JOURNAL OF DIFFERENTIAL EQUATIONS 22, 292–304 (1976)

Positively Invariant Closed Sets for Systems of Delay Differential Equations

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Received April 25, 1975

INTRODUCTION

We consider systems of first order ordinary differential equations where the derivatives with respect to the "time" t of the solutions may depend on the values of these solutions for t all the way back to $-\infty$; i.e., systems with infinite delay. The solutions are essentially functions on intervals of the form $(-\infty, T)$ to a finite dimensional real vector space. Our basic purpose is to obtain necessary and sufficient conditions that, given a closed subset of this space, a solution with values in this set for $t \leqslant t_0$ will have values in the set for all $t > t_0$ for which it is defined.

For such systems which model certain processes evolving in time, the states describing these processes may be meaningful only when they are nonnegative. In this case the closed set of interest would be the set of vectors with nonnegative components. Population processes and certain chemical reactions would be examples of such processes.

For finite dimensional systems with no delay, M. Nagumo has obtained a result of very general nature in [1]. For more recent results which follow from Nagumo's result, but were obtained independently; cf. M. G. Crandall [2], P. Hartman [3], H. Brezis [4], and J.-M. Bony [5]. Extensions of these results to equations in a Banach space have been given by R. H. Martin in [6]. The paper by J. A. Yorke [9] is also of interest.

DEFINITIONS AND NOTATIONS

We denote by R^n a real n-space, and by $|x|$ any suitable norm for $x \in R^n$. Let $R = R^1$. We denote by CB the set of functions on $(-\infty, 0]$ to R^n which are continuous and bounded on that interval.

If for $t_1 \in R$, x is a function on $(-\infty, t_1)$ to R^n continuous and bounded on that interval, then for each $t < t_1$ we define $x_t \in CB$ by $x_t(s) = x(t + s)$, $s \leqslant 0$.

Let f be a function on $R \times CB$ to R^n, and $t_0 \in R$ be fixed. If there exists a $T > 0$, $T \leqslant \infty$, and a function x on $(-\infty, t_0 + T)$ to R^n such that $x_t \in CB$ for $t < t_0 + T$, and such that the derivative $x'(t)$ of $x(t)$ exists on $[t_0, t_0 + T)$, is continuous there, and satisfies

$$x'(t) = f(t, x_t) \tag{1}$$

there, we say x is a solution of (1) on this interval. The initial value problem for (1) is given $(t_0, \phi) \in R \times CB$, to find a solution x of (1) such that $x_{t_0} = \phi$. We interpret $x'(t_0)$ to be the right hand derivative at t_0 of x.

Remark 1. CB is a Banach space with norm defined by $\|\phi\| = \sup_{s\leqslant 0} |\phi(s)|$ for $\phi \in CB$. The following example shows that even if f is continuous in t and globally Lipschitzian in ϕ, the initial value problem for (1) may not have a solution in the sense defined above. Define $f(\phi)$: $CB \to R$ by $f(\phi) = \sup_{n\geqslant 1} \phi[-(2\pi n)^{1/2}]$. It is easy to verify that $|f(\phi_1) - f(\phi_2)| \leqslant \|\phi_1 - \phi_2\|$ for $\phi_1, \phi_2 \in CB$. Fix $t_0 = 0$, and define $\phi(s) = \sin s^2$, $s \leqslant 0$. If x is a solution of $x'(t) = f(x_t)$ on $(-\infty, T)$, $T > 0$, such that $x(s) = \sin s^2$, $s \leqslant 0$, then if $0 < t < \min\{T, (2\pi)^{1/2}\}$, then

$$f(x_t) = \sup_{n\geqslant 1} \sin[t - (2\pi n)^{1/2}]^2.$$

If $t = (\pi/2)^{1/2} p/q$, p and q positive integers, $p < q$, relatively prime, and such that $(\pi/2)^{1/2} p/q < T$, then

$$f(x_t) = \sup_{n\geqslant 1} \sin\left(\frac{\pi}{2}\frac{p^2}{q^2} - 2\pi n\frac{p}{q}\right) < 1.$$

However, if $t = (\pi/2)^{1/2}\alpha$, α irrational and $0 < \alpha < \min\{1, (2/\pi)^{1/2} T\}$, it follows easily that $f(x_t) = 1$. Thus $f(x_t)$ cannot be continuous at all points $t = (\pi/2)^{1/2} p/q$, $p < q$, p and q positive integers relatively prime, and therefore no such solution x exists in our sense. We observe that there exists a function on $(-\infty, 1)$ continuous, bounded, and absolutely continuous on $[0, 1)$ such that $x(s) = \sin s^2$, $2 \leqslant 0$, and $x'(t) = f(x_t)$ almost everywhere on $[0, 1)$; i.e., take $x(t) = t$ for $0 \leqslant t < 1$. Whether such an absolutely continuous solution always exists if f is continuous in (t, ϕ) and Lipschitzian in ϕ seems at present unknown. For existence theorems for solutions to the initial value problem for (1) in the sense of our definition, cf. Driver [7].

