha(0,\infty)$, we see that

$$p_j \to f \text{ in } L^2_\alpha(0,\infty), \tag{12.42}$$

where

$$p_j(t) = \sum_{m=0}^{j} c_m L^\alpha_m(t),$$

and $\{c_m\}_{m=0}^\infty$ are the Fourier coefficients of f in $L^2_\alpha(0,\infty)$ defined by

$$c_m = (f, L^\alpha_m) = \int_0^\infty f(t) L^\alpha_m(t) t^a e^{-t} dt \qquad (m \in \mathbb{N}_0).$$

Since $A^n f \in L^2_\alpha(0,\infty)$, we see that

$$\sum_{m=0}^{j} d_m L^\alpha_m \to A^n f \qquad \text{in } L^2_\alpha(0,\infty),$$

where

$$d_m = (A^n f, L^\alpha_m) = (f, A^n L^\alpha_m) = (m+k)^n (f, L^\alpha_m) = (m+k)^n c_m;$$

that is to say,

$$A^n p_j \to A^n f \qquad \text{in } L^2_\alpha(0,\infty).$$

Moreover, from (12.28), we see that

$$\begin{aligned}\|p_j - p_r\|_n^2 &= (A^n(p_j - p_r), p_j - p_r)\\ &\to 0 \qquad \text{as } j,r \to \infty.\end{aligned}$$

That is to say, $\{p_j\}_{j=0}^{\infty}$ is Cauchy in H_n. From Theorem 12.5, we see that there exists $g \in H_n \subset L^2_\alpha(0,\infty)$ such that

$$p_j \to g \qquad \text{in } H_n.$$

Furthermore, by the definition of $(\cdot,\cdot)_n$ and the fact that $b_0(n,k) = k^n$ for $k > 0$, we see that

$$(p_j - g, p_j - g)_n \geqslant k^n(p_j - g, p_j - g),$$

hence

$$p_j \to g \qquad \text{in } L^2_\alpha(0,\infty). \tag{12.43}$$

Comparing (12.42) and (12.43), we see that $f = g \in H_n$; this completes the proof of (ii).

(iii) $\mathscr{D}(A^n)$ is dense in H_n.

Since polynomials are contained in $\mathscr{D}(A^n)$ and are dense in H_n (see Theorem 12.6), it is clear that (iii) is valid. Furthermore, from Theorem 12.6, we see that $\{L_m^\alpha(t)\}_{m=0}^{\infty}$ forms a complete orthogonal set in H_n; see also (12.29).

(iv) $(f,f)_n \geqslant k^n(f,f)$ $(f \in V_n)$.

This is clear from the definition of $(\cdot,\cdot)_n$, the positivity of the coefficients $b_j(n,k)$, and the fact that $b_0(n,k) = k^n$.

(v) $(f,g)_n = (A^n f, g)$ $(f \in \mathscr{D}(A^n), g \in V_n)$.

Observe that this identity is true for any $f, g \in \mathscr{P}$; indeed, this is seen in (12.28). Let $f \in \mathscr{D}(A^n) \subset H_n$ and $g \in H_n$; since polynomials are dense in both H_n and $L^2_\alpha(0,\infty)$ and convergence in H_n implies convergence in $L^2_\alpha(0,\infty)$, there exist sequences of polynomials $\{p_j\}_{j=0}^{\infty}$ and $\{q_j\}_{j=0}^{\infty}$ such that

$$p_j \to f \text{ in } H_n, \qquad A^n p_j \to A^n f \text{ in } L^2_\alpha(0,\infty) \text{ (see the proof of part (ii))},$$

and

$$q_j \to g \text{ in } H_n \text{ and } L^2_\alpha(0,\infty).$$

Hence, from (12.28),

$$(A^n f, g) = \lim_{j\to\infty} (A^n p_j, q_j) = \lim_{j\to\infty} (p_j, q_j)_n = (f, g)_n.$$

This proves (v). The rest of the proof follows immediately from Theorems 3.2 and 3.6 upon noting that $\mathscr{D}(A_n)$, as defined in (12.41), is V_{n+2}, where V_n $(n \in \mathbb{N}_0)$ is as given in (12.40). ∎

The following corollary follows immediately from this theorem. Remarkably, it characterizes the domain of each of the integral powers of A. In particular, the characterization given below of the domain $\mathscr{D}(A)$ of the classical Laguerre differential operator A having the Laguerre polynomials as eigenfunctions seems to be new.

