Involution for upper triangular matrix algebras✩

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Dedicated to Professor Amitai Regev on the occasion of his sixty-fifth birthday

Abstract

In this paper we describe completely the involutions of the first kind of the algebra $UT_n(F)$ of $n \times n$ upper triangular matrices. Every such involution can be extended uniquely to an involution on the full matrix algebra. We describe the equivalence classes of involutions on the upper triangular matrices. There are two distinct classes for $UT_n(F)$ when $n$ is even and a single class in the odd case.

Furthermore we consider the algebra $UT_2(F)$ of the $2 \times 2$ upper triangular matrices over an infinite field $F$ of characteristic different from 2. For every involution $*$, we describe the $*$-polynomial identities for this algebra. We exhibit bases of the corresponding ideals of identities with involution, and compute the Hilbert (or Poincaré) series and the codimension sequences of the respective relatively free algebras.

Then we consider the $*$-polynomial identities for the algebra $UT_3(F)$ over a field of characteristic zero. We describe a finite generating set of the ideal of $*$-identities for this algebra. These generators are quite a few, and their degrees are relatively large. It seems to us that the problem of describing the $*$-identities for the algebra $UT_n(F)$ of the $n \times n$ upper triangular matrices may be much more complicated than in the case of ordinary polynomial identities.

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1. Introduction

The description of the involutions on a given algebra is an important task in ring theory. It is resolved completely for the class of the central simple algebras, see for example [9, Chapter 1], or [14, Chapter 3]. Given an involution, it is natural to ask what the *-polynomial identities satisfied by the algebra are. It is well known that to study the *-polynomial identities of the full matrix algebra $M_n(F)$ over an infinite field $F$ such that $\text{char}(F) \neq 2$ it is sufficient to consider the transpose and the symplectic involutions, the latter only when $n$ is even. We would like to mention that the concrete form of the *-identities of $M_n(F)$ is known only when $n = 1$ (trivial) and when $n = 2$, see [11] for the case of characteristic 0, and [1] for positive characteristic.

In this paper we consider the algebra $\text{UT}_n(F)$ of the $n \times n$ upper triangular matrices over $F$. This algebra plays an important role in the theory of algebras with polynomial identities. Its identities describe in certain sense the subvarieties of the variety of algebras generated by the matrix algebra of order two. These identities are well known, see for example [3, Chapter 5.2]. Recently it was discovered that the algebras of upper triangular matrices, and more generally, the algebras of block-triangular matrices, classify the exponential growth of the codimensions of varieties of algebras in characteristic 0, see [6,7]. The gradings and the graded identities of $\text{UT}_n(F)$ have been extensively studied, see for example [2,10] and their references. All gradings on $\text{UT}_n(F)$ by a finite abelian group when $F$ is algebraically closed field and $\text{char}(F) = 0$, were described in [15]. Various properties of involutions and involution-like maps for the upper triangular matrices were described in [13].

We classify the involutions on the algebra $\text{UT}_n(F)$ when $F$ is an arbitrary field of characteristic different from 2. We prove that there exist two classes of inequivalent involutions when $n$ is even and a single class otherwise. Then, we give a complete description of the *-polynomial identities of $\text{UT}_2(F)$ with respect to any involution. Moreover, for $\text{char}(F) = 0$, we describe the $\text{GL}_m$-module structure of the corresponding relatively free algebras in $m$ generators. We compute as well the Hilbert (or Poincaré) series of the corresponding relatively free algebras with involution and furthermore we give explicitly the codimension sequences for these algebras. Here we use some ideas from [4] and from [5].

In the last two sections we describe the *-polynomial identities for the algebra $\text{UT}_3(F)$ of the $3 \times 3$ upper triangular matrices when $\text{char}(F) = 0$. We exhibit a finite set of generators of the ideal of these identities. It turns out that this finite set is quite large. It remains an open problem to describe such generators in the general case. It should be noted that when studying the ordinary identities of $\text{UT}_n(F)$ their description is rather straightforward but this is not the case when dealing with *-identities.

2. Involutions

Let $A$ be an associative algebra over a field $F$ of characteristic different from 2. The map $*: A \to A$ is an involution on $A$ if it is an automorphism of the additive group of $A$ such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for every $a, b \in A$. If $*$ is the identity map on the center of $A$ then it is an involution of the first kind; otherwise it is of the second kind. In this paper we consider involutions of the first kind only. An element $a \in A$ is said to be symmetric or skew-symmetric if $a^* = a$ or $a^* = -a$, respectively. Let $X = \{x_1, x_2, \ldots\}$, $X^* = \{x_1^*, x_2^*, \ldots\}$ be two disjoint countable sets of indeterminates and denote by $F(X, *) = F(X \cup X^*)$ the free associative algebra freely generated by $X \cup X^*$ endowed with the involution $x_i \mapsto x_i^*$. A polynomial
Let \((A, \ast), (B, \circ)\) be two algebras with involutions \(\ast\) and \(\circ\), respectively. We say that they are isomorphic as algebras with involution if there exists an algebra isomorphism \(\varphi : A \to B\) such that \(\varphi(a^\ast) = (\varphi(a))\circ\). Clearly, if \(A\) and \(B\) are isomorphic they satisfy the same set of \(\ast\)-polynomial identities in the free algebra with involution. We say that the involutions \(\ast, \circ\) on \(A\) are equivalent if \((A, \ast), (A, \circ)\) are isomorphic as algebras with involution.

**Proposition 2.1.** Let \(\ast\) and \(\circ\) be two involutions on the algebra \(A\). Assume that all automorphisms of \(A\) are inner. Then:

(i) There exists an invertible element \(u \in A\) such that \(a^\circ = ua^\ast u^{-1}\) for all \(a \in A\).

(ii) The involutions \(\ast\) and \(\circ\) are equivalent if and only if there exists an invertible \(v \in A\) such that \(a^\circ = vv^\ast a^\ast (vv^\ast)^{-1}\) for all \(a \in A\).

**Proof.** Both assertions of the proposition are quite standard, one may find them in slightly different form in [14, Chapter 3]. We give the proof for the sake of completeness. In order to prove (i) we note that the composition of two involutions is an automorphism and that the latter is, by assumption, inner. Hence we have \((a^\ast)^\circ = \varphi(a) = waw^{-1}\) for every \(a \in A\). Here \(\varphi\) is the automorphism of \(A\) that acts via conjugation by the invertible element \(w \in A\). Therefore

\[
a^\ast = ((a^\ast)^\circ)^\circ = (waw^{-1})^\circ = (w^{-1})^\circ a^\circ w^\circ = (w^\circ)^{-1} a^\circ w^\circ.
\]

and \(a^\circ = ua^\ast u^{-1}\) for \(u = w^\circ\).

Now we prove the assertion of (ii). Denote \(\gamma_w\) the inner automorphism induced by the invertible element \(w \in A\). Suppose that \(a^\circ = ua^\ast u^{-1}\) where \(u = vv^\ast\) for some invertible \(v \in A\). Then

\[
\gamma_{v^{-1}}(a^\circ) = v^{-1}a^\circ v = v^{-1}(ua^\ast u^{-1})v = v^\ast a^\ast (v^\ast)^{-1} = (v^{-1}av)^\ast = (\gamma_{v^{-1}}(a))^\ast
\]

and \(\gamma_{v^{-1}} : (A, \circ) \to (A, \ast)\) is an isomorphism of algebras with involution.

