Computers and Mathematics with Applications 59 (2010) 2490–2509

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

# Information transmittal, principle of relativity and mass–energy relation

E.A. Galperin

*Departement de mathematiques, Universite du Quebec a Montreal, C.P. 8888, Succ. Centre Ville, Montreal, Quebec H3C 3P8, Canada*

ARTICLE INFO

*Article history:*
Received 25 November 2009
Accepted 22 December 2009

*Keywords:*
Principle of relativity
Concept of mass
The mass–energy relation

ABSTRACT

Relativistic transformations and their application to electrodynamics of a weakly accelerated electron considered in [A. Einstein, Zur Elektrodynamik der bewegter Körper, Annalen der Physik, 17 (1905) 891–921] are further analyzed in their relation to the principle of relativity, the concept of mass, and the mass–energy equation. Alternative consideration in dynamics of weakly accelerated electrons demonstrates that the factors in longitudinal $\mu\beta^3$ and transverse $\mu\beta^2$ masses of an electron with the mass $\mu$ at rest naturally appear in *observed accelerations* as a result of relativistic transformations, without any deviation from the principle of relativity and the second Newton's law of motion. As concerns the special relativity in accelerated motion, this allows us to retain the concept of mass as scalar characteristic of an accelerated body. It is argued that "the principle of equivalence of the mass and energy of rest" (Einstein) depends on the speed of the information transmitting signal by which the observation (measurement) is made (synchronization of clocks is achieved), so that relativistic equation $E = mc^2$ appears as an *image* phenomenon which essentially depends on the propagation of light as a measuring signal and on its speed $V = c$ as the critical parameter of Einstein's relativistic transformations. It is demonstrated that the spherical waves considered by Einstein are distorted in real time, and the fundamental Lorentz invariant is not crisp, but presents a soft interval whose diameter is in the range of 30,000 km for time delays of 0.1 s in transmission of information. The results open new avenues for further research in the theory of relativity and its applications.



## 1. Introduction

Following Einstein's investigation of the impact of special relativity on the concepts of mass and energy and their interrelation for motions in electromagnetic field, some further considerations are presented regarding those fundamental notions and their relativistic interpretation. On this basis, relativistic application of basic principles and laws of physics and mechanics are discussed to clarify some points of interest and importance in theoretical studies of relativistic phenomena and their representations. Interpretation of the fundamental invariant of Lorentz transformations (linear element in 4D geometry) in terms of conservation of the spherical form of waves of light considered by Einstein in isotropic media is justified. It is demonstrated that time uncertainty due to a finite speed of information transmittal perturbs and distorts the spherical form of the waves making the Lorentz invariant *noncrisp* and transforming the 4D geometry into a set-wise affinely connected 4D space structure with the range of coordinate uncertainty about 30,000 km for time delays of the order $\delta = 0.1$s Finally, the role of physical dimension (unit system structure) and the negative effects of very high particle velocities are discussed in regard to the CERN Large Hadron Collider.

The paper is organized as follows. In Section 2, Einstein's definition of simultaneity and his formulation of the principles of relativity and of the constancy of the speed of light are reproduced in quotations from his basic paper [1, Sections 1–2]. Section 3 contains the time transformation derived by Einstein, and an alternative derivation with the use of a linear

*E-mail address:* galperin.efim@uqam.ca.

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doi:10.1016/j.camwa.2009.12.028

function with undetermined coefficients. Section 4 presents Einstein's coordinate transformations from [1, Sections 3,4] with a brief discussion of time and length contraction phenomena, and the $\mu$-meson example. Section 5 contains relativistic transformation of the Maxwell–Hertz equations reproduced from [1, Sections 6,9]. In Section 6, the dynamics of a weakly accelerated electron [1, Section 10] is revisited, and it is demonstrated that relativistic transformations affect the accelerations but *not* the mass as proposed by Einstein. This allows us to preserve the mass as a scalar characteristic of a body, in full compliance with the second law of Newton and the principle of relativity. In Section 7, the mass–energy relation is discussed following [2] for the case of *radiation at rest,* without losing kinetic energy. Section 8 presents the mass–energy relation following [3], in conjunction with the laws of conservation of energy and momentum for the motion of a system of masses in electromagnetic field, and its brief discussion regarding the mix of coordinates and possible directional properties of the energy. In Section 9, the Lorentz transformations are discussed in comparison with the Einstein relativistic transformations and the wave propagation process in real time, leading to the soft set structure of the 4D geometry. In Section 10, the mass–energy relation is further discussed with respect to the rule of dimension and to the safety in particle collisions. Section 11 presents concluding remarks about some general principles and notions involved in relativistic description of physical realities followed by the references immediately relative to the problems considered.

## 2. Definition of simultaneity [1, Sections 1,2]

This is the title of the first section from which we reproduce the original Einstein's description of time and simultaneity in English translation from the Russian edition [4, pp. 8–10]. For a coordinate system "in which are valid the equations of mechanics of Newton," called "still system", or system at rest, the following is written.

"When desired to describe a *motion* of a material point, we specify the values of its coordinates as functions of time. Thereby it should be noted that such mathematical description has physical sense only if it is first understood what is meant by "time". We should pay attention to the fact that all our considerations in which time plays a role are always the considerations about *simultaneous* events." Then we read on page 9 of [4]:

"If at point A of a space there is a clock, then an observer at A can establish the time of events in immediate proximity of A by observing the simultaneous with those events positions of hands of the clock. If at another point B of the space there is also a clock (we add "identical to the one at A"), then in immediate proximity of B it is also possible to make time estimate of events by an observer at B. However, it is impossible without further hypotheses to compare timing of an event at A with an event at B; we have yet defined only "A-time" and "B-time" but not the common for A and B "time". The latter can be established *by introducing a definition* that "time" necessary for passing of a ray of light from A to B is equal to "time" necessary for passing of a ray of light from B to A. Consider that at a moment $t_A$ of "A-time" a ray of light leaves from A to B and is reflected at a moment $t_B$ of "B-time" from B to A returning back at A at a moment $t'_A$ of "A-time". The clocks at A and B will be, by definition, synchronized, if

$$t_B - t_A = t'_A - t_B. \tag{1}$$

We assume that this definition of synchronization can be made in a non-contradictory manner, and furthermore, for as many points as desired, thus, the following statements are valid:

(1) if the clock at B is synchronized with the clock at A, then the clock at A is synchronized with the clock at B;

(2) if the clock at A is synchronized with the clock at B and with the clock at C, then the clocks at B and C are also synchronized with respect to each other.

Thus, using certain (thoughtful) physical experiments, we have established what should be understood as synchronized located in different places still clocks, and thereby we evidently achieved definitions of the concepts: "simultaneity" and "time". "Time" of an event means simultaneous with the event indication of a still clock which is located at the place of the event and which is synchronized with certain still clock, thereby with one and the same clock under all definitions of time.

According to experiments, we also assume that the value

$$2\mathrm{AB}/(t'_A - t_A) = V \quad (\text{AB is the length of a segment}) \tag{2}$$

is a universal constant (the speed of light in vacuum).

It is essential that we have defined time with the help of still clocks in a system at rest; we shall call this time that belongs to a system at rest, "the time of still system".

Further considerations are based on the principle of relativity and on the principle of constancy of the speed of light. We formulate both principles as follows.

1. Laws which govern the changes of state of physical systems do not depend on which of the two coordinate systems, moving with respect to each other with a constant speed along a right line, these changes relate.

2. Every ray of light propagates in a "still" system of coordinates with certain speed V irrespective of whether the ray of light is issued by a resting or moving source.

Thereby, formula (2) applies, and the "segment of time" should be understood in the sense of the above definition."

## 3. Derivation of the time transformation [1, Section 3]

We now quote passages from [4, pp. 13–14] related to the theory of time transformation. "Consider in a "still" space two 3D Cartesian frames with a common origin and parallel axes, each equipped with scales and clocks which are identical in both frames. Now, let the origin of one of those frames $(k)$ be in motion with a constant speed $v$ in direction of increasing $x$ of the other frame $(K)$ which is at rest. Then, to each moment $t$ of still frame $(K)$ corresponds certain position of axes of moving frame $(k)$ whose axes can be assumed parallel to the axes of still frame $(K)$.

Let the space in the still frame $(K)$ be graduated with its scale at rest, and same for the space in the moving frame $(k)$ graduated with its scale, at rest with respect to $(k)$, yielding coordinates $x, y, z$ in $(K)$ and $\xi, \eta, \zeta$ in $(k)$. Using light signals as described in [1, Section 1], see above, let us define time $t$ in $(K)$ and $\tau$ in $(k)$ with the clocks at rest in each frame.

In this way, to the values $x, y, z, t$ which define the place and time of an event in the still frame $(K)$, there will correspond the values $\xi, \eta, \zeta, \tau$ that define the same event in the moving frame $(k)$, and we have to find the system of equations that link those values of coordinates and times.

First of all, it is clear that those equations must be *linear* according to the property of homogeneity which we ascribe to the space and time.

If we denote $x' = x - vt$, then it is clear that to a point at rest in the system $(k)$ will correspond certain, independent of time values $x', y, z$. Let us determine $\tau$ as function of $x', y, z, t$, which would mean that $\tau$ corresponds to the readings of clocks at rest in the moving frame $(k)$ synchronized with the clocks in the still frame $(K)$ by the rule (1)."

Choosing in (1) the point A as the origin of the moving frame $(k)$ and sending at the moment $\tau_0 = t_A$ a ray of light along the $X$-axis to the point $x'$ (point B) which ray is reflected back at the moment $\tau_1 = t_B$ to the origin where it comes at the moment $\tau_2 = t'_A$, we have from (1) the following equation: $\tau_1 - \tau_0 = \tau_2 - \tau_1$ which is written in [1, Section 3], quote from [4, p. 14, the first equation], in the form:

$$\text{“}0.5(\tau_0 + \tau_2) = \tau_1, \tag{3}$$

or, specifying the arguments of the function $\tau$ and using the principle of constancy of the speed of light in the system at rest $(K)$, we have

$$0.5[\tau_0(0, 0, 0, t) + \tau_2(0, 0, 0, \{t + x'/(V - v) + x'/(V + v)\})] = \tau_1[x', 0, 0, t + x'/(V - v)]. \tag{4}$$

If $x'$ is taken infinitesimally small, then it follows

$$0.5[1/(V - v) + 1/(V + v)]\partial\tau/\partial t = \partial\tau/\partial x' + [1/(V - v)]\partial\tau/\partial t, \tag{5}$$

$$\text{or}\quad \partial\tau/\partial x' + [v/(V^2 - v^2)]\partial\tau/\partial t = 0. \tag{6}$$

It must be noted that we could take, instead of the origin, any other point to send a ray of light, therefore, the last equation is valid for all values $x', y, z$.

Since the light along the axes $Y$ and $Z$, if observed from the system at rest, always propagates with the velocity $(V^2 - v^2)^{0.5}$, so the similar argument applied to these axes yields $\partial\tau/\partial y = 0$, $\partial\tau/\partial z = 0$. Since $\tau$ is a *linear* function, so from these equations it follows

$$\tau = a[t - vx'/(V^2 - v^2)], \tag{7}$$

where $a = \varphi(v)$ is yet unknown function, and for brevity it is taken that at the origin of the moving frame $(k)$ if $\tau = 0$, so also $t = 0$." (Einstein's notations, see [4, p. 14–15].)

For more than a century, time and again, different reservations and/or doubts appeared in the literature as to the validity and precision of the classical relativity theory. To dispel any doubt and to make special relativity understandable to everybody, we assume the constancy of $V$ and $v$, $|v| < V$, and Einstein's synchronization method (3)–(4) based on the rays of light, and try to find a linear function with undetermined coefficients

$$\tau(x', y, z, t) = at + bx', \quad a, b = \text{const}, \tag{8}$$

that would satisfy Eq. (4) *identically* with respect to $t$ and $x'$. Substituting (8) into (4) and noting that $y = z \equiv 0$ in (4), for a ray of light along the $X$-axis, we have

$$0.5[at + a\{t + x'/(V - v) + x'/(V + v)\}] \equiv bx' + a[t + x'/(V - v)], \quad \forall t, \forall x'. \tag{9}$$

Multiplying (9) by 2 and canceling the terms with $at$ on both sides, we get

$$a[x'/(V - v) + x'/(V + v)] \equiv 2x'[b + a/(V - v)], \quad \forall x'. \tag{10}$$

Simplifying (10), without division by $x'$, we see that the identity holds if and only if the constants $a$ and $b$ are chosen from the equation

$$aV/(V^2 - v^2) = b + a/(V - v), \quad |v| < V, \tag{11}$$

that is,

$$b = aV/(V^2 - v^2) - a/(V - v) = -av/(V^2 - v^2), \tag{12}$$

yielding in (8)

$$\tau(x', y, z, t) = a[t - vx'/(V^2 - v^2)], \quad |v| < V, \tag{13}$$

which coincides with (7). We see that a linear homogeneous time transformation (13) corresponding to the synchronization equations (3)–(4) exists for all $t, x', |v| < V$, with arbitrary nonzero calibrating factor $a(.)$ to be determined by additional requirements.

Substituting $x' = x - vt$ into (13) yields

$$\tau = a[t - v(x - vt)/(V^2 - v^2)] = a\alpha^2(t - vx/V^2), \quad \alpha^2 = V^2/(V^2 - v^2), \tag{14}$$

so that the time $\tau$ is really homogeneous in $t, x'$ of (13) and in $t, x$ of (14). According to initial conditions, a constant may be added in (8), thus, to (7) and (14), as noted by Einstein [4, p. 16], which constant is cancelled after the substitution of (8) into (3), (4).