Corollary 12.9. *For each $n \in \mathbb{N}$, the domain $\mathscr{D}(A^n)$ of the nth power A^n of the classical self-adjoint operator A, defined in* (12.10), (12.11), *and* (12.12), *is given by*

$$\mathscr{D}(A^n) = V_{2n} = \{f\colon (0,\infty) \to \mathbb{C} \mid f \in AC_{\mathrm{loc}}^{(2n-1)}(0,\infty);\ f^{(2n)} \in L^2_{\alpha+2n}(0,\infty)\}.$$

In particular,

$$\mathscr{D}(A) = V_2 = \{f\colon (0,\infty) \to \mathbb{C} \mid f \in AC_{\mathrm{loc}}^{(1)}(0,\infty);\ tf'' \in L^2_{\alpha}(0,\infty)\}.$$

13. FURTHER EXAMPLES AND CONCLUDING REMARKS

In this last section, we connect – through several remarks – results of this paper to previous work on left-definite theory and the theory of orthogonal polynomials. In addition, we consider some difficult open questions that are related to our work.

Remark 13.1. If B is a densely defined symmetric operator in a Hilbert space $H = (V, (\cdot,\cdot))$ having equal deficiency indices and satisfying

$$(Bx, x) \geqslant k(x, x) \qquad (x \in \mathscr{D}(B))$$

for some constant $k > 0$, then it is known (see [41, pp. 330–335]) that B has a unique self-adjoint extension A in H defined by

$$Ax = B^* x,$$

$$x \in \mathscr{D}(A) = H_1 \cap \mathscr{D}(B^*),$$

and satisfying

$$(Ax, x) \geqslant k(x, x) \qquad (x \in \mathscr{D}(A)).$$

This operator A is called the *Friedrich extension* of B. Here, B^* is the adjoint of B and H_1 is the completion of $\mathscr{D}(B)$ in the topology generated from the inner product $(x, y)_1 = (Bx, y)$. Consequently, the left-definite theory developed in this paper can be applied to A.

REMARK 13.2. In [11] (see also [48]), the authors discuss the right-definite and first left-definite theory for the fourth-order *Legendre-type* differential equation

$$M[y](t) = \lambda y(t) \qquad (t \in (-1, 1)), \tag{13.1}$$

where

$$M[y](t) = ((1 - t^2)^2 y''(t))'' - ((8 + 4A(1 - t^2))y'(t))' + ky(t);$$

Here A and k are fixed, positive constants. For each $n \in \mathbb{N}_0$ and $\lambda = \lambda_n = n(n + 1)(n^2 + n + 4A - 2) + k$, Eq. (13.1) has a polynomial solution $y = P_n^A(t)$ of degree n; the sequence $\{P_n^A(t)\}_{n=0}^{\infty}$ is called the *Legendre-type* polynomials. They form a complete orthogonal set in the Hilbert space $L_\mu^2[-1, 1]$ with inner product

$$(f, g)_\mu = \int_{[-1,1]} f(t)\bar{g}(t)\, d\mu := \int_{-1}^{1} f(t)\bar{g}(t)dt + \frac{1}{A} f(-1)\bar{g}(-1) + \frac{1}{A} f(1)\bar{g}(1) \qquad (f, g \in L_\mu^2[-1, 1]).$$

