On Extremal Connectivity Properties of Unavoidable Matroids

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Ding, Oporowski, Oxley, and Vertigan proved that, for all $n \geq 3$, there is an integer $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor isomorphic to a wheel or whirl of rank $n$, $M(K_3, n)$ or its dual, $U_2(n + 2)$ or its dual, or a rank-$n$ spike. This paper characterizes each of these classes of unavoidable matroids in terms of an extremal connectivity condition. In particular, it is proved that if $M$ is a 3-connected matroid of at least rank 7 for which every single-element deletion or contraction is 3-connected but no 2-element deletion or contraction is, then $M$ is a spike with its tip deleted. It is further proved that if $M$ is a 3-connected matroid of at least rank 4 for which every single-element deletion is 3-connected but no 1-element contraction or 2-element deletion is, then $M \cong M^*(K_3, n)$.

1. INTRODUCTION

A matroid $M$ is said to be $k$-minimally $n$-connected if, for each $X \subseteq E(M)$ with $|X| < k$, the matroid $M \setminus X$ is $n$-connected, but for each $X \subseteq E(M)$ with $|X| = k$, $M \setminus X$ is not $n$-connected. A matroid is said to be $m$-cominimally $n$-connected if its dual is $m$-minimally $n$-connected. We shall be primarily interested here in the case where $n$ is 2 or 3. Usually, 1-minimally $n$-connected matroids are called simply minimally $n$-connected, and $k$-minimally 2-connected matroids are called $k$-minimally connected matroids. Minimally connected matroids have been investigated by several authors including Murty [6], Seymour [12], White [15], and Oxley [8–10]. Moreover, Akkari [1, 2], Akkari and Oxley [3], and Oxley [7] have examined $k$-minimally 3-connected matroids when $k$ is 1 or 2.

Ding et al. [5] identified certain rank-$r$ 3-connected matroids as being unavoidable in the sense that every sufficiently large 3-connected matroid has one of the specified matroids as a minor. Included among these unavoidable matroids are the wheels and whirls, whose fundamental role...
within the class of 3-connected matroids is well known. Perhaps the primary contributor to the notoriety of wheels and whirls is Tutte’s Wheels and Whirls Theorem [14], which asserts that the class of minimally, cominimally 3-connected matroids coincides exactly with the class of wheels and whirls of rank exceeding 2. This paper shows that each of the classes of unavoidable 3-connected matroids noted in [5] can be characterized in terms of an extremal connectivity condition. This fact helps to explain the exact composition of the list of unavoidable matroids.

For \( n \geq 3 \), a matroid \( M \) is called a \( n \)-spike with tip \( p \) [5] if it satisfies the following three conditions:

(i) the ground set is the union of \( n \) lines, \( L_1, L_2, ..., L_n \), all having three points and passing through a common point \( p \);

(ii) for all \( k \) in \( \{1, 2, ..., n-1\} \), the union of any \( k \) of \( L_1, L_2, ..., L_n \) has rank \( k+1 \); and

(iii) \( r(L_1 \cup L_2 \cup ... \cup L_n) = n \).

\( M' \setminus p \) is called a spike without tip. In this paper, we shall only be concerned with spikes without tips and we shall call them simply spikes.

The well-known matroid \( R_{10} \) is a regular matroid represented by the following matrix over every field:

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The matroid \( H_{10} \) is a quartenary matroid represented by the following matrix over the four-element field \( \{0, 1, \omega, 1+\omega\} \):

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & \omega & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & \omega & \omega \\
0 & 0 & 0 & 0 & 1 & 1 & \omega & \omega & \omega \\
\end{bmatrix}
\]
The matroid $H_{12}$ is a binary matroid with 12 elements, represented by

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
\]

Both of the matroids $H_{10}$ and $H_{12}$ are self-dual, but not identically self-dual, 2-minimally, 2-cominimally, 3-connected matroids. In the matrix $[I_6|D]$ representing $H_{12}$, we observe that

\[
D = \begin{bmatrix}
I_2 & J_2 & O_2 \\
O_2 & I_2 & J_2 \\
J_2 & O_2 & I_2
\end{bmatrix},
\]

where $J_2$ and $O_2$ are the $2 \times 2$ matrices of all ones and all zeros, respectively.

In the rest of this paper, the notation and terminology will follow Oxley [11]. Seymour [12] proved that the 4-point line is the unique 2-minimally, 2-cominimally connected matroid. The following are the main results of this paper. The first theorem is the analogue of Seymour's result for 3-connected matroids.

(1.1) **Theorem.** If $M$ is a 2-minimally, 2-cominimally 3-connected matroid with rank greater than or equal to 5, then $M$ is a spike, or $M$ is isomorphic to one of the matroids $H_{10}$, $R_{10}$, and $H_{12}$. Conversely, if $M$ is a spike with $r(M) \geq 4$, then $M$ is 2-minimally, 2-cominimally 3-connected.

(1.2) **Theorem.** A matroid is 2-minimally, 1-cominimally 3-connected if and only if it is isomorphic to $F_7$, $F_7^-$, or $M^*(K_{3,n})$ for some $n \geq 3$.

Theorem 1.1 and 1.2 will be proved in Sections 3 and 4, respectively. Section 2 contains some preliminary results that will be needed in these proofs, while Section 5 examines some of the properties of 2-minimally 3-connected matroids. Finally, Section 6 establishes that lines can be characterized in terms of an extremal connectivity condition. On combining that result with Theorem 1.1 and 1.2 and the main theorem of [5], we obtain the following result.

(1.3) **Theorem.** For every integer $r$ exceeding 6, there is an integer $N(r)$ such that every 3-connected matroid with at least $N(r)$ elements has a minor...
M such that $M$ or $M^*$ is isomorphic to a rank-$r$, $j$-minimally, $k$-cominimally 3-connected matroid for some $(j, k)$ in $\{(1, 1), (1, 2), (2, 2), (1, r)\}$.

2. PRELIMINARIES

In this section, we recall some results from [7, 11], and then prove some new results which will be used to establish Theorem 1.1 and 1.2.

(2.1) **Proposition [11, 2.1.11].** If $C$ is a circuit and $C^*$ is a cocircuit of a matroid $M$, then $|C \cap C^*| \neq 1$.

The last property of matroids is often referred to as orthogonality.

(2.2) **Proposition [11, 8.1.6].** If $M$ is an $n$-connected matroid and $|E(M)| \geq 2(n-1)$, then all circuits and all cocircuits of $M$ have at least $n$ elements.

(2.3) **Corollary.** Let $M$ be a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$, then all circuits and all cocircuits of $M$ have at least 4 elements.

**Proof.** Apply (2.2) to $M\setminus e$ and $M/e$ for some $e \in E(M)$.

(2.4) **Theorem [7, 2.5].** If $C$ is a circuit of a minimally 3-connected matroid $M$ with $|E(M)| \geq 4$, then $M$ has at least two distinct triads intersecting $C$.

(2.5) **Corollary.** Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$, $C$ is a 4-circuit of $M$, and $e \notin C$. Then $C$ intersects at least two distinct 4-cocircuits containing $e$.

(2.6) **Lemma [7, 2.6].** Suppose that $x$ and $y$ are distinct elements of an $n$-connected matroid $M$, where $n \geq 2$ and $|E(M)| \geq 2(n-1)$. Assume that $M\setminus x$ is $n$-connected but that $M\setminus x$ is not $n$-connected. Then $M$ has a cocircuit of size $n$ containing both $x$ and $y$.

(2.7) **Corollary.** Suppose that $x$ and $y$ are distinct elements of a 3-connected matroid $M$, and $|E(M)| \geq 4$. Assume that $M\setminus x$ is 3-connected but that $M\setminus x$ is not. Then $M$ has a triangle containing both $x$ and $y$.

(2.8) **Corollary.** Suppose that $M$ is a 2-cominimally 3-connected matroid with $|E(M)| > 4$, and $x_1, x_2, y$ are distinct elements of $M$. Assume that $M\setminus x_1, x_2 \setminus y$ is 3-connected. Then $M$ has a 4-circuit containing $x_1, x_2, y$. 

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Let $M$ be a minimally 3-connected matroid having at least four elements, and let $U$ be the set of elements of $M$ which are not contained in a triad. If $V$ is a subset of $U$, then $M/V$ is minimally 3-connected.

Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid, $C_1$ and $C_2$ are distinct 4-circuits of $M$, and $|C_1 \cap C_2| = 3$. Then $M|(C_1 \cup C_2) \cong U_{3,5}$.

Proof. Since $|E(M)| > |C_1| = 4$, it follows by (2.3) that all circuits and all cocircuits of $M$ have at least four elements. Let $e$ be an element of $C_1 \cap C_2$. By circuit elimination, and the fact that $|C_1 \cup C_2 - e| = 4$, we deduce that $(C_1 \cup C_2 - e)$ is a 4-circuit of $M$. Hence every 4-element subset of $C_1 \cup C_2$ is a circuit of $M$; that is, $M|(C_1 \cup C_2)$ is isomorphic to $U_{3,5}$.

Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid, and that $r(M) = 3$ or $r^*(M) = 3$. Then $M \cong U_{3,6}$.

Proof. It is easy to check that $|E(M)| > 4$. If $r(M) = 3$, then every subset of $M$ of size four is dependent, and hence is a circuit. Thus $M$ is isomorphic to $U_{3,10}|M|$. Since $U_{3,6}$ is clearly the only rank-3 uniform matroid which is 2-minimally, 2-cominimally 3-connected, we have the required conclusion for this case. In the case when $r^*(M) = 3$, the result follows by duality.

Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid, and that $E_1$ is a subset of $E(M)$ such that $M|E_1 \cong U_{3,6}$. Then $M \cong U_{3,6}$.

Proof. Let $C$ be a 4-circuit of $M|E_1$. Suppose that there is an $x \in E(M) - E_1$. Then, by (2.5), there is a 4-cocircuit containing $x$ and intersecting $C$. By the assumption that $M|E_1 \cong U_{3,6}$, this 4-cocircuit will intersect some 4-circuit of $M|E_1$ in exactly one element, contradicting orthogonality. Thus $E(M) = E_1$ and $M \cong U_{3,6}$.

Define $N_M(e) = \{ x \in E(M) - e : \text{there is no 4-cocircuit containing both } x \text{ and } e \}$. Then we have the following.

Lemma. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$. Then, for each $e \in E(M)$, the set $N_M(e)$ has cardinality at most 2.

Proof. We argue by contradiction. Suppose that we have $x_1, x_2, x_3 \in N(e)$. Then, since $M$ is 2-cominimally 3-connected, none of $M/x_1, x_2, M/x_1, x_3,$ and $M/x_2, x_3$ is 3-connected. However, since $M \setminus e$ is minimally 3-connected,
(2.9) each of $M/x_1, x_2 \setminus e$, $M/x_1, x_3 \setminus e$, and $M/x_2, x_3 \setminus e$ is 3-connected. Hence, by (2.8), there are 4-circuit $C_1$ containing $x_2, x_3$, and $e$, a 4-circuit $C_2$ containing $x_1, x_3$, and $e$, and a 4-circuit $C_3$ containing $x_1, x_2$, and $e$. By (2.5), there are at least two 4-cocircuits containing $e$. Let $C^*$ be one of such 4-cocircuit. By assumption, $C^*$ does not contain any of $x_1, x_2$, and $x_3$. But $C^*$ meets each of $C_1$, $C_2$, and $C_3$. Since $|C_1 \cup C_2 \cup C_3| - \{x_1, x_2, x_3, e\} \leq 3$, it follows by orthogonality that $C^*$ must contain all elements in the set $(C_1 \cup C_2 \cup C_3) - \{x_1, x_2, x_3, e\}$. If this set has cardinality 3, then $C^*$ is the unique 4-cocircuit passing through $e$, a contradiction to (2.5). Therefore, we may assume that $C_1$ and $C_2$ have a common element $f$ other than $x_3$ and $e$. Then, by circuit elimination, $\{x_1, x_2, x_3, e\}$ is a 4-circuit. Thus, by orthogonality, every 4-cocircuit containing $e$ must contain $x_1$, $x_2$, or $x_3$. This contradicts the choice of $x_1$, $x_2$, and $x_3$.}

(2.14) Theorem [7, 4.7, 5.2, 5.6]. Let $M$ be a minimally 3-connected matroid of rank $r$ with $3 \leq r \leq 6$. Then $|E(M)| \leq 2r$. If $M$ has precisely $2r$ elements, then $M$ is isomorphic to $M(\emptyset)$ or $\emptyset^\ast$, or $r(M) = 6$ and $M$ is a disjoint union of four triads.