The analogue of this case is obtained for the $Y$-axis and $Z$-axis with rays of light along those axes propagating with velocity $w = (V^2 - v^2)^{0.5}$, if observed from the system at rest, the same for direct and reflected rays. Instead of Eq. (4), we would have for the $Y$-axis, noting that $\tau$ does not depend on $x', z$ for this case:

$$0.5[\tau_0(0, y, 0, t) + \tau_2(0, y, 0, t + 2y/w)] = \tau_1(0, 0, 0, t + y/w). \tag{15}$$

Comparing (15) with (4), one can see that for a common function $\tau(x', y, z, t)$, Eq. (4) with a ray of light along the $X$-axis does not depend on $y, z$ which implies zeros in the second and third places of $\tau(.)$ in Eq. (4). With a ray of light along the $Y$-axis, Eq. (4) would not depend on $x', z$ which implies zeros in the first and third place in (15). As concerns the second place in $\tau(.)$ of (15), it corresponds to a ray of light sent at a moment $t$ from a point A on $Y$-axis where $y > 0$, see $\tau_0(.)$, to the origin of $(k)$, point B with $y = 0$, see $\tau_1(.)$, then reflected back to A with the same $y > 0$, see $\tau_2(.)$ in (15). In the fourth place of $\tau(.)$ in (15), we see the same entries as in (4) with different time segments for distances covered by the ray of light: $y/w$ for AB in $\tau_1(.)$ and $2yw$ for AB + BA (reflected light) in $\tau_2(.)$, same as in (4) with a difference that along $Y$-axis we observe the velocity $w$, same for direct and reflected rays, so the time $y/w$ in (15) plays the role of terms $x'/(V \pm v)$ in (4).

Now, for $\tau = at + bx' + hy + rz$ we obtain from (15) multiplied by 2, cf. (9), (10):

$$hy + at + hy + a(t + 2y/w) \equiv 2a(t + y/w), \quad \forall t, \forall y, \tag{16}$$

yielding, after cancellation of identical terms, the relation $2hy \equiv 0$, thus $h = 0$, and in the same way for $Z$-axis we would have $r = 0$. Hence, model (8) is valid for all three axes.

## 4. Einstein's coordinate transformations [1, Sections 3,4]

The factor $a(.)$ has been determined by Einstein ([1]; [4, pp. 16–17]) by introducing "…one more, the third coordinate system $(K')$, which with respect to system $(k)$ is in translational motion parallel to $\xi$-axis in such a way that its origin moves with velocity $-v$ along $\xi$-axis." Such choice of $(K')$ implies "that transformation from $(K)$ into $(K')$ must be the identity transformation" [4, p. 17]. Omitting details of derivation which can be found in [5, Section 7, pp. 1563–1564], this yields relativistic transformations [1,4] well known in the literature:

$$\tau = \beta(t - vx/V^2), \qquad \xi = \beta(x - vt), \qquad \eta = y, \qquad \zeta = z, \quad \beta = [1 - (v/V)^2]^{-0.5} \geq 1, \tag{17}$$

where $\beta$ is the calibration factor corresponding to (1), (3), (7), (14). Since $\alpha^2 = \beta^2$ in (14) so $a = \beta^{-1}$ in (7), (13), (14). Note that (17) are invertible with determinant $\Delta = 1$, for the first two equations, if $0 < v < V$. For $v \in [0, V)$ we have $\beta \in [1, \infty)$ monotonically increasing with $v$. If $(K)$ is observed from the moving frame $(k)$, then one has to invert (17) and replace $v$ for $-v$ with which $(K)$ moves with respect to $(k)$ if $(k)$ is considered "at rest", yielding $t = \beta(\tau - v\xi/V^2)$, $x = \beta(\xi - v\tau)$, same as in (17). If $\xi = v\tau$, then observer in $(k)$ "sees" $x = 0$, at rest, but $t = \beta\tau(1 - v^2/V^2) = \tau\beta^{-1} < \tau$, contraction of time in $(K)$ if observed from $(k)$.

The relativistic contraction of time is experimentally confirmed by discovery of $\mu$-mesons at the sea level. These are particles born in cosmic rays that have a short lifetime about 2 μs (in observed $\tau$-time). They are moving with velocity that equals 99.5% of the speed of light which amounts to $v = 2.985 \times 10^{10}$ cm/s $= 2.985 \times 10^8$ m/s. With this velocity and lifetime of $\tau^0 = 2 \times 10^{-6}$ s, these particles could enter the atmosphere not deeper than at $l = v\tau^0 \cong 600$ m. However, the observed $\tau^0$-lifetime actually represents the contracted natural lifetime $t^0 = \beta\tau^0 = (1 - v^2/V^2)^{-0.5}\tau^0 = (1 - 0.990)^{-0.5}\tau^0 = 10\tau^0$, during which the particles would enter the atmosphere at $l^0 = vt^0 = 10v\tau^0 = 6000$ m that corresponds to the sea level at which the $\mu$-mesons have been discovered. It means that they exist not by our observations within the span of $\tau^0$-lifetime, but by their own nature within their natural $t^0$-lifetime.

If we observe a process (clock) unfolding in a moving frame, using rays of light or radar, the unit of time $\Delta t$ in the motion of that process seems shorter, $\Delta\tau = \beta^{-1}\Delta t < \Delta t$. It is instructive that contraction of time happens in exactly the same proportion $\beta^{-1} < 1$ as contraction of the size of a solid in direction of the velocity $v$ of a moving frame, see ([1, Section 4]; [4, p. 18]). It proves the perfect similarity in contraction of time and the relativistic coordinate observed along right line of velocity, in accordance with the assumption (2).

**Remark 4.1.** Note that $\tau, \xi, \eta, \zeta$ are the *observed* time and coordinates in which real processes evolving in $(k)$ are *distorted* when observed from $(K)$, see [5, Section 8]. It means that times $\tau$ and $t$ are *not* the same but present *different* time-entities whereby $\tau$ is the *image* of $t$ if observed from $(K)$ and, according to the principle of relativity, Law 1 in Section 2 above, $t$ is the *proper* time in $(K)$ and in $(k)$ if observed from the same system.

## 5. Transformation of the Maxwell–Hertz equations for vacuum

By letter $V$ is denoted in [1] and in quotations thereof, see (2) and (4)–(7) in Sections 2–4, "a universal constant (the speed of light in vacuum)", which enters the classical relativistic transformations (17) as a parameter with fixed known value $\cong 3 \times 10^{10}$ cm/s and relates to the specific synchronization signals: the rays of light. However, in situations when propagation of light is physically impossible, other signals with speeds lower than the speed of light can be used in (17), see [5]. In contrast, it is the specific electrodynamic constant $c \cong 3 \times 10^{10}$ cm/s (equal to the speed of light in vacuum) which enters the Maxwell–Hertz equations, and Einstein writes, using $V$ instead of $c$ in [1, Section 6], translation from [4, pp. 22–24]: "Let the Maxwell–Hertz equations be true for vacuum in a still system $(K)$; in this case, we have

$$V^{-1}\partial X/\partial t = \partial N/\partial y - \partial M/\partial z, \qquad V^{-1}\partial L/\partial t = \partial Y/\partial z - \partial Z/\partial y, \tag{18}$$

$$V^{-1}\partial Y/\partial t = \partial L/\partial z - \partial N/\partial x, \qquad V^{-1}\partial M/\partial t = \partial Z/\partial x - \partial X/\partial z, \tag{19}$$

$$V^{-1}\partial Z/\partial t = \partial M/\partial x - \partial L/\partial y, \qquad V^{-1}\partial N/\partial t = \partial X/\partial y - \partial Y/\partial x, \tag{20}$$

where $(X, Y, Z)$ — vector of the electric field tension, $(L, M, N)$ — vector of the magnetic field tension." To avoid confusion, we shall use the letter $V$ for both electrodynamic constant and the speed of light, and also for the speed of other synchronization signals, if any, that may be used as carriers of information, thus, entering transformations (17). Thereby $V \leq c$, though in Einstein's considerations always $V = c$.

By the principle of relativity, Eqs. (18)–(20), valid in $(K)$, must be also valid in $(k)$ in the same form since $(k)$ is identical to $(K)$, with the same coordinates $x, y, z, t$, when set in translational motion with a speed $v =$ const. To obtain corresponding equations in *observed* coordinates $\xi, \eta, \zeta, \tau$, the first two equations of (17) can be solved for $t, x$, and writing them in differential form yields $\mathrm{d}t = \beta(\mathrm{d}\tau + v\mathrm{d}\xi/V^2)$, $\mathrm{d}x = \beta(\mathrm{d}\xi + v\mathrm{d}\tau)$. Using this relations with $\mathrm{d}y = \mathrm{d}\eta$, $\mathrm{d}z = \mathrm{d}\zeta$, as per (17), Einstein writes (18)–(20) in observed (transformed) coordinates $\xi, \eta, \zeta, \tau$ with the same tensions $X, Y, Z, L, M, N$. Then Einstein writes: "Principle of relativity requires that the Maxwell–Hertz equations, which are true in system $(K)$ for vacuum, be also true for system $(k)$; this means that for the tension vectors of the electric and magnetic fields $[(X', Y', Z')$ and $(L', M', N')]$, defined in the moving system $(k)$ through their actions on electric charges, or, correspondingly, on magnetic masses, must be valid the following equations: ..." (the same Eqs. (18)–(20) are written), see [4, p. 23], where $(X, Y, Z)$ and $(L, M, N)$ are substituted by $[(X', Y', Z')$ and $(L', M', N')]$, and $x, y, z, t$ are substituted by $\xi, \eta, \zeta, \tau$.

**Remark 5.1.** The supposition that Eqs. (18)–(20) preserve the same form in the *transformed* coordinates $\xi, \eta, \zeta, \tau$ of (17) contains much more than required by the principle of relativity, Law 1, in Section 2. Indeed, Eqs. (18)–(20) are valid in $(K)$ in its natural coordinates $x, y, z, t$ which are also the coordinates for $(k)$ at rest when $(k) \equiv (K)$. When $(k)$ is set in motion at a speed $v =$ const, then, by the principle of relativity, Law 1, those same Eqs. (18)–(20) remain valid in a moving system $(k)$ in that same form, thus, in the same natural coordinates $x, y, z, t$ which are preserved for $(k)$ and called *proper* in $(k)$, in contrast with the transformed coordinates $\xi, \eta, \zeta, \tau$ by which $(k)$ is observed in $(K)$ and in which the Maxwell–Hertz equations (18)–(20) have a *different* form for the same $(X, Y, Z)$ and $(L, M, N)$, as written by Einstein in ([1]; [4, p. 22]). Thus, Einstein introduces here a new assumption that there exist new $[(X', Y', Z')$ and $(L', M', N')]$ for which Eqs. (18)–(20) preserve the *same* form (18)–(20) in *transformed* tensions and coordinates $\xi, \eta, \zeta, \tau$. We accept this additional assumption as an *extension* of the principle of relativity for this case. □

For that system and the system (18)–(20) transformed by (17), see ([1, Section 6]; [4, p. 22]), Einstein writes: "Both systems of equations found for system $(k)$ must, obviously, express exactly the same thing since both systems of equations are equivalent to the Maxwell–Hertz equations for system $(K)$. Furthermore, since the equations for both systems coincide in everything except for symbols, it follows that functions in the corresponding entries for both systems must be equal up to a factor $\psi(v)$, common for all functions, which does not depend on $\xi, \eta, \zeta, \tau$, but can, generally speaking, depend on $v$. Hence,

$$X' = \psi(v)X, \qquad L' = \psi(v)L, \tag{21}$$

$$Y' = \psi(v)\beta(Y - vN/V), \qquad M' = \psi(v)\beta(M + vZ/V), \tag{22}$$

$$Z' = \psi(v)\beta(Z + vM/V), \qquad N' = \psi(v)\beta(N - vY/V). \tag{23}$$

If this system of equations is inverted, first, by a direct solution and, second, by the inverse transformation (from $k$ into $K$) which corresponds to the velocity $-v$, and one takes into account that both resulting systems must be identical, then $\psi(v)\psi(-v) = 1$. Further, from symmetry considerations, it follows that $\psi(v) = \psi(-v)$; therefore, $\psi(v) = 1$ and our equations take the form:

$$X' = X, \qquad L' = L, \tag{24}$$

$$Y' = \beta(Y - vN/V), \qquad M' = \beta(M + vZ/V), \tag{25}$$

$$Z' = \beta(Z + vM/V), \qquad N' = \beta(N - vY/V). \tag{26}$$

For interpretation of these equations, the following can be noted. Consider a point-wise charge, which under measurements in the still system $(K)$ equals "unity", i.e., being at rest in a still system, it acts at the distance of 1 cm with the force of 1 *dyne* onto the same charge. According to the principle of relativity, this electric charge under measurements in a moving system is also equal "unity". If this charge is at rest with respect to a still system, then vector $(X, Y, Z)$, according to definition, equals the force acting on the mentioned charge. But if the charge is at rest with respect to a moving system (at least, at a certain moment of time), then the force acting thereon under measurements from the moving system, equals the vector $(X', Y', Z')$."

As concerns transformation of the Maxwell–Hertz equations with convective currents, Einstein starts with initial equations similar to (18)–(20) where to the partial derivatives at left are added the terms $u_x\rho, u_y\rho, u_z\rho, \rho = \partial X/\partial x + \partial Y/\partial y + \partial Z/\partial z$, in which $\rho$ means density of an electric charge multiplied by $4\pi$, and $(u_x, u_y, u_z)$ is the vector of velocity of the electric charge. These equations, valid in $(K)$, if transformed by (17) to the system $(k)$, produce relations similar to (18)–(20), with $\rho'$ and the partial derivatives of $[(X', Y', Z')$ and $(L', M', N')]$ in respect to $\xi, \eta, \zeta, \tau$, and the transformed values

$$u_\xi = (u_x - v)/[1 - u_x v/V^2], \qquad u_\eta = u_y/\beta[1 - u_x v/V^2], \qquad u_\zeta = u_z/\beta[1 - u_x v/V^2], \tag{27}$$

$$\rho' = \partial X'/\partial\xi + \partial Y'/\partial\eta + \partial Z'/\partial\zeta = \beta[1 - u_x v/V^2]\rho, \quad V = c = 3 \times 10^{10}\ \text{cm/s}. \tag{28}$$

In regard to the transformed equations, Einstein writes ([1, Section 9]; [4, pp. 30–31]): "Thus, as it follows from the theorem of addition of velocities, the vector $(u_\xi, u_\eta, u_\zeta)$ is just the velocity of electric charges measured in system $(k)$. Thereby, it is demonstrated that electrodynamical basis of Lorentz's electrodynamics for moving bodies is subject to the principle of relativity if one starts from our principles of kinematics.