As shown in [8] and [11], the operator A: $L_\mu^2[-1, 1] \to L_\mu^2[-1, 1]$, defined by

$$(Af)(t) = \begin{cases} -8Af'(-1) + kf(-1) & \text{if } t = -1 \\ M[f](t) & \text{if } -1 < t < 1 \\ 8Af'(1) + kf(1) & \text{if } t = 1 \end{cases}$$

with domain

$$\mathscr{D}(A) = \{f\colon (-1, 1) \to \mathbb{C} \mid f^{(j)} \in AC_{\text{loc}}(-1, 1)\ (j = 0, 1, 2, 3);\ f,\ M[f] \in L^2(-1, 1)\},$$

is self-adjoint and satisfies

$$(Af, f)_\mu \geqslant k(f, f)_\mu \qquad (f \in \mathscr{D}(A)).$$

That is to say, A is bounded below by kI in $L^2_\mu[-1,1]$. (We remark that if $f \in \mathscr{D}(A)$, the authors in [11] show that $f, f' \in AC[-1,1]$.) In [11], they define

$$V = \{f\colon [-1,1] \to \mathbb{C} \mid f \in AC[-1,1]; \\ f' \in AC_{\mathrm{loc}}(-1,1); f', (1-t^2)f'' \in L^2(-1,1)\}$$

and

$$(f,g)_1 = \frac{A}{2}\int_{-1}^{1} \{(1-t^2)^2 f''(t)\bar{g}''(t) + (8 + 4A(1-t^2))f'(t)\bar{g}'(t)\}dt \\ + k(f,g)_\mu \qquad (f, g \in V).$$

Combining results from [8] and [11], the authors show that $H = (V, (\cdot,\cdot)_1)$ is the first left-definite space associated with $(L^2_\mu[-1,1], A)$. Furthermore, in [11], they construct the first left-definite operator A_1 in the manner described in the Introduction. Although the domain of A_1 was not specified in [11], we now see that $\mathscr{D}(A_1) = \mathscr{D}(A^{3/2})$, where A is defined above. At the time of this writing, the other left-definite spaces H_r and left-definite operators A_r ($r > 0$, $r \neq 1$) associated with $(L^2_\mu[-1,1], A)$ are not explicitly known.

We note that the first left-definite theory associated with the *Laguerre-type* polynomials, which also satisfy a fourth-order Lagrangian symmetrizable differential equation, is also known; see [9], where the first left-definite space, its associated inner product, and the first left-definite operator are explicitly determined. Wellman [49] followed this work by analyzing, for each $n \in \mathbb{N}_0$, the right-definite and first left-definite properties for the self-adjoint operator $A = A(n)$, generated by the Laguerre-type differential equation of order $2n + 4$ (see [29]), having the generalized *Laguerre-type* polynomials as eigenfunctions. Similarly, the right-definite and first left-definite theory associated with the *Krall polynomials*, which satisfy a sixth-order Lagrangian symmetric equation, was developed and studied by Loveland in [30].

Remark 13.3. The left-definite theory developed in the preceding sections is also applicable to the nonclassical ($\alpha = -2$) Laguerre differential expression

$$\ell_{-2}[y](t) = -ty'' + (t+1)y' + ky \qquad (k > 0). \tag{13.2}$$

For each $n \in \mathbb{N}_0$, $y = L_n^{-2}(t)$ (the nth degree Laguerre polynomial) is a solution of