Assume on the other hand that the involutions \(\ast\) and \(\circ\) are equivalent. Hence \(\gamma_{v^{-1}}(a^\circ) = \gamma_{v^{-1}}(a)^\ast\) for some invertible element \(v \in A\). Since \(a^\circ = ua^\ast u^{-1}\), we have that \(v^{-1}ua^\ast u^{-1}v = v^\ast a^\ast (v^\ast)^{-1}\), and

\[
a^\circ = ua^\ast u^{-1} = v(v^\ast a^\ast (v^\ast)^{-1})v^{-1} = (vv^\ast)a^\ast (vv^\ast)^{-1}.
\]

Thus the proof of the proposition is complete. \(\square\)

Assume that the center \(Z(A)\) of \(A\) is the base field \(F\). Let \(\ast\) be an involution on \(A\) and define the map \(\circ : A \to A\) as \(a \mapsto ua^\ast u^{-1}\). Note that \(\circ\) is also an involution on \(A\) if and only if \(u^\ast = \pm u\).
Consider now the algebra $UT_n = UT_n(F)$ of the $n \times n$ upper triangular matrices over $F$. For every matrix $A \in UT_n$ we define $A^* = J A' J$ where $A \mapsto A'$ denotes the usual matrix transpose and $J$ is the following permutation matrix:

$$J = \begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 1 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
1 & \ldots & 0 & 0
\end{pmatrix}.$$

Note that for the matrix units we have $e^*_{ij} = e_{n+1-j,n+1-i}$ and the involution $*$ can be extended to the full matrix algebra $M_n$ in the natural way. Actually we have the following proposition.

**Proposition 2.2.** Every involution on $UT_n$ can be uniquely extended to an involution on $M_n$.

**Proof.** Let $\circ$ be an involution on $UT_n$. Since all automorphisms of $UT_n$ are inner [8], we may apply Proposition 2.1. Hence there exists an invertible matrix $B \in UT_n$ such that $A^\circ = BA^* B^{-1}$ for all $A \in UT_n$. Note that $Z(UT_n) = F$ and therefore $B^* = \pm B$. Then the map $\circ$ can be extended to an involution on $M_n$ in the natural way.

Suppose now that $\varphi$ and $\phi$ are extensions to $M_n$ of the involution $\circ$ on $UT_n$. Since all automorphisms of $M_n$ are inner, there exist invertible matrices $B, C \in \text{GL}_n$ such that $A^\varphi = BA^* B^{-1}$ and $A^\phi = CA^* C^{-1}$ for all $A \in M_n$. By definition $BA^* B^{-1} = CA^* C^{-1}$ for every $A \in UT_n$ and hence the matrix $C^{-1} B$ belongs to the centralizer of $UT_n$ in the algebra $M_n$ which is the field $F$. Thus $B = \lambda C$ for some $0 \neq \lambda \in F$, and $\varphi = \phi$. \qed

**Definition 2.3.** Let $n = 2m$ be an even integer and consider the matrix

$$D = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix} \in M_n.$$

We define an involution $s$ on $UT_n$ by putting $A^s = DA^* D^{-1}$ for all $A \in UT_n$, and we call $s$ the symplectic involution on $UT_n$.

**Lemma 2.4.** Let $B = (b_{ij})$ be a matrix of $UT_n$. Assume that $n = 2m$ is an even integer, and that $B = B^s$ (respectively $B = B^*$), then $B$ factorizes as $B = CC^*$ (respectively $B = CC^s$), for some $C \in UT_n$. When $n = 2m + 1$ is odd, if $B = B^*$ and $b_{m+1,m+1} = 1$ then $B = CC^*$.\hfill

**Proof.** Let $n = 2m$ and denote:

$$B = \begin{pmatrix} X & Z \\ 0 & Y \end{pmatrix}$$

where $X, Y \in UT_m$ and $Z \in M_m$. By definition we have that

$$B^* = \begin{pmatrix} Y^* & Z^* \\ 0 & X^* \end{pmatrix}, \quad B^s = \begin{pmatrix} Y^* & -Z^* \\ 0 & X^* \end{pmatrix}.$$
Now consider the matrix

$$C = \begin{pmatrix} I_m & Z/2 \\ 0 & Y \end{pmatrix}.$$ 

If $B = B^*$ then $X^* = Y$, $Z^* = Z$ and hence $B = CC^*$. In the same way, if $B = B^s$ then $X^* = Y$, $Z^* = -Z$ and $B = CC^s$.

Assume now $n = 2m + 1$ and let

$$B = \begin{pmatrix} X & a & T \\ 0 & 1 & b \\ 0 & 0 & Y \end{pmatrix}$$

where $X, Y \in \text{UT}_m$, $T \in M_m$ and $a, b$ are column and row matrices with $m$ entries. More precisely

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}, \quad b = \begin{pmatrix} b_m & \ldots & b_1 \end{pmatrix}$$

where $a_i, b_i \in F$. Assuming $B = B^*$ we have $X^* = Y$, $T^* = T$ and $a_i = b_i$. Then we have $B = CC^*$ by putting:

$$C = \begin{pmatrix} I_m & 0 & T/2 \\ 0 & 1 & b \\ 0 & 0 & Y \end{pmatrix}.$$ 

Thus the proof of the lemma is complete. \(\Box\)

Note that when $n = 2m + 1$ and $b_{m+1,m+1} \neq 1$, in order to keep the factorization $B = CC^*$ it is sufficient to assume that a square root of $b_{m+1,m+1}$ is contained in the field $F$.

**Proposition 2.5.** Every involution on $\text{UT}_n$ is equivalent either to $*$ or to $s$.

**Proof.** Let $B \in \text{UT}_n$ be an invertible matrix such that $B^* = \pm B$ and consider the involution $A^0 = BA^*B^{-1}$, for all $A \in \text{UT}_n$. Assume first that $B^* = B$. Then in order to prove that $*$ and $\circ$ are equivalent it is sufficient to show that $B = CC^*$ for some invertible matrix $C \in \text{UT}_n$. If $n = 2m$ then this follows immediately from Lemma 2.4. When $n = 2m + 1$, since $B$ is a non-singular triangular matrix we can assume $b_{m+1,m+1} = 1$ without loss of generality in the definition of $\circ$. Hence the claim follows once again from Lemma 2.4.

For the case $B^* = -B$ note that $n$ has to be an even integer, $n = 2m$. Otherwise the entry $b_{m+1,m+1}$ should be zero but $B$ is non-singular, and hence all diagonal entries of $B$ are nonzero. Then

$$A^0 = BA^*B^{-1} = BD^{-1}DA^*D^{-1}DB^{-1} = UA^sU^{-1}$$

where $U^s = (BD^{-1})^s = D(BD^{-1})^sD^{-1} = U$. Hence $U = CC^s$ by Lemma 2.4 and this implies that $\circ$ is equivalent to the symplectic involution $s$. \(\Box\)
We have the following easy to apply criterion for determining which class of equivalent involutions some $\circ$ belongs to.

**Proposition 2.6.** The involution $\circ$ is equivalent to $s$ if and only if $e_{1n}^{o} = -e_{1n}.$

**Proof.** Let $a, b$ be the entries of $B$ in position $(1, 1)$ and $(n, n)$ respectively. We have that

$$e_{1n}^{o} = ae_{11}e_{1n}b^{-1}e_{nn} = ab^{-1}e_{1n}.$$

Then $ab^{-1} = -1$ if and only if the matrix $B^{*} = -B.$ Using the same argument as in the previous proposition we get the result. $\blacksquare$

3. $*$-polynomial identities of $UT_2$

Let $(A, *)$ be an algebra with involution (of first kind). Denote by $T(A, *)$ the ideal of all $*$-polynomial identities satisfied by $A.$ Note that it is a $T(*)$-ideal that is an ideal invariant under all endomorphisms of $F(X, *)$ commuting with the involution. Then $F(X, *) / T(A, *)$ is the relatively free algebra in the variety of algebras with involution satisfying all $*$-polynomial identities of $A.$ It is useful to consider $F(X, *)$ as generated by symmetric and skew-symmetric variables. More precisely we have $F(X, *) = F(Y, Z),$ where $y_i = x_i + x_i^{*}$ and $z_i = x_i - x_i^{*}$ for all $i.$

Let $f = f(y_1, \ldots, y_r, z_1, \ldots, z_s) \in F(Y, Z)$ be a polynomial. By the Poincaré–Birkhoff–Witt theorem it can be written as a linear combination of polynomials of the type

$$y_1^{a_1} \cdots y_r^{a_r} \cdot z_1^{\beta_1} \cdots z_s^{\beta_s} \cdot u_1^{\gamma_1} \cdots u_k^{\gamma_k},$$

where $u_i$ are higher commutators (of degree $\geq 2$) in the variables $y_i$ and $z_i,$ and $a_i, \beta_i, \gamma_i$ are non-negative integers. We shall assume that the commutators are left normed, that is $[a, b] = ab - ba$ and $[a, b, c] = [[a, b], c].$

The polynomial $f$ is said to be $Y$-proper if $a_1 = \cdots = a_r = 0$ in every summand of the linear combination. In other words, one has $\partial f / \partial y_i = 0$ for all $y_i.$ Here the partial derivative is defined via $\partial y_j / \partial y_i = \delta_{ij}.$ Denote $B$ the subalgebra of $F(Y, Z)$ generated by all $Y$-proper polynomials. Since $F$ is infinite, from [4, Lemma 2.1] it follows that every $T(*)$-ideal is generated by its $Y$-proper polynomials.

