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Embedded desingularization of toric varieties[☆]

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ABSTRACT

We present a new method to achieve an embedded desingularization of a toric variety.

Let W be a regular toric variety defined by a fan Σ and $X \subset W$ be a toric embedding. We construct a finite sequence of combinatorial blowing-ups such that the final strict transforms $X' \subset W'$ are regular and X' has normal crossing with the exceptional divisor.

Introduction

Fix a polynomial ring, a toric (not necessarily normal) variety is defined by a prime ideal generated by binomials. Such varieties can be considered as combinatorial, in fact all the information they carry can be expressed in terms of combinatorial objects. This gives a way of computing geometric invariants of the toric variety. There are also many applications of the theory of toric varieties, see for example Cox (1997). For an introduction to toric varieties, see Danilov (1978), Oda (1988) or Fulton (1993).

This paper is devoted to construct an algorithm of desingularization of toric varieties and log-resolution of binomial ideals. Given a binomial prime ideal, corresponding to a toric variety X, we construct a sequence of combinatorial blowing-ups such that the strict transform of the variety X' is nonsingular and has normal crossings with the exceptional divisor. Our algorithm is valid if the ground field is perfect of any characteristic. In principle a more general field could be used in our case, but some problems arise even in the combinatorial case when computing equimultiple points; see Theorem 3.1 in Bierstone and Milman (2006).

In González Pérez and Teissier (2002) an algorithm of desingularization of toric varieties is given. In this paper the authors construct a toric map $X' \to X$, which is a (non embedded) desingularization

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of X. The map $X' \rightarrow X$ is not a sequence of blowing-ups along nonsingular centers, it is a toric morphism defined in one step in terms of the fan of X.

There is another desingularization method of toric varieties (Bierstone and Milman, 2006), producing a sequence of blowing-ups along nonsingular centers. It uses the Hilbert function of the variety and is valid for binomial ideals without embedding components.

In Blanco (in press, 2011) an algorithm of log-resolution of binomial ideals is constructed based on the computation of an ordering function E-ord. The function E-ord is the order of the ideal defining the variety along a normal crossing divisor E. To achieve a log-resolution, this algorithm needs a step called locally monomial resolution (the transform ideal is locally generated by monomials). The step from the locally monomial resolution to the log-resolution increases the difficulty of the process. The algorithm presented in Blanco (in press, 2011) is valid for binomial ideals without any restriction. In fact it is valid for ideals generated by monomials and binomials, but it depends on a choice of a Gröbner basis of the ideal. Both disadvantages are avoided in this new approach.

The algorithm presented here depends on the ordering function E-ord and a codimension function Hcodim. The use of the codimension function in the resolution function does not appear in the previous approaches and makes this algorithm independent of any choice. This algorithm can be implemented at the computer and we expect to have a working implementation shortly.

On the other hand, we prove an equivalence of the geometric notion of transversality of a variety with respect to a normal crossing divisor and a new notion of transversality of $\mathbb{Z}$-modules. Then we are able to translate geometric notions to combinatorial terms.

This can be considered as a first step on a more ambitious program translating notions from toric varieties, in terms of dual cones and fans, to notions in terms of the binomial equations of the variety.

We would like to thank Ignacio Ojeda for useful suggestions and conversations which helped to improve the presentation of this paper.

The paper is structured as follows: first we recall known facts on toric varieties. There is a bijection between toric varieties and saturated $\mathbb{Z}$-submodules of $\mathbb{Z}^n$. We define a notion of transversality of $\mathbb{Z}$-submodules which will be equivalent to the usual notion of transversality of the toric variety with respect to a normal crossing divisor. Given an affine toric variety X, we will prove also the existence of a minimal regular toric variety V transversal to E (the normal crossing divisor) and containing X. This minimal embedding $X \subset V$ will define a function Hcodim which is the first coordinate of our resolution function. The rest of the coordinates of this resolution function comes from the process of E-resolution constructed in Blanco (in press, 2011). In the last section we will construct the algorithm of embedded desingularization of a toric variety $X \subset W$, where W is a smooth toric variety.

Notation and first definitions

Remark 1. Fix a perfect field k. We will denote the affine space of dimension n as usual $\mathbb{A}^n = \operatorname{Spec}(k[x_1, \ldots, x_n])$. The torus of dimension n is $\mathbb{T}^n = \operatorname{Spec}(k[x_1^{\pm}, \ldots, x_n^{\pm}])$. Note that $\mathbb{A}^n \setminus \mathbb{T}^n$ is a union of n hypersurfaces having only normal crossings. Let E be a set of hypersurfaces such that every $H \in E$ is an irreducible component of $\mathbb{A}^n \setminus \mathbb{T}^n$. The set E corresponds to a subset of indexes $\mathbf{E} \subset \{1, \ldots, n\}$. We set

$$\mathbb{T}\mathbb{A}^n_E = \mathbb{A}^n \setminus \bigcup_{H \notin E} H = \operatorname{Spec}\left(k[x_1, \ldots, x_n]_{\prod_{i \notin \mathbf{E}} x_i}\right)$$

where $\operatorname{Spec}(k[x_1, \ldots, x_n]_{\prod_{i \notin \mathbf{E}} x_i})$ is the localization of the polynomial ring with respect to the product $\prod_{i \notin E} x_i$.

Note that E is a set of hypersurfaces of $\mathbb{T}\mathbb{A}^n_E$ having only normal crossings.

A morphism $\mathbb{T}^n \to \mathbb{T}^1$ which is a group homomorphism is called a *character*.

For example if $a \in \mathbb{Z}^n$ then the morphism defined by $T \to X_1^{a_1} \cdots X_n^{a_n}$ is a character. In fact, every character of $\mathbb{T}^n$ is as above (Cox et al., 2011; Humphreys, 1975; Eisenbud and Sturmfels, 1996). So that the group of characters of a n-dimensional torus is a free abelian group of rank n.

There is a bijection between morphisms $\mathbb{T}^d \to \mathbb{T}^n$ which are also group homomorphisms and homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^d$ of $\mathbb{Z}$-modules (Humphreys, 1975). Moreover, closed reduced immersions $\mathbb{T}^d \to \mathbb{T}^n$ correspond to surjective homomorphisms $\mathbb{Z}^n \to \mathbb{Z}^d$.

We recall some basic definitions and well known results on toric varieties in order to make the article self-contained.

Definition 2. An affine toric variety is an affine variety X of dimension d, such that X contains the torus $\mathbb{T}^d$ as a dense open set and the action of the torus extends to an action of $\mathbb{T}^d$ to X.

Theorems 3 and 4 are well known results.

Theorem 3 (Cox et al., 2011; Miller and Sturmfels, 2005; Eisenbud and Sturmfels, 1996)**.** *Let X be a scheme of dimension d. These facts are equivalent:*

(1) *X is an affine variety.*
(2) $X \cong \mathrm{Spec}(k[t^{a_1}, \dots, t^{a_n}])$, *where $a_1, \dots, a_n \subset \mathbb{Z}^d$.*
(3) *$X \subset \mathbb{A}^n$ and $I(X)$ is prime and generated by binomials.*

Theorem 4. *Let X be an affine toric variety of dimension d.*
X is regular if and only if $X \cong \mathbb{T}\mathbb{A}^d_E$ for some $\mathbf{E} \subset \{1, \dots, d\}$.

