A rapid solution of a kind of 1D Fredholm oscillatory integral equation

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ABSTRACT

How to solve oscillatory integral equations rapidly and accurately is an issue that attracts special attention in many engineering fields and theoretical studies. In this paper, a rapid solution method is put forward to solve a kind of special oscillatory integral equation whose unknown function is much less oscillatory than the kernel function. In the method, an improved-Levin quadrature method is adopted to solve the oscillatory integrals. On the one hand, the employment of this quadrature method makes the proposed method very accurate; on the other hand, only a small number of small-scaled systems of linear equations are required to be solved, so the computational complexity is also very small. Numerical examples confirm the advantages of the method.

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1. Introduction

How to solve integral equations rapidly and accurately is an important issue arising in many fields. Among the integral equations, the Fredholm integral equations of the 2nd kind have attracted much attention, and they can be uniformly expressed as [1,2]

$$\psi(x) = h(x) + \int_a^b \psi(y)k(x,y)\mathrm{d}y, \tag{1}$$

where $k(x,y)$ is the kernel function, and $\psi(x)$ is the unknown function. Many methods (such as the Nyström method [1]) have been developed to solve this kind of integral equation, and they are mainly concerned with the integral equation with a non-oscillatory kernel function. However, in many fields such as the electromagnetics and quantum mechanism, the phenomenon of oscillation actually exists widely. This poses a serious challenge to the conventional methods.

While the oscillation is taken into account, the above integral equation can be rewritten as the following form:

$$\psi(x) = h(x) + \int_a^b f(x,y)\psi(y)\mathrm{e}^{\mathrm{i}\omega g(x,y)}\mathrm{d}y, \tag{2}$$

where $k(x,y) = f(x,y)\mathrm{e}^{\mathrm{i}\omega g(x,y)}$ is the oscillatory kernel function. In this work, we concern the integral equation with smooth functions f, g, h and ψ. With the increase of frequency ω, the kernel function becomes more and more oscillatory. Therefore, if this integral equation is to be solved by a conventional method, very fine samples are required for the convergence of the method, and this might make the method computationally prohibitive. In this sense, it is of significance to develop a rapid and accurate solution method for this kind of integral equation [3,4].

The key of solving this kind of integral equation is how to calculate the oscillatory integral accurately and rapidly. In the past decades, some efficient evaluation methods for oscillatory integrals have been developed [5–10], but they are mainly

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concerned with the cases with non-oscillatory amplitude functions and phase functions. However, Ref. [11] shows that the solution of an oscillatory integral equation should also be of oscillatory type. This means that the unknown function $\psi(x)$ in (2) is also highly oscillatory, and the existing rapid solution methods cannot be directly applied to solve the oscillatory integral involved. Fortunately, in many situations of practical interest, the oscillation of the unknown function $\psi(x)$ can be well extracted, and the extracted oscillation can further be transferred to the existing kernel $e^{i\omega g(x,y)}$ to form another highly oscillatory kernel. In this manner, the transformed oscillatory integral equation is with a non-oscillatory unknown function, and the oscillatory integral involved is with a non-oscillatory amplitude function. In this paper, we are mainly concerned with the transformed oscillatory integral equations, i.e., the functions f, g, ψ, h are assumed to be much less oscillatory in comparison with the oscillatory kernel function.

Actually, this transformation technique has a very promising application in computational electromagnetics and some other fields. For example, in the study of the scattering characteristics of a convex scatterer, the following oscillatory integral equation about unknown function $\mu(\mathbf{r})$ is adopted to characterize the scattering behavior [12–14]:

$$\frac{1}{2}\mu(\mathbf{r}) - \int_S \left[\frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}(\mathbf{r})} + i\gamma G(\mathbf{r}, \mathbf{r}') \right] \mu(\mathbf{r}') ds(\mathbf{r}') = \frac{\partial u^{\text{inc}}(\mathbf{r})}{\partial \mathbf{n}(\mathbf{r})} + i\gamma u^{\text{inc}}(\mathbf{r}), \tag{3}$$

where $u^{\text{inc}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}$ is the incident plane wave, $\mu(\mathbf{r}) = \partial u(\mathbf{r})/\partial \mathbf{n}(\mathbf{r})$ is the surface current, $\mathbf{n}(\mathbf{r})$ is the outward unit normal vector at point $\mathbf{r}$ on S, γ is a coupling constant, and $G(\mathbf{r}, \mathbf{r}') = e^{i|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r} - \mathbf{r}'|$ is the free-space Green function. Here, the presence of Green function makes the kernel function $H(\mathbf{r}, \mathbf{r}') = \frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \nu(\mathbf{r})} + i\gamma G(\mathbf{r}, \mathbf{r}')$ highly oscillatory. Fortunately, according to the physical phenomenon of high frequency scattering [13,15], we can introduce an ansatz

$$\mu(\mathbf{r}) = \mu_{\text{slow}}(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}} \tag{4}$$

to (3) to yield an integral equation with respect to a slowly variational function μ_{slow}:

$$\frac{1}{2}\mu_{\text{slow}}(\mathbf{r}) - \int_S H(\mathbf{r}, \mathbf{r}')e^{i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})}\mu_{\text{slow}}(\mathbf{r}') ds(\mathbf{r}') = i(\mathbf{k} \cdot \mathbf{n}(\mathbf{r}) + \gamma), \quad \mathbf{r} \in S. \tag{5}$$

After the transformed integral equations have been obtained, the efficiency of solving the integral equation relies on the calculation of the oscillatory integral involved. Bruno, et al. have developed a new convergent quadrature method based on the localized integration around the critical points (such as stationary phase points) [12]. The algorithm is well established, but the transformed oscillatory integrals over the local intervals are calculated with a conventional quadrature method. Higher efficiency is expected if a better quadrature method is applied. Huybrechs developed a numerical steepest descent method to solve the oscillatory integrals [14]. This method has a relatively high accuracy, but the integrand must be analytic in the complex plane and the process of finding the steepest descent path might be relatively complicated. In [16,17] Li, et al. proposed an improved-Levin method to compute the oscillatory integrals. The present paper intends to construct an alternative solution method for the oscillatory integral equations based on this quadrature method.

