Multiplicity of positive solutions for semilinear elliptic problems in unbounded domains

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Abstract

In this paper, we study the effect of domain shape on the multiplicity of positive solutions for the semilinear elliptic equations. We prove a Palais–Smale condition in unbounded domains and assert that the semilinear elliptic equation in unbounded domains has multiple positive solutions.

Keywords: Semilinear elliptic equations; Palais–Smale; Multiple positive solutions

1. Introduction

Let $N \geq 2$ and $2 < p < 2^*$, where $2^* = \frac{2N}{N-2}$ for $N \geq 3$ and $2^* = \infty$ for $N = 2$. Consider the semilinear elliptic equation

$$\begin{cases} -\Delta u + u = |u|^{p-2}u & \text{in } \Omega, \\ u \in H^1_0(\Omega), \end{cases}$$

where $\Omega$ is a domain in $\mathbb{R}^N$ and $H^1_0(\Omega)$ is the Sobolev space in $\Omega$ with dual space $H^{-1}(\Omega)$. Associated with Eq. (1.1), we consider the energy functionals $a, b$ and $J$ in $H^1_0(\Omega)$

$$a(u) = \int_\Omega (|\nabla u|^2 + u^2), \quad b(u) = \int_\Omega |u|^p, \quad J(u) = \frac{1}{2} a(u) - \frac{1}{p} b(u).$$

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It is well known that the solutions of Eq. (1.1) and the critical points of the energy functional $J$ in $H^1_0(\Omega)$ are the same. The minimax method is a typical approach for solving problem of this kind, that is
\[
\alpha_{\Gamma}(\Omega) = \inf_{\gamma \in \Gamma(\Omega)} \max_{t \in [0,1]} J(\gamma(t)),
\]
where
\[
\Gamma(\Omega) = \{ \gamma \in C([0,1], H^1_0(\Omega)) \mid \gamma(0) = 0, \gamma(1) = e \},
\]
$J(e) = 0$ and $e \neq 0$. By the well-known mountain pass lemma due to Ambrosetti–Rabinowitz [1], we called the nonzero critical point $u \in H^1_0(\Omega)$ of $J$ is a ground state solution of Eq. (1.1) in $\Omega$ if $J(u) = \alpha_{\Gamma}(\Omega)$. We remark that ground state solutions of Eq. (1.1) in $\Omega$ can also be obtained by the Nehari minimization problem
\[
\alpha(\Omega) = \inf_{v \in M(\Omega)} J(v),
\]
where $M(\Omega) = \{ u \in H^1_0(\Omega) \setminus \{0\} \mid a(u) = b(u) \}$. Note that $\alpha_{\Gamma}(\Omega) = \alpha(\Omega) > 0$ (see Willem [24]).

That the existence of ground state solutions of Eq. (1.1) is affected by the shape of the domain $\Omega$ has been the focus of a great deal of research in recent years. By the Rellich compactness theorem and the minimax method, it is easy to obtain a ground state solution for Eq. (1.1) in bounded domains. For a general unbounded domain $\Omega$ and under various conditions, several authors have established the existence of ground state solutions. We mention, in particular, results by Berestycki–Lions [3], Lien–Tzeng–Wang [17], Chen–Wang [8] and Del Pino–Felmer [10,11]. In [3], the domain $\Omega = \mathbb{R}^N$. Actually, Kwong [16] proved that the positive solution of Eq. (1.1) in $\mathbb{R}^N$ is unique. In [17], the domain $\Omega$ is a periodic domain. In [8,17], the domain $\Omega$ is required to satisfy the following condition:

\begin{itemize}
  \item[(Ω1)] $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 \cap \Omega_2$ is bounded and $\alpha(\Omega) < \min(\alpha(\Omega_1), \alpha(\Omega_2))$.
\end{itemize}

Let $1 \leq l \leq N - 1$, $\mathbb{R}^N = \mathbb{R}^l \times \mathbb{R}^{N-l}$ and $z = (x, t) \in \mathbb{R}^l \times \mathbb{R}^{N-l}$. Denote the projection of $\Omega$ onto $\mathbb{R}^{N-l}$ as $\Omega^x = \{ t \in \mathbb{R}^{N-l} \mid (x, t) \in \Omega \}$. In [10,11], the domain $\Omega$ is required to satisfy the following conditions:

\begin{itemize}
  \item[(Ω2)] $\Omega$ is a smooth subset of $\mathbb{R}^N$ and the projections $\Omega^x$ are bounded uniformly in $x \in \mathbb{R}^l$;
  \item[(Ω3)] there exists a nonempty closed set $F \subset \mathbb{R}^{N-l}$ such that $F \subset \Omega^x$ for all $x \in \mathbb{R}^l$;
  \item[(Ω4)] for each $\delta > 0$ there exists a $K > 0$ such that
    \[
    \Omega^x \subset \{ t \in \mathbb{R}^{N-l} \mid \text{dist}(t, F) < \delta \}
    \]
    for all $|x| \geq K$.
\end{itemize}

Moreover, when $\Omega = \mathbb{R}^N \setminus \omega$ is an exterior domain, where $\omega$ is a bounded domain, it is well known that Eq. (1.1) in $\mathbb{R}^N \setminus \omega$ does not admit any ground state solution (see Benci–Cerami [4]). However, Bahri–Lions [2] and Benci–Cerami [4] asserted that Eq. (1.1) in $\mathbb{R}^N \setminus \omega$ has a positive higher energy solution. When $\Omega$ is an Esteban–Lions domain, Eq. (1.1) in $\Omega$ does not admit any nontrivial solution (see Esteban–Lions [15]).

By the above results, we know that the existence of solutions (or ground state solutions) of Eq. (1.1) depends on the geometry and topology of domain $\Omega$. First, we state our main results in
this paper, which improve the main results in [8,17]. Suppose \( k \geq 2 \) and assume that the domains \( \Theta_1, \Theta_2, \ldots, \Theta_k \) are satisfying the following conditions:

\((D1)\) \( \Theta_i \cap \Theta_j = A_{i,j} \) is bounded (possible empty) for each \( i \neq j \);
\((D2)\) \( \Theta_1, \Theta_2, \ldots, \Theta_m \) are bounded domains and \( \Theta_{m+1}, \Theta_{m+2}, \ldots, \Theta_k \) are unbounded domains for \( 0 \leq m \leq k-1 \);
\((D3)\) \( \bigcup_{i=1}^{k} \Theta_i \) is a domain in \( \mathbb{R}^N \).

Let \( \Theta = \bigcup_{i=1}^{k} \Theta_i \). Then we have the following Palais–Smale (simply by (PS)) condition in \( H^1_0(\Theta) \) for \( J \).