Definition 5. An affine toric embedding is a reduced closed subscheme $X \subset W$ where

- W is a regular affine toric variety.
- X is an affine toric variety.
- The inclusion is toric, which means that one has a group homomorphism from the torus of X to the torus of W.

Toric varieties are related to $\mathbb{Z}$-submodules of $\mathbb{Z}^n$.

Definition 6. Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-module. The saturation of M is:

$$\mathrm{Sat}(M) = \{\alpha \in \mathbb{Z}^n \mid \lambda\alpha \in M \text{ for some } \lambda \in \mathbb{Z}\}.$$

We say that a $\mathbb{Z}$-module $L \subset \mathbb{Z}^n$ is saturated if $\mathrm{Sat}(L) = L$.

Note that $L \subset \mathbb{Z}^n$ is saturated if and only if the quotient $\mathbb{Z}^n/L$ is a free $\mathbb{Z}$-module.
Note also that $M \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{Sat}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$.
The following theorem is based on known results.

Theorem 7 (Eisenbud and Sturmfels, 1996)**.** *Let be $n \in \mathbb{N}$, $r \le n$ and $d \le n$. There is a bijection correspondence between the following sets:*

(1) *The set of affine toric embeddings $X \subset \mathbb{T}\mathbb{A}^n_E$, with $d = \dim X$.*
(2) *The set of closed and reduced immersions $\mathbb{T}^d \to \mathbb{T}^n$.*
(3) *The set of surjective homomorphisms of $\mathbb{Z}$-modules, $\mathbb{Z}^n \to \mathbb{Z}^d$.*
(4) *The set of saturated $\mathbb{Z}$-submodules $L \subset \mathbb{Z}^n$ of rank $n - d$.*

1. Sublattices of $\mathbb{Z}^n$

In this section we introduce the notion of transversality of a $\mathbb{Z}$-module with respect to a subset $\mathbf{E} \subset \{1, \dots, n\}$. We will prove that there exists always a maximal transversal submodule of any saturated $\mathbb{Z}$-submodule of $\mathbb{Z}^n$.

Lemma 8. *Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ with $\alpha \neq 0$.*
The following facts are equivalent:

(1) $\gcd\{\alpha_1, \dots, \alpha_n\} = 1$.
(2) *The $\mathbb{Z}$-module (α) is saturated (Definition 6).*

Moreover assume that $\alpha_{m+1} = \cdots = \alpha_n = 0$ for some $m \leq n$ and that (α) is saturated, then there is a surjective homomorphism $\psi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ with $\operatorname{Ker}\psi = (\alpha)$ and such that $\psi(e_j^{(n)}) = e_{j-1}^{(n-1)}$ for $j = m+1, \ldots, n$, where $e_j^{(n)} \in \mathbb{Z}^n$ is the j-th element of the canonical base of $\mathbb{Z}^n$.

Proof. The equivalence of (1) and (2) is an easy exercise.

The last assertion follows from the fact that the Smith normal form of the column matrix α is $(1, 0, \ldots, 0)$. There is a (non unique) invertible integer matrix P such that $P\alpha = (1, 0, \ldots, 0)$. In fact if $\alpha_{m+1} = \cdots = \alpha_n = 0$ then P may be chosen as follows

$$P = \begin{pmatrix} P' & 0 \\ 0 & I_{n-m} \end{pmatrix}.$$

Set A the $(n-1) \times n$ matrix obtained by deleting the first row of P. The matrix A defines the required homomorphism $\mathbb{Z}^n \to \mathbb{Z}^{n-1}$. □

Definition 9. Let $\mathbf{E} \subset \{1, \ldots, n\}$ be a subset. We define $\mathbb{Z}^n_{\mathbf{E}}$ and $\mathbb{Z}^n_{\mathbf{E}^+}$ to be

$$\mathbb{Z}^n_{\mathbf{E}} = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \mid \alpha_j \geq 0 \ \forall j \in \mathbf{E}\}$$
$$\mathbb{Z}^n_{\mathbf{E}^+} = \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n \mid \alpha_j > 0 \ \forall j \in \mathbf{E}\}.$$

Definition 10. Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $\mathbf{E} \subset \{1, \ldots, n\}$ be a subset.

We say that M is weak-transversal to $\mathbf{E}$ if M admits a system of $\mathbb{Z}$-generators $\alpha_1, \ldots, \alpha_\ell$ with $\alpha_j \in \mathbb{Z}^n_{\mathbf{E}}$ (Definition 9).

We say that a $\mathbb{Z}$-module $L \subset \mathbb{Z}^n$ is transversal to $\mathbf{E}$ if it is weak-transversal to $\mathbf{E}$ and L is saturated.

Definition 11. Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $\mathbf{E} \subset \{1, \ldots, n\}$ be a subset. Consider $pr_j : \mathbb{Z}^n \to \mathbb{Z}$ the j-projection, $j = 1, \ldots, n$. Set

$$\mathbf{E}_M = \{j \in \mathbf{E} \mid pr_j(M) \neq 0\}.$$

Remark 12. The set $\mathbf{E}_M$ depends on the module M, but we can reduce the study of transversality of M to the smaller subset $\mathbf{E}_M$. Note that for any generator system $\alpha_1, \ldots, \alpha_\ell$ of M we have that $\alpha_{i,j} = 0$ for all $i = 1, \ldots, \ell$ and any $j \in \mathbf{E} \setminus \mathbf{E}_M$.

Propositions 13 and 14 come from discussions with Ignacio Ojeda.

Proposition 13. *Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $\mathbf{E} \subset \{1, \ldots, n\}$ be a subset.*

The $\mathbb{Z}$-module M is weak transversal to $\mathbf{E}$ (Definition 10) if and only if M is weak transversal to $\mathbf{E}_M$.

Proof. One implication is obvious. Assume that M is weak transversal to $\mathbf{E}_M$. So that there exists a generator system $\alpha_1, \ldots, \alpha_\ell$ such that $\alpha_{i,j} \geq 0$ for $i = 1, \ldots, \ell$ and $j \in \mathbf{E}_M$. The result follows from Remark 12. □

Proposition 14. *Let $M \subset \mathbb{Z}^n$ be a $\mathbb{Z}$-submodule and $\mathbf{E} \subset \{1, \ldots, n\}$ be a subset.*

The $\mathbb{Z}$-module M is weak transversal to $\mathbf{E}$ if and only if there is $\gamma \in M \cap \mathbb{Z}^n_{\mathbf{E}^+_M}$.

Proof. Assume that M is weak transversal to E. There is a generator system $\alpha_1, \ldots, \alpha_\ell$ of M with $\alpha_i \in \mathbb{Z}^n_{\mathbf{E}}$, $i = 1, \ldots, \ell$.