The Main Result

In what follows, $\mathscr{E}$ is a closed subset of R^n. We say that $\mathscr{E}$ is positively invariant with respect to (1) if given $(t_0, \phi) \in R \times CB$ with $\phi(s) \in \mathscr{E}$ for $s \leqslant 0$, then for any solution x of (1) such that $x_{t_0} = \phi$, we have $x(t) \in \mathscr{E}$

for $t > t_0$ and all t for which it exists. This property of $\mathscr{E}$ has also been described as "forward" or "flow" invariance [2, 4].

We introduce the following hypotheses:

(H1) *There exists a function* F *on* $R \times CB \times [0, \epsilon_0]$ *to* R^n *such that*

(a) $\lim_{\epsilon \to 0+} F(t, \phi, \epsilon) = f(t, \phi)$;

(b) *if for* $(t_0, \phi) \in R \times CB$, x *is a solution of* (1) *on* $[t_0, t_0 + T)$ *such that* $x_{t_0} = \phi$, *then for each interval* $[t_0, t_0 + T_0]$, $0 < T_0 < T$, *there exists for all* ϵ *sufficiently small a solution* $y(t, \epsilon)$ *of*

$$y'(t) = F(t, y_t, \epsilon), \qquad y_{t_0} = \phi, \tag{1_ϵ}$$

such that $\lim_{\epsilon \to 0+} y(t, \epsilon) = x(t)$ *uniformly for* $t \in [t_0, t_0 + T_0]$;

(c) *given* $(t, \phi) \in R \times CB$ *with* $\phi(s) \in \mathscr{E}$ *for* $s \leqslant 0$, *and* ϵ, $0 < \epsilon \leqslant \epsilon_0$, *there exists an* $\alpha = \alpha(\epsilon, t, \phi) > 0$ *such that if* $0 < h \leqslant \alpha$ *and* $u \in R^n$ *is such that* $|u| \leqslant \alpha$, *then*

$$\phi(0) + hF(t, \phi, \epsilon) + hu \in \mathscr{E}. \tag{2}$$

(H2) *For* $(t, \phi) \in R \times CB$, $\phi(s) \in \mathscr{E}$ *for* $s \leqslant 0$,

$$\lim_{h \to 0+} h^{-1} \operatorname{dist}(\phi(0) + hf(t, \phi), \mathscr{E}) = 0 \tag{3}$$

We observe that if (c) of (H1) holds for $F(t, \phi, \epsilon) = f(t, \phi)$, $\epsilon \in (0, \epsilon_0]$, then clearly (H2) holds; we have merely to choose $u = 0$ in (2). On the other hand, (H2) and (a) and (b) of (H1) may hold, but (c) of (H1) need not; in fact, it is clear that (c) of (H1) cannot hold if $\mathscr{E}$ does not have an interior.

An example of a closed set $\mathscr{E}$ for which (H2) and additional conditions on f will imply (H1) is as follows: If $x \in R^n$, $x = (x_1, ..., x_n)$, we say $x \geqslant 0$ if $x_i \geqslant 0$ for $i = 1, 2, ..., n$. Define $\mathscr{E}+$ to be the set of x in R^n such that $x \geqslant 0$. We now introduce the following hypotheses.

(H1′) (a) *For any function* x *continuous and bounded on any interval* $(-\infty, T]$, $T < \infty$, $f(t, x_t)$ *is continuous on that interval*;

(b) f *is locally Lipschitzian* (*in the usual sense*) *in* ϕ.

It follows from known results (cf. Driver [7]) that given $(t_0, \phi) \in R \times CB$, (1) will have a unique solution x satisfying $x_{t_0} = \phi$ provided (H1′) holds. It then follows easily that (1_ϵ) with $F(t, \phi, \epsilon) = f(t, \phi) + \epsilon e_1$, $e_1 = (1, 1, ..., 1)$, will also have a unique solution, and a standard argument shows that (H1) (b) holds. Since (H1) (a) clearly holds, it remains to show that (H2) implies (H1) (c). To this end, if (H2) holds, it follows that for a fixed $(t, \phi) \in R \times CB$,

$\phi(s) \in \mathscr{E}+$ for $s \leqslant 0$, there exists for $h > 0$ and sufficiently small a $v(h) \in R^n$ such that $v(h) \to 0$ as $h \to 0+$ and

$$\phi(0) + hf(t, \phi) + hv(h) \geqslant 0. \tag{4}$$

For ϵ, $0 < \epsilon \leqslant 1$, choose $\alpha \leqslant \epsilon/2$ so small that $0 < h \leqslant \alpha$ implies $| v | \leqslant \epsilon/2$ and (4). So for $u \in R^n$, $| u | \leqslant \alpha$, $0 < h \leqslant \alpha$, we have

$$\phi(0) + h[f(t,\phi) + \epsilon e_1 + u] = \phi(0) + h[f(t,\phi) + v + (\epsilon e_1 - v + u)] \geqslant 0 \tag{5}$$

using $\epsilon e_1 - v + u \geqslant 0$ and (4). So (H1) (c) follows. We have proved the following

LEMMA 1. *Let* (H1′) *and* (H2) *hold and* $\mathscr{E} = \mathscr{E}+$. *Then* (H1) *follows.*

The referee has in fact pointed out that Lemma 1 actually holds in case $\mathscr{E}$ is any closed convex subset of R^n with nonempty interior. We sketch his proof the appendix of this paper.