**Theorem 1.1.** If \( \{u_n\} \) is a (PS)\( \beta \)-sequence in \( H^1_0(\Theta) \) for \( J \) with \( \alpha(\Theta) \leq \beta < \min \{ 2\alpha(\Theta), \alpha(\Theta_{m+1}), \alpha(\Theta_{m+2}), \ldots, \alpha(\Theta_k) \} \), then there exist a subsequence \( \{u_n\} \) and \( u_0 \neq 0 \) such that \( u_n \rightharpoonup u_0 \) strongly in \( H^1_0(\Theta) \) and \( J(u_0) = \beta \). Furthermore, \( u_0 \) is a positive (or negative) solution of Eq. (1.1) in \( \Theta \).

Next, we will apply Theorem 1.1 to prove the multiplicity of positive solutions for Eq. (1.1) in an unbounded domain. We assume that the domains \( \Theta_1, \Theta_2, \ldots, \Theta_{m+1} \) are satisfying the following conditions:

\((D4)\) the domain \( \Theta_{m+1} \) satisfies condition (\( \Omega 1 \)) or conditions (\( \Omega 2 \))–(\( \Omega 4 \));
\((Dr)\) there exist points \( z_1, z_2, \ldots, z_m \) in \( \Theta_{m+1} \) such that
\[ B^N(z_i; r) \subseteq \Theta_i \subseteq B^N(z_i; r+1) \text{ for all } i = 1, 2, \ldots, m, \]
and \( |z_i - z_j| > 3r \) for \( i \neq j \), where \( B^N(z_i; r) = \{ z \in \mathbb{R}^N \mid |z - z_i| < r \} \).

Clearly, the domains \( \Theta_1, \Theta_2, \ldots, \Theta_{m+1} \) satisfy conditions (\( D1 \))–(\( D3 \)). Let \( \Omega(r) = \bigcup_{i=1}^{m+1} \Theta_i \). Then we have the following result.

**Theorem 1.2.** For each domain \( \Theta_{m+1} \) satisfies condition (\( D4 \)), there exists an \( r_0 > 0 \) such that for \( r \geq r_0 \), the domains \( \Theta_1, \Theta_2, \ldots, \Theta_{m+1} \) satisfy condition (\( Dr \)). Then Eq. (1.1) in \( \Omega(r) \) has \( m \) positive solutions \( u^1_0, u^2_0, \ldots, u^m_0 \) such that
\[ \int_{\Theta_i} |u^i_0|^p \leq \frac{p}{p-2} \alpha(\mathbb{R}^N) \text{ for all } i = 1, 2, \ldots, m. \]

**Remark 1.1.** For each \( r > 0 \) the domain \( \Omega(r) \) satisfies condition (\( \Omega 1 \)) or conditions (\( \Omega 2 \))–(\( \Omega 4 \)). We have that the domain \( \Omega(r) \) is a special case in Chen–Wang [8], Lien–Tzeng–Wang [17] or Del Pino–Felmer [10,11]. However, we improve their results.

In particular, for \( k = 3 \) and \( m = 2 \). Let \( z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R} \) and assume that the domains \( \Theta_1, \Theta_2 \) and \( \Theta_3 \) satisfy conditions (\( D4 \)), (\( Dr \)) and
\[(D5)\) \( \Theta_1 = \{ z = (x, y) \mid (x, -y) \in \Theta_2 \} \);
\[(D6)\) \( (x, -y) \in \Theta_3 \) for all \( (x, y) \in \Theta_3 \).
Then \( \Omega(r) \) is axially symmetric in \( y \)-axis. By consequence of Theorem 1.2, it is easy to prove that Eq. (1.1) in \( \Omega(r) \) has two positive solutions \( u_0^1 \) and \( u_0^2 \) which are nonaxially symmetric in \( y \)-axis.

Finally, we will apply Theorem 1.2 and the symmetric Palais–Smale theory (see Wang–Wu [21] or Willem [24]) to improve the result of Wang–Wu [21], their domain \( \Theta_3 \) is separated by an axially symmetric bounded domain. However, we do this without assumption in Wang–Wu [21].

**Theorem 1.3.** For each domain \( \Theta_3 \) satisfies conditions \((D4)\) and \((D6)\), there exists an \( r_0 > 0 \) such that for \( r \geq r_0 \), the domains \( \Theta_1, \Theta_2 \) and \( \Theta_3 \) satisfy conditions \((D5)\) and \((Dr)\). Then Eq. (1.1) in \( \Omega(r) \) has three positive solutions in which one is axially symmetric in \( y \)-axis and other two are nonaxially symmetric in \( y \)-axis.

In this paper, our domain \( \Omega(r) \) is an unbounded dumbbell type domain and the solutions \( u_0^1, u_0^2, \ldots, u_0^m \) are single-peak. If \( \Omega(r) \) is replaced by a bounded dumbbell type domain, Byeon [7] and Dancer [9] have proved the existence of single-peak solutions and multi-peak solutions. Moreover, Del Pino–Felmer [12], Del Pino–Felmer–Wei [13] and Wei [22,23] have considered the effect of domain topology on the existence of single-peak solutions and multi-peak solutions. Roughly speaking, if \( \Omega \) has a “rich” topology and is not necessarily bounded, then the singular perturbation problem

\[
\begin{aligned}
-\varepsilon \Delta u + u &= |u|^{p-2}u \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

has single-peak solutions and multi-peak solutions provided that \( \varepsilon \) is sufficiently small.

This paper is organized as follows. In Section 2 we describe various preliminaries. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 via a series of lemmas, while in Section 5 we prove Theorem 1.3.

## 2. Preliminary

In this section, we recall several known results will be used for later sections. First, we define the (PS)-sequences, (PS)-values, and (PS)-conditions in \( H_0^1(\Omega) \) for \( J \) as follows.

**Definition 2.1.** We define

(i) for \( \beta \in \mathbb{R} \), a sequence \( \{u_n\} \) is a (PS)\(_\beta\)-sequence in \( H_0^1(\Omega) \) for \( J \) if \( J(u_n) = \beta + o(1) \) and \( J'(u_n) = o(1) \) strongly in \( H^{-1}(\Omega) \) as \( n \to \infty \);

(ii) \( \beta \in \mathbb{R} \) is a (PS)-value in \( H_0^1(\Omega) \) for \( J \) if there exists a (PS)\(_\beta\)-sequence in \( H_0^1(\Omega) \) for \( J \);

(iii) \( J \) satisfies the (PS)\(_\beta\)-condition in \( H_0^1(\Omega) \) if every (PS)\(_\beta\)-sequence in \( H_0^1(\Omega) \) for \( J \) contains a convergent subsequence.