Note that for any $j \in \mathbf{E}_M$ there is an index $i \in \{1, \ldots, \ell\}$ such that $\alpha_{i,j} > 0$. Set $\gamma = \alpha_1 + \cdots + \alpha_\ell$ and it is clear that $\gamma \in \mathbb{Z}^n_{\mathbf{E}^+_M}$.

Conversely, assume that there is $\gamma \in M \cap \mathbb{Z}^n_{\mathbf{E}^+_M}$. Consider $\beta_1, \ldots, \beta_\ell$ a generator system of M. There are integers $a_1, \ldots, a_\ell \in \mathbb{Z}$ with $\gamma = a_1\beta_1 + \cdots + a_\ell\beta_\ell$. We may assume that $\gcd\{a_1, \ldots, a_\ell\} = 1$. Note that we may complete γ to a generator system of M, say, $\gamma, \gamma_2, \ldots, \gamma_\ell$. This is a consequence of the fact that the Smith normal form of the row matrix $(a_1, \ldots, a_\ell)$ is $(1, 0, \ldots, 0)$.

Now we may choose positive integers $\lambda_2, \ldots, \lambda_\ell$ such that $\gamma_i + \lambda_i\gamma \in \mathbb{Z}^n_{\mathbf{E}}$. Set $\alpha_1 = \gamma$ and $\alpha_i = \gamma_i + \lambda_i\gamma$, $i = 2, \ldots, \ell$. It is clear that $\alpha_1, \ldots, \alpha_\ell$ is a generator system of M and $\alpha_i \in \mathbb{Z}^n_{\mathbf{E}}$, $i = 1, \ldots, \ell$. In fact we may assume that $\alpha_i \in \mathbb{Z}^n_{\mathbf{E}^+_M}$. □

Proposition 15. *Let* $\mathbf{E} \subset \{1, \ldots, n\}$ *and let* $M \subset \mathbb{Z}^n$ *be a* $\mathbb{Z}$*-module.*
If M *is weak-transversal to* $\mathbf{E}$ *then* $\mathrm{Sat}(M)$ *is transversal to* $\mathbf{E}$*.*

Proof. Note that $\mathbf{E}_M = \mathbf{E}_{\mathrm{Sat}(M)}$. Then Proposition 15 is a direct consequence of Propositions 13 and 14. □

Proposition 16. *Let* $\mathbf{E} \subset \{1, \ldots, n\}$ *be a subset. Let* $L \subset \mathbb{Z}^n$ *be a saturated* $\mathbb{Z}$*-submodule.*
There exists a unique $\mathbb{Z}$*-module* L_0 *such that*

- $L_0 \subset L$,
- L_0 *is transversal to* $\mathbf{E}$,
- *If* $L_0' \subset L$ *and* L_0' *is transversal to* $\mathbf{E}$ *then* $L_0' \subset L_0$.

Proof. Consider all $\mathbb{Z}$-submodules $\{M_\lambda\}_{\lambda \in \Lambda}$ such that $M_\lambda \subset L$ and M_λ is weak-transversal to $\mathbf{E}$ for every $\lambda \in \Lambda$.

Set $M = \sum_{\lambda \in \Lambda} M_\lambda$. Note that M is weak-transversal to $\mathbf{E}$ and $M \subset L$.

By Proposition 15 $L_0 = \mathrm{Sat}(M)$ is transversal to $\mathbf{E}$ and we have also that $M \subset L_0 \subset L$. In fact by construction $M = L_0$ and it is the biggest $\mathbb{Z}$-module with this property. □

2. Affine toric varieties and transversality

In this section we will prove the equivalence of the new notion of transversality of $\mathbb{Z}$-submodules (Definition 10) and the geometric usual notion of transversality of a variety with respect to a normal crossing divisor.

Let W be a regular affine toric variety of dimension n. It follows from Theorem 4 that $W \cong \mathbb{TA}^n_E$ (notation as in Remark 1). Recall that E is a set of regular hypersurfaces in W having only normal crossings. Using the isomorphism $W \cong \mathbb{TA}^n_E$ we may identify E with a set $\mathbf{E} \subset \{1, \ldots, n\}$. With this identification

$$W = \mathbb{TA}^n_E = \mathrm{Spec}(k[x_1, \ldots, x_n]_{\prod_{i \notin \mathbf{E}} x_i}).$$

The variety $W = \mathbb{TA}^n_E$ has a distinguished point $\xi_0 \in W$

$$\xi_0 \in \bigcap_{H \in E} H$$

with coordinates $\xi_0 = (\xi_{0,1}, \ldots, \xi_{0,n})$ where $\xi_{0,i} = 0$ if $i \in \mathbf{E}$ and $\xi_{0,i} = 1$ if $i \notin \mathbf{E}$.

Definition 17 (*Blanco, in press, 2011*)**.** Let W be a regular affine toric variety of dimension n and let $\mathcal{J} \subset \mathcal{O}_W$ be a sheaf of ideals. For any point $\xi \in W$ consider E_ξ the intersection of all hypersurfaces $H \in E$ with $\xi \in H$:

$$E_\xi = \bigcap_{\xi \in H \in E} H.$$

The ideal $I(E_\xi) \subset \mathcal{O}_W$ is generated by all the equations of hypersurfaces H with $\xi \in H \in E$.

We define the function $\text{E-ord}(J) : W \to \mathbb{N}$ as follows:

$$\text{E-ord}(J)(\xi) = \max\{b \in \mathbb{N} \mid J \subset I(E_\xi)^b\}$$

where J is an ideal in W.

Note that the function $\text{E-ord}(J)$ is constant along the strata defined by E. In fact $\text{E-ord}(J)(\xi)$ is the (usual) order of the ideal J at the generic point of E_ξ. The function $\text{E-ord}(J) : W \to \mathbb{N}$ is upper-semi-continuous, see Blanco (in press, 2011) for a proof and more details.

Definition 18. An ideal $J \subset \mathcal{O}_W$, $W = \mathbb{TA}^n_E$, is binomial if J can be generated, as ideal, by binomials: $x^\alpha - x^\beta$, with $\alpha, \beta \in \mathbb{N}^n$.

Lemma 19. *Let* $J \subset \mathcal{O}_W$ *be a binomial ideal,* $W = \mathbb{TA}^n_E$*, and let* $\xi_0 \in W$ *be the distinguished point.*
Then $\xi_0 \in \underline{\mathbf{Max}}\, \text{E-ord}(J) = \{\xi \in W \mid \text{E-ord}(J)(\xi) = \max \text{E-ord}(J)\}$.

Proof. Note that $E_{\xi_0} \subset E_\xi$ for any $\xi \in W$. □

The following definition is general for any variety.

Definition 20. Let $X \subset W$ be an embedded variety and let E be a set of regular hypersurfaces of W having only normal crossings.

We say that X is transversal to E at a point $\xi \in X$ if there is a regular system of parameters of $\mathcal{O}_{W,\xi}$, $x_1, \ldots, x_n \in \mathcal{O}_{W,\xi}$, such that

- $I(X)_\xi = (x_1, \ldots, x_r)$ for some $r \leq n$ and
- for all $H \in E$ with $\xi \in H$, then $I(H)_\xi = (x_i)$ for some i with $r < i \leq n$.