(2.15) Corollary. Let $M$ be a 2-minimally, 2-cominimally 3-connected matroid of rank $r$. If $3 \leq r \leq 6$, then

$$|E(M)| = 2r = 2r^\ast.$$ 

Moreover, if $r \geq 7$, then $|E(M)| \geq 14$.

Proof. First, suppose that $3 \leq r \leq 6$. Let $e$ be an element of $M$. Since $M^\ast e$ is minimally 3-connected, it follows by (2.14) that $|E(M^\ast e)| \leq 2r$; that is,

$$r + r^\ast(M) - 1 \leq 2r, \quad \text{or} \quad r^\ast \leq r + 1.$$ 

Since $M$ has no triangles, $M^\ast e$ is not a wheel or a whirl. Since $M$ has a 4-circuit passing through $e$ but has no triads, it follows by orthogonality that $M^\ast e$ cannot have four disjoint triads. Therefore, by (2.14),

$$|E(M^\ast e)| \neq 2r,$$

and hence, $r^\ast < r + 1$.

Using $M^\ast$ in place of $M$ in the above argument, we deduce that $r < r^\ast + 1$. Thus $r = r^\ast$ and $|E(M)| = 2r = 2r^\ast$. Finally, if $r \geq 7$ but $r^\ast \leq 6$, then $|E(M)| = 2r^\ast = 2r$, a contradiction. Thus $r^\ast \geq 7$, and $|E(M)| \geq 14$.

The next lemma sharpens the bound on $N_M(e)$ given in Lemma 2.13.
(2.16) Lemma. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $|E(M)| > 4$. Then, for each $e \in E(M)$, the sets $N_{M}(e)$ has cardinality at most one. Moreover, if $3 \leq r(M) \leq 5$, then $N_{M}(e)$ is empty.

Proof. Suppose that $r(M) = 3$. Then it follows by (2.11) that $M \cong U_{3, 3}$. Thus $N_{M}(e)$ is empty.

Suppose that $r(M) = 4$, and $x \in E(M) - e$. By (2.3), every circuit and every cocircuit of $M$ has size at least 4. Thus $M \backslash x$, $x$ has no loops, coloops, or 2-circuits. Since $M$ is 2-minimally 3-connected, $M \backslash x$, $x$ is not 3-connected. Combining this with the fact that $r(M \backslash x)e = 2$, we deduce that $M \backslash x$, $x$ has 2-cocircuits; that is, there is a 4-cocircuit containing both $e$ and $x$. Thus $x \notin N_{M}(e)$, and $N_{M}(e)$ is empty.

Suppose that $r(M) = 5$, and $x \in N_{M}(e)$. Then $x$ is not in any triad of $M \backslash x$. Since $M \backslash x$ is minimally 3-connected, it follows by (2.9) that $M \backslash x \backslash e$ is minimally 3-connected. By (2.15), $|E(M)| = 10$. Hence $|E(M \backslash x)| = 8$. Since $r(M / x \backslash e) = 4$, it follows by (2.14) that $M \backslash x \backslash e$ is a wheel or a whirl.

Therefore, we may assume that $D^{\uparrow}_{e} = \{a_{1}, a_{2}, a_{3}\}$, $D^{\downarrow}_{e} = \{a_{3}, a_{4}, a_{5}\}$, $D^{\uparrow}_{x} = \{a_{4}, a_{5}, a_{7}\}$, and $D^{\downarrow}_{x} = \{a_{1}, a_{5}, a_{1}\}$ are the only triads of $M \backslash x$, while $\{e, x, a_{1}, a_{2}, \ldots, a_{6}\}$ is the ground set of $M$. Since $x \in N_{M}(e)$, the matroid $M \backslash x$, $x$ has no 2-cocircuits. By the relations among the triads of $M \backslash x$, the geometrical representation of $(M \backslash x)x)^{*}$ is not the union of two lines. Therefore, $M \backslash x, x)^{*}$ is 3-connected, as it has rank 3. This contradicts the assumption that $M$ is 2-minimally 3-connected. Thus $N_{M}(e)$ is empty.

Now we suppose that $r(M) \geq 6$. By (2.15), $|E(M)| \geq 12$. By (2.13), we have $|N_{M}(e)| \leq 2$. Suppose that $N_{M}(e) = \{x_{1}, x_{2}\}$. Then, since $M$ is 2-cominimally 3-connected, $M / x_{1}, x_{2}$ is not 3-connected. Since $M \backslash e$ is minimally 3-connected, and $x_{1}$ and $x_{2}$ are in $N(e)$, it follows by (2.9) that $M / x_{1}, x_{2} \backslash e$ is 3-connected. Thus, by (2.8), there is a 4-circuit $C$ containing $x_{1}$, $x_{2}$, and $e$. Let $C = \{x_{1}, x_{2}, e, f\}$. Then, by the choice of $x_{1}$ and $x_{2}$, all 4-cocircuits containing $e$ must contain $f$. If two of these 4-cocircuits meet in exactly three elements, then, by (2.10), $M^{*}$ restricted to their union will be isomorphic to $U_{3, 5}$, and hence $M$ will have a 4-cocircuit containing only one of $e$ and $f$, a contradiction. Therefore, no two 4-cocircuits containing $e$ meet in exactly three elements. Thus the set $E(M) - \{x_{1}, x_{2}, e, f\}$ can be labeled $\{a_{i}, \ b_{i} \}_{i \in \{1, 2, \ldots, n\}}$ such that $\{e, f, a_{i}, b_{i}\}$ is a 4-cocircuit for each $i$ in $\{1, 2, \ldots, n\}$. By circuit elimination, for each pair of distinct elements $i$ and $j$ in $\{1, 2, \ldots, n\}$, the set $\{e, a_{i}, b_{i}, a_{j}, b_{j}\}$ contains a cocircuit. Since $C$ is a circuit, by orthogonality, we deduce that $\{a_{i}, a_{j}, b_{i}, b_{j}\}$ is a cocircuit. Since $|E(M)| \geq 12$, we deduce that $n \geq 4$. It follows by orthogonality that every 4-circuit containing $a_{i}$ contains $b_{i}$ for each $i$ in $\{1, 2, \ldots, n\}$.

If there is a 4-circuit containing both $a_{i}$ and $a_{j}$, then, by orthogonality, we deduce that this 4-circuit must be $\{a_{i}, a_{j}, b_{i}, b_{j}\}$. By (2.13), $N_{M^{*}}(a_{i}) \leq 2$. Therefore, we may assume that $\{a_{1}, a_{2}, a_{3}, b_{1}\}$ is a 4-circuit of $M$ for each
\( i \in \{ 2, 3, \ldots, n - 1 \} \). If \( \{a_1, b_1, a_n, b_n\} \) is not a 4-circuit, it follows by (2.13) that \( \{a_1, b_2, a_n, b_n\} \) is a 4-circuit. By applying circuit elimination to \( \{a_1, b_1, a_2, b_2\} \) and \( \{a_2, b_2, a_n, b_n\} \), we obtain that \( \{a_1, b_1, a_n, b_n, a_2\} \) contains a circuit. Orthogonality now implies that \( \{a_3, b_1, a_n, b_n\} \) must be a circuit, a contradiction. Therefore, we conclude that \( \{a_1, b_1, a_n, b_n\} \) is a 4-circuit for each pair of distinct elements \( i \) and \( j \) in \( \{1, 2, \ldots, n\} \).

By (2.13), we may assume that there is a 4-cocircuit containing both \( x_1 \) and \( a_1 \). It follows by orthogonality that this 4-cocircuit must contain \( b_1 \) and either \( f \) or \( x_2 \). Since this 4-cocircuit cannot meet the 4-cocircuit \( \{e, f, a_1, b_1\} \) in exactly three elements, it must be \( \{x_1, x_2, a_1, b_1\} \). By circuit elimination, \( \{e, f, x_1, x_2, a_1\} \) contains a cocircuit. By orthogonality, \( a_1 \) is not in this cocircuit. Thus we deduce that \( \{e, f, x_1, x_2\} \) is a 4-cocircuit, a contradiction to the assumption that \( x_1 \in N_M(e) \). We deduce that \( N_M(e) \) has cardinality at most one.

If \( M \) is 2-minimally, 2-cominimally 3-connected, then so is its dual. Hence the last lemma implies that \( N_M(e) \) also has cardinality at most one.

(2.17) Theorem. Suppose that \( M \) is a 2-minimally, 2-cominimally 3-connected matroid with \( |E(M)| > 4 \). If \( C_1 \) and \( C_2 \) are two 4-circuits such that \(|C_1 \cap C_2| = 3\), then \( M \cong U_{3,6} \).

Proof. We argue by contradiction. Suppose that \( M \not\cong U_{3,6} \). Then, by (2.11), \( r(M) > 3 \). By (2.10), \( M/(C_1 \cup C_2) \cong U_{3,5} \). Let \( C_1 \cup C_2 = \{e_1, e_2, e_3, e_4, e_5\} \). If \( r(M) \geq 6 \), then, by (2.14), \( |E(M)| \geq 12 \). There are at least seven elements of \( M \) not in \( C_1 \cup C_2 \). By (2.16), at least six of them have the property that they lie in a 4-cocircuit with \( e_1 \). However, if a 4-cocircuit intersects \( C_1 \cup C_2 \) then, by orthogonality, it has at least three elements in \( C_1 \cup C_2 \) since \( M/(C_1 \cup C_2) \cong U_{3,5} \). Hence, \( M \) has at least six distinct 4-cocircuits containing \( e_1 \). If there are exactly six, then, there is an element, say \( x \), of \( E(M) - (C_1 \cup C_2) \), such that there is no 4-cocircuit containing both \( x \) and \( e_1 \). Thus, by (2.16), for each \( e_i \) with \( i \in \{2, 3, 4, 5\} \), there is a 4-cocircuit containing \( x \) and \( e_i \). As each 4-cocircuit intersecting \( C_1 \cup C_2 \) intersects it in at least three elements, there is an \( e_i \), say \( e_2 \), such that \( \{x, e_2\} \) is contained in at least two 4-cocircuits. Since there are at least six elements other than \( x \) and the \( e_i \)'s, at least five of these elements lie in some 4-cocircuit with \( e_2 \). Moreover, none of these 4-cocircuits contains \( x \). Therefore, we have at least seven distinct 4-cocircuits that contain \( e_2 \) and meet \( E(M) - (C_1 \cup C_2) \). Since the number of 3-element subsets of \( \{e_1, e_2, e_3, e_4, e_5\} \) containing \( e_2 \) is six, among these seven 4-cocircuits, there are at least two that have the same 3-element intersection with \( C_1 \cup C_2 \). Thus, by circuit elimination, there is a 4-cocircuit intersecting \( C_1 \cup C_2 \) in exactly two elements. This contradicts the fact that \( M/(C_1 \cup C_2) \cong U_{3,5} \). Therefore,
there are at least seven distinct 4-cocircuits that contain \( e_1 \) and meet \( E(M) - (C_1 \cup C_2) \). In this case, an argument similar to the above produces the same contradiction. Thus \( r(M) \leq 5 \).