For any \( \beta \in \mathbb{R} \), a (PS)\(_\beta\)-sequence in \( H_0^1(\Omega) \) for \( J \) is bounded. Moreover, a (PS)-value \( \beta \) should be nonnegative.

**Lemma 2.2.** Let \( \beta \in \mathbb{R} \) and \( \{u_n\} \) be a (PS)\(_\beta\)-sequence in \( H_0^1(\Omega) \) for \( J \), then there exists \( c > 0 \) such that \( \|u_n\|_{H^1} \leq c \) for all \( n \in \mathbb{N} \). Furthermore,
\[ a(u_n) = b(u_n) + o(1) = \frac{2p}{p-2} \beta + o(1) \]

and \( \beta \geq 0 \).

**Proof.** See Willem [24]. \( \square \)

**Lemma 2.3.** Let \( \beta > 0 \) and \( \{u_n\} \) in \( H_0^1(\Omega) \setminus \{0\} \) be a sequence for \( J \) such that \( J(u_n) = \beta + o(1) \) and \( a(u_n) = b(u_n) + o(1) \). Then there is a sequence \( \{s_n\} \subset \mathbb{R}^+ \) such that \( s_n = 1 + o(1) \), \( \{s_n u_n\} \) is in \( M(\Omega) \) and \( J(s_n u_n) = \beta + o(1) \).

**Proof.** By the routine computations, there is a sequence \( \{s_n\} \subset \mathbb{R}^+ \) such that \( \{s_n u_n\} \) in \( M(\Omega) \):

\[ s_n^2 a(u_n) = s_n^p b(u_n) \]

for each \( n \). Since \( a(u_n) = b(u_n) + o(1) \) and \( J(u_n) = \beta + o(1) \) implies \( s_n = 1 + o(1) \). Therefore, \( J(s_n u_n) = \beta + o(1) \). \( \square \)

Consider the Nehari minimization problem

\[ \alpha(\Omega) = \inf_{v \in M(\Omega)} J(v). \]

In fact, if \( u_0 \in M(\Omega) \) achieves \( \alpha(\Omega) \), then \( u_0 \) is a ground state solution of Eq. (1.1) in \( \Omega \) (see Willem [24]). Moreover, we have the following useful lemmas, whose proofs can be found in Benci–Cerami [5] and Wang–Wu [21].

**Lemma 2.4.** Let \( \{u_n\} \) be in \( H_0^1(\Omega) \). Then \( \{u_n\} \) is a \((\text{PS})_{\alpha(\Omega)}\)-sequence in \( H_0^1(\Omega) \) for \( J \) if and only if \( J(u_n) = \alpha(\Omega) + o(1) \) and \( a(u_n) = b(u_n) + o(1) \).

**Lemma 2.5.** If the domain \( \Omega \) satisfies conditions \((\Omega 1)\) or \((\Omega 2)\)–\((\Omega 4)\), then \( \alpha(\Omega) > \alpha(\mathbb{R}^N) \).

**Lemma 2.6.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \). Then the \((\text{PS})_{\alpha(\Omega)}\)-condition holds in \( H_0^1(\Omega) \) for \( J \).

**Lemma 2.7.** Let \( u \in H_0^1(\Omega) \) be a change sign solution of Eq. (1.1) in \( \Omega \). Then \( J(u) > 2\alpha(\Omega) \).

### 3. Proof of Theorem 1.1

The proof will be accomplished by a series of lemmas.

**Lemma 3.1.** Let \( u_n \rightharpoonup u \) weakly in \( H_0^1(\Omega) \) and

\[ J'(u_n) = -\Delta u_n + u_n - |u_n|^{p-2} u_n = o(1) \text{ in } H^{-1}(\Omega). \]

Then

(i) \( |u_n - u|^{p-2}(u_n - u) - |u_n|^{p-2} u_n + |u|^{p-2} u = o(1) \) in \( H^{-1}(\Omega) \);

(ii) \( J'(p_n) = -\Delta p_n + p_n - |p_n|^{p-2} p_n = o(1) \) in \( H^{-1}(\Omega) \), where \( p_n = u_n - u \);

(iii) if \( \{u_n\} \) is a \((\text{PS})_{\beta}\)-sequence, then \( \{p_n\} \) is a \((\text{PS})_{(\beta - J(u))}\)-sequence.

**Proof.** (i), (ii) see Bahri–Lions [2]. (iii) Since \( u_n \rightharpoonup u \) weakly in \( H_0^1(\Omega) \) and \( \{u_n\} \) is a \((\text{PS})_{\beta}\)-sequence, by Lemma 2.2, Brézis–Lieb [6] and part (ii), we have \( \{p_n\} \) is a \((\text{PS})_{(\beta - J(u))}\)-sequence. \( \square \)
Let \( \Omega \) be any unbounded domain and \( \xi \in C^\infty([0, \infty)) \) such that \( 0 \leq \xi \leq 1 \) and
\[
\xi(t) = \begin{cases} 
0, & \text{for } t \in [0, 1], \\
1, & \text{for } t \in [2, \infty). 
\end{cases}
\]

Let
\[
\xi_n(z) = \xi\left(\frac{2|z|}{n}\right),
\]

Then we have the following result.

**Lemma 3.2.** Let \( \{u_n\} \) be a (PS)\( _\beta \)-sequence in \( H^1_0(\Omega) \) for \( J \) satisfy \( u_n \to 0 \) weakly in \( H^1_0(\Omega) \) and let \( v_n = \xi_n u_n \). Then there exists a subsequence \( \{u_n\} \) such that \( \|u_n - v_n\|_{H^1} = o(1) \) as \( n \to \infty \). Furthermore, \( a(v_n) = b(v_n) + o(1) \) and \( J(v_n) = \beta + o(1) \).