Our basic result is

THEOREM 1. *Let* (H1) *hold. Then* $\mathscr{E}$ *is positively invariant with respect to* (1). *If* $\mathscr{E}$ *is positively invariant with respect to* (1), *then* (H2) *holds.*

COROLLARY 1. *Let* $\mathscr{E}+$ *be as previously defined, and* f *satisfy* (H1′). *Then* $\mathscr{E}+$ *is positively invariant with respect to* (1) *if and only if for each* $(t, \phi) \in R \times CB$ *with* $\phi(s) \geqslant 0$ *for* $s \leqslant 0$ *and* $\phi_i(0) = 0$, *we have* $f_i(t, \phi) \geqslant 0$.

Proof of Corollary 1. Fix $(t, \phi) \in R \times CB$, $\phi(s) \geqslant 0$ for $s \leqslant 0$. If $\phi_i(0) > 0$, then for some $h > 0$ and small

$$\phi_i(0) + hf_i(t, \phi) \geqslant 0. \tag{6}$$

On the other hand if $\phi_i(0) = 0$, and $f_i(t, \phi) \geqslant 0$, then (6) holds for all $h > 0$. So $\phi(0) + hf(t, \phi) \geqslant 0$ for h sufficiently small; i.e., (H2) holds. By Lemma 1, (H1) follows, so $\mathscr{E}+$ is by the theorem positively invariant with respect to (1).

Now let $\mathscr{E}+$ be positively invariant with respect to (1). By the theorem (H2) holds; i.e., for fixed $(t, \phi) \in R \times CB$, $\phi(s) \geqslant 0$ for $s \leqslant 0$, there exists for $h > 0$ and sufficiently small a $u = u(t, \phi, h) \in R^n$ such that $u \to 0$ as $h \to 0+$ and

$$\phi(0) + hf(t, \phi) + hu \geqslant 0. \tag{6.1}$$

Suppose $\phi_i(0) = 0$. From (6.1) it follows that $f_i(t, \phi) + u_i \geqslant 0$, and since $u \to 0$ as $h \to 0+$, $f_i(t, \phi) \geqslant 0$. This proves the corollary.

Proof of Theorem 1. Assume (H1), and suppose $\mathscr{E}$ is not positively invariant with respect to (1). Then there exists $(t_0, \phi^0) \in R \times CB$ with $\phi^0(s) \in \mathscr{E}$ for $s \leqslant 0$, and a solution $x(t)$ of (1) such that $x_{t_0} = \phi^0$, and $x(t_1) \notin \mathscr{E}$ for some $t_1 > t_0$. Using (H1) (b) there exists a $\epsilon > 0$, $\epsilon \leqslant \epsilon_0$, and a corresponding solution $y(t, \epsilon)$ of (1_ϵ) with $\phi = \phi^0$ such that $y(t_1, \epsilon) \notin \mathscr{E}$. Since y is continuous in t and $\mathscr{E}$ is closed, there exists a $\bar{t}_0 \geqslant t_0$, $\bar{t}_0 < t_1$, such that $y(s, \epsilon) \in \mathscr{E}$ for $s \leqslant t_0$, while $y(\bar{t}_0 + h_j, \epsilon) \notin \mathscr{E}$ for some sequence $\{h_j\}, j = 1,...,$ $h_j > 0$, $h_j \to 0$ as $j \to \infty$. From (1_ϵ) it follows that

$$y(\bar{t}_0 + h, \epsilon) = y(\bar{t}_0, \epsilon) + hF(\bar{t}_0, y_{\bar{t}_0}, \epsilon) + h\delta(h, \epsilon) \tag{7}$$

for $h > 0$ and sufficiently small and $\delta \to 0$ as $h \to 0+$. Let j be so large that $|\delta(h_j, \epsilon)| < \alpha(\epsilon)$, and $h_j < \alpha(\epsilon)$, where $\alpha(\epsilon)$ is as defined in (H1) (c) with $(t, \phi) = (\bar{t}_0, y_{\bar{t}_0})$. Then by (2) of (H1) (c),

$$y(\bar{t}_0, \epsilon) + h_j F(\bar{t}_0, y_{\bar{t}_0}, \epsilon) + h_j \delta(h_j, \epsilon) \in \mathscr{E}.$$

But this with (7) implies $y(\bar{t}_0 + h_j, \epsilon) \in \mathscr{E}$, a contradiction. Therefore $\mathscr{E}$ must be positively invariant with respect to (1).