Consider $W = \mathbb{TA}^n_E$, for some $\mathbf{E} \subset \{1, \ldots, n\}$. The derivatives with poles along E is a free $\mathcal{O}_W$-module of rank n and a natural basis of this module is

$$x_i^{\epsilon_i} \frac{\partial}{\partial x_i} \qquad i = 1, \ldots, n$$

where $\epsilon_i = 0$ if $i \in \mathbf{E}$ and $\epsilon_i = 1$ if $i \notin \mathbf{E}$.

Lemma 21. *Let $X \subset W = \mathbb{TA}^n_E$ be an affine toric embedding with $d = \dim(X)$.*

Consider any set of binomial generators of the ideal $I(X) \subset \mathcal{O}_W$

$$I(X) = (x^{\alpha_1^+} - x^{\alpha_1^-}, \ldots, x^{\alpha_m^+} - x^{\alpha_m^-}) = (f_1, \ldots, f_m).$$

Fix a point $\xi \in X$. The variety X is transversal to E at the point ξ (Definition 20) if and only if the Jacobian matrix:

$$\left(x_i^{\epsilon_i} \frac{\partial f_j}{\partial x_i} \right)_{i,j}$$

has rank $n - d$ at the point ξ.

Proof. This lemma is a direct consequence of a general fact on algebraic varieties. □

Proposition 22. *Let $X \subset W = \mathbb{TA}^n_E$ be an affine toric embedding.*

If $\max \text{E-ord}(I(X)) > 0$ *then X is not transversal to E.*

Proof. It follows from Lemmas 21 and 19. At the distinguished point ξ_0, the Jacobian matrix in Lemma 21 is zero modulo the maximal ideal at ξ_0. □

Theorem 23. *Let $V \subset W = \mathbb{TA}^n_E$ be an affine toric embedding.*

They are equivalent:

(1) *V is transversal to E.*
(2) *The ideal $I(V)$ is generated by hyperbolic equations*

$$I(V) = (x^{\alpha_1} - 1, \ldots, x^{\alpha_\ell} - 1)$$

where $\ell = n - \dim V$, $\alpha_1, \ldots, \alpha_\ell \in \mathbb{Z}^n_{\mathbf{E}}$ and they generate a saturated lattice of rank ℓ.

Proof. Set $\ell = n - \dim V$.

Let $\alpha_1, \ldots, \alpha_\ell \in \mathbb{Z}^n_{\mathbf{E}}$ be such that $\alpha_1, \ldots, \alpha_\ell$ they generate a saturated lattice of rank ℓ. Assume that $I(V) = (x^{\alpha_1} - 1, \ldots, x^{\alpha_\ell} - 1)$. Consider the Jacobian matrix (Lemma 21)

$$\left(x_i \frac{\partial}{x_i} (x^{\alpha_j} - 1) \right) = \begin{pmatrix} \alpha_{1,1} x^{\alpha_1} & \cdots & \alpha_{\ell,1} x^{\alpha_\ell} \\ \vdots & & \vdots \\ \alpha_{1,n} x^{\alpha_1} & \cdots & \alpha_{\ell,n} x^{\alpha_\ell} \end{pmatrix}.$$

Note that the rank of this matrix at any point $\xi \in V$ is the rank of the matrix $(\alpha_1 | \cdots | \alpha_\ell)$ having α_i as columns. And the rank of this matrix is ℓ (independently of the characteristic of the ground field k). So that V is transversal to E.

Conversely assume that V is transversal to E. We may assume that $\mathbf{E} = \{r+1, \ldots, n\}$.

Let us show first the codimension one case: $\ell = 1$. So that $I(V) = (x^{\alpha_1^+} - x^{\alpha_1^-})$ where $\alpha_1 = \alpha_1^+ - \alpha_1^-$ and $\alpha_1^+, \alpha_1^- \in \mathbb{Z}^r \times \mathbb{N}^{n-r}$. By Proposition 22 we have that max E-ord$(I(V)) = 0$ so that we may assume that $I(V) = (x^{\alpha_1} - 1)$ with $\alpha_1 \in \mathbb{Z}^r \times \mathbb{N}^{n-r}$. Since V is a toric variety the ideal $I(V)$ is prime and $\gcd\{\alpha_{1,1}, \ldots, \alpha_{1,n}\} = 1$. Lemma 8 gives the result.

We now prove the general case, codimension $\ell > 1$. By Proposition 22 there is an hyperbolic equation $x^{\alpha_1} - 1 \in I(V)$. Since V is toric, we may assume that $\gcd\{\alpha_{1,1}, \ldots, \alpha_{1,n}\} = 1$. Set W_1 the toric hypersurface defined by $x^{\alpha_1} - 1$. After reordering the last $n-r$ coordinates we may assume that $\alpha_{1,m+1} = \cdots = \alpha_{1,n} = 0$ and $\alpha_{1,i} > 0$ for $i = r+1, \ldots, m$, where $r \le m$. We do not assume anything on $\alpha_{1,1}, \ldots, \alpha_{1,r}$.

Let $\psi : \mathbb{Z}^n \to \mathbb{Z}^{n-1}$ be the homomorphism given by Lemma 8. Note that $W_1 \cong \mathbb{T}^{m-1} \times \mathbb{A}^{n-m}$. We have $V \subset W_1$ and by induction there are $\bar{\beta}_2, \ldots, \bar{\beta}_\ell \in \mathbb{Z}^{m-1} \times \mathbb{N}^{n-m}$ such that the ideal of V in W_1 is generated by $y^{\bar{\beta}_2} - 1, \ldots, y^{\bar{\beta}_\ell} - 1$, and $\bar{\beta}_2, \ldots, \bar{\beta}_\ell$ generate a saturated lattice in $\mathbb{Z}^{n-1}$ of rank $\ell - 1$. Let $\beta_2, \ldots, \beta_\ell \in \mathbb{Z}^n$ such that $\psi(\beta_i) = \bar{\beta}_i$ for $i = 2, \ldots, \ell$. We have that $\alpha_1, \beta_2, \ldots, \beta_\ell$ generate a saturated lattice of rank ℓ.

It is clear that $\beta_i \in \mathbb{Z}^m \times \mathbb{N}^{n-m}$, but in general $\beta_i \notin \mathbb{Z}^r \times \mathbb{N}^{n-r}$. Since $\alpha_{1,i} > 0$ for $i = r+1, \ldots, m$ there are natural numbers $\lambda_2, \ldots, \lambda_\ell$ such that $\alpha_i = \beta_i + \lambda_i \alpha_1 \in \mathbb{Z}^r \times \mathbb{N}^{n-r}$, $i = 2, \ldots, \ell$. And $\alpha_1, \alpha_2, \ldots, \alpha_\ell$ generate the same lattice. Finally we have the equality of ideals

$$I(V) = (x^{\alpha_1} - 1, x^{\alpha_2} - 1, \ldots, x^{\alpha_\ell} - 1). \quad \square$$

Lemma 24. *Let $J = (x^{\beta_1} - 1, \ldots, x^{\beta_s} - 1)$ be an ideal generated by hyperbolic binomials, $\beta_i \in \mathbb{Z}^n_{\mathbf{E}}$, $i = 1, \ldots, s$. Assume that J is a prime ideal.*

If $\gamma \in \mathbb{Z}^n$ then $x^{\gamma^+} - x^{\gamma^-} \in J$ if and only if γ belongs to the $\mathbb{Z}$-module generated by $\beta_1, \ldots, \beta_s$ in $\mathbb{Z}^n$.