2. A new rapid solution method for Fredholm integral equations of oscillatory kind

2.1. Discrete form of the Fredholm integral equation

To solve the Fredholm integral equation is to work out the expression of the unknown function. However, in many situations the function values are even more useful, so this study focuses on obtaining the function values of $\psi(x)$ on the given nodes.

First, discretizing the integral equation on the given nodes $\{x_j\}_{j=0,1,\ldots,M}$ yields

$$\psi(x_j) = h(x_j) + \int_a^b f(x_j, y)\psi(y)e^{i\omega g(x_j,y)} dy. \tag{6}$$

Obviously, the key of solving these discrete equations is how to compute the following oscillatory integrals rapidly and accurately:

$$I_j[f, g] = \int_a^b f(x_j, y)\psi(y)e^{i\omega g(x_j,y)} dy, \quad j = 0, 1, \ldots, M. \tag{7}$$

These integrals are difficult to calculate with a conventional method, especially while singularity is involved. Here, we first give a brief discussion on the singularity of the integral.

2.2. Singularity of the integral

A singular integral is an integral with unbounded integrand over the interval. The presence of singularity could pose even more challenges to the calculation of oscillatory integrals. It is known that the types of singularity is numerous and

each of them deserves special attention. In [18], we have studied a special singularity—the Cauchy principal value integral of oscillatory kind:

$$I[f, \tau] = \int_{-1}^{1} \frac{f(x)}{x - \tau} e^{i\omega g(x)} dx, \quad \tau \in (-1, 1).$$

This integral can be separated into two sub-integrals:

$$I[f, \tau] = \int_{-1}^{1} \frac{f(x) - f(\tau)}{x - \tau} e^{i\omega g(x)} dx + f(\tau) \int_{-1}^{1} \frac{e^{i\omega g(x)}}{x - \tau} dx. \tag{8}$$

The former is non-singular, so it can be handled as a regular oscillatory integral. The latter is singular, but it has an analytical result in a closed form. Consequently, the singular oscillatory integral is well determined.

It should be noted that a different type of singularity may require a different technique to handle it, so a comprehensive analysis of all the singularities is very difficult. Fortunately, in many situations of practical interest, the singularities are removable. For example, the integral in (5) is with a second-order singularity which is caused by the factor $\partial G(\mathbf{r}, \mathbf{r}')/\partial \mathbf{n}(\mathbf{r})$, but the singularity can be removed if the integral is expressed in the spherical coordinate system. Therefore, in this study we are mainly concerned with the oscillatory integrals free of singularities.

2.3. Calculation of the oscillatory integrals

Efforts have been made in the past decades to develop special solution methods for oscillatory integrals, and the existing representative ones include the asymptotic expansion method [5], the Levin(-type) method [6,7], the Filon(-type) method [8,9], and the numerical steepest descent method [10]. Among them, the Levin(-type) method has attracted much attention for its good applicability to oscillatory integrals with complicated phase functions, but it is very susceptible to the ill-conditioning [19]. Based on the Levin method, we have developed an improved-Levin quadrature method in [16,17]. The new method is numerically stable and has broken through the bottleneck that the Levin method is susceptible to the ill-conditioning. The present study intends to construct a new solution method for the oscillatory integral equations based on this quadrature method.

As has been stated in many references, the stationary phase points play an important role in the calculation of the oscillatory integrals. Therefore, we here study the oscillatory integrals with respect to the presence of stationary phase points or not.

2.3.1. Calculation of the oscillatory integrals free of stationary phase point

The method proposed in [16,17] can be directly used to calculate the oscillatory integrals free of stationary phase point. According to the Levin theory, the calculation of an oscillatory integral like (7) can be reduced to solving an ordinary differential equation (ODE) without boundary condition:

$$p'(y) + i\omega g'(x_j, y) p(y) = f(x_j, y) \psi(y). \tag{9}$$

If the unknown function $p(y)$ is solved from (9), then the integral result is obtained as

$$I_j = p(b) e^{i\omega g(x_j, b)} - p(a) e^{i\omega g(x_j, a)}. \tag{10}$$

In [16], the Chebyshev pseudo-spectral method is employed to solve the differential equation (9), and the nodes used are of Chebyshev–Lobatto type, i.e., $y_k = \frac{b-a}{2} \cos(\frac{\pi k}{N}) + \frac{b+a}{2}, k = 0, 1, \ldots, N$. In the process, the Chebyshev differentiation matrix $\mathbf{D}$ is adopted to obtain the derivative of a function [16,20,21]. To be specified, if the function values of $y = f(x)$ on the given nodes compose a vector $\mathbf{f}$, the function values of $f'(x)$ on these nodes can be well approximated by $\mathbf{f}' = \mathbf{D}\mathbf{f}$. Consequently, applying the differentiation matrix $\mathbf{D}$ on ODE (9) yields the following system of linear equations [16]:

$$\left(\frac{2}{b-a} \mathbf{D} + i\omega \mathbf{\Sigma}_j \right) \mathbf{P}_j = \mathrm{diag}(\mathbf{F}_j) \mathbf{\Phi}, \tag{11}$$