**Proof.** See the proof of Lemma 3.1 in Wu [25]. \( \square \)

Now, we begin to show the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \{u_n\} \) be a (PS)\( _\beta \)-sequence in \( H^1_0(\Omega) \) for \( J \) with
\[
\alpha(\Theta) \leq \beta < \min\{2\alpha(\Theta), \alpha(\Theta_{m+1}), \alpha(\Theta_{m+2}), \ldots, \alpha(\Theta_k)\}.
\]

By Lemma 2.2, there exist a subsequence \( \{u_n\} \) and \( u_0 \in H^1_0(\Theta) \) such that \( u_n \rightharpoonup u_0 \) weakly in \( H^1_0(\Theta) \) and \( u_n \to u_0 \) a.e. in \( \Theta \). Moreover, \( u_0 \) is a solution of Eq. (1.1) in \( \Theta \). If \( u_0 \equiv 0 \), by Lemma 3.2 there exists a subsequence \( \{u_n\} \) such that \( a(\xi_n u_n) = b(\xi_n u_n) + o(1) \) and \( J(\xi_n u_n) = \beta + o(1) \), where \( \xi_n \) is as in (3.1). Let \( v_n = \xi_n u_n \). Since the sets \( \Theta_i \cap \Theta_j \) and \( \bigcup_{i=1}^m \Theta_i \) are bounded. Thus, there exists a \( n_0 \in \mathbb{N} \) such that for \( n > 2n_0 \)
\[
v_n = 0 \qquad \text{in} \qquad \bigcup_{i \neq j} (\Theta_i \cap \Theta_j) \cup \bigcup_{i=1}^m \Theta_i.
\]

Moreover, \( v_n = v_n^{m+1} + v_n^{m+2} + \cdots + v_n^k \), where
\[
v_j^i(z) = \begin{cases} 
v_n(z), & \text{for } z \in \Theta_j, \\
0, & \text{for } z \notin \Theta_j,
\end{cases}
\]

for each \( j = m + 1, m + 2, \ldots, k \) and \( J(v_n) = J(v_n^{m+1}) + J(v_n^{m+2}) + \cdots + J(v_n^k) \). Similar to the proof of Lemma 3.1 in Wu [25], we have \( a(v_j^i) = b(v_j^i) + o(1) \) for each \( j = m + 1, m + 2, \ldots, k \). Moreover, by Lemma 2.3 there exists a sequence \( \{s_n^j\} \subset \mathbb{R}^+ \) such that \( s_n^j v_n^j \in M(\Theta_j) \) and \( s_n^j = 1 + o(1) \) for each \( j = m + 1, m + 2, \ldots, k \). By the fact that \( J(v_n) = \beta + o(1) \). We may assume there exist \( j_0 \in \{m + 1, m + 2, \ldots, k\} \) and a positive number \( c_{j_0} \leq \beta \) such that \( J(s_n^{j_0} v_n^{j_0}) = J(v_n^{j_0}) + o(1) = c_{j_0} + o(1) \). By the definition of Nehari minimization problem, we can conclude that \( \alpha(\Theta_{j_0}) \leq c_{j_0} \leq \beta \), which contradicts to \( \beta < \min\{\alpha(\Omega_{m+1}), \alpha(\Omega_{m+2}), \ldots, \alpha(\Omega_k)\} \). Consequently, \( u_0 \neq 0 \) and \( \beta \geq J(u_0) \geq \alpha(\Theta) \). Let \( p_n = u_n - u_0 \). By Lemma 3.1(iii), \( \{p_n\} \) is a (PS)\( _{\beta - J(u_0)} \)-sequence in \( H^1_0(\Omega) \) for \( J \). Since \( \beta < 2\alpha(\Theta) \), \( J(u_0) \geq \alpha(\Theta) \) and \( \alpha(\Theta) \) is the smallest positive (PS)-value in \( H^1_0(\Omega) \) for \( J \). Thus, \( \beta - J(u_0) = 0 \). This implies that \( u_n \rightharpoonup u_0 \) strongly in \( H^1_0(\Theta) \) and \( J(u_0) = \beta \). Moreover, by Lemma 2.7 and the maximum principle, we have \( u_0 \) is a positive solution. \( \square \)
Corollary 3.3. If \( \{u_n\} \) is a (PS)\(_{\beta}\)-sequence in \( H_{0}^{1}(\Theta) \) for \( J \) with

\[
\alpha(\Theta) \leq \beta < \min\{ \alpha(\Theta_{m+1}), \alpha(\Theta_{m+2}), \ldots, \alpha(\Theta_{k}) \},
\]
then there exist a subsequence \( \{u_n\} \) and \( u_0 \neq 0 \) such that \( u_n \rightharpoonup u_0 \) weakly in \( H_{0}^{1}(\Theta) \) and \( J(u_0) \leq \beta \).

4. Multiple positive solutions

Throughout this section, we assume that the domains \( \Theta_1, \Theta_2, \ldots, \Theta_{m+1} \) satisfy conditions (D4) and (Dr). For \( i = 1, 2, \ldots, m \), let

\[
M_i(r) = \left\{ u \in M(\Omega(r)) \left| \int_{[B^N(z_i; r)]^c} |u|^p < \frac{p}{p-2} \alpha(\mathbb{R}^N) \right. \right\},
\]

\[
N_i(r) = \left\{ u \in M(\Omega(r)) \left| \int_{[B^N(z_i; r)]^c} |u|^p = \frac{p}{p-2} \alpha(\mathbb{R}^N) \right. \right\}.
\]

It is easy to verify that \( M_i(r) \) is nonempty set for all \( i = 1, 2, \ldots, m \). Define the minimization problems in \( M_i(r) \) and \( N_i(r) \) for \( J \),

\[
\beta_i(r) = \inf_{v \in M_i(r)} J(v) \quad \text{and} \quad \gamma_i(r) = \inf_{v \in N_i(r)} J(v).
\]

Clearly, \( \beta_i(r), \gamma_i(r) \geq \alpha(\Omega(r)) \) for all \( i = 1, 2, \ldots, m \). Let \( \overline{M_i(r)} \) be denoted the closure of \( M_i(r) \), then we have \( \overline{M_i(r)} = M_i(r) \cup N_i(r) \) and \( N_i(r) \) is the boundary of \( \overline{M_i(r)} \) for all \( i = 1, 2, \ldots, m \). Furthermore, we have the following results.

Lemma 4.1. For each \( r \geq 2 \), we have \( \overline{M_i(r)} \cap \overline{M_j(r)} = \emptyset \) for all \( i \neq j \).