Proof. The $\mathbb{Z}$-module L generated by $(\beta_1, \ldots, \beta_s)$ is associated to the toric variety defined by J.

It is well known that $\beta \in L$ if and only if $x^{\beta^+} - x^{\beta^-} \in J$. $\square$

Proposition 25. *Let $V \subset \mathbb{T}\mathbb{A}^n_E$ be an affine toric embedding. Set $L \subset \mathbb{Z}^n$ be the lattice associated to V.*

V is transversal to E if and only if L is transversal to **E**.

Proof. It is a consequence of Theorem 23 and Lemma 24. $\square$

Theorem 26. *Let $X \subset W = \mathbb{T}\mathbb{A}^n_E$ be an affine toric embedding.*

There is a unique toric variety V such that the embeddings $X \subset V \subset W$ are toric and V is the minimum toric variety containing X and transversal to E.

Proof. It follows from Propositions 16 and 25. $\square$

3. Embedded toric varieties

In the previous sections we have reduced to the case of affine toric varieties. We generalize here to (non affine) toric varieties, and we define the first coordinate of our resolution function.

Let W be a regular toric variety defined by a fan Σ (Fulton, 1993). Let $T \subset W$ be the torus of W, which is open and dense in W. Set E the simple normal crossing divisor given by $W \setminus T$.

For every $\sigma \in \Sigma$ the open set $W_\sigma \subset W$ is an affine toric variety, so that $W_\sigma \cong \mathbb{T}\mathbb{A}^n_E$.

Definition 27. A toric embedding is a closed subscheme $X \subset W$ such that for every $\sigma \in \Sigma$ if $X_\sigma = X \cap W_\sigma$ then $X_\sigma \subset W_\sigma$ is an affine toric embedding (Definition 5).

For every $\sigma \in \Sigma$ there is a unique toric affine variety $V_\sigma \subset W_\sigma$ transversal to E and such that $X_\sigma \subset V_\sigma$ (Theorem 26).

In fact, this toric affine variety V_σ is a regular toric affine variety.

Remark 28. Note that for any $\xi \in W$, there is a unique $\sigma \in \Sigma$ such that $\xi \in W_\sigma$ and the affine open set W_σ is minimum with this property. In fact ξ belongs to the orbit of the distinguished point of W_σ.

Definition 29. Let $\xi \in X \subset W$ be a point. Let X_σ be the minimum affine open set containing the point ξ.

The hyperbolic codimension of X at ξ is

$$\mathrm{Hcodim}(X)(\xi) = \dim V_\sigma - \dim X$$

where $V_\sigma \subset W_\sigma$ is the minimum toric affine variety such that $V_\sigma \supset X_\sigma$ and it is transversal to E (Theorem 26).

Remark 30. The hyperbolic codimension $\mathrm{Hcodim}(X)$ (Definition 29) can be understood as a toric embedding dimension. The number $\mathrm{Hcodim}(X)$ at ξ is the minimum dimension of a regular toric variety V including X.

In the case $V_\sigma = W_\sigma$, then $\mathrm{Hcodim}(X)(\xi) = \mathrm{codim}_W(X)$, the codimension of X in W.

Remark 31 (*Bierstone and Milman, 2006*)**.** Let $\Delta \in \Sigma$ be an element of the fan Σ defining the regular variety W. The cone Δ defines a smooth closed subvariety $Z_\Delta \subset W$ as follows:

The toric variety W is covered by affine toric varieties W_σ with $\sigma \in \Sigma$. So that Z_Δ is covered by affine pieces $(Z_\Delta)_\sigma = Z_\Delta \cap W_\sigma, \sigma \in \Sigma$.

If Δ is not a face of σ then $(Z_\Delta)_\sigma = \emptyset$.

If Δ is a face of σ, note that $W_\Delta \subset W_\sigma$ is an open inclusion. Then $(Z_\Delta)_\sigma$ is the (closure) of the orbit of the distinguished point of W_Δ.

The smooth closed center Z_Δ we will say that it is a *combinatorial center* of W. In fact note that at every affine chart $W_\sigma \cong \mathbb{TA}^n_E$, for some E, the combinatorial center $Z_\Delta \cap W_\sigma$ is defined by some coordinates x_i with $i \in \mathbf{E}$.

Remark 32. Note that if $X \subset W \cong \mathbb{TA}^n_E$ is an affine toric embedding and $Z_\Delta \subset W$ is a combinatorial center, then the strict transforms $X' \subset W'$ give an affine toric embedding.

Proposition 33. *Let $\Delta \in \Sigma$ and $Z_\Delta \subset W$ the combinatorial center associated to Δ (Remark 31).*

Let $W' \to W$ be the blow-up with center Z_Δ. Set $X' \subset W'$ the strict transform of X. If $\xi' \in X'$ then

$$\mathrm{Hcodim}(X')(\xi') \leq \mathrm{Hcodim}(X)(\xi)$$

where ξ' maps to ξ.

Proof. Let W_σ be the minimum affine open set of W containing the point ξ and let V_σ be the minimum toric affine variety in W_σ such that $V_\sigma \supset X_\sigma$ and it is transversal to E (Theorem 26).

Let $W'_{\sigma'}$, with $\sigma' \in \Sigma'$, be the minimum affine open set of W' containing the point ξ'. Let $V'_{\sigma'}$ be the minimum toric affine variety such that $V'_{\sigma'} \supset X'_{\sigma'}$ and it is transversal to E'. Note that $X'_{\sigma'} \subset (X_\sigma)'$ is an open immersion, where $(X_\sigma)' \subset X'$ is the strict transform of $X_\sigma \subset X$.

Let $(V_\sigma)'$ be the strict transform of V_σ. Note that $(V_\sigma)'$ is smooth and transversal to E'. So that $(V_\sigma)' \cap W'_{\sigma'} \supset V'_{\sigma'}$. And the result follows from the last inclusion. □

4. *E*-resolution of binomial ideals

In Blanco (in press, 2011) were given some notions in terms of *binomial basic objects along E*, where E was a normal crossing divisor in the ambient space W. In terms of the E-ord (Definition 17) one may construct a sequence of combinatorial blowing-ups such that the transform of a given binomial ideal has maximal E-order equal to zero.

We remind here the main results. For more details on the several constructions and proofs, see Blanco (in press) and Blanco (2011). All these notions work for general binomial ideals, without any restriction.