Denote now by $(UT_2, *)$ and by $(UT_2, s)$ the algebra of 2 by 2 upper triangular matrices endowed with the involutions $*$ and $s$ respectively, introduced in the previous section. More precisely, we have:

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^* = \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}, \quad \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}^s = \begin{pmatrix} b & -c \\ 0 & a \end{pmatrix}.$$

Note that the symmetric and skew-symmetric subspaces of $(UT_2, *)$ are spanned respectively by $\{e_{11} + e_{22}, e_{12}\}$ and by $\{e_{11} - e_{22}\}.$ Therefore, it is clear that the algebra $(UT_2, *)$ satisfies the following set of $*$-identities:

$$[y_1, y_2], \quad [z_1, z_2], \quad [y_1, z_1][y_2, z_2], \quad z_1 y_1 z_2 - z_2 y_1 z_1. \quad (1)$$

It is obvious that $(UT_2, *)$ satisfies the identity $[y_1, z_1 z_2]$ as well. This identity follows from the first two identities above; we shall use it in the sequel.
Theorem 3.1. The ideal \( T(UT_2, \ast) \) is generated as a \( T(\ast) \)-ideal by the set (1). Moreover, a linear basis for \( B/B \cap T(UT_2, \ast) \) is given by the following polynomials:

\[
zi_1 \cdots zi_q \quad (i_1 \leq \cdots \leq i_q), \quad zi_1 \cdots zi_{q-1}[zi_q, y_j] \quad (i_1 \leq \cdots \leq i_{q-1}). \tag{2}
\]

**Proof.** Let \( I \) be the \( T(\ast) \)-ideal generated by the set (1). Since every commutator \([y_i, z_j]\) is a symmetric element in the free algebra then so are the commutators \([y_1, z_1, \ldots, z_k]\). Therefore the ideal \( I \) contains the polynomial \([y_1, z_1, \ldots, z_k, y_2]\) for every \( k \geq 0 \). Hence, modulo \( I \), one has that every \( Y \)-proper polynomial is a linear combination of elements of the type

\[
\begin{aligned}
&zi_1 \cdots zi_q \quad (i_1 \leq \cdots \leq i_q), \\
&zi_1 \cdots zi_{q-1}[zi_q, y_j] \quad (i_1 \leq \cdots \leq i_{q-1}).
\end{aligned}
\]

Note that \([z_1, y_1, z_2] + z_1[z_2, y_1] + z_2[z_1, y_1] = [z_1z_2, y_1] \in I \) and hence \( B/B \cap I \) is spanned by the following polynomials:

\[
\begin{aligned}
&zi_1 \cdots zi_q \quad (i_1 \leq \cdots \leq i_q), \\
&zi_1 \cdots zi_{q-1}[zi_q, y_j] \quad (i_1 \leq \cdots \leq i_{q-1}).
\end{aligned}
\]

On the other hand one easily verifies that

\[
z_2[z_1, y] - z_1[z_2, y] = [z_2, z_1]y_1 + (z_1y_2 - z_2yz_1) \in I.
\]

In this way we conclude that the set (2) spans \( B/B \cap I \).

Note that these generators are all multihomogeneous and of different multidegrees. Since the ideal \( T = T(UT_2, \ast) \) is multigraded, in order to prove their linear independence modulo \( T \) it is sufficient to show that none of them is an identity for \( UT_2 \). But this is straightforward since the skew-symmetric elements may be chosen to be invertible matrices in \( UT_n \). Then \( B \cap I = B \cap T \) and hence \( I = T \). \( \square \)

Now we consider the case of \( (UT_2, s) \). First we note that the subspace of the symmetric under \( s \) elements coincides with the center. Moreover, the subspace of the skew-symmetric elements is spanned by \( e_{11} - e_{22} \) and by \( e_{12} \). Then the algebra \( (UT_2, s) \) satisfies the following set of \( \ast \)-identities:

\[
[y_1, y_2], \quad [z_1, y_1], \quad [z_1, z_2][z_3, z_4], \quad z_1z_2z_3 - z_3z_2z_1. \tag{3}
\]

**Theorem 3.2.** The ideal \( T(UT_2, s) \) is generated by the set (3). Moreover, a linear basis for \( B/B \cap T(UT_2, s) \) consists of the following polynomials:

\[
zi_1 \cdots zi_q \quad (i_1 \leq \cdots \leq i_q), \quad [z_j, zi_1]zi_2 \cdots zi_q \quad (j > i_1 \leq \cdots \leq i_q). \tag{4}
\]

**Proof.** Let \( I \) be the \( T(\ast) \)-ideal generated by the set (3). Since \([y_1, y_2], [y_1, z_1] \in I \) we have that \( B/B \cap I \) is spanned by all monomials in the variables \( z_i \). Note that

\[
z_1[z_2, z_3] + [z_2, z_3]z_1 = (z_1z_2z_3 - z_3z_2z_1) + (z_2z_3z_1 - z_1z_3z_2) \in I.
\]
Since \([z_1, z_2][z_3, z_4] \in I\) it follows that, modulo \(I\), every monomial is a linear combination of polynomials of the types

\[
\begin{cases}
z_{i_1} \cdots z_{i_q} & (i_1 \leq \cdots \leq i_q), \\
[z_j, z_l]z_{i_1} \cdots z_{i_{q-1}} & (j > i, i_1 \leq \cdots \leq i_{q-1}).
\end{cases}
\]

Observe that \(I\) contains the polynomial

\[
[z_3, z_2]z_1 - [z_3, z_1]z_2 + [z_2, z_1]z_3 = \frac{1}{2}z_3z_2z_1 - z_2z_3z_1 - z_3z_1z_2 + z_1z_3z_2 + z_2z_1z_3 - z_1z_2z_3
\]
due to the last identity of (3). Therefore the set (4) spans \(B/B \cap I\).

Fix the multidegree \((m_1, \ldots, m_k)\) and consider the generators of this multidegree, say:

\[
f = z_1^{m_1} \cdots z_k^{m_k}, \quad g_j = [z_j, z_1]z_1^{m_1-1}z_2^{m_j-1} \cdots z_k^{m_k}.
\]

Assume that \(\alpha f + \sum_j \beta_j g_j \in T(UT_2, s)\) for some \(\alpha, \beta_j \in F\). If we evaluate \(z_i\) to \(e_{11} - e_{22}\) we get immediately \(\alpha = 0\). Moreover, for \(z_j = e_{11} - e_{22} + e_{12}\) and \(z_i = e_{11} - e_{22}\) when \(i \neq j\), we obtain \(\beta_j = 0\). In fact, after the evaluation, one has that \(g_j = (-1)^{m_1+\cdots+m_k} 2e_{12}\) and \(g_i = 0\) for \(i \neq j\). As in the previous theorem, we can conclude \(I = T(UT_2, s)\).