where

- $\mathbf{P}_j = [p(y_0), \ldots, p(y_N)]^T, \mathbf{\Phi} = [\psi(y_0), \ldots, \psi(y_N)]^T$, and $\mathbf{F}_j = [f(x_j, y_0), \ldots, f(x_j, y_N)]^T$ are numerical vectors composed of the different function values,
- $\mathbf{\Sigma}_j = \mathrm{diag}\left(g'(x_j, y_0), g'(x_j, y_1), \ldots, g'(x_j, y_N) \right)$ is a diagonal matrix,
- $\mathrm{diag}(\mathbf{F}_j)\mathbf{\Phi} = \mathbf{F}_j \otimes \mathbf{\Phi}$ (symbol '$\otimes$' denotes the Hadamard product) is a vector with the entries being $f(x_j, y_k)\psi(y_k), k = 0, 1, \ldots, N$.

As has been stated in [16], Eq. (11) can be well solved by the LU factorization method (or some iteration techniques such as the Conjugate Gradient (CG) method and the Generalized Minimum Residual (GMRES) method) if it is well-conditioned; otherwise, the truncated singular value decomposition (TSVD) method can serve as a proper solution method for it. For simplicity, we uniformly denote the solution of (11) by

$$\mathbf{P}_j = \left(\frac{2}{b-a}\mathbf{D} + i\boldsymbol{\Sigma}_j \right)^{-1} \mathrm{diag}(\mathbf{F}_j)\boldsymbol{\Phi}. \tag{12}$$

Considering that the first and last entries of $\mathbf{P}_j$ are $p(b)$ and $p(a)$ respectively, the integral result can then be obtained from (10) and (12):

$$I_j = \mathbf{Q}_j\mathbf{P}_j = \mathbf{Q}_j \left(\frac{2}{b-a}\mathbf{D} + i\boldsymbol{\Sigma}_j \right)^{-1} \mathrm{diag}(\mathbf{F}_j)\boldsymbol{\Phi}, \tag{13}$$

where $\mathbf{Q}_j = [e^{i\omega g(x_j,b)}, 0, \ldots, 0, -e^{i\omega g(x_j,a)}]$ is a vector with the entries being 0 except for the first and last entries.

Because we have assumed that $f(x,y)$, $g(x)$, and $\psi(x)$ are all smooth and non-oscillatory functions, a small number of nodes are adequate to well interpolate these functions. This means that the computational complexity of (13) is very small. As will be illustrated in Section 2.5, this property makes the total computational complexity of the proposed method very small.

At the same time, the quadrature method samples at the endpoints (the nodes used are of Chebyshev–Lobatto type), so the asymptotic order of the error should be $\mathcal{O}(\omega^{-2})$ at least [9]. This asymptotic order is higher than that of the integral itself ($\mathcal{O}(\omega^{-1})$) [5,7,22], so the accuracy of the quadrature method on a given number of nodes becomes increasingly better as the frequency ω increases. However, the corresponding asymptotic order of a conventional method (such as the Gauss quadrature method) is $\mathcal{O}(1)$ while fixed nodes are adopted, so it becomes less and less accurate with the increase of frequency. This makes the newly proposed method much more advisable while a large frequency ω is involved.

Now, the calculation of an oscillatory integral free of stationary phase point has been well described. The case with stationary phase points are a little complicated, and the following section pays attention on this effect.

2.3.2. Calculation of the oscillatory integrals with stationary phase points

We have studied the impact of stationary phase points on the quadrature algorithm in [17]. As stated there, if stationary phase points are involved in the interval and the frequency ω is large enough, the integral result will be mainly determined by the property of the integrand around the stationary phase points. The presence of stationary phase points requires fine samples around them to obtain a good description of the oscillator's behavior. In our method, the discrete nodes are of Chebyshev–Lobatto type which clusters around the endpoints. Hence, if the stationary phase points are around the endpoints of the interval, fine samples around the stationary phase points can be easily obtained; otherwise, special techniques should be employed. For the latter case, Ref. [17] has provided two candidates to handle it: one is to increase the total number of nodes, and another is to divide the interval into several sub-intervals according to positions of the stationary phase points. The second candidate is proved to be of higher efficiency and accuracy because the stationary phase points are located at the endpoints of the sub-intervals. For example, if the integral $I_j[f,g] = \int_a^b f(x_j,y)\psi(y)e^{i\omega g(x_j,y)}dy$ has a single stationary phase point at point $y = \tau(x_j)$, it can be divided into the following two integrals:

$$I_j[f,g] = \int_{\tau(x_j)}^b f(x_j,y)\psi(y)e^{i\omega g(x_j,y)}dy + \int_a^{\tau(x_j)} f(x_j,y)\psi(y)e^{i\omega g(x_j,y)}dy \triangleq I_j^1 + I_j^2,$$

then each of them can be well calculated with a relatively small number of nodes because the stationary phase point is now located at the endpoints. The case of multiple stationary phase points can be handled in the same way. There is no need to give more details.

