Diquarks and antiquarks in exotics: A ménage à trois and a ménage à quatre

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Abstract

A ménage à trois is very different from an ordinary family. Similarly, exotic hadrons with both qq and $q\bar{q}$ pairs have important color-space correlations that are completely absent in ordinary mesons and baryons. The presence of both types of pairs requires attention to the basic QCD physics that the $q\bar{q}$ interaction is much stronger than the qq interaction. This new physics in multiquark systems produces color structures totally different from those of normal hadrons, for example the ud system is utterly unlike the ud diquark in the uds Λ baryon. The color-space correlations produce unusual experimental properties in tetraquarks with heavy quark pairs which may be relevant for newly discovered mesons like the $X(3872)$ resonance. Tetraquark masses can be below the two-meson threshold for sufficiently high quark masses. A simple model calculation shows the $bq\bar{b}\bar{u}$ and $bq\bar{c}\bar{q}$ tetraquarks below the $B\bar{B}$ and $B\bar{D}$ thresholds. Some of these states have exotic electric charge and their decays might have striking signatures involving monoenergetic photons and/or pions.
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1. Color, space and spin correlations in multiquark systems

A reliable treatment of multiquark states should include the very different physics arising in systems containing both quarks and antiquarks. The effects of color-space correlations generally neglected were already pointed out in one of the first multiquark treatments [1]. These can change the energy of a state at least as much if not more than the color-flavor-spin correlations commonly used [2,3,5]. A consistent treatment of confinement in multiquark states must also consider that only color is confined and that color-singlet clusters are not confined. New additional states having other spin, color and space couplings not present in standard meson and baryon spectroscopy must be included in realistic calculations. Such states may even now be observed as exotic hadrons. Much of this essentially new multiquark physics is missed in many treatments.

In three-quark baryons the color coupling of the three quarks to a color singlet is unique and is antisymmetric in color. The Pauli principle then requires the state to be totally symmetric in flavor, space and spin. An isoscalar ud state in a baryon; e.g. the ud in the Λ is required to be a color antitriplet with spin zero.

The physics is completely different in states where both quark–quark and quark–antiquark pairs are present. The color coupling is no longer unique. At least two color couplings are allowed and these can mix. No Pauli principle connects quarks and antiquarks; all combinations of flavor, space and spin symmetries can occur. Since the $q\bar{q}$ interaction in QCD is much stronger than the qq interaction an antiquark between the two quarks in a quark–quark system can completely change its color and space correlations; e.g. by mixing in the color sextet state that does not exist in normal baryons.

This new physics is already evident in the color-space correlations absent in normal hadrons appearing in the simplest multiquark system, an isoscalar $ud\bar{s}$ color triplet "triquark" state. The two allowed color couplings have a space-symmetric ud state either in a color $\bar{3}$ with spin zero, like the diquark in a baryon, or a color 6 with spin one, which does not exist in three-quark baryons. These two couplings have been used

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in calculations for the $uudd\bar{s}$ pentaquark where the $ud\bar{s}$ system must couple to a color triplet which then couples to an additional isoscalar ud diquark to make a color singlet. The pentaquark states with these two couplings are nearly degenerate and therefore mix [6].

The triquark state with the lowest potential energy was already shown to have color couplings with the qq in a color 6, where the interaction is repulsive, with this repulsion overwhelmed by a much stronger $q\bar{q}$ interaction [1,2]. The potential energy can then be lowered even further by breaking maximum space symmetry and choosing a configuration [1] where the mean distance between qq pairs is much larger than the mean distance between $q\bar{q}$ pairs. This is in contrast with the 3q color singlet state in normal baryons where there is only a single state. The use of diquarks in multiquark states with the same color couplings as the diquark in the baryon is therefore unjustified and leaves out important physics.

One of the first treatments [1] of multiquark systems considered a tetraquark system of two quarks and two antiquarks with the Nambu color-exchange interaction [7] now used to describe the spin-independent part of the interaction in most common QCD-motivated models [2,3,5,8,9]. The interaction between two constituents i and j is

$$V_{cx}^{ij} = V \lambda_c^i \cdot \lambda_c^j, \tag{1}$$

where i and j can be either quarks or antiquarks, V is an operator that depends on the space and spin variables of the constituents but is the same for all pairs, independent of i and j, λ_c^i is the generator of the color SU(3) group and the scalar product $\lambda_c^i \cdot \lambda_c^j$ denotes the scalar product in color space.

Nambu had already shown that no multiquark states are bound with this interaction when spin effects are neglected, color and space are factorized and all two-body subsystems have the same two-body density matrix. All color singlet couplings of the same multiquark states have the same potential energy. No potential energy is lost by coupling them into separate noninteracting color-singlet meson and baryon clusters which then separate.

We generalize Nambu's theorem from color singlet states to any multiquark system whose interaction has color dependence described by Eq. (1).

$$V_{cx}(\text{tot}) = \sum_{i \neq j} \frac{V_{cx}^{ij}}{2} = \sum_{i \neq j} \frac{V \lambda_c^i \cdot \lambda_c^j}{2}. \tag{2}$$

For the case where the wave function factorizes into a color factor and a factor depending on the other degrees of freedom the factor V can be taken outside the summation to give

$$V_{cx}(\text{tot}) = \frac{V}{2} \cdot \left[\sum_{ij} \lambda_c^i \cdot \lambda_c^j - \sum_i (\lambda_c^i)^2 \right]$$

$$= \frac{V}{2} \cdot \left[(\lambda_C)^2 - \sum_i (\lambda_c^i)^2 \right], \tag{3}$$

where $\lambda_C = \sum_i \lambda_c^i$ is the generator of the color SU(3) group for the whole multiquark system.

The interaction energy of a multiquark system depends only on the color of the whole system and all states with the same overall color are degenerate. The degeneracy can be removed by breaking factorization with color-spin and color-space correlations. However, all the previously degenerate states remain as physical states and must be included.