4. Module structure of the identities

Let \((A, \ast)\) be an algebra with involution. Let us denote as \(F(Y, Z)_m\) the free algebra \(F(y_1, \ldots, y_m, z_1, \ldots, z_m)\) in finitely many variables and let \(B_m = B \cap F(Y, Z)_m\). Let \(U, V\) be the vector spaces spanned by the sets \(\{y_1, \ldots, y_m\}\) and \(\{z_1, \ldots, z_m\}\) respectively. Then the group \(GL_m \times GL_m = GL(U) \times GL(V)\) acts on the direct sum \(U \oplus V\) and this action can be extended diagonally to the algebra \(F(Y, Z)_m\) and to its subspace \(B_m\). Note that the ideal \(F(Y, Z)_m \cap T(A, \ast)\) is also a \(GL_m \times GL_m\)-module and so are the quotients \(F_m(A, \ast) = F(Y, Z)_m / F(Y, Z)_m \cap T(A, \ast)\) and \(B_m(A, \ast) = B_m / B_m \cap T(A, \ast)\). Moreover, all these structures are multigraded and we can define the Hilbert (or Poincaré) series for the spaces \(F_m(A, \ast), B_m(A, \ast)\). For instance, we define:

\[
H(F_m(A, \ast)) = \sum_{k,l} \dim_F F_m(A, \ast)^{(k, l)} u_1^{k_1} \cdots u_m^{k_m} v_1^{l_1} \cdots v_m^{l_m}
\]

where \(F_m(A, \ast)^{(k, l)}\) is the multihomogeneous component of multidegree \((k, l) = (k_1, \ldots, k_m, l_1, \ldots, l_m)\).

From now on, unless otherwise stated, we assume that the field \(F\) is of characteristic zero. Then, by the representation theory of the general linear groups it is well known that

\[
H(F_m(A, \ast)) = \sum_{\lambda, \mu} a_{\lambda, \mu} S_\lambda(u_1, \ldots, u_m) S_\mu(v_1, \ldots, v_m),
\]

\[
H(B_m(A, \ast)) = \sum_{\lambda, \mu} b_{\lambda, \mu} S_\lambda(u_1, \ldots, u_m) S_\mu(v_1, \ldots, v_m),
\]

(5)

where \(S_\lambda, S_\mu\) are the Schur functions. Their product is the Hilbert series of the irreducible \(GL_m \times GL_m\)-module associated to the partitions \(\lambda\) and \(\mu\).
Denote $P_m = \bigoplus_{k,l} F(Y,Z)^{(k,l)}_m$ where $k,l$ run over the multidegrees satisfying $k + l = (1, \ldots, 1)$. In other words, $P_m$ is the span of the monomials of degree $m$ such that for any $i = 1, \ldots, m$, either the variable $y_i$ or $z_i$ but not both, occurs exactly once. We define also $\Gamma_m = P_m \cap B_m$, $\Gamma_m(A,*) = P_m / P_m \cap T(A,*)$ and $\Gamma_m(A,*) = \Gamma_m / \Gamma_m \cap T(A,*)$. The dimension of $P_m(A,*)$ over the field $F$ is the $m$th codimension of the algebra $(A,*)$ and is denoted by $c_m(A,*)$. In a similar way, we define $\gamma_m(A,*) = \dim_F \Gamma_m(A,*)$ and call it $m$th $Y$-proper codimension.

The relationship between the $Y$-proper and all $*$-polynomial identities is described in the following result:

**Theorem 4.1.** [4, Theorem 2.3] Let $(A,*)$ be an algebra with involution.

(i) Consider the polynomial ring $F[y_1, \ldots, y_m]$ endowed with the canonical $\text{GL}(U)$-action and the trivial $\text{GL}(V)$-action. Then there is a $\text{GL}_m \times \text{GL}_m$-module isomorphism

$$F_m(A,*) \approx F[y_1, \ldots, y_m] \otimes B_m(A,*) .$$

(ii) The multiplicities $a_{\lambda,\mu}$, $b_{\lambda,\mu}$ in the Hilbert series (5) satisfy $a_{\lambda,\mu} = \sum \nu b_{\nu,\mu}$ where for a fixed $\lambda = (\lambda_1, \ldots, \lambda_m)$ the summation runs over all partitions $\nu = (\nu_1, \ldots, \nu_m)$ such that $\lambda_1 \geq \nu_1 \geq \cdots \geq \lambda_m \geq \nu_m$.

(iii) The ordinary and the $Y$-proper codimensions satisfy the relation

$$c_m(A,*) = \sum_{i=0}^{m} \binom{m}{i} \gamma_i(A,*) .$$

**Remark 4.2.** We recall that the statements of (i) and of (iii) in the above Theorem 4.1, hold for every infinite field, see for example [3, pp. 46, 47] for the case of ordinary polynomial identities.

Concerning the module structure of $B_m(UT_2,*)$, $B_m(UT_2,s)$ we obtain the following results.

**Theorem 4.3.**

$$H(B_m(UT_2,*)) = \sum_q S(q)(v_1, \ldots, v_m) + \sum_q S(1)(u_1, \ldots, u_m)S(q)(v_1, \ldots, v_m),$$

$$H(B_m(UT_2,s)) = \sum_q S(q)(v_1, \ldots, v_m) + \sum_q S(q-1,1)(v_1, \ldots, v_m).$$

**Proof.** Note that the polynomials $z_1^q$ and $z_1^{q-1}[z_1, y_1]$ of the set (2) are generators of two irreducible $\text{GL}_m \times \text{GL}_m$-submodules of $B_m(UT_2,*)$. They are associated to the pairs of partitions $((\emptyset, q))$ and $((1), (q))$ respectively. Moreover, we have a natural one-to-one correspondence between pairs of semistandard tableaux in $m$ letters of shapes (1), (q), and the basis elements $z_{i_1} \cdots z_{i_{q-1}}[z_{i_q}, y_j]$ of (2), namely:

$$[j \mid i_1 \cdots i_q] \mapsto z_{i_1} \cdots z_{i_{q-1}}[z_{i_q}, y_j].$$

A similar result holds for $((\emptyset, (q)))$ and the basis elements $z_{i_1} \cdots z_{i_q}$. 

In the symplectic case, in the set (4) we found the polynomials $z_1^q$ and $[z_2, z_1]z_1^{q-1}$ which are the highest weight vectors for irreducible modules corresponding to the pairs of partitions respectively $(\emptyset, (q))$ and $(\emptyset, (q, 1))$. Moreover, we have the following bijection:

$$[i_1 \ldots i_q] \leftrightarrow [z_j, z_{i_1}]z_{i_2} \cdots z_{i_q}.$$ 

In a similar manner a bijection is defined between semistandard tableaux in $m$ letters of shape $(q)$ and the basis elements $z_{i_1} \cdots z_{i_q}$ of the set (4). The result now follows by Theorems 3.1 and 3.2. □

**Corollary 4.4.** The Hilbert series of the algebra $UT_2$ with its involutions are the following functions:

$$H(F_m(UT_2, *)) = \prod_{i=1}^{m}(1 - u_i)^{-1} \left(1 + \sum_{i=1}^{m} u_i\right) \prod_{i=1}^{m}(1 - v_i)^{-1},$$

$$H(F_m(UT_2, s)) = \prod_{i=1}^{m}(1 - u_i)^{-1} \left(1 + \sum_{i=1}^{m} v_i\prod_{i=1}^{m}(1 - v_i)^{-1}\right).$$

**Proof.** We use the results of Theorems 4.3 and 4.1. When $*$ is the transpose-like involution, one has that:

$$H(B_m(UT_2, *)) = \sum_{q} S(q)(v_1, \ldots, v_m) + \sum_{q} S(1)(u_1, \ldots, u_m)S(q)(v_1, \ldots, v_m) = (1 + S(1)(u_1, \ldots, u_m)) \sum_{q} S(q)(v_1, \ldots, v_m) = (1 + u_1 + \cdots + u_m) \prod_{i=1}^{m}(1 - v_i)^{-1}.$$ 

Here we have used the fact that $\sum_{q} S(q)(v_1, \ldots, v_m) = \sum_{q} \sum_{i_1, \ldots, i_m=q} v_1^{i_1} \cdots v_m^{i_m}$ and then the formula for the summation of a geometric progression, see for example [12, Chapter 1.3, (3.9)], or [4] for similar computations. Then by Theorem 4.1, and by the equalities $H(A \otimes B) = H(A)H(B)$ and $H(F[u_1, \ldots, u_m]) = \prod_{i=1}^{m}(1 - u_i)^{-1}$ one obtains the formula for $H(F_m(UT_2, *))$.