For each new integral, we need to specify new nodes in the sub-interval, and they are generally different from the global nodes. It has been mentioned that we are concerned with the unknowns on the global nodes, so the interpolation technique should be adopted to establish a relationship between the function values on the global nodes and those on the new nodes. Among the interpolation methods, the Barycentric interpolation is a very stable and accurate one [23,24]. For a function $u = u(x)$, when the nodes and the function values on them are given as $\{(\zeta_j, u(\zeta_j))\}_{j=0,1,\ldots,n}$, the function value at a fixed inner point can be interpolated as [24]

$$u(\zeta) = \frac{\displaystyle\sum_{j=0}^n \frac{\varpi_j}{\zeta - \zeta_j} u(\zeta_j)}{\displaystyle\sum_{j=0}^n \frac{\varpi_j}{\zeta - \zeta_j}} \tag{14}$$

where the weights ϖ_j follow

$$\varpi_j = \frac{1}{\displaystyle\prod_{k \neq j} (\zeta_j - \zeta_k)}, \quad j,k = 0,1,\ldots,n.$$

In the present study the nodes used are of Chebyshev–Lobatto type, so the above weights can be simplified as

$$\varpi_j = (-1)^j \delta_j, \quad \delta_j = \begin{cases} 1/2, & j = 0 \text{ or } j = n, \\ 1, & \text{otherwise.} \end{cases}$$

From (14) it is seen that the interpolated function value can be expressed in a matrix form:

$$u(\xi) = \begin{bmatrix} l_0(\xi) & l_1(\xi) & \cdots & l_n(\xi) \end{bmatrix} \begin{bmatrix} u(\zeta_0) & u(\zeta_1) & \cdots & u(\zeta_n) \end{bmatrix}^T,$$

where coefficients are $l_j = \dfrac{\frac{\varpi_j}{\zeta - \zeta_j}}{\sum_{j=0}^{n} \frac{\varpi_j}{\zeta - \zeta_j}}$. Consequently, the function values on new $\nu + 1$ nodes $(\xi'_k, k = 0, 1, \ldots, \nu)$ are expressed as

$$\begin{bmatrix} u(\xi'_0) \\ u(\xi'_1) \\ \vdots \\ u(\xi'_\nu) \end{bmatrix} = \begin{bmatrix} l_0(\xi'_0) & l_1(\xi'_0) & \cdots & l_n(\xi'_0) \\ l_0(\xi'_1) & l_1(\xi'_1) & \cdots & l_n(\xi'_1) \\ \vdots & \vdots & \vdots & \vdots \\ l_0(\xi'_\nu) & l_1(\xi'_\nu) & \cdots & l_n(\xi'_\nu) \end{bmatrix} \begin{bmatrix} u(\zeta_0) \\ u(\zeta_1) \\ \vdots \\ u(\zeta_n) \end{bmatrix} \triangleq \mathbf{L}\mathbf{Y}.$$

We assume that the new nodes in the two sub-intervals are $y_t^1 = \frac{b - \tau(x_j)}{2} \cos(\frac{\pi t}{N_1}) + \frac{b + \tau(x_j)}{2}, t = 0, 1, \ldots, N_1$ and $y_t^2 = \frac{\tau(x_j) - a}{2} \cos(\frac{\pi t}{N_2}) + \frac{\tau(x_j) - a}{2}, t = 0, 1, \ldots, N_2$, respectively. If they are different from the global nodes $y_k = \frac{b-a}{2} \cos(\frac{\pi k}{N}) + \frac{b+a}{2}, k = 0, 1, \ldots, N$, then the interpolated function values can be yielded as

$$\boldsymbol{\Phi}_1 = \begin{bmatrix} \psi(y_0^1) \\ \psi(y_1^1) \\ \vdots \\ \psi(y_{N_1}^1) \end{bmatrix} = \begin{bmatrix} l_0(y_0^1) & l_1(y_0^1) & \cdots & l_n(y_0^1) \\ l_0(y_1^1) & l_1(y_1^1) & \cdots & l_n(y_1^1) \\ \vdots & \vdots & \vdots & \vdots \\ l_0(y_{N_1}^1) & l_1(y_{N_1}^1) & \cdots & l_n(y_{N_1}^1) \end{bmatrix} \begin{bmatrix} \psi(y_0) \\ \psi(y_1) \\ \vdots \\ \psi(y_N) \end{bmatrix} \triangleq \mathbf{L}_1 \boldsymbol{\Phi}$$

and

$$\boldsymbol{\Phi}_2 = \begin{bmatrix} \psi(y_0^2) \\ \psi(y_1^2) \\ \vdots \\ \psi(y_{N_2}^2) \end{bmatrix} = \begin{bmatrix} l_0(y_0^2) & l_1(y_0^2) & \cdots & l_n(y_0^2) \\ l_0(y_1^2) & l_1(y_1^2) & \cdots & l_n(y_1^2) \\ \vdots & \vdots & \vdots & \vdots \\ l_0(y_{N_2}^2) & l_1(y_{N_2}^2) & \cdots & l_n(y_{N_2}^2) \end{bmatrix} \begin{bmatrix} \psi(y_0) \\ \psi(y_1) \\ \vdots \\ \psi(y_N) \end{bmatrix} \triangleq \mathbf{L}_2 \boldsymbol{\Phi}.$$

After the interpolated function values in the sub-intervals have been obtained, the integral results I_j^1 and I_j^2 can then be worked out in the same way as (13). For example, the integral I_j^1 is obtained as

$$I_j^1 = \mathbf{Q}_j^1 \left(\frac{2}{b - \tau(x_j)} \mathbf{D}_{N_1} + i\omega \boldsymbol{\Sigma}_j^1 \right)^{-1} \operatorname{diag}(\mathbf{F}_j^1) \boldsymbol{\Phi}_1 = \mathbf{Q}_j^1 \left(\frac{2}{b - \tau(x_j)} \mathbf{D}_{N_1} + i\omega \boldsymbol{\Sigma}_j^1 \right)^{-1} \operatorname{diag}(\mathbf{F}_j^1) \mathbf{L}_1 \boldsymbol{\Phi} \tag{15}$$

with

- $\mathbf{Q}_j^1 = [e^{i\omega g(x_j, b)}, 0, \ldots, 0, -e^{i\omega g(x_j, \tau(x_j))}],$
- $\boldsymbol{\Sigma}_j^1 = \operatorname{diag}\left(g'(x_j, y_0^1), g'(x_j, y_1^1), \ldots, g'(x_j, y_{N_1}^1)\right)$ a diagonal matrix,
- $\mathbf{D}_{N_1}$ the Chebyshev differentiation matrix of order $N_1 + 1$,
- and $\mathbf{F}_j^1 = [f(x_j, y_0^1), f(x_j, y_1^1), \ldots, f(x_j, y_{N_1}^1)]^T$ a numerical vector.