Using this structure of binomial basic object along E, and the language of *mobiles* (see Encinas and Hauser (2002)), it is possible to construct a resolution function involving the E-order of certain ideals computed by induction on the dimension of W.

Remark 34. Roughly speaking, given $(W, (J, c), H, E)$, where J is a binomial ideal, c is a positive integer, and H is the set of exceptional hypersurfaces, by induction on the dimension of W, construct ideals J_i defined in local flags $W = W_n \supseteq W_{n-1} \supseteq \cdots \supseteq W_i \supseteq \cdots \supseteq W_1$, and then objects $(W_i, (J_i, c_{i+1}), H_i, E_i)$ in dimension i, where each $E_i = W_i \cap E$, $H_i = W_i \cap H$. The integer numbers c_{i+1} are computed as the E-order of certain ideals P_{i+1} coming from the previous dimension $i + 1$, that is $c_{i+1} = \max\ \text{E-ord}(P_{i+1})$ is the *critical value* in dimension i. Denote $c_{n+1} = c$.

If the *E-singular locus* of (J_i, c_{i+1}) is non empty, then factorize the ideal $J_i = M_i \cdot I_i$, where each ideal M_i is defined by a normal crossing divisor supported by the current exceptional locus H_i.

To make this induction on the dimension of W, in Blanco (in press, 2011) the existence of hypersurfaces of *E-maximal contact* at any stage of the resolution process is proved. These hypersurfaces are always coordinate hyperplanes, and produce a combinatorial center to be blown up. Combinatorial centers are convenient to preserve the binomial structure of the ideal after blow-up.

Definition 35. A *binomial basic object along E* is a tuple $B = (W, (\mathcal{J}, c), H, E)$ where

- W is a regular toric variety defined by a fan Σ.
- E is the simple normal crossing divisor given by $W \setminus T$, where $T \subset W$ is the torus of W.
- $(\mathcal{J}, c)$ is a *binomial pair*, this means that $\mathcal{J} \subset \mathcal{O}_W$ is a coherent sheaf of binomial ideals with respect to E, and c is a positive integer number. Note that for any $\sigma \in \Sigma$ the sheaf of ideals $\mathcal{J}$ restricted to the open affine subset $W_\sigma \subset W$ is a binomial ideal $J \neq 0$ in $k[x_1, \ldots, x_n]_{\prod_{i \notin \mathbf{E}} x_i}$.
- $H \subset E$ is a set of normal crossing regular hypersurfaces in W.

Definition 36. Let $J \subset \mathcal{O}_W$ be a binomial ideal, c a positive integer. We call *E-singular locus* of J with respect to c to the set,

$$E\text{-Sing}(J, c) = \{\xi \in W /\ \text{E-ord}_\xi(J) \geq c\}.$$

Remark 37. The E-singular locus is a closed subset of W.

Definition 38. Let $J \subset \mathcal{O}_W$ be a binomial ideal. Let $\xi \in W$ be a point such that $\text{E-ord}_\xi(J) = \max\ \text{E-ord}(J) = \theta_E$. A hypersurface V is said to be a hypersurface of *E-maximal contact* for J at the point ξ if

- V is a regular hypersurface, $\xi \in V$,
- $E\text{-Sing}(J, \theta_E) \subseteq V$ and their transforms under blowing up with a combinatorial center $Z_\Delta \subset V$ also satisfy $E\text{-Sing}(J', \theta_E) \subseteq V'$, whereas the E-order, θ_E remains constant.
  That is, $\text{E-ord}_\xi(J') = \text{E-ord}_\xi(J)$, where J' is the *controlled* transform of J and V' is the strict transform of V.

Remark 39. The controlled transform of J is the ideal $J' = I(Y')^{-\theta_E} \cdot J^*$ where Y' is the exceptional divisor and J^* is the total transform of J under blowing up.

Proposition 40. *Let $J \subset \mathcal{O}_W$ be a binomial ideal. There exists a hypersurface of E-maximal contact for J.*

Definition 41. Let $(W, (J, c), H, E)$ be a binomial basic object along E. For all points $\xi \in E\text{-Sing}(J, c)$ the resolution function $E\text{-inv}_{(J,c)}$ will have n components with lexicographical order, and will be of one of the following types:

$$E\text{-inv}_{(J,c)}(\xi) = \begin{cases} \left(\frac{\text{E-ord}_\xi(I_n)}{c_{n+1}}, \frac{\text{E-ord}_\xi(I_{n-1})}{c_n}, \ldots, \frac{\text{E-ord}_\xi(I_{n-r})}{c_{n-r+1}}, \infty, \infty, \ldots, \infty\right) & (a) \\ \left(\frac{\text{E-ord}_\xi(I_n)}{c_{n+1}}, \frac{\text{E-ord}_\xi(I_{n-1})}{c_n}, \ldots, \frac{\text{E-ord}_\xi(I_{n-r})}{c_{n-r+1}}, \Gamma(\xi), \infty, \ldots, \infty\right) & (b) \\ \left(\frac{\text{E-ord}_\xi(I_n)}{c_{n+1}}, \frac{\text{E-ord}_\xi(I_{n-1})}{c_n}, \ldots, \frac{\text{E-ord}_\xi(I_{n-r})}{c_{n-r+1}}, \ldots, \frac{\text{E-ord}_\xi(I_1)}{c_2}\right) & (c) \end{cases}$$

where the ideals I_i and the integer numbers c_i are as in Remark 34.

In the case $J_i = 1$, for some $i < n$, define $(E\text{-inv}_{(J,c)_i}(\xi), \ldots, E\text{-inv}_{(J,c)_1}(\xi)) = (\infty, \ldots, \infty)$ in order to preserve the number of components.

If $\text{E-ord}_\xi(I_i) = 0$, for some $i < n$, then $E\text{-inv}_{(J,c)_i}(\xi) = \Gamma(\xi)$, where Γ is the resolution function corresponding to the *monomial case*, see Encinas and Villamayor (2000). And complete the resolution function with the needed number of ∞ components.

Remark 42. The $E\text{-inv}_{(J,c)}$ function is an upper-semi-continuous function, see Blanco (in press, 2011). We also denote $E\text{-inv}_{(J,c)}(\xi) = E\text{-inv}_\xi(J, c)$.

As a consequence of the upper-semi-continuity of the $E\text{-inv}_{(J,c)}$ function,

$$\underline{\mathbf{Max}}(E\text{-inv}_{(J,c)}) = \{\xi \in E\text{-Sing}(J, c) |\ E\text{-inv}_{(J,c)}(\xi) = \max\ E\text{-inv}_{(J,c)}\}$$

is a closed set. In fact, it is the center of the next blow-up.

It can be proven that the $E\text{-inv}_{(J,c)}$ function drops lexicographically after blow-up, (Blanco, in press, 2011).