Proof. Fix \( i \in \{ 1, 2, \ldots, m \} \). Assume the contrary, there exist a \( v_0 \in M(\Omega(r)) \) and \( i \neq j \) such that \( v_0 \in \overline{M_i(r)} \cap \overline{M_j(r)} \). Then

\[
\int_{[B^N(z_i; r)]^c} |v_0|^p \leq \frac{p}{p-2} \alpha(\mathbb{R}^N) \quad \text{and} \quad \int_{[B^N(z_j; r)]^c} |v_0|^p \leq \frac{p}{p-2} \alpha(\mathbb{R}^N).
\]

Since \( B^N(z_i; r) \cap B^N(z_j; r) = \emptyset \), we have

\[
\int_{\Omega(r)} |v_0|^p \leq \int_{[B^N(z_i; r)]^c} |v_0|^p + \int_{[B^N(z_j; r)]^c} |v_0|^p \leq \frac{2p}{p-2} \alpha(\mathbb{R}^N).
\]

Therefore,

\[
\alpha(\Omega(r)) \leq J(v_0) = \left( \frac{p-2}{2p} \right) \int_{\Omega(r)} |v_0|^p \leq \alpha(\mathbb{R}^N),
\]

which contradicts to the fact of Lemma 2.5. \( \square \)

Lemma 4.2. For each \( \varepsilon > 0 \) there exists an \( r_1 > 0 \) such that

\[
\beta_i(r) < \min\{ \alpha(\mathbb{R}^N) + \varepsilon, \alpha(\Theta_{m+1}) \}
\]

for all \( i = 1, 2, \ldots, m \) and \( r \geq r_1 \).
Proof. By the Lien–Tzeng–Wang [17, Lemma 2.2], there exists an \( r_1 > 0 \) such that
\[
\alpha(B^N(0; r_1)) < \min\{\alpha(\mathbb{R}^N) + \varepsilon, \alpha(\Theta_{m+1})\}.
\]
By Lien–Tzeng–Wang [17, Theorem 2.10], if \( \Omega \) is a domain of \( \mathbb{R}^N \), then \( \alpha(\Omega) \) is invariant by rigid motions. Thus,
\[
\alpha(B^N(z_i; r_1)) < \min\{\alpha(\mathbb{R}^N) + \varepsilon, \alpha(\Theta_{m+1})\}
\](4.1)
for all \( i = 1, 2, \ldots, m \). By Lemma 2.6, Eq. (1.1) in \( B^N(z_i; r_1) \) has a positive solution \( v_i \) such that
\[
J(v_i) = \alpha(B^N(z_i; r_1))
\]
for all \( i = 1, 2, \ldots, m \) and \( r \geq r_1 \). By (4.1) and (4.2), we can conclude that
\[
\beta_i(r) < \min\{\alpha(\mathbb{R}^N) + \varepsilon, \alpha(\Theta_{m+1})\}
\]
for all \( i = 1, 2, \ldots, m \) and \( r \geq r_1 \).

Lemma 4.3. There exist positive numbers \( \delta, r_2 \) such that for each \( i = 1, 2, \ldots, m \),
\[
\gamma_i(r) > \alpha(\mathbb{R}^N) + \delta
\]
for all \( r \geq r_2 \).

Proof. Fix \( i \in \{1, 2, \ldots, m\} \). Assume the contrary, there exist \( r_n \to \infty \) as \( n \to \infty \), \( \{z_{i,n}\} \subset \Theta_{m+1} \) and \( \{u_n\} \subset N_i(r_n) \subset M(\Omega(r_n)) \) such that
\[
J(u_n) = \alpha(\mathbb{R}^N) + o(1)
\]
and
\[
\int_{B^N(z_{i,n}; r_n)} |u_n|^p = \frac{p}{p - 2} \alpha(\mathbb{R}^N).
\]
By Lemma 2.4, \( \{u_n\} \) is a \((PS)_{\alpha(\mathbb{R}^N)}\)-sequence in \( H^1(\mathbb{R}^N) \) for \( J \). From the concentration compactness principle of Lions [18], there exist \( R > 0, d > 0 \) and \( \{y_n\} \in \mathbb{R}^N \) such that
\[
\int_{B^N(y_n; R)} |u_n|^p \geq d
\]
for all \( n \).

Let \( v_n(z) = u_n(z + y_n) \), then \( \{v_n\} \) is a \((PS)_{\alpha(\mathbb{R}^N)}\)-sequence in \( H^1(\mathbb{R}^N) \) for \( J \) and \( \{v_n\} \subset M(\mathbb{R}^N) \). Thus, there is a \( u_0 \in H^1(\mathbb{R}^N) \) such that
\[
v_n \rightharpoonup u_0 \quad \text{weakly in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad n \to \infty,
\]
\[
v_n \to u_0 \quad \text{a.e. in} \quad \mathbb{R}^N \quad \text{as} \quad n \to \infty
\]
and
\[
\int_{B^N(0; R)} |v_n|^p \to \int_{B^N(0; R)} |u_0|^p \geq d \quad \text{as} \quad n \to \infty.
\]
Moreover, \( u_0 \) is a nonzero solution of Eq. (1.1) in \( \mathbb{R}^N \) and \( J(u_0) = \alpha(\mathbb{R}^N) \). By Lemmas 2.7 and 3.1, we may assume
\[
v_n \to u_0 \quad \text{strongly in} \quad H^1(\mathbb{R}^N) \quad \text{as} \quad n \to \infty
\]
and \( u_0 \) is a positive solution. We complete the proof by establishing the contradiction that
\[
\int_{B^N(z_{i,n}; r_n)} |u_n|^p = \frac{p}{p - 2} \alpha(\mathbb{R}^N) \quad \text{for all} \quad n.
\](4.3)
Consider the sequence \( \{ z_{i,n} - y_n \} \). By passing to a subsequence if necessary, we may assume that one of the following cases occurs:

(a) \( \{ z_{i,n} - y_n \} \) is bounded;
(b) \( \{ z_{i,n} - y_n \} \) is unbounded and for each \( R > 0 \) there exists a \( n(R) \in \mathbb{N} \) such that
\[
B^N(0; R) \cap B^N(z_{i,n} - y_n; r_n) = \emptyset \quad \text{for all } n \geq n(R);
\]
(c) \( \{ z_{i,n} - y_n \} \) is unbounded and there exists an \( R_0 > 0 \) such that
\[
B^N(0; R_0) \cap B^N(z_{i,n} - y_n; r_n) \neq \emptyset \quad \text{for all } n.
\]

Since
\[
v_n \to u_0 \quad \text{strongly in } H^1(\mathbb{R}^N) \quad \text{as } n \to \infty
\]
and \( u_0 \) is a positive solution with \( J(u_0) = \alpha(\mathbb{R}^N) \). By the compact imbedding theorem and the Vitali convergence theorem, for each \( \varepsilon > 0 \) there exists an \( R(\varepsilon) > 0 \) such that
\[
\int_{|z| > R(\varepsilon)} |v_n|^p < \varepsilon \quad \text{for all } n. \quad (4.4)
\]

In case (a) we may assume \( z_{i,n} - y_n \to z_0 \). Since
\[
\int_{B^N(z_{i,n}; r_n)^c} |u_n|^p = \frac{p}{p - 2} \alpha(\mathbb{R}^N).
\]