Conclusively, the total integral result is obtained as

$$I_j = I_j^1 + I_j^2 = \left[\mathbf{Q}_j^1 \left(\frac{2}{b - \tau(x_j)} \mathbf{D}_{N_1} + i\omega \boldsymbol{\Sigma}_j^1 \right)^{-1} \operatorname{diag}(\mathbf{F}_j^1) \mathbf{L}_1 + \mathbf{Q}_j^2 \left(\frac{2}{\tau(x_j) - a} \mathbf{D}_{N_2} + i\omega \boldsymbol{\Sigma}_j^2 \right)^{-1} \operatorname{diag}(\mathbf{F}_j^2) \mathbf{L}_2 \right] \boldsymbol{\Phi}. \tag{16}$$

It should be noted that if a fixed stationary phase point is involved, the situation could be much simpler. In this situation, the stationary phase point $y = \tau$ is irrelative to x_j, and then the interpolation matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ remain unchanged.

The following section intends to solve the discrete integral equations (6) using the result obtained in (13) or (16).

2.4. Solving the unknown function from the discrete equations

In (13) and (16) the result of an oscillatory integral has been expressed in a matrix form. For the convenience of description, we uniformly denote them by

$$I_j = \mathbf{U}_j \, \boldsymbol{\Phi} \tag{17}$$

with $\mathbf{U}_j \in \mathbb{C}^{1 \times (N+1)}$ and $\boldsymbol{\Phi} \in \mathbb{C}^{(N+1) \times 1}$. That is to say, if no stationary phase point is involved, there should be

$$\mathbf{U}_j = \mathbf{Q}_j \left(\frac{2}{b-a} \mathbf{D} + i\omega \boldsymbol{\Sigma}_j \right)^{-1} \mathrm{diag}(\mathbf{F}_j), \tag{18}$$

and if a single stationary phase point is involved, there should be

$$\mathbf{U}_j = \mathbf{Q}_j^1 \left(\frac{2}{b-\tau(x_j)} \mathbf{D}_{N_1} + i\omega \boldsymbol{\Sigma}_j^1 \right)^{-1} \mathrm{diag} \left(\mathbf{F}_j^1 \right) \mathbf{L}_1 + \mathbf{Q}_j^2 \left(\frac{2}{\tau(x_j)-a} \mathbf{D}_{N_2} + i\omega \boldsymbol{\Sigma}_j^2 \right)^{-1} \mathrm{diag} \left(\mathbf{F}_j^2 \right) \mathbf{L}_2. \tag{19}$$

The case of multiple stationary phase points has a similar expression.

Consequently, substituting the integral result (17) into (6) gives the matrix form of the discrete equation

$$\psi(x_j) = h(x_j) + \mathbf{U}_j \, \boldsymbol{\Phi}. \tag{20}$$

In (20), the unknown function values $\boldsymbol{\Phi}$ is based on the Chebyshev–Lobatto nodes in the y direction: $y_k = \frac{b-a}{2} \cos(\frac{\pi k}{N}) + \frac{b+a}{2}$, $k = 0, 1, \ldots, N$. If the nodes in the x direction is specified to be identical with those in the y direction ($M = N$ and $x_j = y_j$), there should be $[\psi(x_0), \psi(x_1), \ldots, \psi(x_N)]^T = \boldsymbol{\Phi}$. For a different j, (20) leads to a different linear equation, then the combination of them results in the following system of linear equations:

$$\boldsymbol{\Phi} = \mathbf{H} + \mathbf{U}\boldsymbol{\Phi}, \tag{21}$$

where

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_N \end{bmatrix} \in \mathbb{C}^{(N+1) \times (N+1)}, \qquad \mathbf{H} = \begin{bmatrix} h(x_0) \\ h(x_1) \\ \vdots \\ h(x_N) \end{bmatrix} \in \mathbb{C}^{(N+1) \times 1}.$$

Finally, the system of linear equation (21) is transformed as

$$(\mathbf{I} - \mathbf{U})\boldsymbol{\Phi} = \mathbf{H}. \tag{22}$$

While solving this system of linear equation, the behavior of $\mathbf{I} - \mathbf{U}$ plays a very important role to the stability and accuracy of the solution. In the present study we mainly concern the high frequency problems ($\omega \gg 1$). Under this assumption, it is easy to observe from (18) and (19) that the entries of $\mathbf{U}_j$ should be of the order of $\mathcal{O}(\omega^{-1})$, and then the entries of coefficient matrix $\mathbf{U}$ should also be of the order of $\mathcal{O}(\omega^{-1})$. In this sense, the matrix $\mathbf{I} - \mathbf{U}$ tends to be well-conditioned when a high frequency is involved ($\omega \gg 1$). Consequently, the unknown numerical vector $\boldsymbol{\Phi}$ can be easily worked out from (22) using a conventional method such as the LU factorization, and then the integral equation (2) is solved.