We also note that from the proven equations can be readily derived the following important theorem: if an electrically charged body is moving in space arbitrarily and if its charge, observed from the coordinate system moving with that body, does not vary, then this charge remains unchanged also when observed from a "still" system $(K)$."

**Remark 5.2.** If a charged body and its coordinate system $(k)$ are moving "arbitrarily", then their relative velocity $v \neq \text{const}$ with respect to a "still" system $(K)$, in which case equations (17) fail, so the last statement about invariance of the charge "when observed from a "still" system $(K)$" cannot be justified by reference to the results obtained with the use of those Eqs. (17), invalid for the case. Now, with $v = 0$, Eqs. (24)–(26) become identical to (18)–(20) since $\beta = 1$, thus, same in $(K)$ and in $(k)$, which equations, in agreement with the principle of relativity, are preserved in the same form when $(k)$ is set in motion at some velocity $v = \text{const}$. Hence, Eqs. (24)–(26) present an *image* of the same tensions as in Eqs. (18)–(20) for $(K)$ and $(k)$, which image is given by observations in $(K)$ from $(k)$ produced by signals transmitting the information and depends on velocity $V$ of the signals employed and on the relative velocity $v$ of $(k)$ with respect to $(K)$, as exhibited in relations (24)–(28). For this image, Eqs. (18)–(20) preserve their form in *observed* (transformed) coordinates $\xi, \eta, \zeta, \tau$, cf. Remark 5.1.

## 6. Motion in electromagnetic field, the second Newton's Law and the notion of mass

In Section 10 of [1] entitled "Dynamics of weakly accelerated electron" Albert Einstein writes (translation from [4, pp. 32-34], notations and format by Einstein):

"Suppose that in electromagnetic field a point-wise particle is moving with electrical charge $\varepsilon$ (called "electron" in what follows), and about the law of its motion we shall assume only the following.

If an electron is at rest during certain interval of time, then at immediately following time moment the motion of the electron, since it is slow, will be described by equations:

$$\mu \mathrm{d}^2x/\mathrm{d}t^2 = \varepsilon X, \qquad \mu \mathrm{d}^2y/\mathrm{d}t^2 = \varepsilon Y, \qquad \mu \mathrm{d}^2z/\mathrm{d}t^2 = \varepsilon Z, \tag{29}$$

where $x, y, z$ are coordinates of the electron, and $\mu$ is the mass of the electron.

Further, suppose that the electron during certain interval of time has velocity $v$. Let us find a law according to which the electron is moving at immediately following thereafter time moment.

Without loss of generality, we can assume, and we assume indeed, that at that moment, when we begin observation, our electron is at the origin and is moving along the $X$-axis of system $(K)$ with velocity $v$. In this case, it is clear that at that moment of time $(t = 0)$ the electron is at rest with respect to coordinate system $(k)$ moving parallel to the $X$-axis with constant velocity $v$.

From the above assumption combined with the principle of relativity, it follows that equations of motion of the electron observed from system $(k)$ during time immediately following after $t = 0$ (at small values of $t$), have the form:

$$\mu \mathrm{d}^2\xi/\mathrm{d}\tau^2 = \varepsilon X', \qquad \mu \mathrm{d}^2\eta/\mathrm{d}\tau^2 = \varepsilon Y', \qquad \mu \mathrm{d}^2\zeta/\mathrm{d}\tau^2 = \varepsilon Z', \tag{30}$$

where denoted by $\xi, \eta, \zeta, \tau, X', Y', Z'$ values are related to system $(k)$. If we also set that for $t = x = y = z = 0$ we have $\tau = \xi = \eta = \zeta = 0$, then the formulae of transformation from Sections 3 and 6 will be valid, and thus, the following equations will hold:

$$\begin{aligned}
&\tau = \beta(t - vx/V^2),\\
&\xi = \beta(x - vt), \qquad X' = X,\\
&\eta = y, \qquad Y' = \beta(Y - vN/V),\\
&\zeta = z, \qquad Z' = \beta(Z + vM/V).
\end{aligned} \tag{31}$$

Making use of these equations, we transform Eqs. (30) from system $(k)$ to system $(K)$, yielding

$$\begin{aligned}
&\mathrm{d}^2x/\mathrm{d}t^2 = \varepsilon\mu^{-1}\beta^{-3}X,\\
&\mathrm{d}^2y/\mathrm{d}t^2 = \varepsilon\mu^{-1}\beta^{-1}(Y - vN/V), \quad (A)\\
&\mathrm{d}^2z/\mathrm{d}t^2 = \varepsilon\mu^{-1}\beta^{-1}(Z + vM/V).
\end{aligned} \tag{32}$$

Using the usual course of argumentation, let us define now the "longitudinal" and "transverse" masses of a moving electron. Let us write equations $(A)$ in the following form:

$$\begin{aligned}
&\mu\beta^3\mathrm{d}^2x/\mathrm{d}t^2 = \varepsilon X = \varepsilon X',\\
&\mu\beta^2\mathrm{d}^2y/\mathrm{d}t^2 = \varepsilon\beta(Y - vN/V) = \varepsilon Y',\\
&\mu\beta^2\mathrm{d}^2z/\mathrm{d}t^2 = \varepsilon\beta(Z + vM/V) = \varepsilon Z'.
\end{aligned} \tag{33}$$

Now, we note, first of all, that $\varepsilon X'$, $\varepsilon Y'$, $\varepsilon Z'$ are components of electromagnetic force acting upon the electron, whereby those components are considered in the coordinate system which at a given moment is moving together with the electron with the same, as for the electron, velocity. (This force could be measured, for example, by a spring scale at rest in that system.) If now we shall call this force simply "a force acting upon the electron", and preserve the equation (for numeric values)

$$\text{Mass} \times \text{Acceleration} = \text{Force},$$

and if we further define that accelerations must be measured in the still system $(K)$, then from the above equations we obtain:

$$\begin{aligned}
&\text{longitudinal mass } = \mu[1 - (v/V)^2]^{-1.5} = \mu\beta^3,\\
&\text{transverse mass} = \mu[1 - (v/V)^2]^{-1} = \mu\beta^2.
\end{aligned} \tag{34}$$

Of course, we shall get different values for masses under different definitions of forces and accelerations; thus, it is clear that in comparison of different theories of motion of an electron, one should be very careful. We note that these results about the mass are valid also for neutral material points since such a point can be treated as electron (in our sense) by adjoining an arbitrarily small electrical charge.

Let us determine the kinetic energy of an electron. If an electron is moving from the origin of system $(K)$ with initial velocity 0 along the $X$-axis under the action of electrostatic force $X$, then it is clear that the energy taken from the electrostatic field is equal $\int \varepsilon X\mathrm{d}x$. Since the electron is accelerating slowly and because of that does not have to give away energy in the form of radiation, so the energy taken from electrostatic field must be set equal to the energy of motion $W$ of the electron. Taking into account that during the entire process of motion the first of equations $(A)$ is valid, we obtain:

$$W = \int \varepsilon X\mathrm{d}x = \int \mu\beta^3 v\mathrm{d}v = \mu\int_0^v v[1 - (v/V)^2]^{-3/2}\mathrm{d}v = \mu V^2\{[1 - (v/V)^2]^{-0.5} - 1\}. \tag{35}$$

For $v = V$ the value $W$ becomes, thus, infinitely large. Like in previous results and here as well, velocities greater than the speed of light cannot exist. This expression for kinetic energy must be valid also for any masses due to the above mentioned argument."

N.B. The integral in (35) can be taken by substitution $v/V = \sin\varphi$, thus, $\beta = \sec\varphi$, and the value at right can be written simply as $\mu V^2(\beta - 1) = \mu V^2(\sec\varphi - 1)$.

**Alternative consideration.** The double value for the mass in (34) was defined in order to preserve the same form of Newton's Second Law of motion in (29), (30) for the *proper* $x, y, z, t$ coordinates in both $(K)$ and $(k)$ frames (according to the principle of relativity), and for *transformed* coordinates $\xi, \eta, \zeta, \tau$ as *observed* in $(K)$ from $(k)$, whereby the *observed* forces $(X', Y', Z')$ in $(k)$ and original forces $(X, Y, Z)$ in $(K)$ are *different,* according to (31). The preservation of physical laws in transformed coordinates is *not* included in the principle of relativity as formulated by Einstein in Section 2, Law 1. Indeed, intuitively it seems clear that the laws of nature stay unchanged in all *inertial* systems at rest and systems moving with constant velocities, with respect to the proper time and coordinates of those inertial systems (frames). Transformations (17), however, are based on a signal propagating with some finite speed, and it is questionable that the same laws might be expressed by the same formulae in the *observed* (transformed) coordinates $\xi, \eta, \zeta, \tau$ conditioned on the relative speed $v$ and on the speed $V$ of the signal propagation and quite different from the proper coordinates $x, y, z, t$; cf. Remark 5.1. Also, the notion of *mass* is usually *not* associated with directions or coordinate transformations, and it would be expedient to preserve the mass as a scalar characteristic of a body.

Let us demonstrate that, in fact, it is the change in *observed* accelerations to which the difference of values in (34) can be attributed, without any deviation from the principle of relativity and from the substance of the Second Law of motion as formulated by Newton. Considering derivatives of *observed* coordinates with respect to *observed* time in (30) and comparing them with derivatives of the *proper* coordinates with respect to the *proper* time in (29), we have, due to (17):

$$\mathrm{d}\xi/\mathrm{d}\tau = \xi_\tau = (\mathrm{d}\xi/\mathrm{d}t)/(\mathrm{d}\tau/\mathrm{d}t) = (x_t - v)/(1 - vx_t/V^2), \quad x_t = w(t) \tag{36}$$

$$\mathrm{d}^2\xi/\mathrm{d}\tau^2 = \xi_{\tau\tau} = [x_{t\tau}(1 - vx_t/V^2) + (x_t - v)vx_{t\tau}/V^2]/(1 - vx_t/V^2)^2. \tag{37}$$

One has to note that in (17) the point $(x, y, z) \in (K)$ is at rest in $(K)$, however, in (29) that same point $(x, y, z) \in (K)$, the electron, is moving, and accelerating to velocity $v$, same as the constant velocity $v$ of $(k)$ along the $0x$ axis of $(K)$, see Einstein's explanation after (29) which are Newtonian equations in *proper* time coordinates of $(K)$. This means that transformation (17) is being done in its *continuous superposition,* with accelerating point $(x, y, z) \in (K)$. In order not to confuse the constant velocity $v$ of $(k)$ with respect to $(K)$ which enters (17), (36), (37) and the velocity $v$ of the accelerating electron (same notation for *different* entities) we denoted the latter by $x_t = w(t)$, as indicated in (36). At some moment, it happens that $w(t) = v$, and at this moment the electron is at rest with respect to $(k)$. For this reason, coordinate $x(t)$ of the electron can be differentiated in (17) as it is done for $x$ in (36)–(37), since it varies with the moving electron. Thus, we have at the moment that velocity $v = \mathrm{d}x/\mathrm{d}t = x_t$ is achieved by the electron:

$$x_{t\tau} = d(\mathrm{d}x/\mathrm{d}t)/\mathrm{d}\tau = x_{tt}/(\mathrm{d}\tau/\mathrm{d}t) = x_{tt}/\beta(1 - vx_t/V^2) = x_{tt}/\beta(1 - v^2/V^2) = x_{tt}\beta. \tag{38}$$

Continuing (37) and using (38) at the moment when $x_t = v$, we obtain

$$\xi_{\tau\tau} = x_{t\tau}\beta^{-2}\beta^4 = x_{t\tau}\beta^2 = x_{tt}\beta^3. \tag{39}$$

Repeating (36)–(39) for coordinates $\eta, \zeta$, we have at that same moment $x_t = v$

$$\mathrm{d}\eta/\mathrm{d}\tau = \eta_\tau = (\mathrm{d}\eta/\mathrm{d}t)/(\mathrm{d}\tau/\mathrm{d}t) = (\mathrm{d}y/\mathrm{d}t)/\beta(1 - vx_t/V^2) = y_t\beta, \tag{40}$$

$$\mathrm{d}^2\eta/\mathrm{d}\tau^2 = \eta_{\tau\tau} = (\mathrm{d}\eta_\tau/\mathrm{d}t)/(\mathrm{d}\tau/\mathrm{d}t) = y_{tt}\beta/\beta(1 - vx_t/V^2) = y_{tt}\beta^2, \tag{41}$$

and in the same way we obtain $\mathrm{d}^2\zeta/\mathrm{d}\tau^2 = z_{tt}\beta^2$. Comparing these values with values in (33), (34), we see that there is no need to attribute the factors $\beta^3, \beta^2$ to the mass in (34) since they naturally occur in the accelerations in (33).