Now assume $\mathscr{E}$ positively invariant with respect to (1). Then for any $(t_0, \phi) \in R \times CB$ with $\phi(s) \in \mathscr{E}$ for $s \leqslant 0$, we have

$$x(t_0 + h) = \phi(0) + hf(t_0, \phi) + h\delta(h, t_0, \phi) \in \mathscr{E}$$

for all $h > 0$ and sufficiently small; here $\delta \to 0$ as $h \to 0+$, and x is a solution of (1) such that $x_{t_0} = \phi$. But then

$$h^{-1} |\phi(0) + hf(t_0, \phi) - x(t_0 + h)| = |\delta(h, t_0, \phi)| \to 0 \quad \text{as} \quad h \to 0+.$$

Thus (H2) holds, and the theorem is proved.

Remark 2. If f is such that each initial value problem for (1) has a unique solution, then whether (H2) is sufficient that $\mathscr{E}$ be positively invariant with respect to (1) seems at present an open question. We note that the hypotheses (H1) does not assume this about (1) but only that if (1) has a solution, (1_ϵ) must also have solutions with the same initial values. We also note that uniqueness of such solutions is not assumed.

A Special Case

We consider next a special case of (1) where (H2) will be sufficient for the positive invariance of a closed set $\mathscr{E}$ which is assumed to have a special property, but not necessarily an interior as is assumed in Theorem 1. This special case includes Volterra integrodifferential equations, and is loosely speaking characterized by the condition that the dependence of f on ϕ does not extend to the values ϕ has all the way back to $-\infty$.

Let $\alpha(t)$ be a function continuous on R to R such that $\alpha(t) \geqslant 0$, $t \in R$. If $(t, \phi) \in R \times CB$, we denote by $\phi(\cdot, \alpha(t))$ the function on $-\alpha(t) \leqslant s \leqslant 0$ to R^n defined by $\phi(s, \alpha(t)) = \phi(s)$, $s \in [-\alpha(t), 0]$. We now consider (1) with $f(t, \phi) = F(t, \phi(\cdot, \alpha(t)))$ where F is defined for each $t \in R$ and $\phi(\cdot, \alpha(t))$ as defined above.

If we are interested in solutions for $t \geqslant 0$ of the Volterra integrodifferential equation

$$x'(t) = \int_0^t G(t, s, x(s))\, ds + h(t)$$

$$= \int_{-t}^0 G(t, t+s, x(t+s))\, ds + h(t)$$

we can choose $\alpha(t) = -t$, $t \geqslant 0$, $\alpha(t) = 0$, $t < 0$, and have this form of (1).

If $\alpha(t) = -r$, r a real positive constant, this form of (1) is a so-called functional differential equation of retarded type (cf. e.g., Hale [8]). In this case it follows easily that instead of CB we may consider the set C_r of functions continuous on $[-r, 0]$ to R^n.

We now show that under certain smoothness conditions on $f = F$, given in hypotheses (H3) and (H4) below, if $\mathscr{E}$ is convex, then (H2) is sufficient for the positive invariance of $\mathscr{E}$ with respect to (1).

(H3) *If* $\phi \in CB$, *then* $F(t, \phi(\cdot, \alpha(t)))$ *is continuous for* $t \in R$.

(H4) *Given* $(t_0, \phi^0) \in R \times CB$, *there exists* $L > 0$ *and* $\delta_0 > 0$ *such that*

$$|F(t, \phi(\cdot, \alpha(t))) - F(t, \psi(\cdot, \alpha(t)))| \leqslant L \,\|\phi - \psi\|_{\alpha(t)}$$

for $|t - t_0| \leqslant \delta_0$, $\|\phi - \phi^0\|_{\alpha(t)} \leqslant \delta_0$, $\|\psi - \phi\|_{\alpha(t)} \leqslant \delta_0$, $\phi, \psi \in CB$; *here and henceforth*:

$$\|\phi - \psi\|_{\alpha(t)} = \sup_{-\alpha(t) \leqslant s \leqslant 0} |\phi(s) - \psi(s)|.$$

LEMMA 2. *Let* (H3) *and* (H4) *hold. Then given* $T \in R$ *and a function* x *continuous and bounded on* $(-\infty, T]$ *to* R^n, $F(t, x_t(\cdot, \alpha(t)))$ *is continuous on* $(-\infty, T]$.