By the change of variable,
\[
\int_{B^N(z_{i,n} - y_n; r_n)^c} |v_n|^p = \frac{p}{p - 2} \alpha(\mathbb{R}^N).
\]

From (4.4), take \( \varepsilon_0 = \frac{p}{(p - 2)} \alpha(\mathbb{R}^N) \), then there exists an \( R(\varepsilon_0) > 0 \) such that
\[
\int_{|z| > R(\varepsilon_0)} |v_n|^p < \frac{p}{p - 2} \alpha(\mathbb{R}^N) \quad \text{for all } n.
\]

Since \( z_{i,n} - y_n \to z_0 \) and \( r_n \to \infty \) as \( n \to \infty \), there exists a \( n_0 \) such that for \( B^N(0; R(\varepsilon_0)) \subset B^N(z_{i,n} - y_n; r_n) \) for all \( n \geq n_0 \). Thus, for each \( n \geq n_0 \)
\[
\int_{B^N(z_{i,n} - y_n; r_n)^c} |v_n|^p \leq \int_{|z| > R(\varepsilon_0)} |v_n|^p < \frac{p}{p - 2} \alpha(\mathbb{R}^N),
\]
which contradicts to (4.3).

In case (b) from (4.4), let \( \varepsilon_0 = \frac{p}{p - 2} \alpha(\mathbb{R}^N) \), then there exists an \( R(\varepsilon_0) > 0 \) such that
\[
\int_{|z| > R(\varepsilon_0)} |v_n|^p < \frac{p}{p - 2} \alpha(\mathbb{R}^N) \quad \text{for all } n.
\]

By the hypothesis, there exists a \( n_0 = n(R(\varepsilon_0)) \) such that
\[
B^N(0; R(\varepsilon_0)) \cap B^N(z_{i,n} - y_n; r_n) = \emptyset \quad \text{for all } n \geq n_0.
\]
Thus,
\[
\int_{B^N(z_i,n-\gamma_n;r_n)} |v_n|^p \leq \int_{|z|>R(\varepsilon_0)} |v_n|^p < \frac{p}{p-2} \alpha(\mathbb{R}^N) \quad \text{for all } n \geq n_0.
\]
(4.5)

Since \( \{u_n\} \subset \mathcal{M}(\Omega(r_n)) \), this means
\[
\int_{\Omega(r_n)} |u_n|^p \geq \left( \frac{2p}{p-2} \right) \alpha(\Omega(r_n)) \quad \text{for all } n.
\]
(4.6)

From (4.5) and (4.6), we obtain
\[
\int_{B^N(z_i,n-\gamma_n;r_n)} c |u_n|^p = \int_{\Omega(r_n)} |u_n|^p - \int_{B^N(z_i,n-\gamma_n;r_n)} |v_n|^p > \frac{p}{p-2} \alpha(\mathbb{R}^N) \quad \text{for all } n \geq n_0,
\]
which contradicts to (4.3).

In case (c) from (4.4) and \( \{v_n\} \subset \mathcal{M}(\mathbb{R}^N) \), we may take positive numbers \( \varepsilon_0 \) and \( R(\varepsilon_0) \geq R_0 \) such that
\[
\int_{|z| \leq R(\varepsilon_0)} |v_n|^p > \frac{3p}{2(p-2)} \alpha(\mathbb{R}^N) \quad \text{for all } n.
\]
(4.7)

First, we claim that there exists \( n_0 \) such that \( B^N(0; R(\varepsilon_0)) \subseteq B^N(z_i,n-\gamma_n;r_n) \) for all \( n \geq n_0 \). Since the domain \( \Theta_{m+1} \) satisfies condition \((D4)\), by Lemma 2.5 and Lien–Tzeng–Wang [17, Lemma 2.5] there exists \( \bar{R} \geq R(\varepsilon_0) \) such that
\[
B^N(0; \bar{R}) \setminus [\Theta_{m+1} - \gamma_n] \neq \emptyset \quad \text{for all } n.
\]
(4.8)

Since \( v_n \equiv 0 \) in \( \Omega(r_n)^c \),
\[
\int_{|z| \leq R(\varepsilon_0)} |v_n|^p \geq \frac{3p}{2(p-2)} \alpha(\mathbb{R}^N) \quad \text{for all } n.
\]
(4.7)

and \( u_0 \) is a positive solution of Eq. (1.1) in \( \mathbb{R}^N \), we have
\[
\lim_{n \to \infty} [\Omega(r_n) - \gamma_n] = \mathbb{R}^N.
\]
(4.9)

Since \( |z_i,n - z_j,n| > 3r_n \) for \( i \neq j \), and \( r_n \to \infty \) as \( n \to \infty \). Thus, for each \( R \geq R(\varepsilon_0) \) there exists \( n_2 \in \mathbb{N} \) such that
\[
B^N(0; R) \cap [\Theta_j - \gamma_n] = \emptyset
\]
(4.10)

for all \( j = 1, 2, \ldots, i-1, i+1, \ldots, m \) and \( n \geq n_2 \). By (4.8)–(4.10), there exists \( n_0 \in \mathbb{N} \) such that \( B^N(0; R(\varepsilon_0)) \subseteq B^N(z_i,n-\gamma_n;r_n) \) for all \( n \geq n_0 \). From (4.7), we can conclude that for \( n \geq n_0 \)
\[
\int_{B^N(z_i,n-\gamma_n;r_n)} |v_n|^p \geq \int_{|z| \leq R(\varepsilon_0)} |v_n|^p > \frac{3p}{2(p-2)} \alpha(\mathbb{R}^N)
\]
or
\[
\int_{B^N(z_n; r_n)} |u_n|^p > \frac{3p}{2(p - 2)} \alpha(R^N).
\]

Since \( \{u_n\} \) is a \((PS)_{\alpha(R^N)}\)-sequence in \( H^1(R^N) \) for \( J \). Thus,
\[
\int_{B^N(z_n; r_n)} |u_n|^p = \int_{\mathbb{R}^N} |u_n|^p - \int_{B^N(z_n; r_n)} |u_n|^p < \frac{2p}{p - 2} \alpha(R^N) - \frac{3p}{2(p - 2)} \alpha(R^N) + o(1)
\]
\[
= \frac{p}{2(p - 2)} \alpha(R^N) + o(1)
\]

for all \( n \geq n_0 \), which contradicts to (4.3). Therefore, we have completed our proof.

2

Here, we will use the idea of Ni–Takagi [19] to get the following results.