Let us consider again the energy conservation law embodied in Eq. (35) which, if written with indication of all limits in the integrals, takes the form:

$$W = \int_{x1}^{x2} \varepsilon X \mathrm{d}x = \int_0^v \mu\beta^3 v \mathrm{d}v = \mu \int_0^v v[1 - (v/V)^2]^{-3/2}\mathrm{d}v = \mu V^2\{[1 - (v/V)^2]^{-0.5} - 1\}. \tag{42}$$

With $x1$ and $x2$ being constant and such that $x_t = 0$ at $x1, x_t = w(t) = v$ at $x2$, the first integral does not depend on $V$ nor on constant relative velocity $v$ of $(k)$, whereas the second and third integrals in (42) depend on $V$ and $v = x_t$. It means that one and the same "energy taken from electrostatic field" takes different numerical values if observed at different velocities $V$ and $v$ (by different signals at different relative speeds). With respect to the energy conservation law, this fact can be explained only as scaling by the signals and instruments produced by observation in the presence of relativistic effects.

The right-hand side of (42) tends to infinity as $v \to V$, so the observed kinetic energy tends to infinity with the mass $\mu$ and the "energy taken from electrostatic field" (the first integral) remaining constant. Respecting the energy conservation law, it means that relativistic transformations produce distortion of the image in observation as argued earlier in [5]. In this case, the observed time $\tau \to \infty$, which means that the observation cannot be accomplished in finite time, thus the electron in motion at velocities close to the speed of light becomes undetectable by signals propagating at same velocities.

**Remark 6.1.** Of course, all formulae (33) to (41) are different if $\mathrm{d}x/\mathrm{d}t = x_t = w(t) \neq v$ at some moment $t$ indicated above. Also, for a spacecraft driven by reactive forces, the representation of the Second Newton's Law of motion is modified [6, Section 6], and the observed decreasing mass of the spacecraft is distorted by relativistic transformations as well as all parameters and physical laws that depend on velocity directly or indirectly.

## 7. Radiation at rest and the mass–energy relation [2]

If for small ratios $v/V$, the brace in (42) is expanded in Taylor series, we obtain

$$W = \mu V^2(\beta - 1) = \mu V^2\{[1 - (v/V)^2]^{-0.5} - 1\} = \mu V^2[0.5(v/V)^2 + \cdots] = 0.5\mu v^2 + \cdots, \tag{43}$$

which represents the usual kinetic energy of an electron gained during the motion in electrostatic field given at left in (35), (42). In accordance with the energy conservation law, up to a small relativistic distortion, relation (43) describes the transformation of energy from "the energy taken from the electrostatic field" ([1, Section 10]; [4, p. 34]) at left in (35), (43) into the kinetic energy of the electron at right in (43).

It remains to see what happens if an electron gives away the energy, e.g., in the form of rays of light or electromagnetic waves (radiation), *without losing its kinetic energy*, which is the case of a body that radiates uniformly in all directions (spherical waves) being at rest. This case is considered in [2] where Einstein writes (translation from [4, pp. 36–38], notations and formulae by Einstein): "Let a system of plane waves of light related to coordinate system $(x, y, z)$, possess the energy $l$ and let the direction of a ray (normal to the front of the wave) make an angle $\varphi$ with the $x$-axis of the system. If we introduce a new coordinate system $(\xi, \eta, \zeta)$, in the uniform translational motion along a right line of the system $(x, y, z)$, and if the origin of that system is moving with velocity $v$ along the $x$-axis, then the mentioned energy of light, measured in the system $(\xi, \eta, \zeta)$, will be

$$l^* = l[1 - (v/V)\cos\varphi][1 - (v/V)^2]^{-0.5} = l\beta[1 - (v/V)\cos\varphi], \tag{44}$$

where $V$ is the speed of light. In what follows, we shall use this result.

Suppose that in the system $(x, y, z)$ there is a body at rest, the energy of which related to the system $(x, y, z)$ equals $E_0$. Let the energy of the same body related to the system $(\xi, \eta, \zeta)$ moving, as above, with velocity $v$, be equal $H_0$.

Let the body send in direction that makes an angle $\varphi$ with the $x$-axis a plane wave of light with the energy $L/2$ [measured with respect to system $(x, y, z)$] and simultaneously send the same quantity of light in the opposite direction. Thereby the body remains at rest with respect to the system $(x, y, z)$. For this process, the law of conservation of energy must hold, and (according to the principle of relativity) with respect to both coordinate systems. If we denote by $E_1$ the energy of the body after the radiation of light, measured with respect to the system $(x, y, z)$, and correspondingly by $H_1$ the same energy with respect to the system $(\xi, \eta, \zeta)$, then, using the above relation, we find

$$E_0 = E_1 + (L/2 + L/2),$$
$$H_0 = H_1 + \{(L/2)\beta[1 - (v/V)\cos\varphi] + (L/2)\beta[1 + (v/V)\cos\varphi]\} = H_1 + L\beta. \tag{45}$$

Subtracting the second equality from the first one, we obtain

$$(H_0 - E_0) - (H_1 - E_1) = L\{[1 - (v/V)]^{-0.5} - 1\} = L(\beta - 1). \tag{46}$$

In this relation, both differences of the form $H - E$ have simple physical sense. The values $H$ and $E$ represent the quantities of energy of one and the same body related to the two coordinate systems moving with respect to each other, whereby the body is at rest in one of the system [in the system $(x, y, z)$].

Thus, it is clear that the difference $H - E$ can deviate from kinetic energy $K$ of the body, taken with respect to the other system [system $(\xi, \eta, \zeta)$], only by some additive constant $C$, which depends on the choice of arbitrary additive constants in the expressions of energies $H$ and $E$. Hence, we can set

$$H_0 - E_0 = K_0 + C, \qquad H_1 - E_1 = K_1 + C, \tag{47}$$

since a constant $C$ does not change with radiation of light. In this way, we get

$$K_0 - K_1 = L\{[1 - (v/V)]^{-0.5} - 1\} = L(\beta - 1). \tag{48}$$

Kinetic energy of a body with respect to the system $(\xi, \eta, \zeta)$ is decreasing with radiation of light by a quantity not depending of the nature of the body. Besides, the difference $K_0 - K_1$ depends on the velocity in the same way as kinetic energy of an electron (see [1, Section 10] of the above cited work).

Discounting the values of the fourth and higher orders, one can get

$$K_0 - K_1 = L\{[1 - (v/V)]^{-0.5} - 1\} \cong (L/V^2)(v^2/2). \tag{49}$$

From this equation, it immediately follows that if a body gives away the energy $L$ in the form of radiation, then its mass decreases by the value $L/V^2$. Thereby, obviously, it is immaterial that the energy taken from the body is directly transferred into the energy of radiation, so that we come to the more general conclusion.

The mass of a body is a measure of energy contained in it; if the energy is changed by the value $L$, the mass is changed correspondingly by the value $L/(9 \times 10^{20})$, whereby the energy is measured here in ergs, and the mass — in grams.

A possibility is not excluded that the theory could be verified for elements, the energy of which is changing by larger amounts (for example, in compounds of radium).

If the theory is supported by facts, then radiation is transferring the inertia between the radiating and absorbing bodies." (Submitted 27 September 1905)

**Remark 7.1.** One can notice that if $v = 0$ (so that $\beta = 1$), then (44)–(45) become identical, and right-hand sides of (46), (48), (49) are all zeros. Thus, in Einstein's explanation after (49), an important indication seems missing which we expose by quoting respective parts of his text after (49) with our additions in italics: "From this equation, it immediately follows that if a body gives away the energy $L$ in the form of radiation, then its mass decreases by the value $L/V^2$," *as observed in the system* $(x, y, z)$ *from the system* $(\xi, \eta, \zeta)$, *system* $(k)$, *moving with velocity* $v$ *with respect to the body being at rest in the system* $(x, y, z)$, *system* $(K)$. "Thereby, obviously, it is immaterial that the energy taken from the body is directly transferred into the energy of radiation, so that we come to the more general conclusion.

The mass of a body is a measure of energy contained in it; if the energy is changed by the value $L$, the mass is changed correspondingly by the value $L/(9 \times 10^{20})$, whereby the energy is measured here in ergs, and the mass — in grams," *as observed in* $(K)$ *from the system* $(\xi, \eta, \zeta)$, system $(k)$, *where the energy* $H_1$ *is measured by the rays of light, with kinetic energy of* (46)–(49) *computed in* $(K)$ *as* $L(\beta - 1)$.

With these comments, it becomes clear that the mass as a measure of energy appears in the context of the above paper as an *image phenomenon* in $(K)$ observed from the moving system $(k)$, and *not* in the still system $(K)$ where the body remains at rest.

**Remark 7.2.** If in (49) we denote $K_0 - K_1 = W$, $L/V^2 = \mu$, then (49) coincides with (43), see the first and last terms. The difference is that in (43) the energy of electrostatic field has been transformed into kinetic energy of the moving electron, and in (49) the energy of radiation from the body at rest has been given away without a change in its zero kinetic energy in $(K)$, e.g., as a loss or addition to its kinetic energy due to reactive forces. Since a loss in its own energy content (of its temperature) is presumed zero (or not accounted), so by default, the result is *interpreted* as a loss of mass (as a measure of energy) in the radiating body with respect to the moving system $(k)$ from which the change of energy is observed in $(K)$ by rays of light propagating with velocity $V = c$.

**Remark 7.3.** The energy relation (44) taken from [1, Section 8, Theory of the pressure of light on ideal mirror] is the same as the relation for energies and frequencies in ([1, Section 8]; [4, pp. 28–29]) and for frequencies in ([1, Section 7]; [4, p. 26, Theory of aberration and of Doppler effect]). So, if a body sends waves of sound at velocity $V = 340$ m/s, not at the speed of light, then, taking into account that relativistic transformations (17) are valid for *any* information transmitting signal that propagates at a constant velocity [5], we obtain that the same mass is decreased by the value $L/V^2 = L/(1.2 \times 10^9)$, much less than for waves of light. This supports (49), if the speed $V$ of observation signals in (17) is equal to the speed of energy rays emitted by the body. However, if we take $v = 0$ or do not introduce a new coordinate system $(\xi, \eta, \zeta, \tau)$, then coordinate transformations (17) are not applicable, $\beta = 1$, and $K_0 = K_1$ in (48)–(49), which vacates the result, although the body is still radiating energy, thereby being at rest with zero kinetic energy and the same mass in the system $(x, y, z, t)$, system $(K)$, but cooling off due to radiation.

## 8. Conservation of momentum and inertia of energy [3]

In [3], Einstein writes: "In the paper published last year (see [2], Section 7 above), we have shown that the electromagnetic equations of Maxwell together with the principle of relativity and the law of conservation of energy lead to the conclusion that the mass of a body is changing with a change of its energy whatever may be the character of those changes. We have shown that the change of energy by the value $\Delta E$ must correspond to the equivalent change of the mass by the value $\Delta E/V^2$ where $V$ is the speed of light.

In this paper, we shall demonstrate that this statement is the necessary and sufficient condition in order that, at least in the first approximation, the law of conservation of momentum be satisfied for the center of mass of a system in which, apart from mechanical processes, also electromagnetic processes are unfolding. Despite that a simple formal consideration which should be done for a proof of this statement is contained mainly in a paper by A. Poincare (Lorentz–Festschrift, 1900, 252), we shall not, for the sake of transparency, base our arguments on that paper."

Skipping "Section 1. Particular case" of [3], we reproduce "Section 2. On the law of conservation of momentum of the center of mass", in exact translation from [4, pp. 41–44] as follows: "Consider a system of $n$ material points with masses $m_1, m_2, \ldots, m_n$, and coordinates $x_1, \ldots, z_n$. These material points in thermodynamic and electric sense are not elementary objects (an atom, a molecule) and must be considered as bodies of small dimensions in the usual sense the energy of which is not defined by velocity of the center of mass. These masses can affect each other in electromagnetic way and also by conservative forces (for example, force of weight, elastic constraints). In spite of that, we shall assume that both potential energy of conservative forces and kinetic energy of the motion of the center of mass should be always considered infinitesimally small with respect to the "internal" energy of the masses $m_1, m_2, \ldots, m_n$.

In the whole space, one can assume that the Maxwell–Lorentz equations are true

$$V^{-1}(u\rho + \partial X/\partial t) = \partial N/\partial y - \partial M/\partial z, \qquad V^{-1}\partial L/\partial t = \partial Y/\partial z - \partial Z/\partial y, \tag{50}$$

$$V^{-1}(v\rho + \partial Y/\partial t) = \partial L/\partial z - \partial N/\partial x, \qquad V^{-1}\partial M/\partial t = \partial Z/\partial x - \partial X/\partial z, \tag{51}$$

$$V^{-1}(w\rho + \partial Z/\partial t) = \partial M/\partial x - \partial L/\partial y, \qquad V^{-1}\partial N/\partial t = \partial X/\partial y - \partial Y/\partial x, \tag{52}$$

where $\rho = \partial X/\partial x + \partial Y/\partial y + \partial Z/\partial z$ means density of the electrical charge multiplied by $4\pi$. (Note that Eqs. (50)–(52) are identical to (18)–(20) with $u_x\rho$, $u_y\rho$, $u_z\rho$ added to partial derivatives at left (convective currents), and $V$ means the speed of light, E.G.)

If Eqs. (50)–(52), multiplied respectively by $VXx/4\pi, VYx/4\pi, \ldots, VNx/4\pi$, are summed up and integrated over the whole space, then after several integrations by parts, we obtain

$$\int (\rho/4\pi)x(uX + vY + wZ)\mathrm{d}\tau + \mathrm{d}\left\{\int (x/8\pi)(X^2 + Y^2 + \cdots + N^2)\mathrm{d}\tau\right\}\Big/\mathrm{d}t - (V/8\pi)\int (YN - ZM)\mathrm{d}\tau = 0. \tag{53}$$

The first term of this relation represents the energy of the electromagnetic field of the bodies $m_1, m_2, \ldots, m_n$. Since, according to our hypothesis about dependence of mass and energy, it is necessary that the first term of the sum be equal to the expression

$$V^2 \sum x_v(\mathrm{d}m_v/\mathrm{d}t), \tag{54}$$

so, according to the above considerations, we assume that the energy and, consequently, the mass of a separate material point $m_v$ is changing *only* when the electromagnetic energy is absorbed.