Proof. Fix $t_0 \in (-\infty, T]$. Then for $t \leqslant T$,

$$|F(t, x_t(\cdot, \alpha(t))) - F(t_0, x_{t_0}(\cdot, \alpha(t_0)))|$$

$$\leqslant |F(t, x_t(\cdot, \alpha(t))) - F(t, x_{t_0}(\cdot, \alpha(t)))|$$

$$+ |F(t, x_{t_0}(\cdot, \alpha(t))) - F(t_0, x_{t_0}(\cdot, \alpha(t_0)))|$$

By (H3), the second term on the right can be made arbitrarily small by choosing $| t - t_0 |$ small. We have clearly

$$\| x_t - x_{t_0} \|_{\alpha(t)} = \sup_{-\alpha(t) \leqslant s \leqslant 0} | x(t + s) - x(t_0 + s)| < \delta_0$$

for $| t - t_0 |$ sufficiently small, where δ_0 is as in (H4) corresponding to (t_0, ϕ^0), $\phi^0 = x_{t_0}$. Using (H4) we can also then make the first term on the right as small as we please for $| t - t_0 |$ sufficiently small. This proves the lemma.

The proof of the following theorem involves basically the same method used by Nagumo [1], and apparently independently by Crandall [2] and Martin [6]. The presence of a time delay however seems to require certain nontrivial modifications.

THEOREM 2. *Let F satisfy* (H3), (H4), *and $\mathscr{E}$ be convex as well as closed. Then $\mathscr{E}$ is positively invariant with respect to* (1) *with $f = F$ if and only if* (H2) *holds.*

Proof. The fact that (H2) is necessary for the positive invariance of $\mathscr{E}$ follows from Theorem 1.

Let (H2) hold. If $(t_0, \phi^0) \in R \times CB$ with $\phi^0(s) \in \mathscr{E}$ for $s \leqslant 0$, let δ_0 be as in (H4). Let T and b be positive numbers each less than δ_0. Define the function x^0: $(-\infty, t_0 + T] \to R^n$ by

$$\begin{aligned} x^0(t) &= \phi^0(t - t_0),\ t \leqslant t_0 \\ &= \phi^0(0),\ t_0 < t \leqslant t_0 + T. \end{aligned}$$

We denote by S_{bT} the set of all functions x: $(-\infty, t_0 + T] \to R^n$ such that $| x(t) - x^0(t)| \leqslant b$ for $t_0 < t \leqslant t_0 + T$, and $x(t) = x^0(t)$ for $t \leqslant t_0$. If $x \in S_{bT}$ we note that for $t_0 \leqslant t \leqslant t_0 + T$, $\| x_t - x_t^0 \|_{\alpha(t)} \leqslant \delta_0$, and it follows from (H4) that

$$| F(t, x_t(\cdot, \alpha(t))) - F(t, x_t^0(\cdot, \alpha(t)))| \leqslant L \| x_t - x_t^0 \|_{\alpha(t)}$$

for $x \in S_{bT}$, and $t \in [t_0, t_0 + T]$, and hence

$$| F(t, x_t(\cdot, \alpha(t)))| \leqslant Lb + M_1 \equiv M \tag{8}$$

for such t and x; here $M_1 = \sup_{t \in [t_0, t_0 + T]} | F(t, x_t^0(\cdot, \alpha(t)))|$, which exists because of Lemma 2. It is clearly no loss of generality to assume that T satisfies $MT < b$; if not, we may replace it by T_1, $0 < T_1 < T$, so small that $MT_1 < b$ holds, and observe that (8) obviously still holds with T replaced by T_1.

For each $\epsilon > 0$ such that $(M + \epsilon) T < b$, we now define a function x^ϵ continuous on $(-\infty, t_0 + T]$ to R^n as follows:

$$\begin{aligned} x^\epsilon(t) &= x^0(t) \quad \text{for } t \leqslant t_0, \\ &= x_j + (t_{j+1} - t_j)^{-1}(t - t_j)(x_{j+1} - x_j) \quad \text{for } t_j < t \leqslant t_{j+1}, \\ j &= 0, 1,..., N - 1; \end{aligned} \tag{8.1}$$

here $x_0 = x^0(t_0) \in \mathscr{E}$, and x_j and t_j are defined as follows: assuming $x^\epsilon(t) \in \mathscr{E}$ defined for $t \leqslant t_j$, put $x_j = x^\epsilon(t_j)$, and choose $x_{j+1} \in \mathscr{E}$ and $t_{j+1} > t_j$ such that $t_{j+1} - t_j \leqslant \epsilon$ and

$$|(t_{j+1} - t_j)^{-1} (x_{j+1} - x_j) - F(t_j, x^\epsilon_{t_j}(\cdot, \alpha(t_j)))| < \epsilon. \tag{9}$$

Such a choice of x_{j+1} and t_{j+1} follows from (H2). We assume $t_N \leqslant t_0 + T$. The fact that $x^\epsilon_{t_{j+1}}(s) \in \mathscr{E}$ for $s \leqslant 0$ follows obviously from the definition of x^ϵ and the convexity of $\mathscr{E}$. Thus by induction, $x^\epsilon(t) \in \mathscr{E}$ for $t \leqslant t_N$.