The simple tetraquark model [1] broke the color-space factorization and gained potential energy over the separated clusters by choosing a color coupling where the qq and $\bar{q}\bar{q}$ couplings were repulsive and making the mean distance between qq and $\bar{q}\bar{q}$ larger than the mean distance between $q\bar{q}$ pairs. When the mean distances were equal the enhanced $q\bar{q}$ attraction exactly overcompensated for the qq and $\bar{q}\bar{q}$ repulsion and gave this configuration the same potential energy as two separated color singlet clusters. Making the distances unequal reduced the qq and $\bar{q}\bar{q}$ repulsion relative to the $q\bar{q}$ attraction and lowered the potential energy.

The present Letter treats this tetraquark model in more detail and shows that it can be relevant to presently observed mesons. A soluble toy model which includes the kinetic energy and describes the spatial dependence of the interaction (2) by a harmonic oscillator potential exhibits the effects of differences in spatial structure between exotics and normal hadrons. These effects are often neglected or improperly treated in common treatments of multiquark states which focus on color, flavor and spin and do not consider spatial correlations.

2. Color couplings in the ($ud\bar{s}$) system

We first investigate the isoscalar color-triplet three-body $ud\bar{s}$ system. In the simplest approximation with the Nambu two-body color-exchange interaction (1), (2) and kinetic energies neglected, there are two degenerate states and no unique definition of a state that can be called a "diquark".

We consider the interactions in three possible color couplings: the two orthogonal states denoted by $|\bar{\mathbf{s}}\bar{\mathbf{3}}\rangle$ and $|\bar{\mathbf{s}}\mathbf{6}\rangle$ which have an isoscalar ud pair coupled either to a color $\bar{\mathbf{3}}$ or a color $\mathbf{6}$ and their "quark–meson" linear combination $|dK^+\rangle$ with the $u\bar{s}$ coupled to a color singlet K^+. We do not consider the $|uK^o\rangle$ state as it has the same color physics as $|dK^+\rangle$. The state $|\bar{\mathbf{s}}\mathbf{6}\rangle$ has a ud pair in a color sextet state which does not arise in ground state baryons, because it cannot couple with the third quark to a color singlet. The qq interaction in this state is repulsive; however, the attractive interaction of the antiquark overcomes this repulsion in the three-body ($ud\bar{s}$) system.

Eq. (1) relates the various two-body couplings to color singlet, antitriplet, sextet and octet states.

$$V_{cx}^1 = 2V_{cx}^{\bar{3}} = -4V_{cx}^6 = -8V_{cx}^8. \tag{4}$$

The potential energy in a potential model with the color-exchange two-body force (1) is given by

$$\langle \bar{\mathbf{s}}\bar{\mathbf{3}} | V | \bar{\mathbf{s}}\bar{\mathbf{3}} \rangle = \frac{1}{2} \cdot \langle V_{ud} \rangle + \frac{1}{4} \cdot \langle V_{u\bar{s}} + V_{d\bar{s}} \rangle$$

$$= \langle V_{u\bar{s}} \rangle + \frac{1}{2} \cdot \langle V_{ud} - V_{u\bar{s}} \rangle, \tag{5}$$

$$\langle \bar{\mathbf{s}}\mathbf{6}|V|\bar{\mathbf{s}}\mathbf{6}\rangle = -\frac{1}{4} \cdot \langle V_{ud}\rangle + \frac{5}{8} \cdot \langle V_{u\bar{s}} + V_{d\bar{s}}\rangle$$

$$= \langle V_{u\bar{s}}\rangle - \frac{1}{4} \cdot \langle V_{ud} - V_{u\bar{s}}\rangle$$

$$= \frac{5}{4}\langle V_{u\bar{s}}\rangle - \frac{1}{4} \cdot \langle V_{ud}\rangle, \tag{6}$$

$$\langle dK^+|V|dK^+\rangle = \langle V_{u\bar{s}}\rangle, \tag{7}$$

where $\langle V_{ud}\rangle$, $\langle V_{u\bar{s}}\rangle$ and $\langle V_{d\bar{s}}\rangle$ denote the expectation values of the three two-body potentials, the coefficients are determined by the color algebra and we have used the isospin relation

$$\langle V_{u\bar{s}}\rangle = \langle V_{d\bar{s}}\rangle. \tag{8}$$

We see again the Nambu result that all couplings have the same color-electric energy if the spatial wave functions are the same for all pairs; i.e. $\langle V_{ud}\rangle = \langle V_{u\bar{s}}\rangle = \langle V_{d\bar{s}}\rangle$. In this approximation, commonly used in most quark models for the pentaquark, kinetic energies and color-space correlations are neglected and an energy difference between different color couplings can only be due to the color magnetic interaction.

There is no unique diquark–antiquark wave function for the three-body system. The color sextet diquark, which does not exist in the three-quark color singlet system, appears in the $qq\bar{q}$ system on the same footing as the color antitriplet.

The $q\bar{q}$ interaction in the state $|\bar{\mathbf{s}}\mathbf{6}\rangle$ is seen to be 25% stronger than the $q\bar{q}$ interaction in the separated "quark–meson" state $|dK^+\rangle$. This additional attraction is balanced exactly by the ud repulsion if the spatial wave functions are the same for all pairs. We thus see that additional attraction can be obtained by making the mean ud distance larger than the mean $q\bar{q}$ distance. We attempt to obtain a quantitative estimate of this additional attraction by using a toy-model for the spatial dependence of the interaction.

We now go beyond this oversimplified treatment with a simple model which includes kinetic energies and color-space-correlations. These drastically modify the color states described by Eqs. (5) and (6). These two states as well as the separated quark–kaon state (7) may no longer be eigenstates of the new interaction. Our treatment does not include this mixing and simply calculates the expectation values of the model Hamiltonian including the color-space correlations for these three states with the same color couplings. By the variational principle these give upper bounds for the exact eigenvalues of the model Hamiltonian.