**Lemma 4.4.** For any \( u^i \in M_i(r) \), there exist \( \epsilon > 0 \) and a differentiable function \( t^i : B(0; \epsilon) \subset H^1_0(\Omega(r)) \rightarrow \mathbb{R}^+ \) such that \( t^i(0) = 1 \), the function \( z^i = t^i(w)(u^i - w) \in M_i(r) \) and
\[
\left\{ (t^i)'(0), v \right\} = \frac{2\int_{\Omega(r)} \nabla u^i \nabla v + u^i v - p \int_{\Omega(r)} |u^i|^{p-2} u^i v}{\int_{\Omega(r)} |\nabla u^i|^2 + (u^i)^2 - (p - 1) \int_{\Omega(r)} |u^i|^p} \quad \text{for all } v \in H^1_0(\Omega(r)).
\]

**Proof.** Define a function \( F : \mathbb{R} \times H^1_0(\Omega(r)) \rightarrow \mathbb{R} \) given by
\[
F(t, w) = t \int_{\Omega(r)} |\nabla(u^i - w)|^2 + (u^i - w)^2 - t^{p-1} \int_{\Omega(r)} |u^i - w|^p.
\]

Since \( u^i \in M_i(r) \), we have \( F(1, 0) = 0 \) and
\[
\frac{d}{dt} F(1, 0) = \int_{\Omega(r)} |\nabla u^i|^2 + (u^i)^2 - (p - 1) \int_{\Omega(r)} |u^i|^p < 0.
\]

According to the implicit function theorem, there exists a function \( t^i : B(0; \epsilon) \subset H^1_0(\Omega(r)) \rightarrow \mathbb{R}^+ \) such that \( t^i(0) = 1 \) and \( F(t^i(w), w) = 0 \) for \( w \in B(0; \epsilon) \). This is equivalent to
\[
\left\{ J'(t^i(w)(u^i - w)), t^i(w)(u^i - w) \right\} = 0.
\]

Furthermore, by the continuity of the functional \( t^i \), we have
\[
\int_{[B^N(z_i; r_i)]^c} |t^i(w)(u^i - w)|^p < \frac{p}{(p - 2)} \alpha(R^N),
\]
if \( \epsilon \) is sufficiently small.

**Proposition 4.5.** There exists an \( r_0 > 0 \) such that for each \( r \geq r_0 \) and \( i \in \{1, 2, \ldots, m\} \), we have \( \beta_i(r) < \min\{2\alpha(\Omega(r)), \alpha(\Theta_{m+1})\} \) and \( \beta_i(r) \) has a minimizing sequence \( \{u^i_n\} \subset M_i(r) \) is satisfying
\[
J(u^i_n) = \beta_i(r) + o(1) \quad \text{and} \quad J'(u^i_n) = o(1) \quad \text{in } H^{-1}(\Omega(r)).
\]
Proof. From Lemmas 4.2, 4.3 and the fact that $\alpha(\Omega(r)) > \alpha(\mathbb{R}^N)$ for all $r > 0$, there exists an $r_0 > 0$ such that for $r \geq r_0$,

$$\beta_i(r) < \min\{2\alpha(\Omega(r)), \alpha(\Theta_{m+1}), \gamma_i(r)\}. \quad (4.12)$$

It follows that for $r \geq r_0$,

$$\beta_i(r) = \inf_{v \in \overline{M_i(r)}} J(v). \quad (4.13)$$

Since $\overline{M_i(r)}$ is a closure of $M_i(r)$. By (4.13) and the Ekeland variational principle [14], there exists a minimizing sequence $\{u^i_n\} \subset \overline{M_i(r)}$ such that

$$J(u^i_n) < \beta_i(r) + \frac{1}{n} \quad (4.14)$$

and

$$J(u^i_n) < J(w) + \frac{1}{n} \left\| w - u^i_n \right\|_{H^1} \quad \text{for any } w \in \overline{M_i(r)}. \quad (4.15)$$

Using (4.12) we may assume that $u^i_n \in M_i(r)$ for all $n$. By application of Lemma 4.4 for $u^i_n = u^i_n$, we obtain the functions $t^i_n: B(0; \epsilon_n) \to \mathbb{R}^+$ for some $\epsilon_n > 0$, such that $t^i_n(w)(u^i_n - w) \in M_i(r)$. Choose $0 < \rho < \epsilon_n$. Let $u \in H^1_0(\Omega(r))$ with $u \neq 0$ and let $w_\rho = \frac{\rho u}{\|u\|_{H^1}}$. We set $z^i_\rho = t^i_n(w_\rho)(u^i_n - w_\rho)$. Since $z^i_\rho \in M_i(r)$, we deduce from (4.15) that

$$J(z^i_\rho) - J(u^i_n) \geq -\frac{1}{n} \left\| z^i_\rho - u^i_n \right\|_{H^1} \quad \text{and by the mean value theorem, we have}$$

$$\left\langle J'(u^i_n), z^i_\rho - u^i_n \right\rangle + o\left(\|z^i_\rho - u^i_n\|_{H^1}\right) \geq -\frac{1}{n} \left\| z^i_\rho - u^i_n \right\|_{H^1}. \quad (4.16)$$

Thus,

$$\left\langle J'(u^i_n), -w_\rho \right\rangle + \left( t^i_n(w_\rho) - 1 \right) \left\langle J'(u^i_n), (u^i_n - w_\rho) \right\rangle \geq -\frac{1}{n} \left\| z^i_\rho - u^i_n \right\|_{H^1} + o\left(\|z^i_\rho - u^i_n\|_{H^1}\right). \quad (4.16)$$

By the fact that $t^i_n(w_\rho)(u^i_n - w_\rho) \in M_i(r)$ and (4.16), we have

$$-\rho \left( J'(u^i_n), \frac{u}{\|u\|_{H^1}} \right) + \left( t^i_n(w_\rho) - 1 \right) \left\langle J'(u^i_n) - J'(z^i_\rho), (u^i_n - w_\rho) \right\rangle \geq -\frac{1}{n} \left\| z^i_\rho - u^i_n \right\|_{H^1} + o\left(\|z^i_\rho - u^i_n\|_{H^1}\right).$$

Thus,

$$\left\langle J'(u^i_n), \frac{u}{\|u\|_{H^1}} \right\rangle \leq \frac{\|z^i_\rho - u^i_n\|_{H^1}}{\rho} \left( \frac{n \rho}{o}\left(\|z^i_\rho - u^i_n\|_{H^1}\right) \right) + \frac{(t^i_n(w_\rho) - 1)\left\langle J'(u^i_n) - J'(z^i_\rho), (u^i_n - w_\rho) \right\rangle}{\rho}. \quad (4.16)$$

On the other hand, by (4.11) we can find a constant $C > 0$, independent of $\rho$, such that

$$\|z^i_\rho - u^i_n\|_{H^1} \leq \rho + |t^i_n(w_\rho) - 1|C \quad (4.16)$$
and
\[
\lim_{\rho \to 0} \frac{|t_n'(w_\rho) - 1|}{\rho} \leq \|t_n'(0)\| \leq C.
\]
If we let \(\rho \to 0\) in (4.15) for a fixed \(n\) and use the fact that \(z_n'(\rho) \to u_n'\) in \(H^1_0(\Omega)\), we get
\[
\left\langle J'(u_n'), \frac{u}{\|u\|_{H^1}} \right\rangle \leq \frac{C}{n}.
\]
This shows that \(\{u_n'\}\) is a (PS)\(\beta_i(r)\)-sequence in \(H^1_0(\Omega)\) for \(J\). \(\square\)

Now, we begin to show the proof of Theorem 1.2.