It also follows easily that $x^\epsilon \in S_{bT_N}$, $T_N = t_N - t_0$; we have, using the definition of x^ϵ, and (9), that for $t_j < t \leqslant t_{j+1}$:

$$\begin{aligned} | x^\epsilon(t) - x_0 | &\leqslant | x_{j+1} - x_j | + | x_j - x_{j-1} | + \cdots + | x_1 - x_0 | \\ &\leqslant (t_{j+1} - t_0)(M + \epsilon) \leqslant T(M + \epsilon) < b. \end{aligned}$$

We now show that there exists a choice of (t_j, x_j), $j = 1, 2,..., N, N + 1$, as above in (9) but where $t_{N+1} \geqslant t_0 + T$.

If $t_N < t_0 + T$, $x_N \in \mathscr{E}$, then there exists $t_{N+1} \in (t_0, t_0 + T]$ and a corresponding $x_{N+1} \in \mathscr{E}$ such that $t_{N+1} - t_N \leqslant \epsilon$, and (9) holds with $j = N$ and x^ϵ as defined in (8.1). This shows that $\bar{t} \equiv \sup\{t_N\} = t_0 + T_t$ the supremum taken over alll possible choices of $t_N \leqslant t_0 + T$. This also essentially shows that there exists a sequence $\{(t_j, x_j)\}$, $j = 0, 1,...$, such that $(t_j, x_j) \to (\bar{t}, \bar{x})$ as $j \to \infty$, $0 < t_{j+1} - t_j \leqslant \epsilon$, $\bar{x} \in \mathscr{E}$, and (9) holds for $j = 0, 1,...$; recall that $x^\epsilon \in S_{bT_j}$, $j = 0, 1,...$, where $T_j = t_j - t_0$.

We now define in terms of this sequence $\{(t_j, x_j)\}$ a function $\bar{x}^\epsilon$ on $(-\infty, \bar{t}]$ as follows: We use (8.1) for $t < \bar{t}$, and define $\bar{x}^\epsilon(\bar{t}) = \bar{x}$. It follows that there exists a $(\bar{t}_1, \bar{x}_1)$ such that (9) holds with $t_j = \bar{t}$, $x_j = \bar{x}$, $x^\epsilon = \bar{x}^\epsilon$, $t_{j+1} = \bar{t}_1$, $x_{j+1} = \bar{x}_1 \in \mathscr{E}$, and $0 < \bar{t}_1 - \bar{t} < \epsilon$. Using the fact (Lemma 2) that $F(t, \bar{x}_t^\epsilon(\cdot, \alpha(t)))$ is continuous at $\bar{t}$, it follows that there exists a n such that $\bar{t}_1 - t_n \leqslant \epsilon$, and

$$|(\bar{t}_1 - t_n)^{-1} (\bar{x}_1 - x_n) - F(t_n, \bar{x}^\epsilon_{t_n}(\cdot, \alpha(t_n)))| < \epsilon \tag{9.1}$$

holds. If we now take $n = N$, $t_{N+1} = \bar{t}_1$, and $x_{N+1} = \bar{x}_1$, our above assertion follows.

We now define a function x^ϵ on $(-\infty, t_{N+1}]$ by (8.1) with $j = 0, 1,..., N$. Clearly this function is defined on $(-\infty, t_0 + T]$ and is in S_{bT}, and by relabeling indices, we can take $t_N = t_0 + T$.

For each ϵ as above we then can take a corresponding function x^ϵ on $(-\infty, t_0 + T]$ to R^n; this function clearly has a derivative on each interval (t_j, t_{j+1}), $j = 0,..., N - 1$, given by

$$x^{\epsilon\prime}(t) = (t_{j+1} - t_j)^{-1} (x_{j+1} - x_j)$$

there.

By an arbitrary extension of the definition of $x^{\epsilon\prime}(t)$ to the points $t = t_j$, $j = 0,..., N - 1$, it follows that

$$\begin{aligned} x^\epsilon(t) - x^\epsilon(t_0) &= \int_{t_0}^{t} x^{\epsilon\prime}(s)\, ds \\ &= \int_{t_0}^{t} [F(s, x_s^\epsilon(\cdot, \alpha(s))) + \Delta^\epsilon(s)]\, ds \end{aligned} \tag{11}$$

where

$$\Delta^\epsilon(s) = x^{\epsilon\prime}(s) - F(s, x_s^\epsilon(\cdot, \alpha(s))).$$

Clearly, using (9):

$$\begin{aligned} \left| \int_{t_j}^{t_{j+1}} \Delta^\epsilon(s)\, ds \right| &= \left| x_{j+1} - x_j - \int_{t_j}^{t_{j+1}} F(s, x_s^\epsilon(\cdot, \alpha(s)))\, ds \right| \\ &\leqslant \int_{t_j}^{t_{j+1}} | F(t_j, x_{t_j}^\epsilon(\cdot, \alpha(t_j))) - F(s, x_s^\epsilon(\cdot, \alpha(s)))|\, ds \\ &\quad + \epsilon(t_{j+1} - t_j), \qquad j = 0, 1,..., N - 1 \end{aligned} \tag{12}$$

We now use the fact that the set $\{x^\epsilon\}$ is equicontinuous and uniformly bounded on $[t_0, t_0 + T]$, and assert the existence of a sequence $\{x^{\epsilon_n}\}$, $n = 1, 2,...$, of these functions, with $\epsilon_n \to 0$ as $n \to \infty$, and a function x on this interval, such that $x^{\epsilon_n}(t) \to x(t)$ uniformly on $(-\infty, t_0 + T]$.