If we further ascribe to the electromagnetic field a density of mass $(\rho_e)$ which differs from the density of energy by the factor $V^{-2}$, then the second term of relation (53) takes the form

$$V^2\mathrm{d}\left\{\int x\rho_e\mathrm{d}\tau\right\}\Big/\mathrm{d}t. \tag{55}$$

If we denote as $J$ the integral that enters the third term of relation (53), then this relation can be written in the form

$$\sum x_v(\mathrm{d}m_v/\mathrm{d}t) + \mathrm{d}\left\{\int x\rho_e\mathrm{d}\tau\right\}\Big/\mathrm{d}t - J/4\pi V = 0. \tag{56}$$

We have now to clarify the physical sense of the integral $J$. Multiplying the second, third, fifth and sixth equations of (50)–(52), correspondingly by $NV$, $-MV$, $-ZV$, $YV$, adding them and integrating over the space, we obtain, after certain integrations by parts

$$\mathrm{d}J/\mathrm{d}t = -4\pi V \int (\rho/4\pi)(X + vN/V - wM/V)\mathrm{d}\tau = -4\pi VR_x, \tag{57}$$

where $R_x$ is the algebraic sum of $x$-components of all forces with which electromagnetic field acts on the masses $m_1, m_2, \dots, m_n$. Since the corresponding sum of all forces, born by conservative interactions, equals zero, so the value $R_x$ presents simultaneously the sum of $x$-components of *all* forces acting on masses $m_v$.

Now we have to consider Eq. (57) which is not linked to the hypothesis of dependence of mass and energy. First, if we ignore the dependence of mass and energy and denote by $X_v$ the resulting force of $x$-components of all forces acting on the mass $m_v$, then for the mass $m_v$ we get the equation of motion

$$m_v(\mathrm{d}^2x_v/\mathrm{d}t^2) = \mathrm{d}\{m_v(\mathrm{d}x_v/\mathrm{d}t)\}/\mathrm{d}t = X_v, \tag{58}$$

and also

$$\mathrm{d}\left[\sum m_v(\mathrm{d}x_v/\mathrm{d}t)\right]\Big/\mathrm{d}t = \sum X_v = R_x. \tag{59}$$

From Eqs. (59) and (57) we have

$$J/4\pi V + \sum m_v(\mathrm{d}x_v/\mathrm{d}t) = \mathrm{const.} \tag{60}$$

Introducing again the hypothesis about dependence of the mass $m_v$ on energy, and consequently, on time, we encounter a difficulty that for this case the equations of mechanics are not yet known; the first equality in relations (58) does not hold. Nevertheless, it must be noted that the difference

$$\begin{aligned}\mathrm{d}\{m_v(\mathrm{d}x_v/\mathrm{d}t)\}/\mathrm{d}t - m_v(\mathrm{d}^2x_v/\mathrm{d}t^2) &= (\mathrm{d}m_v/\mathrm{d}t)(\mathrm{d}x_v/\mathrm{d}t)\\ &= V^{-2}\int (\rho/4\pi)(\mathrm{d}x_v/\mathrm{d}t)(uX + vY + wZ)\mathrm{d}\tau\end{aligned} \tag{61}$$

is proportional to the second power of velocity. Therefore, if all velocities are so small that the terms of the second order can be ignored, then, when the mass $m_v$ is changing, the equation

$$\mathrm{d}\{m_v(\mathrm{d}x_v/\mathrm{d}t)\}/\mathrm{d}t = X_v \tag{62}$$

remains valid up to the considered precision. Then, Eqs. (59) and (60) will hold, and from Eqs. (60) and (56) we obtain

$$\mathrm{d}\left[\sum (m_v x_v) + \int x\rho_e \mathrm{d}\tau\right]\Big/\mathrm{d}t = \mathrm{const.} \tag{63}$$

Denoting by $\xi$ the $x$-coordinate of the center of mass of heavy masses together with the mass of electromagnetic field, we have

$$\xi = \left[\sum (m_v x_v) + \int x\rho_e \mathrm{d}\tau\right]\Big/\left[\sum m_v + \int \rho_e \mathrm{d}\tau\right], \tag{64}$$

thereby, in accordance with the law of conservation of energy, the denominator of the right-hand side of this equality does not depend on time. Therefore, Eq. (63) can be written also in the form

$$\mathrm{d}\xi/\mathrm{d}t = \mathrm{const.} \tag{65}$$

Hence, if to every energy $E$ one ascribes the inertial mass $E/V^2$, then, at least in the first approximation, the law of conservation of the center of mass will be valid also for systems in which electromagnetic processes are unfolding.

From the above considerations, it follows that we have either to abandon the basic principle of mechanics according to which a body initially at rest and not subject to actions of external forces cannot set off in translational motion, or agree that inertia of a body depends in the indicated manner on the energy contained in it." (Submitted 17 May, 1906)

**Remark 8.1.** It is worth noting that Eq. (50) to (52) and (54) are all in the *proper* coordinates $x, y, z, t$ whereas Eqs. (53) and (55)–(57) are written partly in proper and partly in *observed* (transformed) coordinates $\xi, \eta, \zeta, \tau$. Equations of motion (58), (59), (62) are all in proper coordinates whereas $J$ and integrals in (55)–(57), (61), (63), (64) are in observed coordinates, thus, Eqs. (53) and (55)–(57), (60), (61)–(65) contain a *mix* of coordinates, time derivatives with respect to the *proper* time $t$ and integrals with respect to the *observed* time $\tau$. By the principle of relativity, the net amounts of mass, energy, and momentum of a body, in any form, do not change if system of reference (space) is at rest or in uniform translational motion, but only if counted with respect to the *proper* coordinates of a still or moving (inertial) frame. Since $\mathrm{d}\tau/\mathrm{d}t \neq 1$, and may even be non-constant as demonstrated in (36) to (41) in Section 6 above, the conservation laws in calculated quantities ascribed to observed motions or processes are valid only *after* the conversion of all those quantities back into the proper coordinates of the inertial frame under consideration. *Shortly*: the quantities pertaining to conservation laws, if calculated with respect to the observed coordinates or to a mixture of observed and proper coordinates, should be rectified and de-contracted by transforming them into the proper coordinates of some inertial system, see [5]. In Einstein's consideration, the validity of conservation laws in mixed coordinates is assured by *ascribing* certain properties to some quantities and/or processes which may change in motion or if observed by other signals transmitting the information. As an *instrument of observation*, relativistic transformations do not affect in any way the physics of matter or waves as such — they produce the *images* thereof and affect the *interaction* of processes. As Einstein has put it in his comment to "different values for masses under different definition of forces and accelerations" in [1], see his statement after (34) in Section 6 above, "one should be very careful" when operating with equations in different relativistic coordinates. In this case, we have to accept the argumentation in [3] as an *extension* of the principle of relativity; cf. Remark 5.1.

**Remark 8.2.** Regarding Einstein's consideration of longitudinal and transverse masses, see (34) in Section 6, together with definition that "The mass of a body is a measure of energy contained in it" [2], see (49) in Section 7 above, and comparing it with the well known equation $E = mc^2$, it is clear that treating the mass as a vector means treating the energy in the same way since $c^2$ is a scalar. The same we can see in (42)–(43) where electrostatic field energy is transformed into kinetic energy of an electron. Albeit the consideration of energy as a vector does not comply with current views, the problem is not so simple. Indeed, kinetic energy $E_k = 0.5mv^2$ is clearly directed along the velocity vector $v$, and along this direction can be used to produce electricity in hydroelectric plants. The same directional effects can be observed in gyroscopic systems. Thus, retaining the notion of mass as a scalar, as postulated by Newton: "The change of motion is proportional to the motive force impressed and is made in the direction of the right line in which the force is impressed." [7], and respecting Einstein's relation $E = mc^2$, we have to emphasize that this energy $E$ is the energy of electromagnetic waves which propagate as *spherical* waves with the *speed of light* as specified by Einstein in [3], see above.

## 9. Lorentz transformations, mass–energy equivalence, and wave invariants [8]

To the derivation of the equation $E = mc^2$, Albert Einstein returns several times [8–10] in relation to different aspects of the theory and its applications, making connections and comparisons that are of much interest. Here, the links to the Lorentz transformations are discussed in quotations from [8], translated from [11, pp. 416–423], in Einstein's notations with our remarks and formula numbers.

In [8], Einstein writes: "The special theory of relativity came out of the Maxwell equations of electromagnetic field. It so happened that even in derivation of principal laws and notions of mechanics, a significant role was played by the laws of electromagnetic field. The question about independence of these laws is quite natural, since the Lorentz transformations, being, as a matter of fact, the basis of the special theory of relativity, are not, in themselves, linked directly to the theory of Maxwell and because we do not know to what degree the notion of energy in the theory of Maxwell may change under the influence of molecular physics. In considerations given below, we shall take as a basis, apart from the Lorentz transformations, only the laws of conservation of energy and impulse.

We start with an attempt to justify the expressions for the energy and the impulse of a material particle in a well known way. The fundamental invariant of the Lorentz transformations is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2, \tag{66}$$

or $$ds = dt(1 - u^2)^{0.5}, \tag{67}$$

where $$u^2 = (dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2 = u_1^2 + u_2^2 + u_3^2. \tag{68}$$

If components of the contravariant vector $(dt, dx, dy, dz)$ are divided by $ds$, then we get the vector

$$(1 - u^2)^{-1/2}, \qquad u_1(1 - u^2)^{-1/2}, \qquad u_2(1 - u^2)^{-1/2}, \qquad u_3(1 - u^2)^{-1/2}. \tag{69}$$

Let the vector $(dt, dx, dy, dz)$ be directed along the world line of a particle with the mass $m$. We shall get the vector related with its motion, if we multiply by $m$ the 4-vector of velocity which is just written above. Thus, we obtain

$$(\eta^\sigma) = [m(1 - u^2)^{-1/2}, mu_i(1 - u^2)^{-1/2}], \tag{70}$$

where index $i$ takes the values from 1 to 3. Ignoring the third power of velocity, we can express the components of this vector as follows

$$(\eta^\sigma) = [m + 0.5mu^2, mu_i]. \tag{71}$$

Space components $(\eta^\sigma)$ in this approximation coincide with components of the impulse in classical mechanics, and the time component, up to an additive constant $m$, coincides with the kinetic energy of a material point.

Returning again to the exact expression for $(\eta^\sigma)$, it is natural to consider

$$mu_i(1 - u^2)^{-1/2}, \tag{72}$$

as impulse, and

$$m[(1 - u^2)^{-1/2} - 1] \tag{73}$$

as the kinetic energy of the particle. But how do we have to interpret the time component $m(1-u^2)^{-1/2}$, the expression of which has quite real sense? Here, it is reasonable to directly ascribe to it the sense of energy, and thus, to ascribe to a still particle the *energy of rest* $m$ ($mc^2$ in usual units).

This conclusion, of course, cannot be considered as a proof, since it does not follow at all that under interaction of several identical particles with each other this impulse agrees with the law of conservation of impulse, and this energy — with the law of conservation of energy; a priori, it could happen that other expressions for velocity enter the laws of conservation.

Besides, it is not quite clear what is to be understood under the *energy of rest,* since the energy is defined only up to uncertain additive constant; in this respect, however, it is worth noting the following. Any system can be considered as a material point, until we do not deal with any other processes, apart from changes of translational velocity as a whole. However, there is quite clear sense in consideration of changes of the energy at rest in case of processes that cannot be reduced to a simple change of translational velocity. Then the interpretation given above requires that in such processes the mass of a material point be changing as energy at rest; this requirement, of course, needs a proof."

**Remark 9.1.** First of all, the word "impulse" is used above in the sense of "momentum" as in [3], see Section 8 above, and exact expression in (73) equals $0.5mu^2$ of (71), up to the third order of small $u$, which presents the kinetic energy of motion if $m$ in (71) is considered as *energy of rest.* Now, the terms in (66) are to be understood as squares of small segments (differentials): $dx^2 = (dx)^2 \neq d(x^2) = 2xdx$. In this sense, the terms in (68) are projections of a 3D-vector of velocity $u$ obtained by division of (66) by $dt^2$, and then taking the square root of (66) presented in (67). Further, there is a *rule of dimension* (denomination) in physics (not to be confused with geometric or topological dimension), which rule states that in any formula related to *physical* values, denominations (units) of all additive terms must be the same. For this reason, the term $dt^2$ in (66) should include a factor $V^2 \neq u^2$ (the speed of a signal propagating uniformly in all directions: a spherical wave carrying the time component $dt^2$, see (84)–(86) below). Indeed, in the CGS system the term $dt^2$ has denomination $\sec^2$ whereas other terms in (66) have denomination $cm^2$. The terms $u^2$, $u_i^2$ ($i = 1, 2, 3$) in (68) all have denomination of velocity $(cm/s)^2$, thus, in relations (66)–(68) the *rule* will be respected, if we assume in (66) the factor $(V\,cm/s)^2 = 1$ that multiplies $dt^2$. In fact, Albert Einstein tells about it noting "the *energy of rest* $m$ ($mc^2$ in usual units)", see above. However, if $V^2 = c^2 = 9 \times 10^{20}$ $(cm/s)^2$, then setting it equal to 1 distorts the scale of dimensions (comparative units), although such normalization is used in the literature, see, e.g. [12, Ch. IX].

**Remark 9.2.** Taking $c^2 = 1$ is equivalent to dividing (66) by $\cong 10^{21}$ which justifies the approximation up to the third order of velocity $u$ in the expansion of $(1-u^2)^{-1/2}$, used by Einstein in (71), since after such division velocities $u_i$ are very small, yielding precision of the order of $\cong 10^{-63}$. However, it is done at a cost of essential distortion in spatial and temporal scales relevant to the terms in (66). □

Further, Einstein writes: "Now we shall demonstrate that if the laws of conservation of energy and impulse are valid in all coordinate systems linked with each other by the Lorentz transformations, then the energy and impulse are really defined by the above mentioned formulae, and supposed equivalence of the mass and energy at rest also exists.