Now we consider the symplectic-like involution $s$ on $UT_2$. We have only to compute $\sum_{q \geq 2} S(q-1, 1)(v_1, \ldots, v_m)$. But this sum equals

$$\sum_{q \geq 2} S(q-1)(v_1, \ldots, v_m)S(1)(v_1, \ldots, v_m) - \sum_{q \geq 2} S(q)(v_1, \ldots, v_m),$$

see [12, Chapter 1.5, (5.16)]. Therefore according to Theorem 4.3 we have
5. Matrices of order three

The following polynomials belong to the ideal \( T(A, *) \):

(i) \( s_3(z_1, z_2, z_3) = z_1[z_2, z_3] - z_2[z_1, z_3] + z_3[z_1, z_2] \).
(ii) \((-1)^{x_1x_2}[x_1, x_2][x_3, x_4] - (-1)^{x_3x_4}[x_3, x_4][x_1, x_2]\),

(iii) \((-1)^{x_1x_2}[x_1, x_2][x_3, x_4] - (-1)^{x_1x_3}[x_1, x_3][x_2, x_4] + (-1)^{x_1x_4}[x_1, x_4][x_2, x_3]\).

(iv) \(z_1[x_3, x_4]z_2 + (-1)^{x_3x_4}z_2[x_3, x_4]z_1\),

(v) \([x_1, x_2]z_2[x_3, x_4]\),

(vi) \(z_1[x_4, x_5]z_2z_3 + (-1)^{x_4}x_3z_1[x_4, x_5]z_2\).

**Proof.** The proof that all the above polynomials are indeed \(*\)-polynomial identities for \(A\) consists of a straightforward (and sometimes tedious) verification, and we omit it. \(\Box\)

Denote by \(I\) the \(T(\ast)\)-ideal generated by the above identities. We deduce some (quite a lot of) consequences of the \(*\)-identities above. In order to refer to them with ease we divide these “new” identities in several lemmas.

**Lemma 5.2.** The polynomial \([x_1, x_2][x_3, x_4][x_5, x_6]\) is an element of \(I\).

**Proof.** By the identity (v) we may consider only the case when \(x_3 = z_3, x_4 = y_4\). Then modulo \(I\) we have

\[
0 = [(x_1, x_2)z_3[x_5, x_6], y_4] \\
= [x_1, x_2]z_3[x_5, x_6, y_4] + [x_1, x_2][z_3, y_4][x_5, x_6] + [x_1, x_2, y_4]z_3[x_5, x_6] \\
= [x_1, x_2][z_3, y_4][x_5, x_6]
\]

and the proof is complete. \(\Box\)

**Lemma 5.3.** The following polynomial belongs to the ideal \(I\):

\([x_1, x_2][x_3, x_4]x_5 + (-1)^{x_5}x_5[x_1, x_2][x_3, x_4]\).

**Proof.** We work modulo \(I\). First we assume that \(x_5 = z\). By the identities (ii), (v) one has

\[
0 = (-1)^{x_1x_2}[x_1, x_2, z][x_3, x_4] - (-1)^{x_3x_4}[x_3, x_4][x_1, x_2, z] \\
= -(-1)^{x_1x_2}z[x_1, x_2][x_3, x_4] - (-1)^{x_3x_4}[x_3, x_4][x_1, x_2]z \\
= -(-1)^{x_1x_2}z[x_1, x_2][x_3, x_4] - (-1)^{x_1x_2}z[x_1, x_2][x_3, x_4]z.
\]

Now let \(x_5 = y\). We use the identity (ii) and obtain

\[
0 = (-1)^{x_1x_2}[x_1, x_2, y][x_3, x_4] - (-1)^{x_3x_4}[x_3, x_4][x_1, x_2, y]. \tag{6}
\]

Similarly we also have \(0 = (-1)^{x_1x_2}[x_1, x_2][x_3, x_4, y] + (-1)^{x_3x_4}[x_3, x_4, y][x_1, x_2]\). Summing up these equalities we get

\[
0 = yg - 2(-1)^{x_1x_2}[x_1, x_2]y[x_3, x_4] + 2(-1)^{x_3x_4}[x_3, x_4]y[x_1, x_2] + gy
\]
where \( g = (-1)^{|x_1|x_2}|x_1, x_2|[x_3, x_4] - (-1)^{|x_3|x_4}|x_3, x_4|[x_1, x_2] \) which is one of the generators of the ideal \( I \). Then we have
\[
0 = -(-1)^{|x_1|x_2}|x_1, x_2|[x_3, x_4] + (-1)^{|x_3|x_4}|x_3, x_4|[x_1, x_2].
\]

As a consequence of the above equality, from Eq. (6) it follows that
\[
0 = (-1)^{|x_1|x_2}|x_1, x_2|[x_3, x_4] - (-1)^{|x_3|x_4}|x_1, x_2|[x_3, x_4]y
\]
\[
= (-1)^{|x_1|x_2}|x_1, x_2|[x_3, x_4] - (-1)^{|x_1|x_2}|x_1, x_2|[x_3, x_4]y.
\]

Thus the lemma is proved. \( \square \)

**Remark 5.4.** Note that by the Jacobi identity and by the identities (ii) and Lemmas 5.2, 5.3, respectively, we have modulo \( I \):

1. \( [z_a, z_b, z_{\sigma(1)}, \ldots, z_{\sigma(n)}] = [z_a, z_b, z_1, \ldots, z_n] \),
2. \( z_{\rho(1)} \cdots z_{\rho(m)}[x_a, x_b, x_{\sigma(1)}, \ldots, x_{\sigma(n)}][x_c, x_d] = z_1 \cdots z_m[x_a, x_b, x_1, \ldots, x_n][x_c, x_d] \),
3. \( [x_a, x_b, x_{\sigma(1)}, \ldots, x_{\sigma(n)}, y_i] = [x_a, x_b, x_1, \ldots, x_n, y_i] \).

In order to prove these identities one needs the following simple fact. The commutator \( u = [x_1, x_2, \ldots, [x_a, x_b], \ldots] \) can be represented as a linear combination of products of two commutators. Namely write \( u = [v, w, \ldots] \) where \( v = [x_1, x_2, \ldots], \ w = [x_a, x_b] \), and then use the Jacobi identity in the form \( [u, v, x] = -[[v, x], u] + [[u, x], v] \). If necessary, iterate this procedure.

**Lemma 5.5.** The ideal \( I \) contains the polynomial
\[
[x_1, x_2, x_5][x_3, x_4] - (-1)^{|x_5|}[x_1, x_2][x_3, x_4, x_5].
\]

**Proof.** We have the following equalities:
\[
[x_1, x_2, x_5][x_3, x_4] = [x_1, x_2]x_5[x_3, x_4] - x_5[x_1, x_2][x_3, x_4],
\]
\[
[x_1, x_2][x_3, x_4, x_5] = [x_1, x_2][x_3, x_4]x_5 - [x_1, x_2]x_5[x_3, x_4].
\]

Now if \( x_5 \) is skew-symmetric we apply the identity (v) and Lemma 5.3; if \( x_5 \) is symmetric then we use only Lemma 5.3. \( \square \)

**Lemma 5.6.** The following polynomial belongs to \( I \):
\[
(-1)^{|x_1|x_3}|x_4, x_3, x_i_1, \ldots, x_i_n|[x_2, x_1] - (-1)^{|x_2|x_3}|x_4, x_2, x_i_1, \ldots, x_i_n|[x_3, x_1]
\]
\[
+ (-1)^{|x_3|x_2}|x_3, x_2, x_i_1, \ldots, x_i_n|[x_4, x_1].
\]