It should be also noted that the solution of (22) has a special physical background. In the light of the operator theory [25], the solution of (22) can be expanded as the following series if $\|\mathbf{U}\| < 1$:

$$\boldsymbol{\Phi} = (\mathbf{I} - \mathbf{U})^{-1}\mathbf{H} = \sum_{j=0}^{\infty} \mathbf{U}^j \mathbf{H}.$$

This expression is called the Born series (or the Neumann series). If the first two terms on the right-hand side is adopted and the rest terms are truncated, the approximation solution is called the first-order Born approximation (or called the Born approximation for simplicity). Similarly, keeping more terms corresponds to a higher-order Born approximation. The technique of Born approximation is very useful for multi-dimensional scattering analysis to save computation cost. In this study, the scale of the system of linear equations is very small, so we directly use the LU factorization method to solve the Eq. (22).

2.5. Analysis of the computational complexity

As shown in (22), if one tries to obtain the unknown function values $\boldsymbol{\Phi}$, he (or she) has to do the following two things: determining the matrix $\mathbf{U}$ and solve the target system of linear equations (22). Because (22) is a small-scaled system of linear

equations, so the computation of it is computationally very easy. In this sense, the main computational complexity of the present method comes from the determination of matrix $\mathbf{U}$ (or the rows of it: $\mathbf{U}_j, j = 0, 1, \ldots, N$).

1. If the integral is free of stationary phase points, the determination of vector $\mathbf{U}_j$ needs for solving a system of linear equations of order $N + 1$, so the computational complexity is $\mathcal{O}((N + 1)^3)$ if a direct solution method (such as the LU factorization method) is adopted. Of course, if a proper iteration technique (such as the Conjugate Gradient (CG) method or the Generalized Minimum Residual (GMRES) method) is employed [23], the computational complexity can be reduced to $\mathcal{O}((N + 1)^2)$.
2. If the integral is with stationary phase points, the situation is a little complicated. Assume the integral has s stationary phase points in the interval, then we should divide the original integral into $s + 1$ sub-integrals and each sub-integral needs for solving a system of linear equations. The interpolation in a sub-interval requires a computational complexity of $\mathcal{O}((N + 1)N_k)$, which is generally much smaller than that of solving the system of linear equations. Therefore, the computational complexity is mainly determined by the calculation of the $s + 1$ systems of linear equations, resulting in a total computational complexity of $\mathcal{O}(\sum_{k=1}^{s}(N_k + 1)^3)$ and $\mathcal{O}(\sum_{k=1}^{s}(N_k + 1)^2)$ for direct solution methods and iteration methods, respectively. The number of nodes in a sub-interval is generally smaller than that in the global interval, so the total computational complexity could be even smaller than the case free of stationary phase points.

Conclusively, the maximum computational complexity of determining the $N + 1$ different $\mathbf{U}_j, j = 0, 1, \ldots, N$ is $\mathcal{O}((N+1)^4)$ and $\mathcal{O}((N + 1)^3)$ for direct solutions and iteration methods, respectively. It has been stated that the unknown function can be well interpolated by a small number of nodes (i.e., N is small), so the total computational complexity is very small.

3. Numerical examples

Three examples are provided in this section to illustrate the performance of the present method: the first one is free of a stationary phase point, the second one is with a fixed stationary phase point, and the third one is with an unfixed stationary phase point.

For a common oscillatory integral equation, its solution is generally not possible to obtain in a closed form. In order to make the analysis feasible, the numerical examples are proceeded in the following way:

- First, the functions $\psi(x)$ and $g(x, y)$ are given, and the integral is given as $\int_a^b f(x, y)\psi(y)e^{i\omega g(x,y)}dy = p(x, y)e^{i\omega g(x,y)}$, then the functions $f(x, y)$ and $h(x)$ can be constructed in the following way: $f(x, y) = \frac{p_y'(x,y)+i\omega p(x,y)g_y'(x,y)}{\psi(y)}$ and $h(x) = \psi(x) - p(x, b)e^{i\omega g(x,b)} + p(x, a)e^{i\omega g(x,a)}$.
- Second, construct an oscillatory integral equation using the resulting functions $f(x, y)$, $h(x)$ and the given functions $g(x, y)$. This integral equation has the form of (7), and its exact solution should be $\psi(x)$.
- Third, numerically work out the "unknown function $\psi(x)$" from the resulting integral equation with the proposed method, then the comparison between the numerical solution and the exact $\psi(x)$ leads to the relative error.

We first study an oscillatory integral equation which is free of stationary phase point.

Example 1. Solve the oscillatory integral equation

$$\psi(x) = h(x) + \int_a^b f(x, y)\psi(y)e^{i\omega g(x,y)}dy,$$

with $g(x, y) = x^2/20 + (y + 6/5)^2$, $h(x) = 1 + (x - 1/2)^2 \cos(10x) - e^{-x^2-7}\left[e^{i\omega(x^2/20+121/25)} - e^{i\omega(x^2/20+1/25)}\right]$, and $f(x, y) = \frac{-28y^3+i\omega(2y+12/5)}{1+(y-1/2)^2\cos(10y)}e^{-x^2-7y^4}$.

The exact solution of this integral equation is $\psi(x) = 1 + (x - 1/2)^2 \cos(10x)$. Obviously, the oscillatory integral $\int_{-1}^{1} f(x_j, y)\psi(y)e^{i\omega g(x_j,y)}dy$ is free of stationary phase point in the y direction.