Let us start from the simple kinematical consequences of the Lorentz transformations:

$$t = (t' + vx')(1-v^2)^{-1/2}, \qquad x = (x' + vt')(1-v^2)^{-1/2}, \qquad y = y', \qquad z = z', \tag{74}$$

where $v$ is the relative velocity of coordinate systems $K$ and $K'$. The same relations hold also for differentials $dx$, etc.. Making the corresponding calculations, it is easy to get the law for transformation of the components of velocity:

$$u_1 = (u_1' + v)/(1 + u_1'v), \qquad u_2 = u_2'(1-v^2)^{1/2}/(1 + u_1'v), \qquad u_3 = u_3'(1-v^2)^{1/2}/(1 + u_1'v). \tag{75}$$

From this, it follows …" [expressions are given for the values in (68), (69) through the velocities $u_i'$ ($i = 1, 2, 3$) of (75)]. Further, Einstein considers a couple of particles of equal mass with velocities in $K'$ equal and opposite in direction and derives expressions for respective velocities of those particles, see [8, or 11, p. 419]. Then Einstein writes:

"Let us pass now to the essence of the problem. Suppose that the impulse and energy of a material point are given by expressions of the form

$$I_\nu = mu_\nu F(u), \qquad E = E_0 + mG(u) \quad (\nu = 1, 2, 3), \tag{76}$$

where $F$ and $G$ are universal even functions of velocity $u$, vanishing as $u = 0$. Then $mG(u)$ will represent the kinetic energy, $E_0$ — the *energy at rest* of a material point, and $m$ — the *mass at rest,* or simply the mass. Here it is assumed that impulse and energy of a *point-wise mass* do not depend on the direction of motion and on the orientation of the *point-wise mass* with respect to its velocity. Further, it is assumed that the expressions for impulse and energy contain *one and the same* constant mass $m$. Later, we shall find a partial justification thereof.

Now, let us consider the elastic non-central impact of two particles of equal masses. One can always choose a system of coordinates $K'$ so that with respect to that system the velocities of masses before the impact would be equal to each other in value and opposite in direction. What are the velocities of particles after the impact with respect to the system $K'$? If the velocities after the impact would not be, as before, equal and opposite in direction, then it would contradict to the law of conservation of impulse. If the equal in value velocities of both masses after the impact would not be equal to the respective velocities before the impact, then it would contradict to the law of conservation of energy. These conclusions do not depend at all on a particular form of dependence of the impulse and energy on the velocity. Thus, the impact changes only direction of motion of two point-wise masses with respect to system $K'$. Shortly, this can be expressed as follows: a couple of particles before an impact is transformed after the impact in a couple of particles with the same velocity $u'$." Equations for velocities before and after an impact follow, and Einstein writes: "These equations are true for the general case of elastic impacts of equal masses, and have the form of conservation laws; therefore, it can be considered proven that there are no other symmetric or anti-symmetric functions of components of velocity which in the considered case of elastic impact of two identical point-wise masses would produce similar relations. Accordingly to this, we have to consider $mu_i(1-u^2)^{-1/2}$ of (72) as impulses and $m[(1-u^2)^{-1/2}-1]$ of (73) as kinetic energy of a particle.

Let us proceed now with the proof that the mass equals the *energy of rest.* For the full energy $E$ of a moving particle, we have to take the expression

$$E = E_0 + m[(1-u^2)^{-1/2} - 1], \tag{77}$$

whereby we shall assume that $E_0$ (energy of rest) and $m$ may be changing in case if the interaction of point-wise masses is not *elastic.*

Now let us consider the non-elastic impact of two particles with equal masses and energies at rest, which before the impact formed a couple of particles with respect to system $K'$ (equal in value and opposite velocities). Further, we shall assume for simplicity that the particles at impact undergo the same internal changes. From the law of conservation of impulse, it follows that in system $K'$ the final velocities of particles must be equal in value and opposite in direction. The law of conservation of energy in systems $K'$ and $K$ implies" that the full energy defined by (77) must be equal in systems $K'$ and $K$ for the couple of particles: $E' = E$. Einstein justifies it by consideration of corresponding equations for velocities and energy, see ([8] or [11, p. 421]), for the couple of particles, which results in the equations:

$$E_0 - m + m(1-u^2)^{-1/2}(1-v^2)^{-1/2} = E_0' - m' + m'(1-u'^2)^{-1/2}(1-v^2)^{-1/2}, \tag{78}$$

$$E_0 - m + m(1-u^2)^{-1/2} = E_0' - m' + m'(1-u'^2)^{-1/2}. \tag{79}$$

Einstein writes: "Multiplying the last Eq. (79) by $(1-v^2)^{-1/2}$ and subtracting the result from (78), we obtain

$$[(E_0' - E_0) - (m' - m)][(1-v^2)^{-1/2} - 1] = 0, \tag{80}$$

$$\text{or} \quad E_0' - E_0 = m' - m. \tag{81}$$

Thus, the energy of rest after non-elastic impact changes additively, like the mass. As to the energy of rest, it is defined, as follows from the very notion of energy, only up to an additive constant, and we can impose the condition that $E_0$ become zero together with $m$:

$$E_0 = m, \tag{82}$$

which presents the proof of the principle of equivalence of the inertial mass and the energy of rest.

From the law of conservation of the $x$-component of the impulse, it follows (for a non-elastic impact): $m(1-u^2)^{-1/2} = m'(1-u'^2)^{-1/2}$. This relation follows also from Eqs. (79), (81) obtained from the law of conservation of energy. If from the very beginning, we had assumed that in the expression of impulse enters a constant mass, then using similar considerations, it could be demonstrated that after non-elastic impact the "impulse mass" changes in the same way as "energy mass". This represents a partial justification of the assumed equality of both masses.

Our results can be summarized as follows. If after an impact of point-wise masses the laws of conservation are satisfied in all (Lorentzian) systems of coordinates, then from this alone it follow the known expressions for impulse and energy, as well as the validity of the principle of equivalence of the mass and energy of rest.

Professor Birkhoff has brought my attention that in his book "Relativity and Modern Physics", written jointly with professor Landger, similar considerations are presented about impacts of particles, and also about energy and impulse. This notwithstanding, it seems to me that derivation given above represents certain interest.

In particular, in the just mentioned book it is essentially employed the notion of *force,* which in relativistic theory does not have such a clear sense as in classical mechanics. This is due to the fact that in the latter the force must be considered as a given function of coordinates of all particles, which obviously is impossible in the relativistic theory.

Besides, I avoided making any assumptions about transformational properties of the energy and impulse with respect to the transformations of Lorentz."

**Discussion.** Comparing (74) with Einstein's transformations (17), one can see that (74) are the inverse of transformations (17) if we set $V = 1$, and denote $(1-v^2)^{-1/2}$ as $\beta$, and $(t', x', y', z')$ as $(\tau, \xi, \eta, \zeta)$. Thus, system $K'$ is, in fact, identical to $(k)$ from [1], see Sections 2 and 3 above. Also, components of velocity in (75) coincide with the formulae given in the "Theorem of addition of velocities" in [1, Section 5] for a point moving in $(k)$ with constant velocity $w = (w_\xi, w_\eta, w_\zeta) = (u_1, u_2, u_3)$ of $K'$. This means that the Lorentz transformations represent a particular case of Einstein's transformations (17) which were derived independently using the time synchronization conditions (1)–(2), see Sections 2–4. For this reason, it is interesting to formulate the Lorentz invariant (66) in terms of Einstein's transformations (17) which respect the rule of dimension and would clarify the physical sense of the Lorentz invariant. Squaring the time–space coordinates in (17) and introducing the *measured* velocity $p = d\xi/dt = -\beta v = \text{const}$, cf. [5, Lemma 9.1], instead of postulated (usually unknown) relative velocity $v = \text{const}$, whereby

$$v/V = -(p/V)[1 + (p/V)^2]^{-0.5}, \qquad \beta(v) = [1 - (v/V)^2]^{-0.5} = [1 + (p/V)^2]^{0.5} = \gamma(p) \geq 1, \tag{83}$$

we get the expressions for spherical wave propagation in Einstein's $\beta$ — representation (17), and in its $\gamma$ — representation [5, p. 1567], based on *measured* velocity $p$, as follows:

$$0 = \xi^2 + \eta^2 + \zeta^2 - V^2\tau^2 = \beta^2(x - vt)^2 + y^2 + z^2 - V^2\beta^2(t - vx/V^2)^2 \tag{84}$$

$$= (\gamma x + pt)^2 + y^2 + z^2 - V^2(\gamma t + px/V^2)^2 = x^2 + y^2 + z^2 - V^2t^2 = 0. \tag{85}$$

This means that the observed in $(k)$ spherical waves (84), are identical to initial spherical waves in $(K)$, the last equality (85), thus, Einstein's transformations preserve the identity of initial and observed light propagation waves — a version of synchronization condition equivalent to (1)–(2). Opening the parentheses in (84)–(85), the reader can verify that parameters $\beta, v, \gamma, p$ algebraically cancel out, thus, the *observed*, (84) left, and *initial*, (85) right, waves depend only on the signal propagation velocity $V$ (the speed of light in Einstein's consideration), in agreement with the physical sense of observation process and the synchronization arrangement in Sections 2 and 3 above.

The simple form of the wave equations in (84)–(85) is due to the choice of zero initial conditions, see [1] or [5, Section 7.1]. For arbitrary initial conditions, zeros in (84)–(85) should be replaced by a constant, and squares of coordinates by differences $(x - x_0)^2, \ldots, (t - t_0)^2$, which in differential form with changed sign in (85) yield the Lorentz invariant (66) for $V = 1$; cf. [12, Ch. IX, (9.3.7)–(9.3.9)]. In addition, if we take in (84) the values $\xi = \eta = \zeta = 0$ at some moment $\tau = t' > 0$, then we have

$$ds'^2 = V^2dt'^2 = V^2dt^2 - dx^2 - dy^2 - dz^2, \quad (p, v, V = \text{const}), \tag{86}$$

and if $t' = 0$, as in Einstein's setting $t = 0$ for $\tau = 0$ at the origin, then $ds'^2 = 0$, and we return from the invariant of (86), similar to (66), to the wave equations in (84), (85). From (84)–(85), it follows that propagation of waves in (84)–(85) is invariant also with respect to the measured velocity $p = \text{const}$, which fact is masked in formula (66). Invariant (86) directly relates to the wave propagation (84)–(85) for *arbitrary* information transmitting signal at constant velocity $V$, and expressly shows some contingencies that may be important in real life processes. However, geometric invariant (66) with normalization condition $V = 1$ (cm/s) applies to all kinds of observation signals, hence, it supports the concept of relativity affecting *all* interacting processes, linked by *any* signals, not just by rays of light — the point of view advanced in [5].

There is strong temptation to regard the metric invariant (66) as a pillar of the general relativistic 4D geometry that defines the structure of the universe. Albeit the importance of this invariant, magnified by the beauty of quaternionic considerations [12, Ch. IX], is quite clear, it is worth noting that this invariant relates only to signals propagating as spherical waves at a constant speed $V = 1$ through isotropic media that are applied to observation of events in two coordinate systems moving in such a way that the relative velocity $v$ between the two systems remains constant. In nonlinear relativity at variable velocities, the Lorentz transformations are invalid, whereas Einstein's transformations (17) can be modified, so as they become applicable to piecewise linear trajectories with *measured average* velocities $p_n$ over small intervals of time $dt_n$, see [13,14].

Any measurements take time. Signal propagation (transmission of information) takes time. It means that the measured or transmitted quantities are never known exactly at the time of measurement, but only with some delay $\delta > 0$ when the measured event has already passed and current state of a process in motion has changed. It implies delay and uncertainty of the *measured* (real) time which, if included in Einstein's transformations (17) of *linear* relativity at constant velocities, yields the complete set of relativistic transformations in *real time* as follows, see [5,13] for details:

$$\tau^* = \tau - a\delta = \beta(t - vx/V^2) - \beta^{-1}\delta, \quad \delta = \tau^o + \delta_B - \delta_A, \tag{87}$$

$$\xi^* = \xi - Va\delta = \beta(x - vt) - V\beta^{-1}\delta, \quad \beta = [1 - (v/V)^2]^{-0.5} \geq 1, \tag{88}$$

$$\eta^* = \eta - Va\delta = y - V\beta^{-1}\delta, \qquad \zeta^* = z - V\beta^{-1}\delta, \quad \beta^{-1} = [1 - (v/V)^2]^{0.5} \leq 1. \tag{89}$$

For uncertain delay of information at the level $\delta = 0.1$ s, the imprecision of all three real time coordinates in (88)–(89) attains 30,000 km (equatorial diameter of Earth is 12,756 km), see [5, Section 10]. Such imprecision is due to unaccounted time uncertainty in transmittal of information, and it is absent in numerous theoretical studies of relativistic phenomena.

Considering the first left equality in relations (84)–(85) with the real time coordinates of (87)–(89), we obtain the *real time* wave equation:

$$\xi^2 + \eta^2 + \zeta^2 - V^2\tau^2 = (\xi^* + Va\delta)^2 + (\eta^* + Va\delta)^2 + (\zeta^* + Va\delta)^2 - V^2(\tau^* + a\delta)^2 = 0, \tag{90}$$

and it is clear that abstract time waves (84)–(85) are distorted in the presence of time delays, with the shift of the observed starting point, which is zero in (84), and in the observed time interval. The real waves coincide with the abstract waves in (84)–(85) only if $\delta = 0$, or if for $\delta \neq 0$ it happens that $\xi^* + \eta^* + \zeta^* = V\tau^* - Va\delta$, $a = \beta^{-1}$. Hence, the observed real time spherical waves in $(k)$ are *not* identical to the initial spherical waves in $(K)$ as per (85), the right equality, which agrees with the fact that time uncertainty distorts the original abstract time synchronization. It means that the abstract time homogeneous 4D geometry is distorted in observation of real time processes, and moreover, the fundamental invariant (66) of Lorentz transformations is *not crisp* but rather presents a *continuum of soft intervals* [15] whose diameter may attain 30,000 km, and more, if $\delta > 0.1$ s and the speed of light is accounted therein.