Lemma 43. *Let $(W, (J, c), H, E)$ be a binomial basic object along E. Let $W \stackrel{\pi}{\leftarrow} W'$ be a blow-up with combinatorial center $Z_\Delta = \underline{\mathbf{Max}}(E\text{-inv}_{(J,c)})$ then*

$$E\text{-inv}_{(J,c)}(\xi) > E\text{-inv}_{(J',c)}(\xi')$$

where $\xi \in Z_\Delta$, $\xi' \in Y' = \pi^{-1}(Z_\Delta)$, $\pi(\xi') = \xi$.

The function $E\text{-inv}_{(J,c)}$ is the resolution function associated to the binomial basic object along E given by $(W, (J, c), H, E)$, and $E\text{-inv}_{(J',c)}$ corresponds to its transform by the blow-up π, $(W', (J', c), H', E')$.

Proof. See Blanco (in press, 2011). □

Remark 44. The $E\text{-inv}_{(J,c)}$ function provides an *E-resolution* of the binomial basic object along E, $(W, (J, c), H, E)$.

Definition 45. Let $(W, (J, c), H, E)$ be a binomial basic object along E, where $E = \{E_1, \ldots, E_r\}$, with $r \le n = \dim W$. Let $H = \{H_1, \ldots, H_s\} \subset E$ be the set of exceptional divisors, for some $s \le r$.

We define a *transformation* of the binomial basic object

$$(W, (J, c), H, E) \leftarrow (W', (J', c), H', E')$$

by means of the blowing up $W \stackrel{\pi}{\leftarrow} W'$, in a center $Z \subset E\text{-Sing}(J, c)$, with

- $H' = \{H_1^\curlyvee, \ldots, H_s^\curlyvee, Y'\}$ where $H_i^\curlyvee, i = 1, \ldots, s$, is the strict transform of H_i and Y' is the exceptional divisor in W'.
- $E' = \{E_1^\curlyvee, \ldots, E_r^\curlyvee, Y'\}$ where $E_i^\curlyvee, i = 1, \ldots, r$, is the strict transform of E_i and Y' is the exceptional divisor in W'.
- $J' = I(Y')^{-c} \cdot J^*$ is the controlled transform of J, where J^* is the total transform of J.

Definition 46. A sequence of transformations of binomial basic objects

$$(W^{(0)}, (J^{(0)}, c), H^{(0)}, E^{(0)}) \leftarrow (W^{(1)}, (J^{(1)}, c), H^{(1)}, E^{(1)}) \leftarrow \cdots \leftarrow (W^{(N)}, (J^{(N)}, c), H^{(N)}, E^{(N)}) \quad (1)$$

is an *E-resolution* of $(W^{(0)}, (J^{(0)}, c), H^{(0)}, E^{(0)})$, or simply an *$E$-resolution* of the pair $(J^{(0)}, c)$, if $E\text{-Sing}(J^{(N)}, c) = \emptyset$.

This E-resolution function, the $E\text{-inv}_{(J,c)}$, works for general binomial ideals, without any restriction, for more details see Blanco (in press, 2011).

The E-resolution constructed in this way is independent of the choice of coordinates.

In Blanco (in press, 2011) was proved that one may use this E-resolution in order to construct an algorithm of log-resolution of binomial ideals and embedded desingularization of binomial varieties. But this algorithm of log-resolution depends on a choice of a Gröbner basis of the original binomial ideal.

In the next section we will construct an Algorithm 1 of embedded desingularization of toric varieties which is independent of the choice of coordinates.

Theorem 47. *Let J be a binomial ideal. If $E\text{-Sing}(J, c) \neq \emptyset$ then there exists an E-resolution of (J, c).*

Proof. The E-resolution of (J, c) is given by the $E\text{-inv}_{(J,c)}$ function, such that $E\text{-Sing}(J^\curlyvee, c) = \emptyset$, this means $\max \text{E-ord}(J^\curlyvee) = 0$. □

Remark 48. At any stage of the E-resolution process, the $E\text{-inv}_{(J,c)}$ function determines the next combinatorial center to be blown-up $Z_\Delta = \underline{\mathbf{Max}}(E\text{-inv}_{(J,c)})$, or equivalently, the cone Δ, (Remark 31).

5. Embedded desingularization

Now we construct an algorithm of embedded desingularization of toric varieties. This algorithm is defined in terms of the function Hcodim and a E-resolution of a suitable ideal, depending on the function E-inv.

Proposition 49. *Let $X \subset W$ be a toric embedding and let $\xi \in X$ be a point. Let X_σ be the minimum affine open set containing the point ξ and let V_σ be the minimum toric affine variety such that $V_\sigma \supset X_\sigma$ and it is transversal to E.*

At any affine open subset, $X_\sigma \subset V_\sigma \subset W_\sigma$. Let $W_\sigma \stackrel{\pi}{\leftarrow} W'_\sigma$ be a blow-up with center Z_Δ then

$$(\mathrm{Hcodim}(X)(\xi), E\text{-inv}_\xi(I_{V_\sigma}(X_\sigma), c)) > (\mathrm{Hcodim}(X')(\xi'), E\text{-inv}_{\xi'}(I_{V'_{\sigma'}}(X'_{\sigma'}), c))$$

where $\xi \in Z_\Delta, \xi' \in Y' = \pi^{-1}(Z_\Delta), \pi(\xi') = \xi, I_{V_\sigma}(X_\sigma)$ is the ideal of X in V, and $c = \max \mathrm{E\text{-}ord}(I_{V_\sigma}(X_\sigma))$.

Proof. It follows from Proposition 33 and Lemma 43. □

Algorithm 1. Let $X \subset W$ be a toric embedding and let $\xi \in X$ be a point. Let W_σ be the minimum affine open set containing the point ξ (Remark 28).

Input: $X \subset W$ a toric embedding.

(1) Compute V_σ, the minimum toric affine variety such that $V_\sigma \supset X_\sigma$ and it is transversal to E (Theorem 26).

(2) Set $\mathrm{Hcodim}(X)(\xi) = \dim V_\sigma - \dim X$.
- If $\mathrm{Hcodim}(X)(\xi) > 0$ then compute $E\text{-inv}_\xi(I_{V_\sigma}(X_\sigma), 1)$, here we set $c = 1$. This determines Z_Δ. Go to step (3).
- If $\mathrm{Hcodim}(X)(\xi) = 0$ the algorithm stops. Note that in this case $V_\sigma = X_\sigma \subset X$.

(3) Perform the blow-up with center Z_Δ and go to step (1).

Output: A collection of affine charts X^j, where each X^j is regular and transversal to E^j.

Correctness of the algorithm follows by construction (Definition 29 and Proposition 33). Termination of the algorithm follows from (Proposition 49) and (Theorem 47).

Remark 50. The algorithm of embedded desingularization given in Blanco (in press) depends on the choice of a system of coordinates. Note that the new Algorithm 1 given here does not depend on the coordinates election. Complexity of this algorithm and also for algorithm in Blanco (in press, 2011) has not been studied yet. Even if all centers are combinatorial there is not an estimation of how long computations are.