3. Spatial and kinetic energy effects in the harmonic oscillator model

We can estimate the spatial and kinetic energy effects by introducing a spatial dependence of the operator V in the interaction (2). We choose the harmonic oscillator potential which enables easy separation of center-of-mass motion and enables simple analytical exact solutions and assume a nonrelativistic kinetic energy. Our model Hamiltonian is thus

$$H = \sum_i \frac{p_i^2}{2m} - \sum_{i \neq j} \frac{V_o}{4} \cdot \lambda_c^i \cdot \lambda_c^j \cdot r_{ij}^2$$

$$= \sum_i \frac{p_i^2}{2m} + \frac{V_o}{2}\left[\sum_{ij} \cdot \lambda_c^i \cdot \lambda_c^j \vec{r}_i \cdot \vec{r}_j \right.$$

$$\left. - \sum_i \lambda_c^i \cdot \sum_j \lambda_c^j r_j^2 \right], \tag{9}$$

where p_i denotes the momentum of particle i, $\vec{r}_{ij}$ denotes the distance between particles i and j, and m denotes the particle mass. In this toy model we assume all masses to be equal.

This model Hamiltonian has been extensively investigated [4] and been used in many multiquark calculations [1,6,10]. Although the harmonic oscillator potential becomes infinite at large distances, these infinities are not relevant to our calculations which use harmonic oscillator Gaussian wave functions confined to regions where the potentials are reasonable.

We now introduce this explicit spatial form (9) for the interactions into Eqs. (5)–(7) while keeping the same color couplings. The potential for the separated dK system becomes

$$\frac{V_{dK}}{V_o} = \frac{1}{2} \cdot \left(r_{\bar{s}u}^K\right)^2. \tag{10}$$

For the triquark states we separate the center-of-mass motion by introducing the relative co-ordinates

$$\vec{r} \equiv \vec{r}_{ud} = \vec{r}_u - \vec{r}_d,$$

$$\vec{R} = \vec{r}_{\bar{s}} - \frac{1}{2} \cdot (\vec{r}_u + \vec{r}_d) = \frac{1}{2} \cdot (\vec{r}_{\bar{s}u} + \vec{r}_{\bar{s}d}),$$

$$\frac{V_{\bar{s}3}}{V_o} = \frac{1}{4} \cdot r^2 + \frac{1}{8} \cdot \left[\vec{R} - (\vec{r}/2)\right]^2 + \left[\vec{R} + (\vec{r}/2)\right]^2$$

$$= \frac{1}{4} \cdot R^2 + \frac{5}{16} \cdot r^2,$$

$$\frac{V_{\bar{s}6}}{V_o} = -\frac{1}{8} \cdot r^2 + \frac{5}{16} \cdot \left[\vec{R} - (\vec{r}/2)\right]^2 + \left[\vec{R} + (\vec{r}/2)\right]^2$$

$$= \frac{5}{8} \cdot R^2 + \frac{1}{32} \cdot r^2. \tag{11}$$

The potentials are seen to be positive definite functions of the variables R and r and therefore bounded from below. Although the color sextet has a repulsive force, the negative term $-(1/8) \cdot r^2$ is overcompensated by the attraction of the antiquark and the resulting dependence upon r^2 has a positive coefficient. That the Hamiltonian (9) is positive definite for overall color singlet and triplet states was pointed out in general for a color-exchange interaction with a harmonic oscillator force [4]. This is easily seen by inspection from Eq. (9) for an overall color singlet since $\sum_{ij} \lambda_c^i \cdot \lambda_c^j \vec{r}_i \cdot \vec{r}_j$ is a square and $\sum_i \lambda_c^i = 0$ for a color singlet.

The expectation values of the model Hamiltonian (9) give the energies of the ground states for these three systems

$$E_{dK} = \langle dK^+|H|dK^+\rangle = \frac{3\hbar}{2} \cdot \omega^{dK},$$

$$E_{\bar{s}3} = \langle \bar{\mathbf{s}}\mathbf{3}|H|\bar{\mathbf{s}}\mathbf{3}\rangle = 1.40 \cdot E_{dK},$$

$$E_{\bar{s}6} = \langle \bar{\mathbf{s}}\mathbf{6}|H|\bar{\mathbf{s}}\mathbf{6}\rangle = 1.22 \cdot E_{dK}, \tag{12}$$
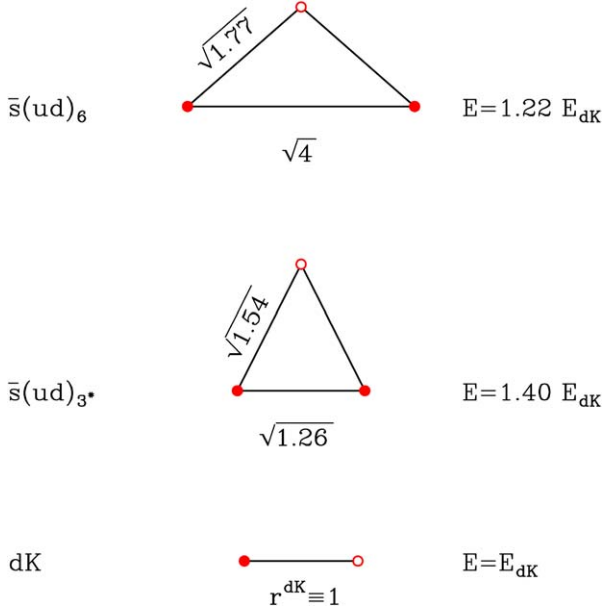
Fig. 1. Schematic depiction of the geometry of the dK, $\bar{s}(ud)_{3*}$ and $\bar{s}(ud)_6$ configurations. Quarks are denoted by full circles, antiquarks by empty circles. The lengths of the triangles' sides are square roots of the corresponding rms distances. In the dK configuration the "outside" d-quark is not shown, as there are no color forces between it and the color-singlet K meson.