**Proof of Theorem 1.2.** It follows from Proposition 4.5 that there exists an \(r_0 > 0\) such that for each \(r \geq r_0\) and \(i \in \{1, 2, \ldots, m\}\), we can find a (PS)\(\beta_i(r)\)-sequence in \(H^1_0(\Omega)\) for \(J\) with \(\beta_i(r) < \min\{2\alpha(\Omega(r)), \alpha(\Theta_{m+1})\}\).

By Theorem 1.1, there exists \(u_0^i \in H^1_0(\Omega(r))\) such that
\[
u_n^i \to u_0^i \quad \text{strongly in } H^1_0(\Omega(r)),
\]
\(J(u_0^i) = \beta_i(r)\) and \(u_0^i\) is a positive solution of Eq. (1.1) in \(\Omega(r)\). Moreover,
\[
\int_{B_N(\xi; r)^c} |u_0^i|^p \leq \frac{p}{(p-2)} \alpha(\mathbb{R}^N),
\]
that is \(u_0^i \in M_i(r)\). By Lemma 4.1, \(u_0^1, u_0^2, \ldots, u_0^m\) are different. \(\square\)

Note that \(\alpha(\Omega(r)) \leq \beta_i(r)\) for all \(i \in \{1, 2, \ldots, m\}\). If \(\alpha(\Omega(r)) < \beta_i(r)\) for each \(i \in \{1, 2, \ldots, m\}\). Then we have the following result.

**Theorem 4.6.** For \(r \geq r_0\). If \(\alpha(\Omega(r)) < \beta_i(r)\) for all \(i = 1, 2, \ldots, m\), then Eq. (1.1) in \(\Omega(r)\) has at least \(m+1\) positive solutions.

**Proof.** By Theorems 1.1 and 1.2. \(\square\)

5. **Proof of Theorems 1.1 and 1.2.**

In this section, we focus on the \(y\)-symmetric Sobolev space \(H_y(\Omega)\) defined as follows: For \(z = (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}\) and \(\Omega\) is an axially symmetric domain for \(y\)-axis. Let \(H_y(\Omega)\) be the \(H^1\)-closure of the space \(\{u \in C_0^\infty(\Omega) \mid u \text{ is axially symmetric in } y\}\) and \(H^{-1}_y(\Omega)\) be the dual space of \(H_y(\Omega)\). Then \(H_y(\Omega)\) is a closed linear subspace of \(H^1_0(\Omega)\).

Consider the \(y\)-symmetric Nehari minimization problem
\[
\alpha_y(\Omega) = \inf_{v \in M_y(\Omega)} J(v),
\]
where \(M_y(\Omega) = \{u \in H_y(\Omega) \setminus \{0\} \mid a(u) = b(u)\}\). Note that, some properties and results of the minimization problem are as in Section 2 so we omitted proofs here. Moreover, by the principle of symmetric criticality (see Palais [20]), every (PS)\(\alpha_y(\Omega)\)-sequence in \(H_y(\Omega)\) for \(J\) is a (PS)\(\alpha_y(\Omega)\)-sequence in \(H^1_0(\Omega)\) for \(J\).
Proof of Theorem 1.3. From Theorem 1.2, we only need to show Eq. (1.1) in \( \Omega \) has a positive solution \( u^s_0 \in H_s(\Omega(r)) \). Let \( \{ u^s_n \} \subset H_s(\Omega(r)) \) be a (PS)\(_{\alpha_s(\Omega(r))}\)-sequence for \( J \). By Lemma 2.2 there exist a subsequence \( \{ u^s_n \} \) and \( u^s_0 \in H_s(\Omega(r)) \) such that \( u^s_n \rightharpoonup u^s_0 \) weakly in \( H_s(\Omega(r)) \). If \( u^s_0 \equiv 0 \), then similar to the argument of proof in Lemma 3.2, there exists a subsequence \( \{ u^s_n \} \) such that \( J(\xi_n u^s_n) = \alpha_s(\Omega(r)) + o(1) \) and \( a(\xi_n u^s_n) = b(\xi_n u^s_n) + o(1) \), where \( \xi_n \) is as in (3.1). Moreover, by Lemma 2.3 there exists a sequence \( \{ \lambda_n \} \subset \mathbb{R}^+ \) such that

\[
a(\lambda_n \xi_n u^s_n) = b(\lambda_n \xi_n u^s_n), \quad J(\lambda_n \xi_n u^s_n) = \alpha_s(\Omega(r)) + o(1) \quad \text{and} \quad \lambda_n = 1 + o(1).
\]

Let \( v_n = \lambda_n \xi_n u^s_n \). Since the domains \( \Theta_1 \) and \( \Theta_2 \) are bounded, there exists \( n_0 \in \mathbb{N} \) such that for \( n > 2n_0 \), \( v_n = 0 \) in \( \Theta_1 \cup \Theta_2 \). Thus, \( v_n \in M_s(\Theta_3) \). By the fact that \( J(v_n) = \alpha_s(\Omega(r)) + o(1) \) and the definition of Nehari minimization problem, we have \( \alpha_s(\Theta_3) \leq \alpha_s(\Omega(r)) \), which contradicts to \( \alpha_s(\Omega(r)) < \alpha_s(\Theta_3) \). Therefore, \( u^s_0 \neq 0 \) and \( J(u^s_0) \geq \alpha_s(\Omega(r)) \). Let \( p_n = u^s_n - u^s_0 \). Similar to the argument of proof in Lemma 3.1, we can conclude that \( u^s_n \rightharpoonup u^s_0 \) strongly in \( H_s(\Omega(r)) \) and \( J(u^s_0) = \alpha_s(\Omega(r)) \). Moreover, by Lemma 2.7 and the maximum principle, we obtain \( u^s_0 \) is a positive solution. \( \square \)

References