Since $x^\epsilon(t) = \phi^0(t - t_0)$ for all $t \leqslant t_0$, it follows that $x(t) = \phi^0(t - t_0)$ for $t \leqslant t_0$.

Using (12) with $\epsilon = \epsilon_n$, and the facts that $\| x_t^{\epsilon_n} - x_t \| \to 0$ as $n \to \infty$ uniformly for $t \in [t_0, t_0 + T]$, and $F(t, x_t(\cdot, \alpha(t)))$ is uniformly continuous on that interval, it follows that

$$\int_{t_0}^{t} \Delta^{\epsilon_n}(s)\, ds \to 0 \qquad \text{as} \quad n \to \infty, \quad t \in [t_0, t_0 + T].$$

Hence using (11) with $\epsilon = \epsilon_n$ and letting $n \to \infty$ we get, using (H4),

$$x(t) - x(t_0) = \int_{t_0}^{t} F(s, x_s(\cdot, \alpha(s)))\, ds, \qquad t \in [t_0, t_0 + T].$$

Thus x is a solution of (1) with $f = F$, $x_{t_0} = \phi^0$.

A routine argument using (H4) assures us that x is also unique, and since $x^\epsilon(t) \in \mathscr{E}$ for $t \leqslant t_0 + T$, and $\mathscr{E}$ is closed, we have $x(t) \in \mathscr{E}$ for all such t.

Finally $x(t) \in \mathscr{E}$ for as long as it exists. This follows from the familiar argument that if $x(t) \in \mathscr{E}$ for $t \leqslant t_0 + \hat{T}_1$, but for some sequence $\{T_n\}$, $T_n > \hat{T}_1$, $T_n \to \hat{T}_1$ as $n \to \infty$, we have $x(t_0 + T_n) \notin \mathscr{E}$, then an application of our theorem with t_0 replaced by $t_0 + \hat{T}_1$ and ϕ^0 by $x_{t_0 + \hat{T}_0}$ would clearly lead to a contradiction.

This completes the proof of Theorem 2.

Remark 3. It is clear that the convexity requirement on $\mathscr{E}$ in Theorem 2 could be replaced by a less restrictive condition; all that is needed for the proof to go through is that $\mathscr{E}$ be such that for all $\epsilon > 0$, ϵ sufficiently small, so-called ϵ-approximate solutions of (1) exist and that $x^\epsilon(t) \in \mathscr{E}$ for all $t \leqslant t_0 + T$, T independent of ϵ. The ϵ-approximate solutions, for example, need not be piecewise linear on the interval $[t_0, t_0 + T]$.

For the special case $\alpha(t) = 0$, (1) reduces for $f = F$ to a system of ordinary differential equations, and in this case it is clear that no additional conditions on $\mathscr{E}$ are required. Nagumo's Theorem 1 in [1] considers such systems but is more general in the sense that no Lipschitz condition on F is assumed there, but the conclusion is only that if $x_0 \in \mathscr{E}$, there exists a solution $x(t) \in \mathscr{E}$ for $t \geqslant t_0$ such that $x(t_0) = x_0$. Thus $\mathscr{E}$ is positively invariant with respect to (1) in this case if solutions to the initial value problem (1) with $x(t_0) = x_0$ are unique, but not necessarily otherwise.

Also for the case where (1) reduces to a system of ordinary differential equations, Corollary 1, an obvious consequence of Nagumo's result for $\mathscr{E} = \mathscr{E}+$, was obtained independently by I. W. Sandberg in an unpublished result. In fact, Sandberg's method suggested the perturbation idea of (H1), which the author would like to acknowledge here.

Application to a Volterra Integrodifferential Equation

Consider the linear homogeneous Volterra integrodifferential equation

$$x'(t) = Ax(t) + \int_0^t B(t - s)\, x(s)\, ds, \qquad t \geqslant 0; \tag{13}$$

here $x(t) \in R^n$, A and B are $n \times n$ matrices, $B(t)$ is defined and continuous for $t \geqslant 0$. Using previously defined notation we may write (13) in the form

$$x'(t) = Ax_t(0) + \int_{-t}^{0} B(-s)\, x_t(s)\, ds, \qquad t \geqslant 0;$$

This has the form of (1) with

$$f(t, \phi) = A\phi(0) + \int_{-t}^{0} B(-s)\, \phi(s)\, ds, \qquad t \geqslant 0, \tag{14}$$

where in the initial value problem we obviously restrict ourselves to the case $t_0 \geqslant 0$. Clearly any solution $x(t)$ of (1) with f given by (14) and $x_{t_0} = \phi \in CB$, $t_0 \geqslant 0$, will also satisfy (13) with $x(t) = \phi(t - t_0)$, $0 \leqslant t \leqslant t_0$. Also if $x(t)$ is a solutions of (13) with $x(t) = x^0(t)$, $0 \leqslant t \leqslant t_0$, then it is a solution of (1) with f given by (14) such that $x_{t_0} = \phi \in CB$ where $\phi(s) = x^0(t_0 + s)$, $-t_0 \leqslant s \leqslant 0$.