## 10. Mass–energy relation and the rule of dimension

Two basic results in the theory of special relativity are still widely discussed in the literature, generating profound interest and curiosity:

*Relativistic expression for the mass* ([9]; [16, pp. 46–47]; [12, Ch. IX, Section 5]; [17, p. 643])

$$m = \beta m_0 = m_0[1 - (v/V)^2]^{-0.5} = m_0 + 0.5 m_0 v^2/V^2 + 0.375 m_0 v^4/V^4 + \cdots, \tag{91}$$

where $m_0$ is the rest mass in $(k)$ and $m = \beta m_0 \geq m_0$ is the mass observed in $(K)$ from $(k)$.

*The mass–energy relation* [1–3,8–10], see (35), (42), (43), (49), (65), (73), (77), (82), that includes kinetic energy $E_k$ as observed in $(K)$ from $(k)$:

$$E_k = m_0 V^2(\beta - 1) = m_0 V^2(0.5 v^2/V^2 + 0.375 v^4/V^4 + \cdots), \tag{92}$$

and the full energy $E$ that includes the energy at rest $E_0$:

$$E = E_0 + E_k = m_0 V^2 + m_0 V^2(\beta - 1) = \beta m_0 V^2 = mV^2. \tag{93}$$

The corresponding notations of Einstein cited in quotations from his papers in preceding sections are $W$ for $E_k$ and $\mu$ for $m_0$. The formulae (91)–(93) are valid under the conditions:

$$\text{relative velocity } v = \text{const}, \qquad \text{signal velocity } V = \text{const}, \quad 0 \leq v < V < \infty. \tag{94}$$

If $v = 0$, then $\beta = 1$, $\tau = t$, $\xi = x$, for any $V \neq 0$, relativistic transformations (17) become trivial identities, $m = m_0$ (the *rest mass*), $E_k = 0$, and the relation $E = E_0 = m_0 V^2$ (the energy *at rest*) implied by those transformations is undefined (arbitrary $V$ of no effect).

If $v > 0$, $V \to \infty$, the case of instantaneous signal propagation, then $\beta \to 1$, so that, in the limit, $\tau = t$, $\xi = x - vt$ in (17), and we return to the absolute time and mechanics of Newton. Thereby $m = m_0$, $E_k = 0.5mv^2$, and the value $E_0 = m_0 V^2$ is undefined.

If $V = \text{const}$, $v \to V$, then $\beta \to \infty$, $\tau \to \infty$, $\xi \to \infty$ for any finite $t$, $x$, thus, a process in $(k)$ cannot be observed in finite time, relativistic transformations (17) as well as the Lorentz transformations (66), (74) become inapplicable, and the values $m = \beta m_0 \to \infty$, $E = mV^2 \to \infty$ are undefined.

If $v \neq$ const or $V \neq$ const, then Einstein's transformations (17) and also the Lorentz transformations (66), (74) are *invalid* as well as all results obtained with the use of those transformations. The complete theory of this general case is yet to be developed. Certain results making use of discretization and piecewise linear approximations based on direct measurements in $(K)$ of the velocity $d\xi/dt$ of a motion $\xi(.) \in (k)$ with respect to the *proper* time $t$ of $(K)$ can be found in [13,14].

If $0 < v < V < \infty$ with *constant* velocities $v$ and $V$ (the standard case), then Einstein's transformations (17) are valid for *any* constant velocity $V$ of an available observation signal (the carrier of information), see [5], so the quantities $m$ in (91), $E_k$ in (92), $E$ and $E_0$ in (93) appear as depending on the velocity of a signal used for synchronization of time, thus, having different values for different signals, which contradicts the reality of mass and energy existing irrespective of the employed observation system. Those values appear as *images* produced by relativistic transformations with the actually employed signals transmitting the information to compute those values. The choice of $V = c$, adopted by Einstein and in the literature, does not change the situation. However, the *action* of one process, in $(k)$, onto another process, in $(K)$, is realized through its *image* or influence from $(k)$ by means of a physical process (observation system) with relativistic effects and real time delays (if accounted) in transmission of information. Therefore, it is the mass $m$ of (91) and the kinetic energy $E_k = (m - m_0)V^2 \cong 0.5 m_0 v^2$ of (92) which are really felt (observed) in $(K)$ as an action from $(k)$. Later, when huge amounts of energy were discovered on atomic levels, the equation $E = mV^2$ of (93) was considered to be experimentally confirmed. The questions reside with proportions and possible kinds of velocity in this formula for the mass–energy relation.

**Rule of dimension.** *Additive terms of physical formulae must be of same denomination.* Indeed, one cannot add apples, cows and nails; however, their weights can be added to check the weight limit before loading a spacecraft for a flight. Einstein's transformations (17) respect the rule of dimension, whereas Lorentz transformations (74) do not respect this rule. Indeed, if $v^2$ in (74) is presumed to be unit-free, like $v/V$ in Einstein's formulae (17), then the first two equations in (74) present denominational inconsistencies because [s] $\neq$ [s] + [cm], [cm] $\neq$ [cm] + [s]. Now, if $v$ as relative velocity is measured in *cm/s,* then, ignoring the unit inconsistency of the factor $(1 - v^2)^{-1/2}$, we have the inconsistency in the first equality of (74): [s] $\neq$ [s] + [cm$^2$/s]. Formulae with denominational inconsistencies, even if used in mathematics, are questionable in physics. Formulae used in physics should be unit-rectified and scale de-contracted *before* making any conclusions or evaluations. For this reason, it is preferable to argue on the basis of Einstein's theory of relativity, and not on the basis of the Lorentz transformations.

Application of special relativity is subject to the rule of dimension too. Addition of time intervals in *proper* $t$ and *observed* $\tau$ time, even in the same units, say, in seconds, does not make sense because $\tau = \beta t$, presenting two different time scales if $\beta > 1$. For this reason, equations in mixed coordinates require the use of normalization to transform the relations into one and the same coordinate system taking into account relativistic effects.

The relation $E = mc^2$ is derived by Einstein in [2,3] from different conservation laws, see Sections 7 and 8 above, then justified by its agreement with Lorentz transformations [8], see Section 9, and discussed also in [9,10]. On the basis of the currently accepted physical unit system supported by centuries of experiments, the form of mass–energy relation follows, in fact, from the rule of dimension. Indeed, in the CGS system we have:

$$[E] = [\text{erg}] = [\text{dyne} \times \text{cm}] = [(\text{g} \times \text{cm/s}^2) \times \text{cm}] = [\text{g} \times (\text{cm/s})^2] = [\text{m w}^2]. \tag{95}$$

This denominational equation suggests the relation $E = \theta m w^2$, where $E, m$ denote quantities of energy and mass, $w$ denotes a kind of velocity, and $\theta$ is a factor of proportionality, cf. the formulation of the Second Law of motion by Newton cited above in Remark 8.2. For the energy (work) that is accumulated in a mass moving with a speed $w = v$ along a right line, we have $\theta = 0.5$ which follows from integration in (35) due to gradual accumulation of kinetic energy in accelerating mass. Its complete transformation into other forms of energy (not into the mass itself) happens when the mass comes to a full stop *without any changes* in the mass as a whole. This case, is considered by Einstein in [1,2] for transformation of the energy of electrostatic field into the kinetic energy of an electron, see Section 6 above where $\theta = 0.5$, $m = \mu$. The formula $E = \theta m w^2$ can accommodate other formulae by adjusting the values of $\theta$ and $w$ according to a process in consideration. With certain modifications, it can be used to represent partial mass and multi-energy transformations, as well as relativistic values in *observed* coordinates of multi-relativistic processes.

The values $\theta = 1$, $w = V = c$, where $V$ is velocity of the signal transmitting the information, correspond to Einstein's formula $E = mV^2 = mc^2$ which suggested that the rest mass $m = m_0$ (if $v = 0$) may itself carry large amounts of internal energy, and the problem is whether and how it could be released and in which form and proportions. In [10], Albert Einstein wrote about the early knowledge (1946) of such processes as follows: "We know only one phenomenon where a quantity of energy at such level is released per one unit of mass; this is radioactive fission. Schematically, the process goes in this way: an atom with a mass $M$ is splitting in two atoms with masses $M'$ and $M''$, which are separating at huge kinetic energy. If we stop those atoms, i.e., take from them the energy of motion, they would have in common much less energy than the initial atom. According to the principle of equivalence, the total mass $M' + M''$ of the products of fission must be a little less than the initial mass $M$ of the splitting atom, which contradicts to the old principle of the conservation of mass. The relative difference of those masses makes approximately one tenth of one per cent." (Translated from [11, pp. 655–656]).

If the speed of separated two atoms is the same and denoted by $v < c$, then their total kinetic energy, with the relativistic increase of mass $m = \beta(v)m_0 = m_0[1 - (v/c)^2]^{-0.5}$, equals $E = 0.5M'v^2 + 0.5M''v^2 \rightarrow 0.5(M' + M'')\beta v^2 = 0.5(M - \Delta M)\beta v^2$, since there is a loss of mass $\Delta M = M - (M' + M'') \cong 0.001M$. Now, using the Einstein formula for the internal energy of mass, we obtain: $0.5(M - \Delta M)\beta v^2 = \Delta M c^2$, which for $\Delta M = 0.001M$ yields the equation $v^4 = 4.00801 \times 10^{-6} c^2(c^2 - v^2)$. The approximate positive solution of this equation is $v \cong (0.002)^{0.5} c \cong 0.04472c = 0.13416 \times 10^{10}$ (cm/s) $=$ 13,416 (km/s), about 4.5% of the speed of light. If the formula $E = \theta m v^2$ is used, then $E = \theta(M - \Delta M)\beta v^2 = \Delta M c^2$, so that $\theta = [1 - (v/c)^2]^{0.5}(c/v)^2[(M/\Delta M) - 1]^{-1}$, thus, for this process of fission, $\theta(\Delta M/M, v/c)$ can be expressed in terms of the relative loss of mass and the fraction of the actual velocity of separating particles with respect to the speed of light.

**Safety in particle collisions.** The proton rest mass is $m_p = 1.672614(11) \times 10^{-27}$ kg [18] with uncertainty in two last digits (11) of mantissa, and the same value is given in [17, p. 1093]. Accordingly, the proton rest energy $E_p = m_p c^2 = 938.2592(52)$ MeV, cited in the same source [17] from which the data below are taken. In the Large Hadron Collider (LHC) built by CERN "underneath the Franco–Swiss border between the Jura Mountains and the Alps near Geneva, Switzerland,... it is intended to collide opposing beams of protons or lead ions, each moving at approximately 99.999999% of the speed of light." [19, 2008-11-19, p. 1].

To compare the levels (orders) of energy that may be released, we take this projected speed in its rounded value, the speed of light $c = 3 \times 10^{10}$ cm/s. We can also assume that each two protons are colliding head-on to the full stop, and that energy released comes from their kinetic energy acquired in the particle accelerator LHC. By the relativistic formula, the total energy released in collision of two protons accelerated to the speed $v$ and then abruptly brought to the full stop is $E_p = 2 \times (0.5\beta m_p v^2) = m_p v^2[1 - (v/c)^2]^{-0.5}$ which for $v$ close to the speed of light, as cited above, is very big since $\beta \rightarrow \infty$ as $v \rightarrow c$, thus, $E_p \rightarrow \infty$. So we compare data given in [19, 2008, p. 3, or 2009, p. 3] where, for two protons each having an energy of 7 TeV, "a total collision energy of 14 TeV (2.2 μJ)" is cited, which equals $14 \times 10^{12}$ eV $= 1.4 \times 10^7$ MeV $= 2.24$ μJ.

To compare it with the energy release per nucleon in a thermonuclear reaction, we cite from [17, p. 1045]: "The most feasible with respect to the necessary temperatures is the reaction between the nuclei of deuterium and tritium: ${}_1D^2 + {}_1T^3 \rightarrow {}_2He^4 + n$. The energy released in this reaction is 17.6 MeV. The energy release per nucleon in such a reaction is ≅3.5 MeV. In the fission of the nucleus ${}_{92}U^{238}$ only 0.85 MeV per nucleon is released." Comparing these numbers with the energy per one collision given in the CERN information cited above, we have *per nucleon*:

Protons collision vs Thermonuclear reaction (fusion) $= 1.4 \times 10^7$ MeV/3.5 MeV $= 4 \times 10^6$;
Protons collision vs Fission of the nucleus ${}_{92}U^{238} = 1.4 \times 10^7/0.85 = 1.65 \times 10^7$.

The numbers are eloquent. A total collision energy of *two* protons in the LHC is expected to be $4 \times 10^6$ greater than the energy release *per nucleon* in the thermonuclear reaction cited above. In fact, more than 2 protons will be colliding and if a stream of protons makes a fraction of a mole, then the same fraction of Avogadro's number $N_A = 6.022 \times 10^{23}$ (1/mole) will multiply the amount of energy released.