**Proof.** By induction on \( n \), it suffices to show that the polynomial \( g \) of degree 5:
\[
(-1)^{|x_1|x_3}|x_4, x_3, x_5|x_2, x_1] - (-1)^{|x_2|x_3}|x_4, x_2, x_5|x_3, x_1] + (-1)^{|x_3|x_2}|x_3, x_2, x_5|x_4, x_1]
\]
belongs to $I$. If $x_5 = z$ is a skew-symmetric variable then immediately $g \in I$ by the identity (v). When $x_5 = y$ we write

$$f = (-1)^{|x_4 x_3|} [x_4, x_3][x_2, x_1] - (-1)^{|x_4 x_2|} [x_4, x_2][x_3, x_1] + (-1)^{|x_3 x_2|} [x_3, x_2][x_4, x_1].$$

Then one easily verifies that

$$f y = (-1)^{|x_4 x_3|} [x_4, x_3][x_2, x_1, y] - (-1)^{|x_4 x_2|} [x_4, x_2][x_3, x_1, y] + (-1)^{|x_3 x_2|} [x_3, x_2][x_4, x_1, y] - g.$$

On the other hand, we have that, modulo $I$, by using the identity of Lemma 5.5,

$$y f = (-1)^{|x_4 x_3|} [x_4, x_3, y][x_2, x_1] + (-1)^{|x_4 x_2|} [x_4, x_2, y][x_3, x_1] - (-1)^{|x_3 x_2|} [x_3, x_2, y][x_4, x_1] + g$$

$$= (-1)^{|x_4 x_3|} [x_4, x_3][x_2, x_1, y] - (-1)^{|x_4 x_2|} [x_4, x_2][x_3, x_1, y] + (-1)^{|x_3 x_2|} [x_3, x_2][x_4, x_1, y] + g.$$

Since $[y, f] \in I$, subtracting the above two equalities, we get $2g \in I$. Hence $g \in I$. \qed

Let $\Gamma_{m,n}(I)$ be the subspace of the $Y$-proper multilinear polynomials in the variables $y_1, \ldots, y_m, z_1, \ldots, z_n$ of the relatively free algebra $F(Y, Z, *)/I$.

The polynomial $[x_1, x_2][x_3, x_4][x_5, x_6] \in \Gamma_{m,n}(I)$ belongs to $I$. Therefore a spanning set of the space $\Gamma_{m,n}(I)$ is given by the proper polynomials

$$z_{i_1} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}][x_{k_1}, \ldots, x_{k_t}]$$

where $r, s, t \geq 0$, $s \neq 1$, $t \neq 1$, and $z_{i_1} < \cdots < z_{i_r}$, $x_{j_1} > x_{j_2} < \cdots < x_{j_s}$, and $x_{k_1} > x_{k_2} < \cdots < x_{k_t}$, see [3, Section 5.2]. We denote this set as $S_1$ and we call its elements $S_1$-standard polynomials. Since such polynomials are in one-to-one correspondence with the monomials

$$z_{i_1} \cdots z_{i_r} x_{j_1}, \ldots, x_{j_s} x_{k_1}, \ldots, x_{k_t}$$

we can order $S_1$ by means of the lexicographical ordering of the monomials induced by $z_1 < z_2 < \cdots < y_1 < y_2 < \cdots$.

**Definition 5.7.** An $S_1$-standard polynomial is said to be $S_2$-standard if $s > 0$ implies that $t = 0$ or 2, and when $t = 2$ we have that $x_{j_1} > x_{k_1}$, $x_{j_2} > x_{k_2}$. The subset of the $S_2$-standard polynomials will be denoted by $S_2 \subseteq S_1$.

**Proposition 5.8.** The $S_2$-standard polynomials span the vector space $\Gamma_{m,n}(I)$.

**Proof.** It suffices to show that the $S_1$-standard polynomials that are equal to the product of two commutators, can be represented as linear combinations of polynomials from $S_2$. The identity
from Lemma 5.5 implies that the length of the second commutator can be reduced to 2. By the identity (ii) we may assume that we can write these polynomials modulo $I$, up to a sign, as

$$[x_c, x_b, \ldots][x_d, x_a]$$

where $x_b > x_a$. Therefore $x_a$ is the least variable that appears in our polynomial. Now suppose that there is a violation of the $S_2$-standardness of the form $x_c < x_d$, hence $x_a < x_b < x_c < x_d$. By Lemma 5.6 we have

$$[x_c, x_b, \ldots][x_d, x_a] = (-1)^{|x_c-x_d|}[x_d, x_b, \ldots][x_c, x_a] = (-1)^{|x_c-x_d|}[x_d, x_c, \ldots][x_b, x_a].$$

Using the Jacobi identity and (2) of Remark 5.4 one can rewrite the second polynomial of the right-hand side as a linear combination of $S_1$-standard polynomials that are higher in the fixed ordering and hence we have the result. In fact, for $[x_d, x_c, \ldots] = [x_d, x_c, x_e, \ldots]$, if $x_c > x_e$ then by the Jacobi identity we obtain

$$[x_d, x_c, x_e, \ldots] = -[x_c, x_e, x_d, \ldots] + [x_d, x_e, x_c, \ldots].$$

Then, starting from the third entry in the above commutators, we can apply the identity (2) of Remark 5.4 in order to obtain $S_1$-standard polynomials. $\Box$

Note that $S_2$-standard polynomials that contain only variables $y$ can be either of type $[y_{j_1}, \ldots, y_{j_m}]$ or $[y_{j_1}, \ldots, y_{j_{m-2}}][y_{k_1}, y_{k_2}]$, where the indices satisfy the respective conditions. In this case, that is for $\Gamma_{m,0}(I)$, we call these polynomials $S_3$-standard.

We define now $S_3$-standardness for the polynomials of $\Gamma_{0,n}(I)$.

**Definition 5.9.** A polynomial $f \in \Gamma_{0,n}(I)$ belongs to the subset $S_3 \subset S_2$ if either $f = z_1 \cdots z_n$ or $f = [z_{j_1}, z_{j_2}, \ldots, z_{j_n}]$, or else $f = z_i[z_{j_1}, \ldots, z_{j_{i-1}}]$ where $i_1 < j_1$. Recall that in the last two cases we require that $j_1 > j_2 < j_3 < \cdots$.

**Lemma 5.10.** The following polynomials are elements of $I$:

1. $[z_1, z_2][x_3, x_4] - z_1[x_3, x_4]z_2$ when $|x_3| = |x_4|$,
2. $[z_1, z_2][z_3, y_4] + z_1[z_2, y_4]z_3 - z_2[z_1, y_4]z_3$.

**Proof.** Denote by $f$ the polynomial (1). Let

$$g_1 = [z_1, z_2][x_3, x_4] - (-1)^{|x_3x_4|}[x_3, x_4][z_1, z_2],$$

$$g_2 = z_1[x_3, x_4]z_2 + (-1)^{|x_3x_4|}z_2[x_3, x_4]z_1,$$

then $g_1, g_2 \in I$. By substituting in the identity (1) of Proposition 5.1, we have

$$g_3 = z_1[z_2, [x_3, x_4]] - z_2[z_1, [x_3, x_4]] + [x_3, x_4][z_1, z_2] \in I$$

when $|x_3| = |x_4|$. Now we have that $2f = g_1 - g_2 + g_3$. (Observe that $|x_3x_4| = 0$ since $|x_3| = |x_4|$.)
Now let $f$ stand for the polynomial (2). We consider the following polynomials:

\begin{align*}
g_1 &= s_3(z_1, z_2, z_3) = z_1[z_2, z_3] - z_2[z_1, z_3] + z_3[z_1, z_2], \\
g_2 &= z_1[z_3, y_4]z_2 - z_2[z_3, y_4]z_1, \\
g_3 &= z_1[z_2, y_4]z_3 - z_3[z_2, y_4]z_1, \\
g_4 &= z_2[z_1, y_4]z_3 - z_3[z_1, y_4]z_2, \\
g_5 &= s_3(z_1y_4, z_2, z_3), \\
g_6 &= s_3(z_1, z_2y_4, z_3), \\
g_7 &= s_3(z_1, z_2, z_3y_4).
\end{align*}

Direct computations show that $2f = g_1y_4 + g_2 + g_3 - g_4 - g_5 - g_6 + g_7$. Hence it is sufficient to prove that $g_5$, $g_6$, $g_7 \in I$. But the polynomial $s_3(z_1y_4 + y_4z_1)$ belongs to $I$ since $z_1y_4 + y_4z_1$ is skew-symmetric. Moreover we have

\begin{equation*}
s_3(z_1y_4 - y_4z_1, z_2, z_3) = ([z_1, y_4][z_2, z_3] + [z_2, z_3][z_1, y_4]) - (z_2[z_1, y_4]z_3 + z_3[z_1, y_4]z_2)
\end{equation*}

and the right-hand side is a linear combination of generators of $I$. Hence $2g_5 \in I$, and we are done. The remaining two polynomials, $g_6$ and $g_7$, are dealt with in the same manner. 