If \( r(M) = 4 \), then, by (2.15), \( E(M) = 8 \), so there are three elements not in \( C_1 \cup C_2 \). By (2.16), there is a 4-cocircuit containing at least two elements of these three. But this 4-cocircuit intersects \( C_1 \cup C_2 \) in one or two elements, a contradiction to orthogonality.

If \( r(M) = 5 \), then \( |E(M)| = 10 \). Let \( E(M) - (C_1 \cup C_2) = F = \{ f_1, f_2, f_3, f_4, f_5 \} \). By orthogonality and (2.16), it is easy to show that \( M^* \setminus \{ f_1, f_2, f_3, f_4, f_5 \} \cong U_{3,5} \). By orthogonality, every 4-circuit intersecting \( F \) intersects it in at least three elements. By (2.16) and the fact that \( |F| = 5 \), it follows that there are at least two distinct 4-circuits passing through \( e_1 \) and intersecting \( F \). We may assume that \( \{ e_1, f_1, f_2, f_3 \} \) is a 4-circuit. Moreover, by (2.16), there is a 4-cocircuit containing \( e_1 \) and \( f_4 \). This 4-cocircuit must have three elements in \( C_1 \cup C_2 \). Hence it intersects the 4-circuit \( \{ e_1, f_1, f_2, f_3 \} \) in exactly one element, a contradiction to orthogonality.

(2.18) Corollary. Suppose that \( M \) is a 2-minimally, 2-cominimally 3-connected matroid with \( |E(M)| > 4 \). If \( M \cong U_{3,6} \), and \( C_1 \) and \( C_2 \) are two 4-circuits of \( M \), then \( |C_1 \cap C_2| \neq 3 \). Hence if \( x \in E(M) \), and \( T_1 \) and \( T_2 \) are two distinct triangles of \( M/x \), then \( |T_1 \cap T_2| \leq 1 \).

3. PROOF OF THEOREM 1.1

Suppose that \( M \) is a 2-minimally, 2-cominimally, 3-connected matroid with rank at least five. We prove in (3.1) that \( M \) is a spike if it has a 2-element subset contained in four 4-circuits. If \( M \) is not a spike, then (3.7) proves that \( M \) is isomorphic to \( R_{10} \) or \( H_{10} \) if \( r(M) = 5 \), and (3.10) proves that \( M \) is isomorphic to \( H_{12} \) if \( r(M) \geq 6 \). Propositions 3.2–3.6 and 3.8–3.9 are preliminary results used to prove (3.7) and (3.10), respectively. The first part of Theorem 1.1 follows immediately on combining (3.1), (3.7), and (3.10). The second part of Theorem 1.1 is straightforward.

(3.1) Lemma. Suppose that \( M \) is a 2-minimally, 2-cominimally 3-connected matroid, and that for some element \( x \) of \( E(M) \), the matroid \( M/x \) has four triangles sharing a common element \( y \). Then \( M \) is a spike.

Proof. If \( M \) is isomorphic to \( U_{3,6} \), the result holds, since \( U_{3,6} \) is a spike. Otherwise, by (2.18), each pair of triangles of \( M/x \) intersect in at most one element. Thus each two of the four triangles containing \( y \) have no other common elements. Thus \( M \) has four 4-circuits that contain \( \{ x, y \} \) but are otherwise disjoint. A 4-cocircuit passing through \( x \) intersects all four of these 4-circuits. Thus, by orthogonality, such a 4-cocircuit must contain \( y \).
Hence every 4-cocircuit containing $x$ also contains $y$. Similarly, each 4-cocircuit containing $y$ also contains $x$. Hence, by (2.16), if $z \in E(M) - \{x, y\}$, then $z$ is in a 4-cocircuit meeting $\{x, y\}$, so $z$ is in a 4-cocircuit containing $\{x, y\}$. 

Thus, by (2.15) and (2.16), we can denote the elements of $E(M) - \{x, y\}$ by $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ for some $n > 3$, such that, for each $i$ in $\{1, 2, \ldots, n\}$, the set $\{x, y, a_i, b_i\}$ is a 4-cocircuit. If $i$ and $j$ are distinct elements of $\{1, 2, \ldots, n\}$, then, by (2.16), there is a 4-circuit containing $a_i$ and either $a_j$ or $b_j$, say $a_i$. By orthogonality, this 4-circuit either contains both $x$ and $y$, or contains neither of these elements. If the latter case, this 4-circuit is $\{a_i, b_i, a_j, b_j\}$. In the former case, $\{x, y, a_i, a_j\}$ is a 4-circuit and, by (2.16) again, there is a 4-circuit containing $b_j$ and $x$ or $y$. By orthogonality, this 4-circuit must contain both $x$ and $y$, and, by circuit elimination and orthogonality, $b_j$ must be in this 4-circuit and $\{a_i, b_i, a_j, b_j\}$ is a 4-circuit. We conclude that in both cases, $\{a_i, b_i, a_j, b_j\}$ is a 4-circuit.

Now consider 4-cocircuits. By (2.16), there is a 4-cocircuit containing $a_i$ and $a_j$ or $b_i$. We may assume that this 4-cocircuit contains $a_i$ and $a_j$. Since, by (2.16), $\{x, y, a_i, a_j\}$ cannot be a 4-cocircuit, and all sets of the form $\{a_x, b_x, a_i, b_i\}$ are 4-circuits, we deduce that $\{a_i, b_i, a_j, b_j\}$ is a 4-cocircuit for each pair of elements $i, j$ of $\{1, 2, \ldots, n\}$. Denote $x$ by $a_{n+1}$ and $y$ by $b_{n+1}$. Then, it follows by the above results that, for each pair of elements $i, j$ of $\{1, 2, \ldots, n+1\}$, the set $\{a_i, b_i, a_j, b_j\}$ is a 4-cocircuit.

Similarly, it is easy to deduce that the set $\{a_i, b_i, a_j, b_j\}$ is also a 4-circuit for each pair of elements $i$, $j$ of $\{1, 2, \ldots, n+1\}$. Let $E_{i,j} = \{a_i, b_i, a_j, b_j\}$ for each pair of elements $i$, $j$ of $\{1, 2, \ldots, n+1\}$. Then each $E_{i,j}$ is both a 4-circuit and a 4-cocircuit. If a circuit $C$ meets three of the sets $E_1 = \{a_1, b_1\}$, $E_2 = \{a_2, b_2\}$, $\ldots$, and $E_{n+1} = \{a_{n+1}, b_{n+1}\}$, then it cannot be any of the $E_{i,j}$'s. Since all $E_{i,j}$'s are cocircuits, it follows by orthogonality that $C$ must meet all the $E_{i,j}$'s. Thus, for each non-empty set $J \subseteq \{1, 2, \ldots, n+1\}$ such that $|J| \neq n$, the set $F_J = \bigcup_{i,j \in J} E_{i,j}$ is a flat of $M$. Let $\mathcal{H}$ be the collection of such $F_J$'s. It is easily checked that $\mathcal{H}$ is a modular cut of $M$. Let $p$ be an element not in $E(M)$. By [11, 7.2.2], the unique extension $N$ of $M$ on $E(M) \cup p$ such that $\mathcal{H}$ consists of those flats $F$ of $M$ for which $F \cup p$ is a flat of $N$ is an $(n+1)$-spike with tip $p$. Thus $M$ is a spike.

The next six results deal with the case where $M$ has rank 5.

(3.2) **Proposition.** Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. Then, for each pair of elements $x$, $y$ of $E(M)$, there is at least one 4-circuit containing both.

**Proof.** Since $M^*$ is also 2-minimally, 2-cominimally 3-connected; it follows by (2.16) that $N_{xy}(x)$ is empty for each $x$ in $E(M)$; that is, each pair of elements $x$, $y$ of $E(M)$ is in at least one 4-circuit of $M$.  


(3.3) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$, and $x \in E(M)$. Then $M/x$ has at least four triangles.

Proof. By (2.15), $|E(M)| = 10$. By (3.2), each element is in some triangle of $M/x$. Since $|E(M/x)| = 9$, there are at least three triangles in $M/x$. If there are exactly three of them, then they are disjoint. We denote them by $T_1 = \{a_1, b_1, c_1\}$, $T_2 = \{a_2, b_2, c_2\}$, and $T_3 = \{a_3, b_3, c_3\}$. There are three 2-element subsets of $T_1$, three of $T_2$, and three of $T_3$. By the dual of (3.2), each of these subsets is contained in a 4-cocircuit of $M$. By orthogonality, each of these 4-cocircuits must contain another 2-element subset of this kind. Since there is an odd number of subsets of this kind, at least one of them, say $\{b_1, c_1\}$, is in at least two 4-cocircuits of $M$. By (2.18), we may assume that these two 4-cocircuits are $\{b_1, c_1, b_2, c_2\}$ and $\{b_1, c_1, b_3, c_3\}$. By applying the circuit elimination axiom to these two cocircuits and using orthogonality, we deduce that $\{b_2, c_2, b_3, c_3\}$ is also a 4-cocircuit of $M$. By (3.2), there is a 4-circuit of $M$ containing $a_1$ and $a_2$. To avoid a contradiction to orthogonality and (2.18), this 4-circuit has to be $\{a_1, a_2, b_3, c_3\}$. In $M/x$, apply circuit elimination to this 4-circuit and $T_3$ to obtain that $\{a_1, a_2, a_3, b_3\}$ contains a circuit of $M/x$. By orthogonality and the fact that $\{b_1, c_1, b_3, c_3\}$ is a cocircuit of $M/x$, we deduce that $\{a_1, a_2, a_3\}$ is a circuit of $M/x$, a contradiction to the original assumption. \qed

(3.4) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. If $M$ is not a spike, and there is a pair of elements of $E(M)$ contained in three distinct 4-circuits, then $M \cong H_{10}$.

Proof. By (2.15), $|E(M)| = 10$. Since $M$ is not a spike, by (3.1), for each $x \in E(M)$, the matroid $M/x$ has at most three triangles sharing a common element. Suppose that $\{x, y\}$ is contained in three distinct 4-circuits. Let $T_1 = \{y, a_1, b_1\}$, $T_2 = \{y, a_2, b_2\}$, and $T_3 = \{y, a_3, b_3\}$ be the three corresponding triangles of $M/x$, and let the remaining elements of $M/x$ be $a_4$ and $b_4$.