where ω^{dK} is the oscillator frequency for the separated quark–kaon system

$$\omega^{dK} = \sqrt{\frac{2V_o}{m_{\text{reduced}}}} = \sqrt{\frac{2V_o}{m}}. \tag{13}$$

The ground state mean square distances are

$$\langle [r^{dK}]^2 \rangle = \frac{3\hbar}{m\omega^{dK}} = \frac{3\hbar}{\sqrt{mV_o}} \cdot \frac{\sqrt{2}}{2},$$

$$\langle [r^{\bar{s}\bar{3}}]^2 \rangle = 1.26 \cdot \langle [r^{dK}]^2 \rangle, \qquad \langle [r^{\bar{s}\bar{3}}_{\bar{s}u}]^2 \rangle = 1.54 \cdot \langle [r^{dK}]^2 \rangle,$$

$$\langle [r^{\bar{s}6}]^2 \rangle = 4 \cdot \langle [r^{dK}]^2 \rangle, \qquad \langle [r^{\bar{s}6}_{\bar{s}u}]^2 \rangle = 1.77 \cdot \langle [r^{dK}]^2 \rangle. \tag{14}$$

The results in Eq. (14) are schematically illustrated in Fig. 1 below.

The $|\bar{s}6\rangle$ triquark has a lower energy than the $|\bar{s}\bar{3}\rangle$ triquark but still higher than the energy of the separated quark–kaon system. The mean square quark–quark and quark–antiquark distances in both triquarks are larger than the quark–antiquark distance in the kaon. They are largest for the $|\bar{s}6\rangle$ triquark.

When color-space factorization is abandoned, the various color recouplings produce different energies and rms distances. The interaction of the two quarks with the antiquark creates a ud structure very different from that of the ud diquark found in three-quark baryons.

These results for the $(ud\bar{s})$ system are not easily tested against experiments which always involve color singlet hadrons. We now extend the basic physics of this approach to treat realistic exotic multiquark color singlet systems; e.g. tetraquarks and pentaquarks, for which there are experimental candidates and for which multiquark models have already been proposed.

The case of the $(uudd\bar{s})$ pentaquark has been treated [10] using a similar harmonic oscillator Hamiltonian with an additional spin-dependent interaction. This system is much more complicated than our triquark because it includes two pairs of quarks with identical flavor and the necessity to properly antisymmetrize the wave function. They are therefore unable to find a simple set of relative co-ordinates in which the spatial problem can be solved exactly and center-of-mass motion can be ignored. They use a model space of single-particle shell-model wave functions in a model space of 15 000 basis functions.

One solution found in Ref. [10] has a color-space correlation similar to that of our triquark. The mean quark–antiquark distance is much smaller than the mean quark–quark distance. In particular they find that the one-body r.m.s. radius measured from the center of mass is 1.10 Fm. for u, d quarks and 0.72 Fm for the $\bar{s}$ antiquark while the corresponding radius is 0.69 Fm for the $(0s)^5$ configuration.

That the one-body r.m.s. radius of the multiquark system is larger than expected from normal hadrons implies that the short-range color-magnetic interaction will be weaker than in normal hadrons. The conventional practice of using spin splittings from normal hadrons to normalize the color-magnetic interaction [2,3,5] is therefore questionable.

4. A tetraquark in the harmonic oscillator model

We consider the simplest multiquark color singlet system which can be treated exactly in the harmonic oscillator model without color mixing and without the need for Pauli antisymmetrization. The tetraquarks treated here exhibit features missed in other models which have been suggested for these states; e.g. the color sextet quark–quark couplings and can provide useful models for mesons containing heavy quarks that are now being discovered.

As discussed in more detail in Section 5, these wave functions may also provide a good basis for further treatments in which color-magnetic and spin effects are treated as perturbations.

4.1. The $(us\bar{d}\bar{c})$ tetraquark

We develop general results for tetraquarks by considering the $(us\bar{d}\bar{c})$ system with four different flavors and four different masses. This will show how the difference between the tetraquark mass and the mass of two separated mesons depends upon the quark masses.

The general multiquark model Hamiltonian (9) used for the interaction used in the three triquark states (5), (6) and (7) is now used for the analogous four-body states. There is a separated two-meson state denoted by $2M$.

$$\frac{V_{2M}}{V_o} = \frac{1}{2} \cdot (r_{u\bar{c}}^2 + r_{s\bar{d}})^2. \tag{15}$$

There are a triplet–antitriplet tetraquark denoted by $\bar{3}3$, and a sextet–antisextet tetraquark denoted by $\bar{6}6$.

$$\frac{V_{\bar{3}3}}{V_o} = \frac{1}{4} \cdot \left(r_{us}^2 + r_{\bar{c}\bar{d}}^2\right) + \frac{1}{8} \cdot \left(r_{u\bar{c}}^2 + r_{s\bar{c}}^2 + r_{u\bar{d}}^2 + r_{s\bar{d}}^2\right),$$

$$\frac{V_{\bar{6}6}}{V_o} = -\frac{1}{8} \cdot \left(r_{us}^2 + r_{\bar{c}\bar{d}}^2\right) + \frac{5}{16} \cdot \left(r_{u\bar{c}}^2 + r_{s\bar{c}}^2 + r_{u\bar{d}}^2 + r_{s\bar{d}}^2\right). \quad (16)$$

Here again we note that when all the spatial separations are equal, $r_{us}^2 = r_{\bar{c}\bar{d}}^2 = r_{u\bar{c}}^2 = r_{s\bar{c}}^2 = r_{u\bar{d}}^2 = r_{s\bar{d}}^2$ the $q\bar{q}$ interaction in the state $\bar{6}6$ is seen to be 25% stronger than the $q\bar{q}$ interaction in the separated two-meson state $2M$. This additional attraction is balanced exactly by the us and $\bar{c}\bar{d}$ repulsions if the spatial wave functions are the same for all pairs. We thus see that additional attraction can be obtained by making the mean us and $\bar{c}\bar{d}$ distances larger than the mean $q\bar{q}$ distance.