For conventional methods, the high oscillation of $e^{i\omega g(x,y)}$ makes the integral equation difficult to be solved with a high efficiency. But for the present method, both rapid computational speed and accurate result can be achieved. We show its performance in the following two ways:

- for a fixed ω and different N, the relative errors (E_r) on the nodes are presented in Fig. 1(a);
- for a fixed N and different ω, the relative errors (E_r) on the nodes are presented in Fig. 1(b).

Here, the relative error is defined as $E_r = |\psi_{\text{num}}/\psi - 1|$ with ψ_{num} denoting the numerical result.

From Fig. 1, two phenomena are observed:

1. Fig. 1(a) shows that for a fixed frequency ω (here $\omega = 100$), the relative error becomes smaller and smaller as the number of nodes increases. This is a very natural phenomenon because more nodes can give a better interpolation of the related functions.
2. In Fig. 1(b), it is found that for a fixed number of nodes (here $N = 30$), the relative error decays with the increase of frequency ω. This is a very interesting phenomenon. For a conventional method, the accuracy with fixed nodes should

(a) ω is fixed at $\omega = 100$ and N varies as $N = 10, 20, 30, 40$.

(b) N is fixed at $N = 30$ and ω varies as $\omega = 100, 200, 300, 400, 500, 600$.
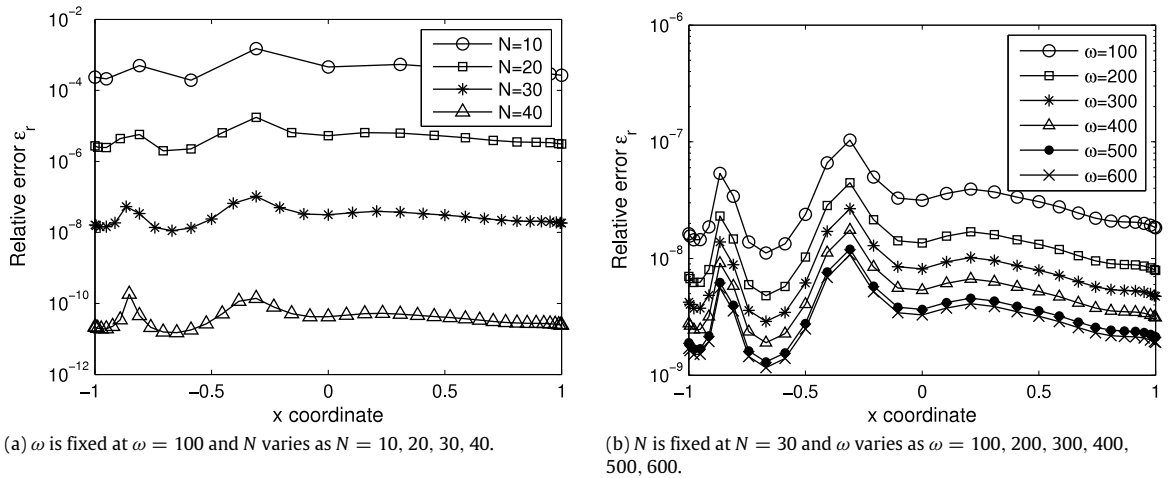
Fig. 1. Performance of the proposed method for the oscillatory integral equations free of stationary point.

be worse and worse with the increase of frequency. But for the present method, an inverse trend is observed. The root cause of this phenomenon is that the error of the improved-Levin quadrature method ($\mathcal{O}(\omega^{-2})$ [5]) has a higher asymptotic order than the integral result itself ($\mathcal{O}(\omega^{-1})$ [5,7]). That is to say, the error decays faster than the integral result. Consequently, the final solution will naturally be more accurate for a large ω.

At the same time, it is also observed that for larger frequencies the gaps between relative error lines become smaller and smaller. This is to say, the relative error of the present method gradually converge to its limit for the fixed number of nodes (the number of nodes in Fig. 1(b) is $N = 30$). Of course, if more nodes are used, the limits of the relative error will be smaller.

The above example shows the good performance of the proposed method when the integral equation is free of stationary phase points. The following examples pay attention to the cases with stationary phase points. First, a fixed stationary phase point case is presented.

Example 2. Solve the oscillatory integral equation

$$\psi(x) = h(x) + \int_a^b f(x, y)\psi(y)e^{i\omega g(x,y)}dy,$$

with $g(x, y) = x^2/20 + (y - 0.3)^2$, $h(x) = 1 + \frac{\sin(5x)}{x^2+1/5} - e^{-2x^3-6}\left[e^{i\omega(x^2/20+49/100)} - e^{i\omega(x^2/20+169/100)}\right]$, and $f(x, y) = \frac{-12y+i\omega(2y-3/5)}{1+\sin(5y)/(y^2+1/5)}e^{-2x^3-6y^2}$.

The exact solution of this integral equation is $\psi(x) = 1 + \frac{\sin(5x)}{x^2+1/5}$. Obviously, the oscillatory integral has a fixed stationary phase point $y = \tau = 0.3$ in the interval $[-1, 1]$. We divide the target interval $[-1, 1]$ into two sub-intervals according to the location of the stationary phase point: $[-1, 0.3]$ and $[0.3, 1]$, and specify Chebyshev–Lobatto nodes to each of them. Obviously, the interpolation matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ in the example need to be calculated only once because the stationary phase point is fixed. The performance of the proposed method is shown in Fig. 2.

It is observed in Fig. 2 that the relative error decays with the increase of the number of nodes N. This is also caused by the fact that more nodes can give a better interpolation of the related functions. At the same time, the relative error of the present method also decays with the increase of frequency; this also well shows the advantage of the present method for high frequency situations.