If there is no triangle of $M/x$ containing both $a_4$ and $b_4$, then, by (2.18), we may assume that $T_4 = \{b_1, b_2, a_4\}$ is a triangle of $M/x$, and $b_4$ is in a triangle $T_5$ of $M/x$. By the dual of (3.2), there is a 4-cocircuit of $M$ containing both $a_4$ and $a_5$. By orthogonality, this 4-cocircuit must be either $\{a_4, a_1, x, y\}$, or $\{a_4, a_1, b_1, b_4\}$. In the first case, it follows by orthogonality that $T_5$ contains $\{b_4, a_1\}$. Since at least one of $a_3$ and $b_3$, say $a_3$, is not in $T_5$, a 4-cocircuit containing both $b_4$ and $a_3$ will intersect some 4-circuit of $M$ in exactly one element, a contradiction to orthogonality. The second case also results in a contradiction in a similar way. Thus $M/x$ has a triangle containing both $a_4$ and $b_4$. By (3.1) and the assumption that $M$ is not a spike, we may assume that this triangle is $T_4 = \{b_1, a_4, b_4\}$. 

\[\]
By the dual of (3.2), each of the sets \( \{y, a_3\}, \{y, b_3\}, \{y, a_2\}, \) and \( \{y, b_2\} \) is in a 4-cocircuit of \( M \). If such a 4-cocircuit contains \( x \), then, by orthogonality, it must contain either \( a_4 \) or \( b_4 \). Thus, by (2.18), there are at most two of these 4-cocircuits containing \( x \). Therefore, we may assume that there is a 4-cocircuit containing \( \{y, a_2\} \) and avoiding \( x \). Since \( T_1 \cup x, \ T_2 \cup x, \ T_3 \cup x, \) and \( T_4 \cup x \) are 4-circuits of \( M \), it follows by orthogonality that, up to relabeling on \( \{a_3, b_3\} \), this 4-cocircuit is \( D_1^y = \{y, a_3, a_2, a_4\} \). If a 4-cocircuit containing \( \{y, b_2\} \) contains \( x \), then, up to relabeling on \( \{a_4, b_4\} \), it follows by orthogonality that this cocircuit is \( \{x, y, b_2, a_4\} \). If a 4-cocircuit containing \( \{y, b_2\} \) also contains \( x \), then, by orthogonality and (2.18), this 4-cocircuit must be \( \{x, y, b_3, b_4\} \). By circuit elimination, the set \( \{x, b_2, b_3, a_4, b_4\} \) contains a cocircuit. By orthogonality, \( x \) is not in this cocircuit. Thus \( \{b_2, b_3, a_4, b_4\} \) is a 4-cocircuit. This 4-cocircuit, which is also a 4-cocircuit of \( M/x \), intersects the set \( \{a_1, b_1, a_2, b_1\} \), which by circuit elimination and (2.18) is a circuit of \( M/x \), in exactly one element, a contradiction to orthogonality. Therefore, we may assume that \( M \) has a 4-cocircuit containing \( \{y, b_2\} \) and avoiding \( x \). By orthogonality and (2.18), this 4-cocircuit is \( D_1^y = \{y, a_3, b_2, b_3\} \). Since \( M \) is not a spike, (3.1) and (3.3) imply that there is a 4-cocircuit of \( M \) containing \( a_4 \) but avoiding \( y \). By orthogonality, it must be \( D_1^y = \{a_1, b_1, a_4, b_4\} \). Applying circuit elimination to \( D_1^y \) and \( D_1^y \), we have, by orthogonality and the fact \( M \) does not have any cocircuit of size less than 4, that \( D_1^y = \{a_2, b_2, a_3, b_3\} \) is also a 4-cocircuit of \( M \). Since there is a 4-cocircuit containing both \( a_4 \) and \( a_2 \), by orthogonality, it is either \( \{a_4, a_2, x, y\} \) or \( \{a_4, a_2, b_4, b_2\} \). In the first case, consider the 4-cocircuit containing both \( a_4 \) and \( a_2 \). By (2.18), this 4-cocircuit must be \( D_1^y = \{a_4, a_3, b_4, b_3\} \). Applying circuit elimination to \( D_1^y \) and \( D_1^y \), we have, by orthogonality and the fact \( M \) does not have any cocircuit of size less than four, \( D_1^y = \{a_4, a_2, b_4, b_2\} \). The second case also implies that the same sets \( D_1^y \) and \( D_1^y \) are cocircuits. Hence these two sets are indeed 4-cocircuits of \( M \). By the dual of (3.2), there is a 4-cocircuit of \( M \) containing both \( a_3 \) and \( b_1 \). By orthogonality, it must be one of \( \{a_2, b_1, x, y\}, \{a_2, b_1, x, a_4\}, \) and \( \{a_2, b_1, x, b_3\} \). If \( \{a_2, b_1, x, y\} \) is a 4-cocircuit, consider the 4-cocircuit of \( M \) containing both \( b_2 \) and \( b_1 \). By (2.18), this 4-cocircuit must be either \( \{b_2, b_1, x, b_3\} \) or \( \{b_2, b_1, x, a_3\} \). Therefore, by symmetry, we may assume that either

(i) \( \{a_2, b_1, x, a_3\} \), or

(ii) \( \{a_2, b_1, x, b_3\} \).

is a 4-cocircuit.

In case (i), \( D_1^y = \{x, b_1, a_2, a_3\} \) is a 4-cocircuit of \( M \). Consider the 4-circuit of \( M \) containing both \( a_4 \) and \( a_2 \). By orthogonality and the existence of
the 4-cocircuits $D_1^y, D_2^y, \ldots, D_8^y$, this 4-circuit must be either $\{a_4, a_2, a_3, b_1\}$ or $\{a_4, a_3, a_5, b_3\}$. In the former case, from considering the 4-circuit of $M$ containing both $b_3$ and $a_2$, we obtain a contradiction to (2.18). Hence $C = \{a_4, b_4, a_2, a_3\}$ is a 4-circuit of $M$. Similarly, $C = \{a_4, b_4, b_2, b_3\}$ is also a 4-Circuit of $M$. By the dual of (3.2), there is a 4-cocircuit $D_1^x$ of $M$ containing both $a_4$ and $x$. By orthogonality and the fact that $T_1 \cup x, T_2 \cup x, T_3 \cup x$, and $T_4 \cup x$ are 4-circuits of $M$, we conclude that $y$ is an element of $D_1^x$. Since $D_1^x$ intersects both $C$ and $C'$, and it already contains $\{x, y, a_4\}$, by orthogonality, the fourth element must be $b_4$; that is, $D_1^y = \{a_4, b_4, x, y\}$ is a 4-cocircuit of $M$. Therefore, the 4-cocircuits $D_1^y, D_2^y, D_3^y$, and $D_8^y$ all share two common elements $a_4$ and $b_4$. By (3.1), $M$ is a spike, a contradiction to the assumption.

In case (ii), $D_1^y = \{x, b_1, a_2, b_3\}$ is a 4-circuit. Consider the 4-cocircuit containing both $a_4$ and $b_1$. By orthogonality and (2.18), it must be either $\{a_4, b_1, x, y\}$, or $\{a_4, b_1, x, b_2\}$. If the former case occurs, consider the 4-cocircuit containing both $b_2$ and $b_1$, it follows by orthogonality that $D_1^y = \{x, b_1, b_2, a_3\}$ or $\{x, b_1, b_2, a_5\}$, a contradiction to (2.18). Therefore, $D_1^y = \{x, b_3, b_2, a_3\}$ is a 4-circuit of $M$. Consider the 4-cocircuit containing $\{y, b_3\}$. By orthogonality, it contains $x$. By (2.18) and the existence of $D_1^y$, this 4-cocircuit must contain either $a_4$ or $b_4$. By symmetry, we may assume that $D_1^y = \{y, b_3, a_4\}$ is this 4-cocircuit. Similarly, consider the 4-cocircuit containing $\{x, a_1\}$. By (2.18), this 4-cocircuit is $D_1^y = \{x, a_1, y, b_4\}$.

Using the obtained information about 4-circuits and 4-cocircuits, we argue similarly to the above and obtain ten 4-circuits of $M$. Applying orthogonality and (2.18), it is routine to show that there are no other 4-circuits and no other 4-cocircuits. It is now straightforward to find all other circuits of $M$ and check that $M \cong H_{10}$.

(3.5) Proposition. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. If $M$ is not a spike, and there is an $x \in E(M)$ such that $M/x$ has exactly four triangles, then $M \cong H_{10}$.

Proof. If there are three triangles sharing a common element, then, by (3.4), $M \cong H_{10}$. Thus we may suppose that each pair of elements of $M$ is in at most two distinct 4-circuits.

Choose two disjoint triangles of $M/x$ and denote them by $T_1 = \{a_1, b_1, c_1\}$ and $T_2 = \{a_2, b_2, c_2\}$. Denote the remaining three elements of $E(M/x)$ by $a_3, b_3,$ and $c_3$. Suppose that $T_3$ is a triangle of $M/x$ containing $a_2$. If it meets $T_1$ but not $T_2$, by (2.18), we may assume it is $\{a_3, b_3, a_1\}$. The element $c_3$ is in the remaining triangle of $M/x$. Up to relabeling, this triangle is one of (i) $\{c_3, a_1, b_1\}$, (ii) $\{a_2, b_3, c_3\}$, and (iii) $\{c_1, a_2, c_3\}$. In these three cases,
we consider the 4-cocircuits of $M$ containing $\{a_2, c_2\}$, $\{a_3, c_3\}$, and $\{c_1, b_1\}$, respectively. In each case, we can find a 4-circuit of $M$ meeting the chosen 4-cocircuit in exactly one element; a contradiction to orthogonality. We conclude that the triangles meeting $\{a_2, b_3, c_3\}$ will be disjoint from $T_1$ and $T_2$, or will intersect both of them. By assumption, there are exactly four triangles, so, one of the remaining two triangles intersects both $T_1$ and $T_2$. By (3.2) and the above argument, the last triangle must be disjoint from both $T_1$ and $T_2$. We may assume that $T_3 = \{a_3, b_3, c_3\}$, and $T_4 = \{a_1, a_2, a_5\}$.

Consider a 4-cocircuit containing $\{b_1, c_1\}$. By orthogonality, it does not contain $x$ and does not intersect $T_4$. Thus we may assume that $D^*_1 = \{b_1, c_1, b_2, c_2\}$. Consider a 4-cocircuit containing $\{b_1, c_1\}$. Similarly, we may assume that $D^*_2 = \{b_1, c_1, b_3, c_3\}$ is the 4-cocircuit of $M$. By circuit elimination and orthogonality, $D^*_1 = \{b_2, c_2, b_3, c_3\}$ is also a 4-cocircuit. By the dual of (3.2), each of the sets $\{a_1, b_1\}$, $\{a_1, c_1\}$, $\{a_2, b_2\}$, $\{a_2, c_2\}$, $\{a_3, b_3\}$, and $\{a_3, c_3\}$ is contained in a 4-cocircuit. By orthogonality, (2.18), and the existence of the $T_i$’s, each of these 4-cocircuits consists of two such 2-element sets. Suppose that some of these 2-element sets are contained in two such 4-cocircuits. Then, by circuit elimination, three of these 2-element sets will occur in two such 4-cocircuits. Thus we may assume that $D^*_2 = \{a_1, b_1, a_2, b_2\}$, $D^*_3 = \{a_1, b_1, a_3, b_3\}$, and $D^*_4 = \{a_2, b_2, a_3, b_3\}$ are 4-cocircuits. This implies that the 4-circuit containing $\{c_1, c_2\}$ has to be $C = \{c_1, c_2, a_3, b_3\}$. Applying circuit elimination to $C$ and $T_3$ in $M/x$, we deduce that $C = \{c_1, c_2, a_3, c_3\}$ contains a circuit of $M/x$. By orthogonality, $a_3$ is not in this circuit. Thus $\{c_1, c_2, c_3\}$ is a triangle of $M/x$. This contradiction implies that each of these six 2-element sets occurs in exactly one 4-cocircuit, and hence we may assume that $D^*_1 = \{a_1, b_1, a_2, b_2\}$, $D^*_3 = \{a_1, c_1, a_3, c_3\}$, and $D^*_4 = \{a_2, c_2, a_3, b_3\}$ are 4-cocircuits of $M$.

Consider a 4-circuit of $M$ containing $\{b_2, c_3\}$. By orthogonality, it contains two of $a_1$, $b_1$, and $c_1$. Suppose that $c_1$ is in this circuit. Then, the remaining element is either $b_1$ or $a_1$. Consider a 4-circuit of $M$ containing $\{b_1, c_3\}$. By orthogonality, it contains $b_3$ and one of $a_1$ and $c_1$. By (2.18), this implies that $C = \{b_1, c_1, b_2, c_3\}$ is a 4-circuit of $M$. Similarly, assuming that $b_3$ is in this 4-circuit, we consider the 4-cocircuit containing $\{c_1, b_2\}$ to draw the same conclusion. Thus, $C$ is indeed a 4-circuit of $M$. Consider a 4-circuit $C$ containing both $c_1$ and $b_3$. By orthogonality, it is easy to show that it contains $c_2$ and one of $c_3$ and $a_5$. Now consider a 4-cocircuit containing both $x$ and $b_3$. By orthogonality, this cocircuit contains one element of each of $T_1, T_2, T_3$. Among these elements, one is $a_i$ for some $i \in \{1, 2, 3\}$. It follows that this cocircuit does not meet $C$, but meets $C$ in two elements. Thus, it must be $D^*_5 = \{x, a_1, c_2, b_3\}$. We now find that $D^*_1, D^*_2, D^*_3, D^*_5$ all share two common elements $c_2$ and $b_3$. By (3.4), $M^* = H_{10}$. Since $H_{10}$ is self-dual, we conclude that $M \cong H_{10}$. \[\]
(3.6) PROPOSITION. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. If, for each pair of elements $x$, $y$ of $E(M)$, there are at most two 4-circuits containing both, then $M \cong R_{10}$.