It is not difficult to verify that if in all of the previous conditions we had restricted t_0 to be nonnegative, all of the previously obtained results remain valid. Using such a suitably modified version of Corollary 1 with $\mathscr{E}+$ as defined there, we can easily establish the following result suggested to the author by R. K. Miller. We omit the proof.

THEOREM 3. *A necessary and sufficient condition that for each* $t_0 \geqslant 0$, *the solution of* (14) *such that* $x(t_0 + s) \in \mathscr{E}+$ *for* $-t_0 \leqslant s \leqslant 0$ *will satisfy* $x(t) \in \mathscr{E}+$ *for* $t \geqslant t_0$ *is that* $a_{ij} \geqslant 0$ *for* $i \neq j$, *and* $b_{ij}(t) \geqslant 0$ *for* $t \geqslant 0$ *and all* i, j; *here* $A = (a_{ij})$ *and* $B(t) = (b_{ij}(t))$.

We conclude with an example which shows that if in the definition of positive invariance t_0 is restricted to be a fixed constant, our basic results fail. In particular, we show that in this case the condition on f given in Corollary 1 is sufficient but not necessary that for a fixed t_0, if $(t_0, \phi) \in R \times CB$, $\phi(s) \in \mathscr{E}+$ for $s \leqslant 0$, the solution $x(t)$ such that $x_{t_0} = \phi$ satisfies $x(t) \in \mathscr{E}+$ for $t \geqslant t_0$ as long as it exists.

In this example $R^n = R$, $t_0 = 0$, and f is defined by

$$\begin{aligned} f(t, \phi) &= 2\phi(0), \qquad t \leqslant 1 \\ &= 2\phi(0) - \int_{1-t}^{0} \phi(s)\, ds, \qquad t > 1. \end{aligned}$$

Then $x'(t) = f(t, x_t)$ becomes eventually

$$\begin{aligned} x'(t) &= 2x(t), \qquad t \leqslant 1, \\ &= 2x(t) - \int_{1}^{t} x(v)\, dv, \qquad t > 1; \end{aligned}$$

the solutions of this equation are given by

$$x(t) = x(0)e^{2t}, \qquad t \leqslant 1$$
$$= x(0)\, te^{t+1}, \qquad t > 1$$

and clearly $x(t) \geqslant 0$ for $t \geqslant 0$ whenever $x(0) \geqslant 0$.

However the condition of the corollary is that if $\phi(s) \geqslant 0$ for $s \leqslant 0$ and $\phi(0) = 0$, then $f(t, \phi) \geqslant 0$ for all t. For $t > 1$ this is clearly not true for our f above and any $\phi(s) > 0$ for $s < 0$.

Appendix

We sketch here a proof due to the referee that if $\mathscr{E}$ is a closed convex subset of R^n with nonvoid interior, then the conclusion of Lemma 1 holds; i.e., (H1′) and (H2) imply (H1).

Let $w \in \mathscr{E}^0$, the interior of $\mathscr{E}$, and let $r > 0$ be such that $| x - w | \leqslant r$ implies $x \in \mathscr{E}$. For $0 < \epsilon < 1$ define

$$F(t, \phi, \epsilon) = f(t, \phi) - \epsilon(\phi(0) - w).$$

Again, (H1) (a) holds trivially, and (H1) (b) follows from (H1′). Since $\mathscr{E}$ is convex, if $0 < \eta < 1$ and $v \in \mathscr{E}$, then

$$| x - [(1 - \eta)v + \eta w]| \leqslant \eta r \tag{15}$$

implies $x \in \mathscr{E}$. If $u \in R^n$, $| u | \leqslant \epsilon r/3$, and $v(h) \in \mathscr{E}$ is such that $\phi(0) + hf(t, \phi) = v(h) + h\beta(h)$, and $| v(h) - \phi(0)| \leqslant r/3$ for $0 < h < \delta$ where $| \beta(h)| \leqslant \epsilon r/3$, then

$$| \phi(0) + h[F(t, \phi, \epsilon) + u] - [(1 - h\epsilon)\, v(h) + h\, \epsilon\, w]| \leqslant h\, \epsilon\, r,$$

so for h sufficiently small,

$$\phi(0) + h[F(t, \phi, \epsilon) + u] \in \mathscr{E}$$

by (15). Thus (H1) (c) holds, and we are done.

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