We also note that the repulsive qq interaction in the color-sextet plays a crucial role in the suppression of binding low-lying exotic hadrons. The additional $q\bar{q}$ attraction arises from the presence of four attractive $q\bar{q}$ interactions in the tetraquark, whereas there are only two attractive $q\bar{q}$ interactions in the separated two-meson system. The larger number of $q\bar{q}$ pairs in multiquark systems as opposed to separated color singlet hadrons would produce bound low-lying exotics if there were no repulsive qq interactions.

We now obtain a quantitative estimate of this additional attraction by using the harmonic oscillator model for the spatial dependence.

We express the interactions $V_{\bar{3}3}$ and $V_{\bar{6}6}$ in terms of the relative co-ordinates

$$\vec{r}_{us} = \vec{r}_u - \vec{r}_s, \qquad \vec{r}_{dc} = \vec{r}_{\bar{d}} - \vec{r}_{\bar{c}},$$

$$\vec{R} = \frac{1}{2} \cdot (\vec{r}_{\bar{c}} + \vec{r}_{\bar{d}}) - \frac{1}{2} \cdot (\vec{r}_u + \vec{r}_s),$$

$$\frac{V_{\bar{3}3}}{V_o} = \frac{1}{4} \cdot \left(r_{us}^2 + r_{cd}^2\right) + \frac{1}{2} \cdot R^2 + \frac{1}{8} \cdot \left(r_{us}^2 + r_{cd}^2\right)$$
$$= \frac{1}{2} \cdot R^2 + \frac{3}{8} \cdot \left(r_{us}^2 + r_{cd}^2\right),$$

$$\frac{V_{\bar{6}6}}{V_o} = -\frac{1}{8} \cdot \left(r_{us}^2 + r_{cd}^2\right) + \frac{5}{4} \cdot R^2 + \frac{5}{16} \cdot \left(r_{us}^2 + r_{cd}^2\right)$$
$$= \frac{5}{4} \cdot R^2 + \frac{3}{16} \cdot \left(r_{us}^2 + r_{cd}^2\right). \quad (17)$$

The ground state energies of these states are given by the sum of the ground state energies of the contributing harmonic oscillators. The ground state energy of an oscillator with co-ordinate r, mass M and potential $V r^2$ is

$$E_g(\text{osc}) = \frac{3}{2} \cdot \hbar\omega = \frac{3}{2} \cdot \hbar\sqrt{\frac{V}{M}}. \quad (18)$$

For the two-meson $(u\bar{c}; s\bar{d})$ system, which has two separated harmonic oscillators with reduced masses denoted by $M(uc)$ and $M(ds)$ and potentials $V_o/2$ from Eq. (15),

$$E_g(2Musdc) = \frac{3}{2} \cdot \hbar\left(\sqrt{\frac{V_o}{2M(uc)}} + \sqrt{\frac{V_o}{2M(ds)}}\right). \quad (19)$$

The ground state energies for the $\bar{3}3$ and $\bar{6}6$ systems are obtained by substituting the potentials in (17) into the expression

(18) along with the reduced masses

$$M(us) = \frac{m_u m_s}{m_u + m_s}, \qquad M(dc) = \frac{m_d m_c}{m_d + m_c},$$

$$M\{R; (us)(dc)\} = \frac{(m_u + m_s) \cdot (m_d + m_c)}{m_u + m_s + m_d + m_c}, \quad (20)$$

$$E_g(\bar{3}3) = \frac{3}{2} \cdot \hbar\left(\sqrt{\frac{V_o}{2M\{R; (us)(dc)\}}} + \sqrt{\frac{3V_o}{8M(us)}}\right.$$
$$\left. + \sqrt{\frac{3V_o}{8M(dc)}}\right), \quad (21)$$

$$E_g(\bar{6}6) = \frac{3}{2} \cdot \hbar\left(\sqrt{\frac{5V_o}{4M\{R; (us)(dc)\}}} + \sqrt{\frac{3V_o}{16M(us)}}\right.$$
$$\left. + \sqrt{\frac{3V_o}{16M(dc)}}\right). \quad (22)$$

The ratios of these energies to the energy of the two meson state are then

$$\frac{E_g(\bar{3}3\,usdc)}{E_g(2M\,usdc)}$$
$$= \sqrt{\frac{3}{4} + \frac{\sqrt{(m_u + m_s + m_d + m_c)m_s m_u m_c m_d}}{m_c m_s (m_d - m_u) + m_d m_u (m_c - m_s)}}$$
$$\times \left[\sqrt{\frac{m_d m_c}{m_d + m_c}} - \sqrt{\frac{m_u m_s}{m_u + m_s}}\right] = 1.18, \quad (23)$$

$$\frac{E_g(\bar{6}6\,usdc)}{E_g(2M\,usdc)}$$
$$= \sqrt{\frac{3}{8}} + \sqrt{\frac{5}{2}} \cdot \frac{\sqrt{(m_u + m_s + m_d + m_c)m_s m_u m_c m_d}}{m_c m_s (m_d - m_u) + m_d m_u (m_c - m_s)}$$
$$\times \left[\sqrt{\frac{m_d m_c}{m_d + m_c}} - \sqrt{\frac{m_u m_s}{m_u + m_s}}\right] = 1.11, \quad (24)$$

where we have set $m_u = m_d$ and substituted the values of the constituent quark masses obtained by fitting the ground state meson and baryon spectra [2].

$$m_u = 360, \qquad m_s = 540,$$

$$m_c = 1710, \qquad m_b = 5050. \quad (25)$$

The ratios of the energy of each tetraquark state to the energy of the separated two-meson system are the sum of two terms. The first terms, respectively $\sqrt{3/4}$ and $\sqrt{3/8}$ are the energies of the qq and $\bar{q}\bar{q}$ systems which respectively have the same reduced masses as the corresponding mesons. They have a lower energy because they are spread over a larger domain in configuration space and have a lower kinetic energy. The second terms depend upon the flavor via the reduced mass of the $(qq) - (\bar{q}\bar{q})$ system. These ratios decrease with increasing quark masses and for sufficiently high quark masses will produce bound tetraquarks below the two-meson threshold, as discussed in Sections 4.3 and 4.4 below.