Proof. Let $x \in E(M)$. By (3.4) and (3.5), $M/x$ has at least five triangles. By mimicking the first part of the proof of (3.5), we may assume that $T_1 = \{a_1, b_1, c_1\}$, $T_2 = \{a_2, b_2, c_2\}$, $T_3 = \{a_3, b_3, c_3\}$, $T_4 = \{a_4, a_2, a_3\}$, and $T_5 = \{c_1, c_2, c_3\}$ are five of the triangles of $M/x$. By the dual of (3.2), $M$ has a 4-cocircuit containing both $a_1$ and $b_1$. By orthogonality, we may assume that $D_1^* = \{a_1, b_1, a_2, b_2\}$. Consider a 4-cocircuit containing $a_3$ and $b_3$. By orthogonality, we may assume that it is $D_5^* = \{a_1, a_3, b_1, b_3\}$. By circuit elimination, $D_1^* = \{a_2, b_2, a_3, b_3\}$ is also a 4-cocircuit. Similarly, $D_2^* = \{b_1, c_1, b_2, c_2\}$, $D_3^* = \{b_1, c_1, b_3, c_3\}$, and $D_4^* = \{b_2, c_2, b_3, c_3\}$ are all 4-cocircuits of $M$. Consider a 4-cocircuit containing both $a_1$ and $c_1$. By (2.18) and orthogonality, this 4-cocircuit must be $D_7^* = \{a_1, c_1, a_2, c_2\}$ or $D_8^* = \{a_1, c_1, a_3, c_3\}$. We may assume that $D_7^*$ occurs. Consider the 4-cocircuit containing $a_3$ and $c_3$. By (2.18) and orthogonality, it must be $D_5^*$ or $D_6^* = \{a_2, c_2, a_3, c_3\}$. By circuit elimination, we conclude that $D_6^*$, $D_7^*$, and $D_8^*$ are all 4-cocircuits of $M$. If $\{b_1, b_2, b_3\}$ is not a triangle of $M/x$, then, consider a 4-circuit containing both $b_1$ and $b_2$. By orthogonality, the nine 4-cocircuits $D_1^*, D_2^*, ..., D_8^*$ force the 4-circuit to be $\{b_1, b_2, a_3, c_3\}$. Applying circuit elimination to this circuit and $T_5$ in $M/x$, we conclude that $\{b_1, b_2, a_3, b_3\}$ contains a circuit of $M/x$. Since $D_8^*$ is a cocircuit of $M/x$, by orthogonality, $a_3$ is not in this circuit of $M/x$. Hence $\{b_1, b_2, b_3\}$ is a triangle of $M/x$. This implies that $M/x$ is isomorphic $M^*(K_{3,3})$ for each $x \in E(M)$. It is routine to check that $M \cong R_{10}$.  

On combining (3.4), (3.5), and (3.6), we immediately obtain the following:

(3.7) LEMMA. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) = 5$. Then $M$ is a spike, or $M$ is isomorphic to either $H_{10}$ or $R_{10}$.

(3.8) PROPOSITION. Suppose that $M$ is a 2-minimally, 2-cominimally 3-connected matroid with $r(M) \geq 6$. Then, for each $e \in E(M)$, the matroid $M^e$ has at least five triads.

Proof. As $r(M) \geq 6$, by (2.15), $|E(M^e)| \geq 11$. By (2.16), $M^e$ has at least four triads. If the union of the triads of $M^e$ has at least eleven elements, and $M^e$ has exactly four triads, then $M^e$ has three disjoint triads. By (2.16), there is a 4-circuit of $M$ containing $e$ and some element not in any of these three triads of $M^e$. This 4-circuit will intersect some 4-cocircuit of $M$ in exactly one element, a contradiction to orthogonality. Hence if $M^e$ has at least eleven elements that are in triads, then $M^e$ has
at least five triads. We now consider the case that \( M \setminus e \) has at most ten elements that are in triads. By (2.16), \(|E(M \setminus e)| \leq 11\). Thus, by (2.16), \( r(M) \leq 6 \). Hence \( r(M) = 6 \), and \(|E(M)| = 12\). Let \( f \) be the element not in any triad of \( M \setminus e \). Then, by (2.9) and (2.14), \( M \setminus e/f \) is isomorphic to \( W^5 \), or \( M \setminus e \) has at least five triads.

(3.9) Proposition. Suppose that \( M \) is a 2-minimally, 2-cominimally 3-connected matroid with \( r(M) > 6 \). Then \( M \) has a pair of elements \( e \) and \( f \) such that there are at least three 4-circuits of \( M \) containing both.

Proof. Assume the contrary. Then, for each pair of elements of \( M \), there are at most two 4-circuits of \( M \) containing both. By (2.15), as \( r(M) \geq 6 \), we have \(|E(M)| \geq 12\). Let \( x \) be an element of \( E(M) \). Since every element of \( E(M/x) \) is in at most two triangles, if there are at least seven triangles, any 4-cocircuit of \( M \) containing \( x \) will intersect some 4-circuit of \( M \) containing \( x \) in exactly one element, a contradiction. If there are exactly six triangles in \( M/x \), then, as \(|E(M/x)| \geq 11\), there are at least two elements of \( M/x \) such that each is in at most one triangle of \( M/x \). Hence, by (2.16), there is a 4-cocircuit of \( M \) containing \( x \) and one such element. This 4-cocircuit in turn will intersect some 4-circuit of \( M \) containing \( x \) in exactly one element, a contradiction. Thus, by (3.8), for every element \( x \in E(M) \), the matroid \( M/x \) has exactly five triangles.

If \( r(M) > 6 \), then, by (2.15), \(|E(M)| > 13\). By (2.16), the union of the triangles of \( M/x \) has cardinality greater than or equal to 12. As \( M/x \) has exactly five triangles, it is easy to find three disjoint triangles. Thus a 4-cocircuit containing \( x \) and an element not in these three disjoint triangles will intersect some 4-circuit in exactly one element, a contradiction. Therefore, we have that \( r(M) = 6 \) and \( M/x \) has exactly five triangles.

If each element of \( M/x \) is in at least one triangle, then, by the fact that \(|E(M/x)| = 11\), there are exactly four elements of \( E(M/x) \) such that each is in exactly two triangles. Suppose there is a triangle such that each of its element is in only one triangle. Then, by (2.16), there is an element \( y \) in exactly one of the other four triangles such that \( y \) is in a 4-cocircuit of \( M \) containing \( x \). It follows that this 4-cocircuit intersects some 4-circuit of \( M \) in exactly one element, a contradiction. We conclude that every triangle of \( M/x \) intersects some other triangle of \( M/x \). It follows by (2.18) that there are four triangles, \( T_1, T_2, T_3, T_4 \), such that \(|T_1 \cap T_2| = 1\), \(|T_3 \cap T_4| = 1\), and \(|(T_1 \cup T_2) \cap (T_3 \cup T_4)| = 0\). Up to relabeling, the remaining triangle will intersect \( T_2 \) and either \( T_1 \) or \( T_3 \). In the former case, by (2.16), one element in \( T_3 - T_4 \) is in a 4-cocircuit of \( M \) containing \( x \), and this contradicts orthogonality. In the latter case, by (2.16) again, \( M \) has a 4-cocircuit containing \( x \) and an element in \((T_2 - T_1) \cup (T_3 - T_4)\) that is only in one
triangle of $M/x$. This 4-cocircuit will intersect some 4-circuit in exactly one element, a contradiction.

We may now assume that $r(M) = 6$, and that, for each $x \in E(M)$, the matroid $M/x$ has exactly five triangles and has an element $y(x)$ such that $y(x)$ is not in any triangle of $M/x$. By (2.14), $M/x\setminus y(x)$ is a wheel or a whirl. Thus $M/x, y(x)$ is isomorphic to $M(K_5)$. This contradicts the assumption that $M$ is 2-cominimally 3-connected and hence proves the proposition.

(3.10) Lemma. Let $M$ be a 2-minimally, 2-cominimally 3-connected matroid with $r(M) \geq 6$. Suppose that, for some element $a_i$ in $E(M)$, the matroid $M/a_i$ has three triangles sharing a common element $a_2$, and that $M$ is not a spike. Then $r(M) = 6$ and $M \cong H_1$.

Proof. Since $M$ is not a spike, by (3.8) and (3.1), for each pair of elements $x, y$ in $E(M)$, there are at least two 4-circuits and two 4-cocircuits of $M$ containing $x$ and avoiding $y$.

Denote the three triangles of $M/a_i$ by $T_1 = \{a_2, a_3, a_4\}, T_2 = \{a_2, b_1, b_3\}$, and $T_3 = \{a_2, b_2, b_4\}$. Then, by (3.1), there are no other triangles of $M/a_i$ containing $a_2$. Since there are two 4-cocircuits $D_1^*$ and $D_2^*$ of $M$ that contain $a_1$ and avoid $a_2$, by orthogonality, we may assume that $D_1^* = \{a_2, a_4, b_3, b_4\}$ is a 4-cocircuit of $M$. If $D_2^* = \{a_2, a_3, b_1, b_2\}$, then, by orthogonality, every other triangle of $M/a_i$ will either intersect the set $\{a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$ in at least two elements or avoid this set. The first case is impossible since it leads to the conclusion that every 4-cocircuit of $M$ containing $a_1$ will contain $a_2$ which implies that $M$ is a spike. The second case is also not possible as it forces the matroid $M/a_i$ to have rank four, contradicting the fact that $r(M) \geq 6$. Therefore, by (2.18), $D_2^* = \{a_2, a_4, b_1, b_2\}$. Applying circuit elimination to $D_1^*$ and $D_2^*$, we conclude by orthogonality that $\{a_2, b_1, b_2, b_3, b_4\}$ contains a cocircuit. By orthogonality and the fact that $T_1 \cup a_i$ is a circuit of $M$, this circuit cannot contain $a_2$. Hence $D_1^* = \{b_1, b_2, b_3, b_4\}$ is also a 4-cocircuit of $M$.

If $M/a_i$ has a triangle disjoint from $T_1 \cup T_2 \cup T_3$, then, by orthogonality, every 4-cocircuit containing $a_4$ must contain $a_2$. Hence $M^*$ is a spike. This contradiction implies that every triangle of $M/a_i$ must intersect the set $\{a_2, a_3, a_4, b_1, b_2, b_3, b_4\}$. Moreover, by orthogonality, every triangle of $M/a_i$ intersecting $\{a_4, b_1, b_2, b_3, b_4\}$ intersects the set in at least two elements. As $r(M/a_i) \geq 5$, there is at least one triangle of $M/a_i$ which contains $a_3$, and avoids $\{a_4, b_1, b_2, b_3, b_4\}$. Let $T_4 = \{a_3, c_1, c_2\}$ be this triangle. If $r(M) \geq 6$, then, by (2.15), $|E(M)| \geq 14$. Since $M$ is not a spike, there are at most three triangles of $M/a_i$ containing $a_3$. Thus there are at least two elements $s, t$ in $E(M/a_i) - (T_1 \cup T_2 \cup T_3)$ that are not in triangles of $M/a_i$ containing $a_3$. By (2.16), we may assume that $s$ is in a triangle $T'$
of $M \setminus a_1$. Consider a 4-cocircuit of $M$ containing $t$ and one of $a_1$ and $a_2$. By orthogonality, it must contain $a_1, a_2,$ and one element of $T_4$. Hence this cocircuit intersects the 4-circuit $T' \cup a_1$ of $M$ in exactly one element, a contradiction. Therefore, $r(M) = 6$, and so $|E(M)| = 12$.