To this, we can add three relevant citations:

1. "Rather than continuous beams, the protons will be bunched together, into 2,808 bunches, so that interactions between the two beams will take place at discrete intervals never shorter than 25 nanoseconds (*ns*) apart." [19, 2008, p. 3]

2. "Loss of only one ten-millionth part ($10^{-7}$) of the beam is sufficient to quench a superconducting magnet, while the beam dump must absorb 362 MJ (87 kg of TNT) for each of the two beams. These immense energies are even more impressive considering how little matter is carrying it: under nominal operating conditions (2808 *bunches per beam*, $1.15 \times 10^{11}$ protons per bunch), the beam pipes contain $1.0 \times 10^{-9}$ g of hydrogen, which, in standard conditions for temperature and pressure, would fill the volume of one grain of fine sand." [19, 2009, p. 7]

3. "The upcoming experiments at the Large Hadron Collider have sparked fears among the public that the LHC particle collisions might produce doomsday phenomena, involving the production of stable microscopic black holes or the creation of hypothetical particles called strangelets." [19, 2008, p. 6; and 2009, p. 7; here is also published "The safety of the LHC", pp. 1–5, and mentioned *Main article: Safety of particle collisions at the Large Hadron Collider*]

Leaving aside the black holes, strangelets, and doomsday for the universe or a planet (excluded in "The safety of the LHC" by references to the LSAG report (2008) and to past experience with cosmic rays), one may have earthly fears about the safety of the City of Geneva and of the people, personnel who work at the LHC. Moreover, the notable LHC advantage of having the highest speed of protons close to the speed of light means also its weakness regarding the possibility of correct observation of those protons in motion. Events in accelerators are observed on moving particles through relativistic time and coordinates that provide a *distorted image* of a physical process really evolving in the accelerator. The motion of particles moving at a given speed cannot be observed with signals of the same or lower speed, but only with signals propagating at a *greater* speed than the speed of the particle. In experiments with protons moving at about 99.999999% of the speed of light and with no available signal of a significantly greater speed, this may lead to illusory effects and misinterpretations caused by relativistic contraction of the *observed* time plus uncertain time delays due to information transmittal, see [5,6]. This may also compromise the timely reaction of alarm systems and shutdown devices.

## 11. Conclusion: Inertial systems, signals, and relativity

1. *Inertial systems.* Natural (physical) laws are formulated verbally or by certain formulae related to a chosen system of reference (a frame or coordinate system). Systems of reference are in motion with respect to each other, thus, the time and coordinate transformations from one frame to another depend on the motion of each frame and on *signal(s)* of communication between each two frames. A frame which is *still,* i.e. at *absolute rest* (a primitive term which cannot be further defined through simpler terms), and frames which are in translational motion at constant velocities with respect to each other and to the still frame are called *inertial.* These intuitive notions come from Galileo or some ancient civilizations lost in the past, but they should be remembered in their correct sense. Since planets are in orbital motions around the Sun, and also around their own axes, and moving with the Sun through the "expanding" universe, the natural inertial systems do not exist. However, such systems are routinely considered with the origin at some point of choice and axes pointing at some directions, and *called* inertial, ignoring unknown motion of the system. Though seemingly absurd, such liberty is used by default in science and technology, with or without evaluation of errors.

Another approach is *to postulate* an inertial Cartesian *still* system with the origin at some point of choice assumed still, and then try to find the formulae transforming its absolute coordinates in the universe into realistic coordinates of some convenient system (*non-inertial*) located on Earth at a point of interest. This problem is beyond the scope of the paper, and at this time we cannot even guess how or if it can be solved, or whether it is decidable at all. So we use realistic systems of reference deemed inertial by default and accept or evaluate errors that may be associated with such choices, by comparison with *measured* (observed) data provided by currently available instrumentation. Thereby, it is important to understand that *experiments and computations cannot give us more than the nature allows us to obtain through the signals employed in those experiments and computations.*

2. *The principle of relativity.* It states that physical laws (and their formulae) are invariant (i.e. expressed in one and the same form) with respect to all inertial frames in their *proper* time–space coordinates $x, y, z, t$, *not* in transformed coordinates $\xi$, $\eta$, $\zeta$, $\tau$ observed in one frame from another. In transformed coordinates $\xi$, $\eta$, $\zeta$, $\tau$, the laws of nature may also be transformed and generally have a different form as compared with the form in proper time–space coordinates. Processes unfolding in a moving frame, inertial or not, are governed by the laws acting in its proper coordinates, not by their transformed *images* in the observed time–space coordinates $\xi$, $\eta$, $\zeta$, $\tau$, and if we want to recover the correct natural laws from their images, we have to rectify and de-contract those images into the proper natural coordinates $x, y, z, t$ of a moving system, as argued in [5]. However, the processes in a moving system and their effects can be known in the system of observation only through the *observed* coordinates $\xi$, $\eta$, $\zeta$, $\tau$, and for this to be possible, the correct relativistic transformations have to be established between two systems of communication, irrespective whether they are inertial or not, moving with constant or variable velocities, observed with the use of rays of light (Einstein) or with other signals at speeds different from the speed of light.

3. *The role of light.* Citation from Albert Einstein: "The theory of relativity is often criticized for giving without justification, a central theoretical role to the propagation of light, in that it founds the concept of time upon the law of propagation of light. The situation, however, is somewhat as follows. In order to give physical significance to the concept of time, processes of some kind are required which enable relations to be established between different places. It is immaterial what kind

of processes one chooses for such a definition of time. It is advantageous, however, for the theory, to choose only those processes concerning which we know something certain. This holds for the propagation of light *in vacuum* in a higher degree than for any other process which could be considered, thanks to the investigations of Maxwell and H.A. Lorentz'' [16, p. 28].

These words of Einstein support the point of view advanced in [5] that relativistic transformations (17) are valid for any constant velocity $V > v$ of the signal-carrier of information between two frames moving at relative velocity $v = \text{const}$ with regard to each other. With due respect to the classical theory of relativity based on rays of light as the signals of communication, we have to recognize that in practice, especially in living organisms, the rays of light are *not* the preferred or even possible means of communication but some other signals are, which signals are to be incorporated into relativistic transformations in order to obtain viable realistic instruments to study the nature and to control processes connected by those signals. *Light or no light, the relativity is all around us and we have to cope with it in real life on Earth, our non-inertial planet.*

4. *Multi-relativity*. Interacting physical processes are linked by different communication signals, not just by rays of light. Due to variable velocities and to diversity of those communication signals, different relativities exist that provide necessary links between intertwined processes in technology and in living organisms. Real life is not constrained by a single relativistic invariant or by a couple of currently adopted transformations with constant relative and signal velocities.

5. *Presumption of physical reality*. To avoid misconceptions and misinterpretations, three simple facts should be recognized in scientific considerations:

*Statement* 5.1. *For every physical notion or process, there is a carrier or carriers* (known or yet unknown). For example, transmission of information is carried by signals.

*Statement* 5.2 *(Extended relativity).* Measurements and computations produce a relativistic *image* through which a physical process becomes known. This real time image depends on *signals* employed in measurement and computation and contains time uncertainty, relativistic distortion, imprecision of instruments employed, and possible errors of computation. To retrieve the realization of a genuine process from its image given in observation, *rectification and de-contraction* of that image should be performed.

Since physical processes actually interact and affect each other only through their *real time* realizations, those realizations are usually accepted as the processes themselves, and in this quality contain the unavoidable imprecision and distortion produced by time uncertainty and relativistic effects at variable velocities [13,14] which should be properly understood and taken into account in the process control, technology and medicine.

*Statement 5.3. Mathematical transformations, formulae and theoretical arguments cannot create physical realities (substances or processes: mass, energy, waves, signals).*

Theoretical considerations are instruments of understanding that may create additional knowledge (or misconceptions) but *not* physical matter, process or quality thereof. For example, the notion of velocity has no sense if there is no carrier (particle, signal, a point as mathematical abstraction) moving with that velocity.

From this point of view, the statement that the speed of light in *vacuum* is not attainable needs clarification. It may be impossible to achieve it by accelerating some lump masses with available physical processes, but there are *carriers* of such speed: the rays of light or electromagnetic waves. The rays of light are carriers of energy, resulting in the pressure of light. The photon (carrier of light) is *ascribed* zero rest mass, so its momentum $p_v$ is defined on the basis of its energy: $p_v = m_v v_v = \beta(E_v/c^2)v_v = [1-(v_v/c)^2]^{-0.5}(E_v/c^2)v_v$ where $v_v = \mathrm{d}x_v/\mathrm{d}t$, cf. (60) in Section 8 above.

As to superluminal velocities, we do not have physical instruments, signals to discover and identify such. The phase velocity of light $c/n$ in a medium with refractive index $n$ can be attained and exceeded by a charged particle, and in this case, the electromagnetic radiation is observed (Vavilov–Cherenkov radiation). In special relativity with the Einstein (17) or Lorentz (74) transformations, the speed of light $V = c$ in (17), or $V = 1$ in (74), or any speed $V$ of the measurement signal, cannot be attained *mathematically,* because of calibrating factor $\beta = [1-(v/V)^2]^{-0.5}$ which tends to infinity as $v \to V$ and does not allow $v \geq V$ in (17) and $v \geq 1$ in (74), since those transformations do not exist in this case. In contrast, if we use the $\gamma$-representation for relativistic transformations which follows from the *observed* wave representation in (85) with $\gamma$ of (83) and can be retrieved by comparison of (84) and (85), we can see that $\gamma(p)$ does not tend to infinity as $p \to V$. The $\gamma$-representation corresponds to a different way of observation in systems $(K)$ and $(k)$, not from the origin A of a moving frame $(k)$, an asteroid, as proposed by Einstein in [1], see the text before (3) in Section 3 above, but in reverse, from point B in $(K)$, on Earth, with reflection at point A in $(k)$, on asteroid, which yields the *measured* velocity $p = \mathrm{d}\xi/\mathrm{d}t$ of the observed object, the asteroid, with respect to the *proper* (natural) time $t$ in $(K)$, for details see [5,13]. If $p$ can be measured (by some signal with velocity $V > p$ whatever it may be), then the observation can be completed. Thus, the natural physical barrier to *observed* velocity is imposed by *physical measurability* of such velocity, and to velocity in accelerated motion by the velocity of propulsion and the power applied by the acting acceleration devices (e.g., by the force of electromagnetic waves propagating at the speed of light), but *not* by the requirement of some finite limit in a mathematical formula. If a moving body $\xi(.)$ cannot be observed by available signals propagating at a speed $V < \mathrm{d}\xi/\mathrm{d}t$, it does not mean that a body or a process $\xi(.)$ does not exist — it may exist being unobservable. Sound signals can be used to observe subsonic flights, but not supersonic. Phase speed of light $c/n$ in a refractive media (water, air) can be exceeded which effect was experimentally observed. Even a vacuum where rays of light (thus, lots of photons) or electromagnetic waves (carrying energy, thus, ''equivalent masses'') are propagating is *not* the vacuum in the exact sense of this word. There is a vast area of further research in this field, with applications of paramount importance.

## References


[1] Albert Einstein, Zur Elektrodynamik der bewegter Körper, Annalen der Physik 17 (1905) 891–921.
[2] Albert Einstein, Ist die Trägheit eines Körpers von seinem Energieinhalt abhängig? Annalen der Physik 18 (1905) 639–641.
[3] Albert Einstein, Das Prinzip von der Erhaltung der Schwerpunktsbewegung und die Trägheit der Energie, Annalen der Physik 20 (1906) 627–633.
[4] Albert Einstein, Collected Scientific Papers in Four Volumes. Vol. 1, Works on Relativity Theory, years 1905–1920, Nauka, Moscow, 1965 (in Russian).
[5] E.A. Galperin, Information transmittal, time uncertainty and special relativity, Computers & Mathematics with Applications 57 (9) (2009) 1554–1573; doi:10.1016/j.camwa.2008.09.048.
[6] E.A. Galperin, Information transmittal and time uncertainty, measuring the speed of light and time of reflection, representations of the second Newton's law, and related problems, Computers & Mathematics with Applications 56 (5) (2008) 1271–1290; doi:10.1016/j.camwa.2008.02.029.
[7] Isaac Newton, Phylosophiae Naturalis Principia Mathematica, printed under the auspices of the London Royal Society, London, 1687, English translation: Mathematical Principles of Natural Philosophy, translated by A. Mott in 1729, University of California Press, 1934.
[8] Albert Einstein, Elementary derivation of the equivalence of mass and energy, Bull. of American Mathematical Society 61 (4) (1935) 223–230. See also under the same title in: Technical J., Vol. V, Haifa, 1946, pp. 16–17.
[9] Albert Einstein, Über die vom Relativitätsprinzip geforderte Trägheit der Energie, Annalen der Physik 23 (1907) 371–384.
[10] Albert Einstein, $E = mc^2$: The urgent problem of our time, Science Illustrated I (1946) 16–17.
[11] Albert Einstein, Collected Scientific Papers in Four Volumes. Vol. 2, Works on Relativity Theory, years 1921–1955, Nauka, Moscow, 1965 (in Russian).
[12] C. Lanczos, The Variational Principles of Mechanics, second edition, University of Toronto Press, Toronto, 1962.
[13] E.A. Galperin, Special relativity for motions with variable speed, Computers & Mathematics with Applications 58 (3) (2009) 536–547; doi:10.1016/j.camwa.2009.03.091.
[14] E.A. Galperin, Relativistic transformations at variable velocities, Computers & Mathematics with Applications 59 (2010) 886–897; doi:10.1016/j.camwa.2009. 10.005.
[15] D. Molodtsov, Soft Set Theory — First Results, Computers & Mathematics with Applications 37 (4–5) (1999) 19–31.
[16] Albert Einstein, The Meaning of Relativity, fifth edition, Princeton University Press, Princeton, NJ, 1954 (Fourth Princeton Printing, 1974).
[17] B. Yavorsky, A. Detlaf, Handbook of Physics, Mir Publishers, Moscow, 1975.
[18] Taylor, Parker, Langenberg, Review of Modern Physics 41 (1969) 375.
[19] http://en.wikipedia.org/wiki/Large_Hadron_Collider (Updates: 2008-11-19; 2009-09-22).