$$\ell_{-2}[y](t) = (n+k)y(t).$$

Expression (13.2) is made formally symmetric when multiplied by the weight function $w(t) = t^{-2}e^{-t}$. The classical Glazman–Krein–Naimark theory of self-adjoint extensions of symmetric differential expressions [33] shows that $\ell_{-2}[\cdot]$ has a unique self-adjoint representation A in the Hilbert space $L^2((0,\infty); w(t))$; in fact, A is bounded below by $kI > 0$. Moreover, the "tail-end" sequence of Laguerre polynomials $\{L_n^{-2}(t)\}_{n=2}^{\infty}$ forms a complete set of orthogonal eigenfunctions of A in $L^2((0,\infty); w(t))$. This raises the question: Is there a self-adjoint operator S, generated by $\ell_{-2}[\cdot]$, in some Hilbert space W having the *entire* sequence $\{L_n^{-2}(t)\}_{n=0}^{\infty}$ of Laguerre polynomials as eigenfunctions? In [14], [26] and [27] the authors show that $\{L_n^{-2}(t)\}_{n=0}^{\infty}$ forms a complete orthogonal sequence in the Hilbert space $W = (V, (\cdot,\cdot))$, where

$$V = \{f\colon [0,\infty) \to C \mid f, f' \in AC_{\mathrm{loc}}[0,\infty);\, f'' \in L^2((0,\infty); e^{-t})\}$$

and

$$(f,g) = f(0)\bar{g}(0) - f'(0)\bar{g}(0) - f(0)\bar{g}'(0) - 2f'(0)\bar{g}'(0)$$
$$+ \int_0^{\infty} |f''(t)\bar{g}''(t)e^{-t}dt \qquad (f, g \in V).$$

Applying the results of this paper, it is the case that $S\colon \mathscr{D}(S) \subset W \to W$ is explicitly given by

$$Sf = \ell_{-2}[f],$$

$$f \in \mathscr{D}(S) = \left\{ f \in W \mid f \in AC^{(3)}_{\mathrm{loc}}[0,\infty);\, \int_0^{\infty} |f^{(4)}(t)|^2 t^2 e^{-t} dt < \infty \right\}.$$

The key to this result is the decomposition $W = W_1 \oplus W_2$ into two orthogonal subspaces W_1 and W_2, where W_1 is finite dimensional and W_2 is isometrically isomorphic to H_2, the second left-definite space associated with the pair $(L^2((0,\infty); w(t)), A)$. Moreover, it is the second left-definite operator A_2 associated with the pair $(L^2((0,\infty); w(t)), A)$ that generates S. A complete discussion of the spectral theory for the Laguerre expression (12.1), when α is a negative integer, is forthcoming in a paper by Everitt *et al.* [12].

In [14], the authors construct a fourth-order self-adjoint differential operator $T\colon \mathscr{D}(T) \subset W \to W$, generated by $(\ell_{-2})^2$, that has the polynomials $\{L_n^{-2}\}_{n=0}^{\infty}$ as eigenfunctions. This operator T is partly generated by the square $(A_2)^2$ of the second left-definite operator associated with the pair $(L^2((0,\infty); w(t)), A)$. In view of Corollary 9.1, we can now say that T is partially generated by the first left-definite operator $(A^2)_1$ associated with the pair $(L^2((0,\infty); w(t)), A^2)$.

REMARK 13.4. The results of this paper have a significant impact on some important unsolved problems in the classification of ordinary differential equations having a sequence of orthogonal polynomial solutions. Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomial solutions to the differential equation

$$L_N[y](t) := \sum_{j=1}^{N} a_j(t)y^{(j)}(t) = \lambda y(t)$$

and that $\{p_n\}_{n=0}^{\infty}$ is orthogonal with respect to some inner product $(\cdot,\cdot)$ (see [29] where, for each $N \in \mathbb{N}$ and $M \in \mathbb{N}_0$, the $BKS(N,M)$ classification problems are discussed). Is there a self-adjoint operator A, generated from $L_N[\cdot]$, in some Hilbert space $(H,(\cdot,\cdot))$ having these polynomials as eigenfunctions? If so, is A bounded below in H? This last question, in all likelihood, is tantamount to showing that $L_N[\cdot]$ is Lagrangian symmetrizable; i.e., $N = 2m$ and $L_N[\cdot]$ has the form

$$L_N[y](t) = w^{-1}(t)\sum_{j=1}^{m} (-1)^j (b_j(t)y^{(j)}(t))^{(j)},$$

where $w(t)$ and each $b_j(t)$ are positive on some interval I of the real line. Along this line, we note that a result of Krall [24] shows that if the polynomials are orthogonal with respect to a bilinear form of the type