4.2. The $cu\bar{c}\bar{u}$ tetraquark

We treat the $cu\bar{c}\bar{u}$ configuration by changing the flavors in Eqs. (23), (24). This gives

$$\frac{E_g(\bar{\mathbf{3}}\mathbf{3}\,cucu)}{E_g(2M\,cucu)} = 1.134, \qquad \frac{E_g(\bar{\mathbf{6}}\mathbf{6}\,cucu)}{E_g(2M\,cucu)} = 1.036, \qquad (26)$$

where we have substituted the constituent quark masses (25).

That the energy of the $cu\bar{c}\bar{u}$ tetraquark with the color sextet–antisextet coupling ($\bar{\mathbf{6}}\mathbf{6}$) is found to be only 4% above the ground state energy of the two-meson $D\bar{D}$ system suggests that such states should be included in all calculations for these states.

That the ($\bar{\mathbf{6}}\mathbf{6}$) state is considerably below the ($\bar{\mathbf{3}}\mathbf{3}$) state casts doubt on all tetraquark calculations which neglect the color space correlations [11] and the ($\bar{\mathbf{6}}\mathbf{6}$) state [12,13]; e.g. those for the X(3872) resonance. These neglect the basic physics seen in the experimental hadron mass spectrum and its description by QCD motivated models [2,3,5,8,9] showing that the attractive $q\bar{q}$ interaction as observed in mesons is much stronger than the attractive qq interaction observed in baryons.

The color-space correlation contributions to the energy may well be more important than the color-magnetic energy neglected here which dominates other tetraquark model calculations [11–13].

The color magnetic energy can be included in the same way as in all other quark model calculations, as a perturbation using the unperturbed wave functions. The correction may not be large here because of the small color-magnetic interaction of the heavy charmed quark and the suppression of this short-range interaction at larger distances. This is discussed in detail in Section 5.

4.3. The $bu\bar{b}\bar{u}$ tetraquark

We treat the $bu\bar{b}\bar{u}$ configuration by changing the flavors in Eqs. (23), (24). This gives

$$\frac{E_g(\bar{\mathbf{3}}\mathbf{3}\,bubu)}{E_g(2M\,bubu)} = 1.042, \qquad \frac{E_g(\bar{\mathbf{6}}\mathbf{6}\,bubu)}{E_g(2M\,bubu)} = 0.891, \qquad (27)$$

where we have substituted the constituent quark masses (25).

The mass of the $bu\bar{b}\bar{u}$ tetraquark with the color sextet–antisextet coupling is found to be well below the two-meson $B\bar{B}$ threshold in this approximation. Such a low-mass threshold may also be found in a more exact calculation including spin effects. That they might be found in the experimental spectrum must be taken seriously.

4.4. The $cu\bar{b}\bar{u}$ tetraquark

We treat the $cu\bar{b}\bar{u}$ configuration by changing the flavors in Eqs. (23), (24). This gives

$$\frac{E_g(\bar{\mathbf{3}}\mathbf{3}\,cubu)}{E_g(2M\,cubu)} = 1.10, \qquad \frac{E_g(\bar{\mathbf{6}}\mathbf{6}\,cubu)}{E_g(2M\,cubu)} = 0.975, \qquad (28)$$

where we have substituted the constituent quark masses (25).

This suggests that a $cu\bar{b}\bar{u}$ tetraquark might exist with a mass at the BD threshold. This could be a narrow state decaying electromagnetically to the B_c and a photon or an electron pair. A spin-zero tetraquark with a mass too low for pion decay to B_c would have to decay by electron pair emission in a $0 \to 0$ transition. It is therefore of interest to search for such tetraquarks in experiments observing the B_c.

4.5. The $su\bar{b}\bar{u}$ tetraquark

We treat the $su\bar{b}\bar{u}$ configuration by changing the flavors in Eqs. (23), (24) and using the constituent quark masses (25) as before. This gives

$$\frac{E_g(\bar{\mathbf{3}}\mathbf{3}\,subu)}{E_g(2M\,subu)} = 1.16, \qquad \frac{E_g(\bar{\mathbf{6}}\mathbf{6}\,subu)}{E_g(2M\,subu)} = 1.077. \qquad (29)$$

5. Inclusion of color magnetic interactions

5.1. Spin quantum numbers of tetraquarks

The spin states of the $qq\bar{q}\bar{q}$ system where all pairs are in relative S-waves can be denoted as $|s_q, s_{\bar{q}}; S\rangle$ where s_q, $s_{\bar{q}}$, and S denote the spins respectively of the quarks, the antiquarks and the total spin. The states of the S-wave four-body system are thus

$$|1, 1; 0\rangle, \qquad |1, 1; 2\rangle \quad \text{and} \quad |0, 0; 0\rangle$$

which can decay into two pseudoscalar mesons, and

$$|1, 1; 1\rangle, \qquad |0, 1; 1\rangle \quad \text{and} \quad |1, 0; 1\rangle$$

whose decay into two pseudoscalars is forbidden by angular momentum and parity.

We now estimate the corrections from color-magnetic and spin effects to the above results in the conventional manner used in constituent quark calculations [5,8]. The expression for the interaction energy (1) is generalised to include an explicit spin dependence [5,8] which is not the same for all pairs. The interaction energy between two constituents i and j is now given by [5]

$$V_{cx}^{ij} = \left(V - \frac{16v}{3}\vec{\sigma}_i \cdot \vec{\sigma}_j\right) \cdot \left(\lambda_c^i \cdot \lambda_c^j\right), \qquad (30)$$

where $\vec{\sigma}$ denotes the Pauli spin operators and v is a strength parameter fixed by experiment.