Let $E(M) - \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_1, c_2\} = \{c_3, c_4\}$. If $c_3$ is in a triangle containing $a_3$, then, by orthogonality and the existence of the cocircuits $D^*_1$ and $D^*_2$, it follows that $c_4$ is the remaining element of this triangle. If $c_3$ is not in a triangle containing $a_3$, then neither is $c_4$. If this is the case, then, by (2.16), we may assume that $c_3$ is in a triangle $T'$ avoiding $a_3$, and $M$ has a 4-cocircuit $C'$ containing $c_4$ and either $a_1$ or $a_2$. It follows by orthogonality that $C'$ contains both $a_1$ and $a_2$, and one element of $T_4$. Hence $C'$ intersects the 4-circuit $T' \cup a_1$ in exactly one element. This contradiction shows that $T_5 = \{a_3, c_3, c_4\}$ is a triangle of $M \setminus a_1$.

Since $M^*$ is not a spike, it follows by (3.8) and (3.1), there are at least two 4-cocircuits containing $a_3$ and avoiding $a_1$. By (2.18), orthogonality and relabeling on $\{c_1, c_2\}$ and $\{c_3, c_4\}$, we may assume that these 4-cocircuits are $D^*_3 = \{a_3, a_4, c_1, c_3\}$ and $D^*_4 = \{a_3, a_4, c_2, c_4\}$. Applying circuit elimination to $D^*_3$ and $D^*_4$, it follows by orthogonality that $D^*_5 = \{c_1, c_2, c_3, c_4\}$ is another 4-cocircuit of $M$. By (2.16), there is a 4-cocircuit of $M$ containing $b_3$ and either $c_1$ or $c_2$. By orthogonality, this cocircuit is $D^*_6 = \{b_1, b_3, c_1, c_2\}$. Similarly, $D^*_7 = \{b_1, b_3, c_3, c_4\}$, $D^*_8 = \{b_2, b_4, c_1, c_2\}$, and $D^*_9 = \{b_2, b_4, c_3, c_4\}$ are all 4-cocircuits of $M \setminus a_1$. By (2.16), there is a 4-circuit $C$ of $M$ containing $b_4$ and either $c_3$ or $c_4$. We may assume that $c_3 \in C$. Then, by orthogonality and the existing 4-cocircuits, we conclude that either $C = \{b_3, b_4, c_1, c_3\}$, or $C = \{b_2, b_4, c_3, c_4\}$. By (2.16), there is a 4-cocircuit containing $a_3$ and one of $b_1$ and $b_2$. By orthogonality, this circuit contains $\{a_1, a_3\}$, one of $\{b_2, b_4\}$ and one of $\{b_1, b_3\}$. This implies that $C = \{b_3, b_4, c_1, c_3\}$ and $D^*_1 = \{a_1, a_3, b_1, b_2\}$ is a 4-cocircuit of $M$. Similarly, $D^*_2 = \{a_1, a_3, b_3, b_4\}$, $D^*_3 = \{a_1, a_2, c_1, c_3\}$, and $D^*_4 = \{a_1, a_2, c_2, c_4\}$ are also 4-cocircuits of $M$. Applying circuit elimination to $D^*_1$ and $D^*_5$, we conclude that $\{a_1, a_2, a_3, a_4, c_1\}$ contains a cocircuit of $M$. Since this cocircuit does not meet $C$ in exactly one element, $c_1$ is not contained in this cocircuit. Thus this cocircuit must be $D^*_6 = \{a_1, a_2, a_3, a_4\}$. Arguing with $M^*$, we will also obtain fifteen 4-circuits of $M$. Moreover, by orthogonality and the existence of the $D^*_i$'s, it is now straightforward to check that $M$ has no 5-circuits and no 7-circuits. Therefore, $M$ is binary, and it is routine to check that the matroid $M$ is isomorphic to $H_{12}$.

**Proof of Theorem 1.1.** The first part of the theorem follows immediately on combining (3.1), (3.7) and (3.10). The check that each spike of rank at least four is 2-minimally, 2-cominimally 3-connected is straightforward and is omitted.
Theorem 1.1 shows that a matroid of rank at least seven is 2-minimally, 2-cominimally connected if and only if it is a spike. Although there are only three 2-minimally, 2-cominimally 3-connected matroids of rank at least five that are not spikes, there are more than thirty 2-minimally, 2-cominimally 3-connected matroids of rank four that are not spikes.

4. 2-MINIMALLY, 1-COMINIMALLY 3-CONNECTED MATROIDS

This section identifies all 2-minimally, 1-cominimally 3-connected matroids by proving Theorem 1.2.

(4.1) Proposition. Suppose that \( M \) is a 2-minimally 3-connected matroid with \(|E(M)| \geq 6\). Then no 4-cocircuit of \( M \) contains a triangle of \( M \).

Proof. Suppose that \( T = \{a, b, c\} \) is a triangle of \( M \), and that \( \{a, b, c, d\} \) is a 4-cocircuit of \( M \). Then \( T \) is both a triangle and a triad of \( M \setminus d \). Let \( r \) be the rank function of \( M \setminus d \). Then, as \(|E(M \setminus d)| \geq 5\) and \( r(T) + r^*(T) - |T| = 1\), it follows that \((T, E(M \setminus d) - T)\) is a 2-separation of \( M \setminus d \). This contradicts the assumption that \( M \) is 2-minimally 3-connected.

(4.2) Proposition. Suppose that \( M \) is a 2-minimally, 1-cominimally 3-connected matroid with \(|E(M)| \geq 6\). Then \( M \) has at most one element not contained in a triangle.

Proof. Suppose that each of \( x, y \) is an element of \( M \) not contained in a triangle. Since \( M \) is 1-cominimally 3-connected, it follows by the dual of (2.9) that \( M \setminus x, y \) is 3-connected. This contradicts the assumption that \( M \) is 2-minimally 3-connected.

(4.3) Proposition. Let \( M \) be a 2-minimally 3-connected matroid with \(|E(M)| \geq 6\). Suppose that \( T_1 \) and \( T_2 \) are two distinct triangles of \( M \). Then \(|T_1 \cap T_2| \leq 1\).

Proof. Suppose that \(|T_1 \cap T_2| = 2\). Then, by circuit elimination and (2.2), \( M'(T_1 \cup T_2) \cong U_{2,4} \). Let \( x \in T_1 \). Then \( M \setminus x \) is minimally 3-connected. It follows by (2.4) that \( M \setminus x \) has a triad. Thus \( M \) has a 4-cocircuit \( D^* \) containing \( x \). By orthogonality and the fact that \( M(T_1 \cup T_2) \cong U_{2,4} \), \( D^* \) contains at least three elements of \( T_1 \cup T_2 \), a contradiction to (4.1).

(4.4) Proposition. Let \( M \) be a 2-minimally 3-connected matroid with \(|E(M)| \geq 6\). Suppose that \( D_T^* \) and \( D_T^* \) are two distinct 4-cocircuits of \( M \) and \( T \) is a triangle meeting \( D_T^* \). Then \(|D_T^* \cap D_T^*| \neq 3\).
Proof. Suppose that \( D_1^* = \{e_1, e_2, e_3, e_4 \} \), and \( D_2^* = \{e_1, e_2, e_4, e_6 \} \). It follows by (2.2) and circuit elimination that \( M^*([D_1^* \cup D_2^*]) \cong U_{1, 5} \). We may assume that \( e_1 \) is in \( T \). By (4.1), \( T \not\subseteq D_1^* \cup D_2^* \). Thus \( T \) meets some 4-cocircuit in \( D_1^* \cup D_2^* \) in exactly one element, a contradiction to orthogonality.

(4.5) PROPOSITION. Let \( M \) be a 2-minimally 3-connected matroid with \( |E(M)| \geq 6 \). Suppose that \( T \) is a triangle of \( M \), \( e \in T \), and \( x \in E(M) - T \). Then \( \{e, x\} \) is contained in a 4-cocircuit of \( M \).

Proof. By (2.4), \( T \) meets at least two triads of \( M \setminus x \). By orthogonality and (4.1), each of these triads contains exactly two elements of \( T \). It follows by (4.4) that every element of \( T \) is contained in at least one of these triads of \( M \setminus x \).

(4.6) LEMMA. Let \( M \) be a 2-minimally, 1-comminimally 3-connected matroid with \( |E(M)| \geq 6 \). Suppose that \( M \) has three triangles sharing a common element. Then \( M \) is isomorphic to either \( F_7 \) or \( F_7^- \).

Proof. By (4.3), we may assume that these three triangles are \( T_1 = \{e_1, e_2, e_3\} \), \( T_2 = \{e_1, e_4, e_6\} \), and \( T_3 = \{e_1, e_5, e_7\} \). If there is an element \( x \) in \( E(M) - \{e_1, e_2, ... , e_7\} \), then, by (4.5), \( e_1 \) is contained in a triad of \( M \setminus x \). It follows that this triad meets one of \( T_1, T_2, \) and \( T_3 \) in exactly one element, a contradiction. Therefore, \( E(M) = \{e_1, e_2, ... , e_7\} \). Moreover, clearly \( r(M) = 3 \).

Since the matroid \( M \setminus e_4 \) is minimally 3-connected, it is not isomorphic to \( U_{3, 6} \), and hence has at least one triangle. By (4.3), we may assume that \( T_4 = \{e_5, e_6, e_7\} \) is a triangle of \( M \). By (4.1), orthogonality, and the fact that \( T_4 \) meets at least two triads of \( M \setminus e_1 \), we conclude that at least two of the sets \( C_1^* = \{e_2, e_5, e_7\} \), \( C_2^* = \{e_3, e_4, e_7\} \), and \( C_3^* = \{e_3, e_5, e_6\} \) are triads of \( M \setminus e_1 \). As \( r(M \setminus e_1) = 3 \) and \( |E(M)| = 7 \), \( M \setminus e_1 \) has corank 3. By (4.4), every triad of \( M \setminus e_1 \) is a cohyperplane of \( M \setminus e_1 \). Hence the complement of a triad of \( M \setminus e_1 \) is a triangle of \( M \setminus e_1 \). Therefore, at least two of the sets \( E(M \setminus e_1) - C_1^* \), \( E(M \setminus e_1) - C_2^* \), and \( E(M \setminus e_1) - C_3^* \) are circuits of \( M \setminus e_1 \); that is, at least two of \( \{e_3, e_4, e_6\}, \{e_2, e_5, e_6\}, \) and \( \{e_2, e_4, e_7\} \) are triangles of \( M \). Thus \( M \) is isomorphic to either \( F_7 \) or \( F_7^- \).

(4.7) PROPOSITION. Let \( M \) be a 2-minimally, 1-comminimally 3-connected matroid with \( |E(M)| \geq 6 \). Suppose that \( T_1, T_2, \) and \( T_3 \) are distinct triads of \( M \) such that \( |T_1 \cap T_2| = 1 \), and \( M \) is not isomorphic to \( F_7 \) or \( F_7^- \). Then \( T_3 \) meets exactly one of \( T_1 \) and \( T_2 \).

Proof. Suppose that \( T_1 = \{e_1, e_2, e_3\} \) and \( T_2 = \{e_1, e_4, e_5\} \). If \( T_3 \) is disjoint from \( T_1 \cup T_2 \), then, by (2.4), \( T_3 \) meets two distinct triads of \( M \setminus e_2 \).
Thus $T_3$ meets two 4-cocircuits of $M$ containing $e_2$. As $|T_3| = 3$, it follows by orthogonality that some element of $T_3$ is contained in both 4-cocircuits. By (4.4) and orthogonality, one of these two 4-cocircuits must contain $e_1$ and two elements of $T_3$. This implies that this 4-cocircuit meets $T_2$ in exactly one element, a contradiction. Therefore, $T_3$ meets at least one of $T_1$ and $T_2$.