\section*{Lemma 5.11.} The polynomials

\begin{equation*}
f_n = z_1[z_3, z_2, x_4, \ldots, x_n] - z_2[z_3, z_1, x_4, \ldots, x_n] + z_3[z_2, z_1, x_4, \ldots, x_n], \quad n \geq 3,
\end{equation*}

belong to the ideal $I$.

\section*{Proof.} We induct on the integer $n \geq 3$. The polynomial $f_3$ is one of the generators of $I$, hence $f_3 \in I$. Suppose that $f_n \in I$, then modulo $I$ we have

\begin{equation*}
0 = [f_n, x_{n+1}] = f_{n+1} + [z_1, x_{n+1}][z_3, z_2, x_4, \ldots, x_n]
- [z_2, x_{n+1}][z_3, z_1, x_4, \ldots, x_n] + [z_3, x_{n+1}][z_2, z_1, x_4, \ldots, x_n].
\end{equation*}

Now Lemma 5.6 and identity (ii) imply that $[f_n, x_{n+1}] - f_{n+1} \in I$ and therefore $f_{n+1} \in I$. 

\section*{Proposition 5.12.} The $S_3$-standard polynomials span the vector space $\Gamma_{0,n}(I)$.

\section*{Proof.} Using the identity (ii) we can rewrite every $S_2$-standard polynomial as a linear combination of $z_1 \cdots z_n$ and polynomials of the type

\begin{equation*}
z_{i_1} \cdots z_{i_r} [z_{j_1}, \ldots, z_{j_r}]
\end{equation*}

where the indices are not necessarily ordered. Using identity (1) of Lemma 5.10 we have

\begin{equation*}
z_{i_1} \cdots z_{i_r} [z_{j_1}, \ldots, z_{j_r}] = (-1)^{r-1} z_{i_r} [z_{j_1}, \ldots, z_{j_r}, z_{i_{r-1}}, \ldots, z_{i_1}].
\end{equation*}
But identity (1) of Remark 5.4 implies that
\[ z_{i_1} [z_{j_1}, \ldots, z_{j_s}, z_{i_{r-1}}, \ldots, z_{i_1}] = z_{i_1} [z_{j_1}, z_{j_2}, z_{k_1}, \ldots, z_{k_t}] \]
where \( k_1 < \cdots < k_t \). Now the Jacobi identity and the above mentioned identity imply that every such polynomial can be written as a linear combination of polynomials of the following type
\[ z_i [z_{l_1}, z_{l_2}, \ldots, z_{l_p}] \]
where the indices are ordered as follows \( l_1 < l_2 < \cdots < l_p \).

Finally assume that \( i > l_1 \). In this case we apply the identity from Lemma 5.11 and we can rewrite, modulo \( I \), this polynomial as a linear combination of polynomials of the type
\[ z_{l_1} [z_i, z_{l_2}, \ldots, z_{l_p}] - z_{l_2} [z_i, z_{l_1}, \ldots, z_{l_p}] \].
We observe that the first polynomial is already \( S_3 \)-standard. On the other hand, the second polynomial can be rewritten as a linear combination of two \( S_3 \)-standard polynomials by using first identity (1) of Remark 5.4, and afterwards, the Jacobi identity (if necessary). \( \square \)

Now we turn to the case of \( \Gamma_{m,n}(I) \) where \( m > 1, n > 0 \).

**Lemma 5.13.** The following polynomial belongs to the ideal \( I \):
\[ z_1 z_2 [y_1, y_2, \ldots, y_m] - z_2 [y_2, y_1, y_3, \ldots, y_m] + z_2 [y_1, z_1, y_2, y_3, \ldots, y_m], \quad m \geq 2. \]

**Proof.** We work modulo \( I \). Denote by \( f_m \) the above polynomial, then
\[ f_m = z_1 z_2 [y_1, y_2, \ldots, y_m] + z_2 [y_1, y_2, z_1, y_3, \ldots, y_m]. \]
We proceed by induction on \( m \geq 2 \). If \( m = 2 \) then according to identity (1) of Lemma 5.10 we have \( f_2 \in I \). Now suppose that \( m \geq 2 \) and that \( f_m \in I \). Then it is immediate that \([f_m, y_{m+1}] = 0 \). On the other hand, by using the identities (v) and Lemma 5.3, we compute
\[
[f_m, y_{m+1}] = f_{m+1} + [z_1 z_2, y_{m+1}] [y_1, \ldots, y_m] + [z_2, y_{m+1}] [y_1, y_2, z_1, y_3, \ldots, y_m] \\
= f_{m+1} + z_1 [z_2, y_{m+1}] [y_1, \ldots, y_m] + [z_2, y_{m+1}] [y_1, \ldots, y_m, z_1] \\
= f_{m+1} + z_1 [z_2, y_{m+1}] [y_1, \ldots, y_m] + [z_2, y_{m+1}] [y_1, \ldots, y_m] z_1 = f_{m+1}.
\]
Hence \( f_{m+1} \in I \). \( \square \)

**Lemma 5.14.** The polynomial
\[ z_1 z_2 [y_1, z_3, y_2, \ldots, y_m] + z_2 [y_1, z_3, z_1, y_2, \ldots, y_m] \]
belongs to the ideal \( I \).
Proof. The polynomial \([y_1, z_3]\) is a symmetric element, hence by the previous lemma and by the identity (ii) we have that modulo \(I\)

\[
0 = z_1 z_2 [y_1, z_3, y_2, \ldots, y_m] - z_2 [y_2, z_1, y_1, z_3, y_3, \ldots, y_m] \\
+ z_2 [y_1, z_3, z_1, y_2, y_3, \ldots, y_m] \\
= z_1 z_2 [y_1, z_3, y_2, \ldots, y_m] + z_2 [y_1, z_3, z_1, y_2, y_3, \ldots, y_m],
\]

and we are done. \(\Box\)

Lemma 5.15. The following polynomial belongs to \(I\):

\[
z_1 z_2 [z_3, z_4, y_1, \ldots, y_m] + z_2 [z_3, z_4, z_1, y_1, \ldots, y_m].
\]

Proof. Note that the commutator \([z_3, z_4, y_1]\) is a symmetric element. Hence by Lemma 5.13, the identities (ii), and (3) of Remark 5.4, we obtain that the following equality holds modulo the ideal \(I\):

\[
0 = z_1 z_2 [z_3, z_4, y_1, \ldots, y_m] - z_2 [y_2, z_1, z_3, z_4, y_1, y_3, \ldots, y_m] \\
+ z_2 [z_3, z_4, z_1, y_1, y_2, y_3, \ldots, y_m] \\
= z_1 z_2 [z_3, z_4, y_1, \ldots, y_m] + z_2 [z_3, z_4, z_1, y_1, y_2, \ldots, y_m]. \Box
\]

Now we define \(S_3\)-standardness for the polynomials of \(\Gamma_{m,n}(I)\), when \(m > 1\) and \(n > 0\).

Definition 5.16. A polynomial \(f \in \Gamma_{m,n}(I), m > 1, n > 0\), belongs to the subset \(S_3 \subseteq S_2\) if it is of one of the following types:

(i) \([x_{j_1}, x_{j_2}, \ldots, x_{j_d}]\);
(ii) \(z_i [x_{j_1}, x_{j_2}, \ldots, x_{j_d-1}]\) where \(z_i < x_{j_1}\);
(iii) \([x_{j_1}, x_{j_2}, \ldots, x_{j_d-2}] [y_{k_1}, z_1]\).