We first note that when this interaction is treated as a first-order perturbation, the color magnetic contribution calculated from Eq. (30) vanishes when suitably averaged over masses of the pseudoscalar $\mathcal{P}$ and vector $\mathcal{V}$ states [2,14] of a quark of flavor i bound to a $\bar{u}$ antiquark.

$$|\mathcal{P}_i\rangle = |q_i\bar{u}\rangle_{S=0}, \qquad |\mathcal{V}_i\rangle = |q_i\bar{u}\rangle_{S=1},$$
$$\tilde{M}(\mathcal{V}_i) = \frac{3M(\mathcal{V}_i) + M(\mathcal{P}_i)}{4}, \qquad (31)$$

where $\tilde{M}$ denotes the linear combination of masses for which the color magnetic interaction (30) cancels out [2,14].

We can immediately relate the calculated two-meson masses like (19) to the experimental masses

$$
\begin{aligned}
&E_g(2Mcubu)\\
&= \tilde{M}(c\bar{u}) + \tilde{M}(b\bar{u})\\
&= \frac{3M(D^*) + M(D)}{4} + \frac{3M(B^*) + M(B)}{4},
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
&E_g(2Mcubu)\\
&= M(D) + M(B) + \frac{3[M(D^*) - M(D)]}{4}\\
&\quad + \frac{3[M(B^*) - M(B)]}{4}.
\end{aligned}
\tag{33}
$$

Thus the statements about the BD threshold must be corrected to include the hyperfine splittings. For the BB system,

$$
\begin{aligned}
&E_g(2Mbubu)\\
&= 2\tilde{M}(b\bar{u}) + \frac{3M(B^*) + M(B)}{2}\\
&= 2M(B) + \frac{3[M(B^*) - M(B)]}{2}.
\end{aligned}
\tag{34}
$$

The $B^* - B$ mass difference is 46 MeV, the color-magnetic mass reduction is 69 MeV for the BB system and 23 MeV for the B^*B system. These are down in the noise of our rough calculation.

The color-magnetic corrections for the tetraquark masses are not easily calculated because they depend upon the spatial wave functions, but the sign of the change vs. ordinary hadrons is clear. A short-range two-body hyperfine interaction depends upon the wave function at the origin, which can be calculated using the harmonic oscillator wave functions. Since the mean square distances are considerably larger in the tetraquarks than in mesons, a simple scaling of the wave function at the origin will scale the hyperfine splittings in the tetraquark to a lower value than those in the observed mesons. The neglect of the hyperfine splittings is therefore justified as a reasonable approximation, as supported by the following estimates.

5.2. Some estimates of the color magnetic interaction

The lowest tetraquark state when color-magnetic interactions are included is expected to be the spin zero state of a spin-one color sextet and a spin-one antisextet, $|1, 1; 0\rangle$. Although the color-magnetic energy is not easily calculated in the general case, the quark–antiquark contribution for this state is simplified because of its symmetry.

In the state $|1, 1; 0\rangle$ the spin of one of the antiquarks must couple with the total spin 1 of the quarks to give a total spin 1/2. The relevant angular momentum algebra gives the expectation values of $\vec{\sigma}_q \cdot \vec{\sigma}_{\bar{q}}$ for a single $q\bar{q}$ pair in this state and for any spin singlet state of the $q\bar{q}$ system

$$
\begin{aligned}
&\langle 1, 1; 0|\vec{\sigma}_q \cdot \vec{\sigma}_{\bar{q}}|1, 1; 0\rangle = -2,\\
&\langle (q\bar{q})_{S=0}|\vec{\sigma}_q \cdot \vec{\sigma}_{\bar{q}}|(q\bar{q})_{S=0}\rangle = -3.
\end{aligned}
\tag{35}
$$

The contribution of the $q\bar{q}$ pairs to the color magnetic energy of this state is obtained by substituting the spin expectation

values (35) into Eq. (30), summing over all quarks and antiquarks and substituting the $SU(3)$ Casimir operators for the sextet $C_3(6) = (40/3)$ and triplet $C_3(3) = (16/3)$.

$$
\begin{aligned}
&\langle 1, 1; 0|V_{CM}(q\bar{q})|1, 1; 0\rangle\\
&= \frac{16v}{3} \cdot 2 \cdot \langle 1, 1; 0| \sum_{i=q} \sum_{j=\bar{q}} \lambda_c^i \cdot \lambda_c^j |1, 1; 0\rangle\\
&= \frac{16v}{3} \cdot 2 \cdot C_3(6).
\end{aligned}
\tag{36}
$$

The ratio of the color-magnetic energy for this state to the color-magnetic energy of two mesons can be obtained for the case where the spatial distances between quark–antiquark pairs is equal in all cases,

$$
\frac{\langle 1, 1; 0|V_{CM}(q\bar{q})|1, 1; 0\rangle}{\langle 2M|V_{CM}(q\bar{q})|2M\rangle} = \frac{C_3(6)}{3 \cdot C_3(3)} = \frac{5}{12}.
\tag{37}
$$

This suggests that the color-magnetic effects reduce the mass of the tetraquark by an amount roughly half of the reduction in mass of the two-meson system.

6. Experimental implications of the existence of tetraquarks

6.1. Tetraquark symmetry quantum numbers and decay selection rules

States which have two quark–antiquark pairs of the same flavor have charge conjugation quantum numbers, and states having a light quark–antiquark pair have isospin and G-parity quantum numbers. $SU(3)$ flavor symmetry is not considered here as this symmetry is badly broken and considering the effects of symmetry-breaking on selection rules requires further analysis.

The $cu\bar{c}\bar{u}$ and $bu\bar{b}\bar{u}$ tetraquarks which contain charge-conjugate sextet–antisextet states have charge conjugation quantum numbers that are conserved in strong and electromagnetic decays. Starting with the spin quantum numbers, we note that the $|s_q, s_{\bar{q}}; S\rangle$ states $|1, 1; 0\rangle$, $|1, 1; 2\rangle$ and $|0, 0; 0\rangle$ are all even under C and therefore also even under CP. The $|1, 1; 1\rangle$ state is odd under C and therefore also odd under CP, while the $|0, 1; 1\rangle$ and $|1, 0; 1\rangle$ are linear combinations of even and odd C and the sum and difference of these states are respectively even and odd under both C and CP.