Suppose that $T_3$ meets both $T_1$ and $T_2$. By (4.3) and the assumption that $M \not\cong F_2$ or $F_7^*$, we may assume that $T_3 = \{e_5, e_6, e_4\}$. Since $M$ is 2-minimally 3-connected, $M \setminus e_6$ is 3-connected. Since $M \setminus (T_1 \cup T_2)$ is not 3-connected, there is an element $e_7$ in $E(M) - (T_1 \cup T_2 \cup T_3)$. Suppose that $x \in E(M) - (T_1 \cup T_2 \cup T_3)$. By (4.5), $M$ has a 4-cocircuit containing both $x$ and $e_2$. By orthogonality, this 4-cocircuit must be either $C^*_1 = \{x, e_1, e_2, e_4\}$ or $C^*_2 = \{x, e_4, e_5, e_6\}$. By (4.5) again, $M$ has a 4-cocircuit containing $x$ and $e_6$. By orthogonality, this 4-cocircuit must be either $C^*_3$ or $C^*_4 = \{x, e_2, e_3, e_4\}$. In other words, at least two of $C^*_1, C^*_2$ and $C^*_4$ are 4-cocircuits of $M$.

It follows by (4.4) that $|E(M) - (T_1 \cup T_2 \cup T_3)| \leq 1$. Therefore, $E(M) = \{e_1, e_2, \ldots, e_6\}$.

By (4.4), it is clear that $r(M) \geq 4$. Thus $r(M) \leq 3$. We conclude by (4.3) that $r(M) = 3$. By (4.5), there is a 4-cocircuit $D^*$ containing both $e_4$ and $e_5$.

By (4.1) and orthogonality, $D^*$ is either $\{e_1, e_5, e_4, e_6\}$ or $\{e_2, e_3, e_4, e_5\}$. Therefore, either $\{e_2, e_3, e_5\}$ or $\{e_1, e_5, e_6\}$ is a hyperplane of $M$. As $r(M) = 3$, this hyperplane is a triangle of $M$. Therefore, either $e_1$ or $e_5$ is contained in three distinct triangles. By (4.6), $M$ is isomorphic to either $F_7^{-}$ or $F_7^{-}$, a contradiction. Therefore, $T_3$ meets exactly one of $T_1$ and $T_2$.

(4.8) Lemma. Let $M$ be a 2-minimally, 1-cominimally 3-connected matroid with $|E(M)| \geq 6$. Suppose that $T_1, T_2$ are triangles of $M$ such that $|T_1 \cap T_2| = 1$, and $M$ is not isomorphic to $F_5$ or $F_7^{-}$. Then $M \cong M^*(K_3, 3)$.

Proof. Let $T_1 = \{e_1, e_2, e_3\}$, $T_2 = \{e_1, e_4, e_5\}$, and $x \in E(M) - (T_1 \cup T_2)$. Since $M \setminus x$ is 3-connected but $M \setminus (T_1 \cup T_2)$ is not, $|E(M)| \geq 7$. By (4.5), $M$ has a 4-cocircuit containing both $x$ and $e_1$. By orthogonality, this 4-cocircuit contains one element of $\{e_2, e_3\}$ and one element of $\{e_4, e_5\}$. As $(\gamma_1)^2 = 4$, it follows by (4.4) that $|E(M) - (T_1 \cup T_2)| \leq 4$. Thus $|E(M)| \leq 9$.

Since $|E(M)| \geq 7$, by (4.2), there is an element $e_6$ in $E(M) - (T_1 \cup T_2)$ which is contained in a triangle $T_3$. By (4.7), we may assume that $T_3 = \{e_2, e_6, e_7\}$. Since $M \setminus e_3$ is 3-connected but $M \setminus (T_2 \cup T_3)$ is not, $M$ has an element, say $e_8$, that is not contained in $T_1 \cup T_2 \cup T_3$. By (4.5), $M$ has a 4-cocircuit $C^*$ containing both $e_1$ and $e_8$. By orthogonality, we may assume that $C^* = \{e_1, e_3, e_4, e_6\}$. By (4.5), $M$ has a 4-cocircuit $D^*$ containing both $e_5$ and $e_8$. By orthogonality and (4.4), $D^*$ contains $\{e_4, e_5, e_8\}$ and one element not in $T_1 \cup T_2 \cup T_3$, say $e_9$. Therefore, $|E(M)| = 9$. 


By (4.2), we may assume that $e_8$ is contained in a triangle $T_4$. Applying (4.7) to the triangles $T_1$, $T_2$, and $T_4$, we conclude that $T_4$ meets exactly one of $T_1$ and $T_2$. Applying (4.7) again, this time to $T_1$, $T_3$, and $T_4$, we conclude that $T_4$ meets exactly one of $T_1$ and $T_3$. Therefore, $T_4$ either meets both $T_2$ and $T_3$, or meets $T_1$ but avoids $T_2 \cup T_3$. Thus, we may assume that $T_4$ is either $\{e_4, e_6, e_8\}$ or $\{e_3, e_5, e_8\}$.

In the former case, consider the set $B = \{e_1, e_2, e_3, e_4, e_5\}$. Since $M$ is 2-minimally 3-connected, it follows by (2.2) that all cocircuits of $M$ have at least four elements. Thus, by orthogonality, it is easy to check that $B$ contains no cocircuits. Therefore, $r^*(M) \geq 5$. By (4.5), $M$ has a 4-cocircuit $D_7^\circ$ containing both $e_1$ and $e_2$. By orthogonality, $D_7^\circ = \{e_1, e_2, e_5, e_7\}$. Similarly, $M$ has a 4-cocircuit $D_6^\circ$ containing both $e_4$ and $e_7$. By orthogonality and (4.4), $D_6^\circ = \{e_4, e_5, e_6, e_7\}$. By (4.5), $M$ has a 4-cocircuit $D_5^\circ$ containing both $e_1$ and $e_6$. By orthogonality and (4.4), $D_5^\circ = \{e_1, e_2, e_4, e_6\}$. Let $H = \{e_1, e_2, e_4, e_5, e_6, e_7\}$. Then $H = D_7^\circ \cup D_6^\circ \cup D_5^\circ$, and $r^*(H) \leq 4$. Thus $T_5 = E(M) - H = \{e_3, e_8, e_9\}$ is dependent and so $T_5$ is a triangle of $M$. By the fact that $r^*(M) \geq 5$, we conclude that $r^*(M) = 5$. Hence $r(M) = 4$. By a similar argument to the above, we conclude that $T_6 = \{e_3, e_7, e_9\}$ is also a triangle. Therefore, $M \cong M^*(K_{3,3})$.

It remains to consider the case where $T_4 = \{e_3, e_5, e_6\}$. In that case, we can apply a similar argument to the above to draw the same conclusion. 

Let $h$ be an integer exceeding one. An $h$-raft $[3]$ is a matroid of rank $2h-2$ whose ground set is the union of $h$ disjoint triangles such that, for all $k < h$, the union of every set of $k$ of these triangles has rank $2k$. Thus, for example, $M^*(K_{3,3})$ is a 3-raft.

(4.9) Lemma. Let $M$ be a 2-minimally, 1-minimally 3-connected matroid with $|E(M)| \geq 6$. Suppose that each pair of distinct triangles of $M$ are disjoint. Then $M$ is a binary raft.

Proof. Suppose that $T = \{a, b, c\}$ is a triangle of $M$, and $x \in E(M) - T$. By (4.5), $M$ has a 4-cocircuit $D^\circ$ containing $x$ and meeting $T$. By (4.1) and orthogonality, we may assume that $D^\circ = \{x, y, a, b\}$, while $y$ is not an element of $T$. By (4.2), $M$ has a triangle $T'$ containing one of $x$ and $y$. By assumption, $T \cap T' = \emptyset$. It follows by orthogonality that $T' \cap D^\circ = \{x, y\}$. Thus, every element of $M$ is contained in a triangle. We conclude that there is a positive integer $n$ such that $E(M) = 3n$, and the ground set of $M$ is the union of $n$ disjoint triangles.

Since $M$ is 2-minimally 3-connected, $M$ cannot be one of $U_{2,6}$, $U_{2,4} \oplus U_{2,4}$, or $U_{2,3} \oplus U_{2,3}$. Thus, $n \geq 3$. Denote the $n$ triangles of $M$ by $T_1 = \{a_1, b_1, c_1\}$, $T_2 = \{a_2, b_2, c_2\}$, ..., $T_n = \{a_n, b_n, c_n\}$. By (2.4), the matroid $M \setminus a_2$ has two triads meeting $T_1$. Thus $M$ has two 4-cocircuits containing $a_2$ and meeting $T_1$. By orthogonality and (4.4), we may assume that these two
4-cocircuits are $C_{2,2}^* = \{a_1, b_1, a_2, b_2\}$ and $D_{1,2}^* = \{a_1, c_1, a_2, c_2\}$. Similarly, $M$ has two 4-cocircuits containing $c_3$ and meeting $T_1$. By orthogonality and (4.4), these two 4-cocircuits are $D_{1,2}^* = \{b_1, c_1, b_2, c_2\}$. Therefore, up to relabeling, we may assume that, for each pair of distinct integers $i, j$ in $\{1, 2, \ldots, n\}$, $C_{i,j}^* = \{a_i, b_i, a_j, b_j\}$, $D_{i,j}^* = \{a_i, c_i, a_j, c_j\}$ and $E_{i,j}^* = \{b_i, c_i, b_j, c_j\}$ are all 4-cocircuits of $M$.

Since $M$ is 2-minimally 3-connected and $|E(M)| > 4$, every cocircuit of $M$ has at least four elements. Thus every cocircuit of $M$ meets at least two triangles. If a set contains a triangle and two elements of another triangle, it contains a 4-cocircuit. Thus a cocircuit cannot contain a triangle. Therefore, a cocircuit is either disjoint from a triangle or meets that triangle in it contains a 4-cocircuit. Thus a cocircuit cannot contain a triangle. Therefore, a cocircuit is either disjoint from a triangle or meets that triangle in two elements. If $X$ is the union of four 2-element sets, each of which is a subset of distinct triangles, then it is clear that $X$ contains a 4-cocircuit. Thus we deduce that $M$ has only 4-cocircuits and 6-cocircuits. Applying circuit elimination to $C_{i,j}^*$ and $D_{i,k}^*$, it follows by orthogonality that $C_{i,j}^* \triangle D_{i,k}^*$ is a 6-cocircuit of $M$. It is now straightforward that for each pair of distinct cocircuits of $M$, their symmetric difference is a disjoint union of cocircuits. Therefore, $M$ is binary.

By orthogonality, the set $\{a_1, a_2, \ldots, a_n\} \cup \{b_1, c_1\}$ contains no cocircuit. Thus $r^*(M) \geq n + 2$. By orthogonality, the set $\bigcup_{2 \leq j \leq n} C_{i,j}^*$ is a coflat. Thus the set $C = \{c_1, c_2, \ldots, c_n\}$ is dependent. By orthogonality, $C$ must be a circuit. Therefore, $\bigcup_{2 \leq j \leq n} C_{i,j}^*$ is a cohyperplane, and $r^*(M) = r^* (\bigcup_{2 \leq j \leq n} C_{i,j}^* + 1) \leq (n + 1) + 1$. We deduce that $r^*(M) = n + 2$, and hence $r(M) = 3n - (n + 2) = 2n - 2$. Moreover, by arguing as for $C$, we deduce that $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are circuits of $M$. By orthogonality, every circuit having more than three elements must meet all triangles. On combining this observation with the fact that $A$, $B$, and $C$ are circuits, we conclude that, for each $k < n$, the union of $k$ distinct triangles has rank $2k$. Thus $M$ is a raft.  