Theorem 51 (*Embedded Desingularization*). *Let $X \subset W$ be a toric embedding (Definition 27). Let E be the simple normal crossing divisor given by $W \setminus T$, where $T \subset W$ is the torus of W.*

There exists a sequence of transformations of pairs

$$(W, E) \leftarrow (W^{(1)}, E^{(1)}) \leftarrow \cdots \leftarrow (W^{(N)}, E^{(N)})$$

which induces a proper birational morphism $\Pi : W^{(N)} \to W$ such that

(1) *The restriction of this morphism Π to the* regular locus of X along E, *defines an isomorphism*

$$Reg_E(X) \cong \Pi^{-1}(Reg_E(X)) \subset W^{(N)}$$

where $Reg_E(X) = \{\xi \in X \mid X$ is regular at ξ and has normal crossings with $E\}$.

(2) *$X^{(N)}$, the strict transform of X in $W^{(N)}$, is regular and has normal crossings with the exceptional divisors $E^{(N)}$.*

Proof. It follows from correctness and termination of Algorithm 1. □

The embedded desingularization of Theorem 51 can be implemented, since the key points are to determine the stratum E_ξ (or the open subset W_σ) where the hyperbolic codimension is maximum, and then to compute a combinatorial blow-up, that can be easily encoded in the computer.

Remark 52. Theorem 51 may be used to achieve a log-resolution of a toric ideal.

With the notation of Theorem 51, let $W^{(N+1)} \to W^{(N)}$ be the blowing up with center $X^{(N)}$, which is a permissible center. Note that the total transform of $I(X)\mathcal{O}_{W^{(N+1)}}$ is locally a monomial ideal (generated by monomials) and we may use an algorithm of log-resolution of monomial ideals as in Goward (2005) or Bierstone and Milman (2006). So that log-resolution of the ideal $I(X)$ follows from Theorem 51 and log-resolution of monomial ideals.

Example 53. We give here an example of the Algorithm 1. All the computations are made by hand.

Input: Let $X \subset W = \mathbb{A}^4$ be a toric embedding and let $\xi \in X$ be a point. Let $E = \{V(x), V(y), V(z), V(w)\}$ be the simple normal crossing divisor given by $W \setminus T$. The toric variety X is given by the equations

$$X = \{x^2 - y^3 = 0\} \cap \{xyz - w^2 = 0\}.$$

The singular locus of the surface X is the z-axis. Compute the hyperbolic codimension at some points of X (steps 1 and 2 of the algorithm), for example:

- If $\xi \notin E$, then in a neighborhood of ξ, $W_\sigma = Spec(k[x^\pm, y^\pm, z^\pm, w^\pm])$ and $E_\xi = \emptyset$. The minimum toric affine variety V_σ such that $V_\sigma \supset X_\sigma$ and it is transversal to E_ξ is $V_\sigma = X_\sigma$. Then $\text{Hcodim}(X)(\xi) = 0$.
- Set $(V(x))^c = W \setminus V(x)$ be the complement of $V(x)$. If $\xi \in (V(x))^c \cap (V(y))^c \cap V(z) \cap V(w)$, then $W_\sigma = Spec(k[x^\pm, y^\pm, z, w])$ and $E_\xi = V(z) \cap V(w)$. It is clear that $V_\sigma = \{x^2 - y^3 = 0\}$ and therefore $\text{Hcodim}(X)(\xi) = 3 - 2 = 1$.
- If $\xi \neq 0$ is a point at the z-axis, $W_\sigma = Spec(k[x, y, z^\pm, w])$ and $E_\xi = V(x) \cap V(y) \cap V(w)$. Assume ξ is the distinguished point of W_σ, in this case $V_\sigma = W_\sigma$ and $\text{Hcodim}(X)(\xi) = 4 - 2 = 2$.
- If ξ is the origin, $W_\sigma = Spec(k[x, y, z, w])$ and $E_\xi = V(x) \cap V(y) \cap V(z) \cap V(w)$. The minimum toric affine variety $V_\sigma = W_\sigma$ and $\text{Hcodim}(X)(\xi) = 4 - 2 = 2$.

It is easy to check that the hyperbolic codimension Hcodim attains its highest value along the z-axis.

If one computes the whole resolution function (Hcodim, E-inv) (here we set $c = 1$) then its maximum value is

$$\max(\text{Hcodim}(X), E\text{-inv}) = (\text{Hcodim}(X)(0), E\text{-inv}(I_{V_\sigma}(X_\sigma), 1)) = (2, 2, 1, 3/2, 2)$$

and it is achieved at the origin, which is the first center to be blown-up.

We denote as *x-th chart* the chart where we divide by x. For simplicity, we will denote each $\frac{y}{x}, \frac{z}{x}, \frac{w}{x}$ again as y, z, w. (Step 3 of the algorithm.)

At the x-th chart, the controlled transform of the ideal $I(X)$ is

$$I(X)' = x^{-1} \cdot (x^2 - x^3y^3, x^3yz - x^2w^2) = x \cdot (1 - xy^3, xyz - w^2).$$

If η' is a point that maps to the origin then $\eta' \notin X'$ and it lies in the first exceptional divisor $V(x)$. Consider $\xi' \in X' \cap V(z) \cap V(w)$, the affine chart $W'_{\sigma'}$ associated to ξ' as in (Remark 28) is $W'_{\sigma'} = Spec(k[x^\pm, y^\pm, z, w])$ and $E'_{\xi'} = V(z) \cap V(w)$. The minimum toric affine variety $V'_{\sigma'}$ containing X' is $V'_{\sigma'} = \{xy^3 - 1 = 0\}$ and $\text{Hcodim}(X')(\xi') = 3 - 2 = 1$. The maximum value of the resolution function is

$$\max(\text{Hcodim}(X'), E\text{-inv}) = (\text{Hcodim}(X')(\xi'), E\text{-inv}_{\xi'}(I_{V'_{\sigma'}}(X'_{\sigma'}), 1)) = (1, 1, 2, \infty, \infty)$$

and it is reached along $Z' = \{z = 0\} \cap \{w = 0\} \cap \{xy^3 - 1 = 0\}$. Inside $V'_{\sigma'}$, the center is given by coordinates, $Z'_\Delta = \{z = 0\} \cap \{w = 0\}$, which is the next combinatorial center to be blown-up. After the blow-up at Z'_Δ, we consider the w-th chart

$$I(X)'' = w^{-1} \cdot (xyzw - w^2) = (xyz - w) \bmod I(V'_{\sigma'}),$$

this means $I(X)'' = (xy^3 - 1, xyz - w)$.

Let $\xi'' \in X''$ mapping to ξ'. At this stage of the resolution process, the maximum value of the resolution function is

$$\max(\text{Hcodim}(X''), E\text{-inv}) = (\text{Hcodim}(X'')(\xi''), E\text{-inv}_{\xi''}(I_{V''_{\sigma''}}(X''_{\sigma''}), 1)) = (1, 1, 1, \infty, \infty)$$

and it is reached along $Z'' = \{z = 0\} \cap \{w = 0\} \cap \{xy^3 - 1 = 0\}$. After the blowing-up at Z''_Δ we obtain two charts and for both max $\text{Hcodim}(X''') = 0$. And X''' is regular and transversal to E'''.

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