The following example intends to testify the benefits of the proposed method in the case with an unfixed stationary phase point.
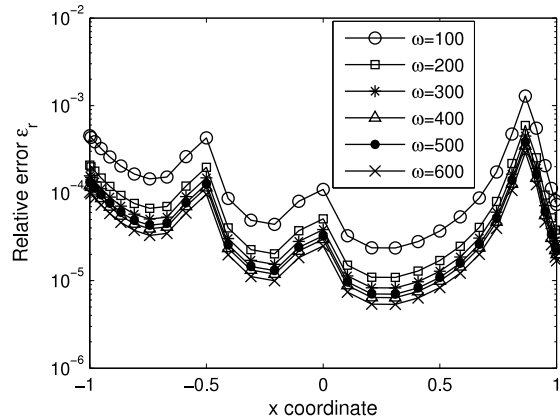
Example 3. Solve the oscillatory integral equation

$$\psi(x) = h(x) + \int_a^b f(x, y)\psi(y)e^{i\omega g(x,y)}dy,$$

with $g(x, y) = (x/20 - y)^2$, $h(x) = 1 + (x - 1/5)^2\cos(10x) - e^{-2x^3-6}\left[e^{i\omega(x/20-1)^2} - e^{i\omega(x/20+1)^2}\right]$, and $f(x, y) = \frac{-12y+i\omega(-x/10+2y)}{1+(y-1/5)^2\cos(10y)}e^{-2x^3-6y^2}$.
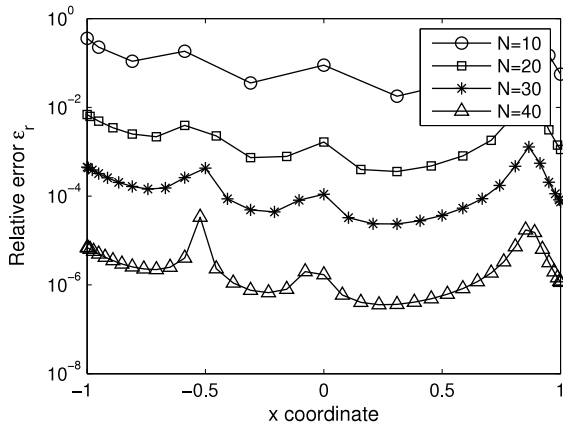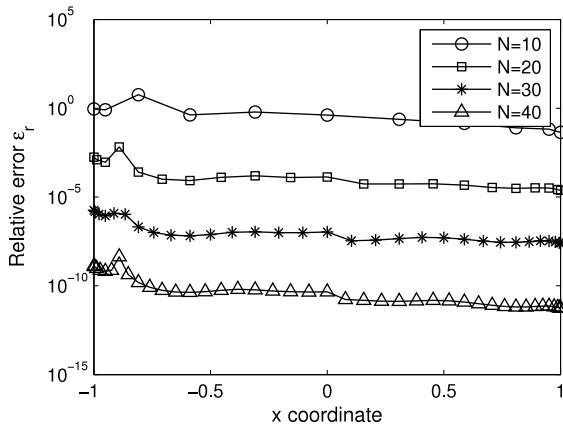
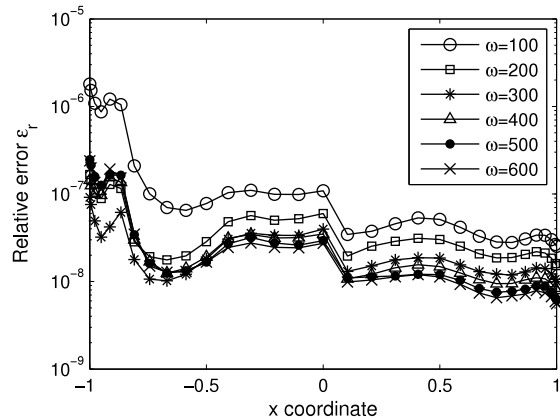(a) ω is fixed at $\omega = 200$ and N varies as $N = 10, 20, 30, 40$.

(b) N is fixed at $N = 30$ and ω varies as $\omega = 100, 200, 300, 400, 500, 600$.

Fig. 2. Performance of the proposed method for the oscillatory integral equation with a fixed stationary phase point.

(a) ω is fixed at $\omega = 200$ and N varies as $N = 10, 20, 30, 40$.

(b) N is fixed at $N = 30$ and ω varies as $\omega = 100, 200, 300, 400, 500, 600$.

Fig. 3. Performance of the proposed method for the oscillatory integral equation with an unfixed stationary phase point.

The exact solution of this integral equation is $\psi(x) = 1 + (x - 1/5)^2 \cos(10x)$. Obviously, the stationary phase point $y = x_j/20$ varies with the variation of point x_j. The performance of the proposed method is shown in Fig. 3. Fig. 3(a) also shows that more nodes can give a solution of better accuracy, and Fig. 3(b) shows that a higher frequency will result in a more accurate solution.

From the above three examples we see that the proposed method can obtain very satisfactory solutions even if the number of nodes is very small. What is more important, the asymptotic order of the error is relatively high, so the method has a much better applicability for high frequency problems in comparison with the conventional methods.

4. Conclusions and future work

A new solution method for highly oscillatory integrals is put forward based on the development of an improved-Levin quadrature method for oscillatory integrals. The new method has the merits of being accurate and having a rapid computational speed. This could provide a positive contribution to the rapid solution of oscillatory integral equations arising in the scattering study and other related fields.

In this paper, one-dimensional oscillatory integral equations are considered. However, in practical physical problems (such as the electromagnetic scattering), the integral equations are generally in multi-dimensional forms, so it is of significance to extend this method from one-dimension to multi-dimension. The authors of this paper are now considering this problem.

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