**Proof of Theorem 1.2.** It is easy to check that there is no 2-minimally, 1-cominimally 3-connected matroid $M$ with $|E(M)| < 6$. Moreover, it is proved in [4] that, for all $n \geq 3$, the only binary $n$-raft is the matroid $M^*(K_{3,n})$; and the last matroid is easily shown to be 2-minimally, 1-cominimally 3-connected. On combining these observations with (4.6), (4.8), and (4.9), we obtain (1.2).

5. 2-MINIMALLY 3-CONNECTED MATROIDS

In the preceding two sections, we showed that both 2-minimally, 2-cominimally 3-connected matroids and 2-minimally, 1-cominimally 3-connected...
matroids have a familiar structure. The combination of (2.16), Theorem 1.1, and Theorem 1.2 implies the following theorem about their 4-cocircuits.

(5.1) Theorem. Let $M$ be a 2-minimally, $k$-cominimally 3-connected matroid with $|E(M)| \geq 5$ and $k \in \{1, 2\}$. Then each pair of distinct elements of $M$ is contained in a 4-cocircuit of $M$.

In [3], Akkari and Oxley proved:

(5.2) Theorem. Let $M$ be a matroid with $|E(M)| \geq 4$. Then $M$ is 2-minimally connected if and only if each pair of distinct elements of $M$ is contained in a triad.

By analogy with Theorem 5.2, one may hope that Theorem 5.1 can be extended to give that in all 2-minimally 3-connected matroids with at least five elements, every pair of distinct elements is contained in a 4-cocircuit. The following example shows that this is false.

(5.3) Example. Let $A$ be the matrix over $GF(11)$ shown below and let $M$ be the matroid represented by $A$. Then every 2-element subset of $E(M)$ except $\{1, 2\}$ is in a 4-circuit. Using this, it is not difficult to check that $M$ is 2-cominimally 3-connected.

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 4 & 0 & 3 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 2 & 6 & 2 & 5 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 4 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 3 & 3 & 3 \\
\end{bmatrix}
\]

In spite of this example, we do have the following result.

(5.4) Theorem. Let $M$ be a 2-minimally 3-connected matroid with $|E(M)| \geq 5$. If $M$ has a triangle, then there is at most one pair of distinct elements of $M$ that is not contained in a 4-cocircuit of $M$.

Proof. It is easy to check that there is no 5-element 2-minimally 3-connected matroid. Thus we may assume that $|E(M)| \geq 6$. Suppose that $T$ is a triangle of $M$ and $x, y \in E(M) - T$. We shall show that $x$ and $y$ are contained in a 4-cocircuit. By (2.4), the matroid $M \setminus x$ has two triads meeting $T$. Since $|E(M \setminus x)| \geq 5$, $T$ cannot be a triad of $M \setminus x$; otherwise $\{ T, E(M \setminus x) - T \}$ is a 2-separation of the 3-connected matroid $M \setminus x$. Thus we deduce that $M$ has two 4-cocircuits, say $D_1^x$ and $D_2^x$, each of which contains $x$ and meets
$T$ in exactly two elements. Similarly, $M$ has two 4-cocircuits, say $D_1^*$ and $D_2^*$, each of which contains $y$ and meets $T$ in exactly two elements. If $y$ is not contained in $D_1^* \cup D_2^*$, and $x$ is not contained in $D_1^* \cup D_2^*$, then, as $T$ has only three distinct 2-element subsets, we may assume that $D_1^* \cap T = D_2^* \cap T$. Applying circuit elimination to $D_1^*$ and $D_2^*$, it follows by orthogonality that the 4-element set $(D_1^* \cup D_2^*) - T$, which contains both $x$ and $y$, is a cocircuit of $M$. Therefore, each pair of distinct elements of $E(M) - T$ is contained in a 4-cocircuit of $M$.

By (4.4), $D_1^* \cap T \neq D_2^* \cap T$. Thus at most one 2-element subset of $T$ is not contained in some 4-cocircuit. Moreover, by (4.5), every 2-element subset of $E(M)$ that meets $T$ in a single element is contained in some 4-cocircuit. We conclude that $M$ has at most one pair of distinct elements that is not contained in a 4-cocircuit, and when such a pair exists, it is a subset of $T$.  

By combining the last sentence of the proof of the preceding theorem with (4.3), we immediately obtain the following:

(5.5) COROLLARY. Let $M$ be a 2-minimally 3-connected matroid with $|E(M)| \geq 5$. If $M$ has two distinct triangles, then every pair of distinct elements of $M$ is contained in a 4-cocircuit of $M$.

6. UNAVOIDABLE MATROIDS

In [5], Ding et al. proved the following:

(6.1) THEOREM. For every integer $n$ exceeding 2, there is an integer $N(n)$ such that every 3-connected matroid with at least $N(n)$ elements has a minor isomorphic to $U_{n+2,2}$, $U_{2,n+2}$, $M(K_{3,n})$, $M^*(K_{3,n})$, the cycle matroid of a wheel with $n$ spokes, the whirl of rank $n$, or an $n$-spike.

By Tutte’s Wheels and Whirls Theorem [14], the minimally, cominimally 3-connected matroids are exactly wheels and whirls. By Theorem 1.1, the 2-minimally, 2-cominimally 3-connected matroids of rank more than six are exactly spikes with their tips deleted. By the dual of Theorem 1.2, the minimally, 2-cominimally 3-connected matroids of rank more than four are exactly the cycle matroids of $K_{3,n}$ with $n \geq 3$. In this section, we prove that, for each $n \geq 3$, the only $n$-minimally, 1-cominimally 3-connected matroid is $U_{2,n+2}$. Using all these results, Theorem 1.3 is just a restatement of Theorem 6.1.

(6.2) PROPOSITION. Let $k$ be an integer exceeding two and $M$ be a $k$-minimally, 1-cominimally 3-connected matroid. If $|E(M)| \geq k + 4$, then no $(k + 2)$-cocircuit of $M$ contains a triangle of $M$. 
Proof. Suppose that \( D^* \) is a \((k + 2)\)-cocircuit of \( M \) and \( T \) is a triangle contained in \( D^* \). Let \( X = D^* - T \). Then \( |X| = k - 1 \) and the matroid \( M \setminus X \) is minimally 3-connected. Clearly, \( T \) is both a triad and triangle of \( M \setminus X \). As \(|E(M)| \geq k + 4\), it follows that \(|E(M \setminus X)| \geq 5\). Thus \((T, E(M \setminus X) - T)\) is a 2-separation of \( M \setminus X \), a contradiction. \( \blacksquare \)

(6.3) PROPOSITION. Let \( k \) be an integer exceeding two and \( M \) be a \( k \)-minimally 1-cominimally 3-connected matroid. Then \(|E(M)| \leq k + 3\).

Proof. We argue by contradiction. Hence assume that \(|E(M)| \geq k + 4\).

Suppose first that \( k \geq 4 \). Since \( M^* \) is minimally 3-connected and \(|E(M)| \geq 4\), it follows by (2.4) that \( M \) has a triangle \( T \). Let \( X \) be a subset of \( E(M) \) such that \( T \subseteq X \) and \(|X| = k - 1\). Since the matroid \( M \setminus X \) is minimally 3-connected and \(|E(M)| \geq 5\), by (2.4), \( M \setminus X \) has a triad \( C^* \). Clearly, \( X \cup C^* \) is a \((k + 2)\)-cocircuit of \( M \) that contains a triangle, a contradiction to (6.2).

We may now suppose that \( k = 3 \). By (2.4), we may assume that \( T_1 \) and \( T_2 \) are distinct triangles of \( M \). If \(|T_1 \cap T_2| = 2\), by circuit elimination, it is easy to show that \(|M|(T_1 \cup T_2) \cong U_{2.4}\). Let \( e, f \) be distinct elements of \( E(M) - (T_1 \cup T_2) \). Then \( M \setminus e, f \) is minimally 3-connected. By (2.4), \( M \setminus e, f \) has a triad \( C^* \) meeting \( T_1 \). By orthogonality and the fact that \(|M|(T_1 \cup T_2) \cong U_{2.4}\), \( C^* \) must be a subset of \( T_1 \cup T_2 \). Thus \( M \) has a 5-cocircuit \( C^* \cup \{e, f\} \) that contains a triangle, a contradiction. Therefore, \(|T_1 \cap T_2| \leq 1\). If \(|T_1 \cap T_2| = 1\), let \( e \) be the element in \( T_1 \cap T_2 \) and \( X = T_1 - e \). Since \( M \setminus X \) is minimally 3-connected, it follows by (2.4) that \( M \setminus X \) has two distinct triads \( C^*_1 \) and \( C^*_2 \) meeting \( T_2 \). Since both \( C^*_1 \cup X \) and \( C^*_2 \cup X \) are 5-cocircuits of \( M \), it follows by (6.2) that \( e \notin C^*_1 \cup C^*_2 \). Thus, by orthogonality, \(|C^*_1 \cap T_2| = |C^*_2 \cap T_2| = 2\), and hence \( C^*_1 \cap T_2 = C^*_2 \cap T_2 \). Let \( x \) be an element of \( C^*_1 \cap T_2 \). Applying circuit elimination to \( C^*_1 \) and \( C^*_2 \), we deduce that \((C^*_1 \cup C^*_2) - x\) contains a cocircuit of \( M \setminus X \). By orthogonality, \((C^*_1 \cup C^*_2) - T_2\) contains a cocircuit of \( M \setminus X \); that is, \( M \setminus X \) has a cocircuit of size at most 2, a contradiction. We conclude that no two distinct triangles of \( M \) meet.

Let \( U \) be the set of elements \( e \) of \( M \) for which \( e \) is not contained in a triangle. By the dual of (2.9), the matroid \( M \setminus V \) is 3-connected for every \( V \subseteq U \). Since \( k = 3 \), \( M \) is 3-minimally 3-connected. Thus \(|U| \leq 2\). If \(|U| = 2\), consider the matroid \( M \setminus U \). Suppose that \( C^* \) is a triad of \( M \setminus U \). Since every element of \( C^* \) is in a triangle of \( M \setminus U \), it follows by orthogonality that \( C^* \) is a triangle, a contradiction. Therefore, \(|U| \leq 1\). Let \( T_1 = \{a_1, b_1, c_1\} \) and \( T_2 \) be two distinct triangles and \( X = \{a_1, b_1\} \). Since the matroid \( M \setminus X \) is minimally 3-connected, it has two triads \( C^*_1 \) and \( C^*_2 \) meeting \( T_2 \). By (6.2), neither \( X \cup C^*_1 \) nor \( X \cup C^*_2 \) contains either \( T_1 \) or \( T_2 \). By orthogonality, both \( C^*_1 \) and \( C^*_2 \) contain two elements of \( T_2 \) and one element that is not contained in any triangle. Since \(|U| \leq 1\), we conclude that \( U = \{e\} \).
and \( e \in C_1^* \cap C_2^* \). Applying circuit elimination to \( C_1^* \) and \( C_2^* \), we deduce that the set \( D^* = (C_1^* \cup C_2^*) - e \) contains a cocircuit of \( M \setminus X \). By the fact that \( |D^*| = 5 \) and \( T_2 \subseteq D^* \), it follows that \( T_2 \) is a triad of \( M \setminus X \), a contradiction.

(6.4) Theorem. Let \( k \) be an integer exceeding two and \( M \) be a \( k \)-minimally, 1-cominimally 3-connected matroid. Then \( M \cong U_{2,k+2} \).

Proof. By (6.3), \( |E(M)| \leq k + 3 \). Let \( X \) be a subset of \( E(M) \) such that \( |X| = k - 1 \). Then \( |E(M \setminus X)| \leq 4 \). But \( M \setminus X \) is minimally 3-connected. Hence \( M \setminus X \cong U_{2,3} \). Therefore, each 3-element set of \( E(M) \) is a triangle. Thus \( M \cong U_{2,k+2} \).

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REFERENCES