The states having a light quark–antiquark pair are found with both isospins, 0 and 1. Since the mesons with one heavy or strange quark and one light quark have isospin 1/2, two-meson states with isospin 0 and 1 exist and no isospin selection rule forbids a tetraquark decay into two mesons. Charmonium and bottomonium states have isospin zero. Thus the $\pi\Upsilon$ and $\pi J\psi$ strong decays are allowed only for tetraquarks with isospin one and are isospin forbidden for isospin-zero tetraquarks.

The $cu\bar{c}\bar{u}$ and $bu\bar{b}\bar{u}$ tetraquarks which contain a heavy quark–antiquark pair of the same flavor can decay into light quark mesons by annihilation of the heavy quark–antiquark pair. Although a single $Q\bar{Q}$ state can decay strongly into light quarks only by annihilating the $Q\bar{Q}$ into two or three gluons, a

$Q\bar{Q}$ constituent in a $Q\bar{Q}d\bar{u}$ tetraquark can annihilate into one gluon which is then absorbed in the light quark system. Such states are apt to be broad. These include all tetraquarks with these constituents, including the molecular states.

States like the $cu\bar{b}\bar{u}$ and $su\bar{b}\bar{u}$ tetraquarks which heavy quark–antiquark pairs of different flavors cannot decay strongly or electromagnetically into light quark mesons. Their strong decays must conserve their heavy flavors and isospin. Thus the strong πB_s and πB_c decays are allowed only for tetraquarks with isospin one and are isospin forbidden for isospin-zero tetraquarks.

6.2. Experimental detection via decay modes

Thus the best candidates for experimental detection are the $cu\bar{b}\bar{u}$ and $su\bar{b}\bar{u}$ tetraquarks which cannot decay strongly or electromagnetically into light quark mesons.

A $bq\bar{c}\bar{q}$ tetraquark with isospin 1 and a mass below the $B\bar{D}$ threshold but above the mass of the $B_c\pi$ system can decay strongly into a $B_c\pi$ and should have a strong width limited by phase space. The $I = 0$ state and the $I = 1$ state below the $B_c\pi$ threshold should be narrow. They can both decay electromagnetically into a B_c and a photon and the $I = 0$ state above the $B_c\pi$ threshold can decay into $B_c\pi$ via isospin violation. The above considerations apply also to states below the $B^*\bar{D}$ threshold that cannot decay into $B\bar{D}$.

If the mass of a $bq\bar{c}\bar{q}$ tetraquark is below the B_c mass it can decay only weakly. The tetraquark with the same charge as the B_c can have the same final state as the B_c; e.g. $J/\psi e\nu$. It would appear in any invariant mass plot of the final state as an additional mass peak along with the B_c. Isovector tetraquarks like $bu\bar{c}\bar{d}$ or $bd\bar{c}\bar{u}$ have exotic final states with wrong charges, like $J/\psi\eta$ or $J/\psi\pi^-\pi^-$.

There is therefore interest in looking for monoenergetic photons or pions emitted together with a B_c meson, a doublet structure of the B_c mass and exotic B_c decays, like $J/\psi\eta$ or $J/\psi\pi^{\pm}\pi^{\pm}$.

Analogous considerations hold for a $bq\bar{s}\bar{q}$ tetraquark with a mass below the BK threshold: strong decay and strong width into $B_s\pi$ for $I = 1$ states above $B_s\pi$ threshold; narrow widths for all other states into $B_s\gamma$ or $B_s\pi$ with isospin violation; weak decays for tetraquarks below the B_s mass, producing a mass doublet with the B_s or exotic final states with wrong charges, like $J/\psi\pi^{\pm}$.

A $cq\bar{c}\bar{q}$ tetraquark with a mass below the $D\bar{D}$ threshold or a $bq\bar{b}\bar{q}$ tetraquark with a mass below the $B\bar{B}$ threshold would very likely decay strongly into light quark hadrons and be very broad. The $I = 1$ states might also decay strongly by pion emission respectively into $\pi J/\psi$ or $\pi\Upsilon$. Isospin violating pion emission might occur for $I = 0$ states above the pion threshold. Electromagnetic decays into $\gamma J/\psi$ or $\gamma\Upsilon$ might also occur.

Since exotic multiquark states are required by color-space correlations to have a larger extension in space than normal hadrons, they may be easily broken up by final state interactions or rescattering by other particles in the same final state. This may make it difficult for such states to be produced and survive in multiparticle final states.

7. Conclusion

Color-space correlations can play a crucial role in multi-quark systems and may be more important than the color-flavor-spin correlations generally dominating other treatments of multiquark states.

In any theoretical treatment of multiquark states containing both quarks and antiquarks the full implications of the basic QCD physics that the $q\bar{q}$ interaction observed in mesons is much stronger than the qq interaction observed in baryons must be considered. Admixtures of quark states that do not exist in normal baryons [2,6] must be included. In the simplest multiquark system containing one antiquark and a quark system the $q\bar{q}$ interaction already destroys completely the spatial and color structures of the quark system that existed in the absence of the antiquark including all diquark structures. Color-space correlated tetraquarks may already be observed in mesons containing heavy quarks.

The most likely signals that would indicate the unambiguous presence of tetraquarks should be searched in experiments focused on the B_c system. The $bq\bar{c}\bar{q}$ tetraquark might be below the $B\bar{D}$ threshold. The possible exotic signatures include strong or electromagnetic decays into a B_c and a pion or photon, weak decays producing additional peaks in the mass spectrum of B_c decay final states or weak decays into states with exotic electric charge, like $J/\psi\eta$ or $J/\psi\pi^-\pi^-$.

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