$$\int_{\mathbb{R}} f(t)\bar{g}(t)\,d\mu,$$

where μ is a (possibly signed) Borel measure, then N is indeed even. Moreover, a result of Kwon and Yoon [28] shows, in this case, that $L_N[\cdot]$ is symmetrizable. It is not clear from their result, however, when the coefficients b_j are positive. Of course, if we can determine when these coefficients are positive, then the results of this paper would apply and we would obtain a continuum of left-definite spaces $\{H_r\}_{r>0}$ and left-definite operators $\{A_r\}_{r>0}$ in H_r associated with (H,A). Moreover, from Theorem 3.7, the polynomial solutions would be orthogonal with respect to each of the left-definite inner products $(\cdot,\cdot)_r$.

REMARK 13.5. The left-definite theory presented in this paper has some connections to the concepts of *positive and negative spaces* presented by Berezanskiĭ in his research monograph [5]. Indeed, in our notation, Berezanskiĭ begins with two Hilbert spaces $H_1 = (V_1,(\cdot,\cdot)_1)$ and $H_2 = (V_2,(\cdot,\cdot)_2)$, with V_2 being a dense subspace of H_1 and $(x,x)_2 \geqslant (x,x)_1$ for all $x \in V_2$. Using the Riesz representation theorem, Beresanskiĭ constructs a bounded, invertible self-adjoint operator $R_1: H_1 \to H_2$ such that $(R_1x, y)_2 = (x, y)_1$. Using analysis similar to that in the Introduction of this paper, Berezanskiĭ finds an (unbounded) self-adjoint operator A_1: $\mathscr{D}(A) \subset H_1 \to H_1$ such that $(A_1x, y)_1 = (x, y)_2$ for all $x \in \mathscr{D}(A)$ and $y \in V_2$. From this, Berezanskiĭ constructs what we have called the right-definite space H and right-definite operator A. He goes on to show that H_1 and H_2 are the first and second left-definite spaces associated with (H, A), as given in our Definition 2.1. It is not difficult to show that, in fact, the operator A_1 is what we have called the first left-definite operator associated with (H, A). Consequently, Berezanskiĭ's work may be seen as a *converse* of the theory presented in this paper. Berezanskiĭ goes on to produce a doubly infinite sequence of Hilbert spaces that, in general, are different than the sequence of left-definite spaces presented in this paper.

REMARK 13.6. Given a self-adjoint differential operator A: $\mathscr{D}(A) \subset H \to H$ generated by a quasi-differential expression $\ell[\cdot]$, it follows from the left-definite theory presented in this paper that if A is a positive operator, then $\ell[\cdot]$ generates self-adjoint operators in *uncountably many different Hilbert spaces*. We remark that Möller and Zettl [31] show that if $\ell[\cdot]$ is a regular quasi-differential expression with positive leading coefficient, then the minimal operator generated by $\ell[\cdot]$ is bounded below; hence the theory developed in this paper applies to their work.

REMARK 13.7. The underlying reason why we were able to explicitly determine the left-definite Hilbert spaces and left-definite operators associated with the Laguerre operator in Section 11 is, undoubtedly, due to the extraordinary properties of the Laguerre polynomials $\{L_m^\alpha(x)\}_{m=0}^\infty$ (most importantly, the orthogonality of their derivatives as seen in (12.5) as well as their completeness in the Hilbert space $L_\alpha^2(0, \infty)$). In general, however, characterizing the left-definite spaces and left-definite operators associated with a positive self-adjoint operator – and, in particular, those that are generated from quasi-differential expressions $\ell[\cdot]$ – appears to be a very formidable and difficult problem. A key paper in the determination of integral powers of general quasi-differential expressions $\ell[\cdot]$ is the contribution by Everitt and Zettl [15].

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