Recall that we require that the variables in the commutators above satisfy the inequalities \(x_{j_1} > x_{j_2} < x_{j_3} < \cdots\), and that in the third type \(x_{j_1} > y_{k_1}, x_{j_2} > z_1\).

Proposition 5.17. The vector space \(\Gamma_{m,n}(I), m > 1, n > 0\), is spanned by the \(S_3\)-standard polynomials.

Proof. We shall prove that every polynomial in \(S_2\) can be rewritten as a linear combination of polynomials from \(S_3\), modulo the ideal \(I\).

First we note that if a product of two commutators is \(S_2\)-standard then it has the variable \(z_1\) as its last entry of the last commutator.

We consider now an element \(f \in S_2\) of the type

\[
f = z_1 \cdots z_r [x_{j_1}, \ldots, x_{j_s}] [x_{k_1}, x_{k_2}]
\]

where \(r > 0\) and the variables satisfy the conditions from Definition 5.7.
Claim. The polynomial $f$, modulo $I$, is a linear combination of polynomials of $S_2$ such that $x_{k_2} = z_1$.

Proof of the Claim. The only case to be considered is when $z_{i_1} = z_1$. In this case, by using the identity (2) of Remark 5.4, (v), and Lemma 5.3, we have that

$$f = -z_{i_2} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}, x_{k_1}, x_{k_2}, z_1] = z_{i_2} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}, x_{k_2}, z_1, x_{k_1}] = z_{i_2} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}, x_{k_1}] [x_{k_2}, z_1, x_{k_1}] = z_{i_2} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}, x_{k_1}] [x_{k_2}, z_1].$$

Now applying Lemma 5.5 we obtain

$$f = z_{i_2} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}, x_{k_1}] [x_{k_2}, z_1] = z_{i_2} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}, x_{k_2}] [x_{k_1}, z_1].$$

Since $x_{j_2} > x_{k_1} > x_{k_2}$ we can rearrange the variables in the first commutators in order to obtain $S_2$-standard polynomials and the claim holds. Recall that while rearranging the variables after the second position in a commutator, we get a third commutator and the corresponding element must vanish due to Lemma 5.2, see also Remark 5.4.

Let us consider the polynomial $f = z_{i_2} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}, x_{k_1}, z_1]$. Suppose that $x_{k_1} = z_{k_1}$. Then in both cases, $r = 0$ and $r > 0$, the polynomial $f$ can be written as follows:

$$f = \pm z_{i_1} \cdots z_{i_r} z_{k_1} [x_{j_1}, \ldots, x_{j_s}] \mp z_{i_1} \cdots z_{i_r} z_{k_1} [x_{j_1}, \ldots, x_{j_s}].$$

where the variables $z$ that precede the commutators may possibly be out of order. Therefore in order to prove the proposition, it suffices to show that every polynomial of the types:

1. $z_{i_1} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}] [y_k, z_1]$;
2. $z_{i_1} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}]$

is a linear combination of $S_3$-standard polynomials. Here the polynomial in (1) is $S_2$-standard; in that in (2), the variables of the monomial $z_{i_1} \cdots z_{i_r}$ are not necessarily ordered. In the first case we move the variables from the monomial $z_{i_1} \cdots z_{i_r}$ to the first commutator, using the identity (v). More precisely we have that, modulo $I$, the polynomial $f$ of type (1) equals

$$f = (-1)^r [x_{j_1}, \ldots, x_{j_s}, z_{i_1}, \ldots, z_{i_r}] [y_k, z_1].$$

By identity (2) of Remark 5.4 we obtain that

$$f = (-1)^r [x_{j_1}, x_{j_2}, x_{l_1}, \ldots, x_{l_r}] [y_k, z_1]$$

where $x_{l_1} < \cdots < x_{l_r}$. If $x_{j_2} < x_{l_1}$ we are done. Otherwise we use the Jacobi identity and once again the identity (2) of Remark 5.4. Therefore the polynomials of type (1) have the required form.

Let $f$ be a polynomial of type (2). In case there is only one variable $z$ in $f = z_{i_1} \cdots z_{i_r} [x_{j_1}, \ldots, x_{j_s}]$ then $f$ must be already $S_3$-standard. Otherwise using Lemma 5.13, we may assume that $x_{j_2} = z_{j_2}$. Then by means of repeated using of Lemmas 5.14, 5.15, we obtain that modulo $I$

$$f = (-1)^{r-1} z_{i_r} [x_{j_1}, z_{j_2}, z_{i_{r-1}}, \ldots, z_{i_1}, x_{j_3}, \ldots, x_{j_s}].$$
Since the number $m$ of the variables $y$ is larger than 1 we know that $x_{j_1} = y_{j_1}$. Then we apply identity (3) of Remark 5.4 and write

$$f = (-1)^{r-1}z_{i_r}[x_{j_1}, z_{j_2}, x_{l_1}, \ldots, x_{l_r}]$$

where $x_{j_1} < \cdots < x_{l_r}$. If $z_{j_2} > x_{l_1}$ then as above we can write $f$ as a linear combination of polynomials of the type

$$g = z_{i_r}[x_{j_1}, x_{p_1}, \ldots, x_{p_{r+1}}]$$

where $x_{j_1} > x_{p_1} < \cdots < x_{p_{r+1}}$. If $z_{i_r} < x_{j_1}$ then $g$ is $S_3$-standard. Otherwise $x_{j_1} = z_{j_1}$ and we apply Lemma 5.11 in order to complete the proof. \qed

Finally we deal with the vector space $\Gamma_{1,n}(I)$ of the $Y$-proper polynomials that contain only one symmetric variable $y$. It turns out that the approach chosen above does not work very well in this situation. Hence we were forced to follow a different one. It is evident that every $Y$-proper multilinear polynomial in $\Gamma_{1,n}(I)$ can be written as a linear combination of polynomials of type

$$z_{l_1} \cdots z_{l_r}[y, z_{j_1}]z_{k_1} \cdots z_{k_s}$$

where $r, s \geq 0$. We call such generators $T_1$-standard and denote by $T_1$ the set formed by them.

**Definition 5.18.** A $T_1$-standard polynomial is called $T_2$-standard if it is of one of the following types:

(i) $z_1 \cdots \hat{z}_j \cdots z_n[y, z_{j_1}], \quad j = 1, 2, \ldots, n$;
(ii) $[y, z_{j_1}]z_1 \cdots \hat{z}_j \cdots z_n, \quad j = 1, 2, \ldots, n$;
(iii) $z_1 \cdots \hat{z}_j \cdots z_{n-1}[y, z_{j_1}]z_{n_1}, \quad j = 1, 2, \ldots, n - 1$;
(iv) $z_1 \cdots \hat{z}_i \cdots \hat{z}_j \cdots z_n z_{j_1}[y, z_{j_1}], \quad 1 \leq i < j \leq n - 1$.

Here the hat over a variable means that the corresponding variable is missing from the expression.

We denote by $T_2 \subset T_1$ the subset of these polynomials.

**Remark 5.19.** Note that from the identity (v) it follows that modulo $I$ one has:

$$x_{\sigma(1)} \cdots x_{\sigma(i)}z_{j_1}[x_{a_1}, x_{b_1}] = x_1 \cdots x_i z_{j_1}[x_{a_1}, x_{b_1}]$$

$$[x_{a_1}, x_{b_1}]z_{j_1}x_{\sigma(1)} \cdots x_{\sigma(i)} = [x_{a_1}, x_{b_1}]z_{j_1}x_1 \cdots x_i.$$

**Proposition 5.20.** The $T_2$-standard polynomials span the vector space $\Gamma_{1,n}(I)$.

**Proof.** First we consider the $T_1$-standard polynomial $f = [y, z_{j_1}]z_{k_1} \cdots z_{k_{n-1}}$. According to Remark 5.19 we obtain that $f = [y, z_{j_1}]z_{k_1}z_{l_2} \cdots z_{l_{n-1}}$ where one has $l_2 < \cdots < l_{n-1}$. If $k_1 < l_2$ then $f$ is already $T_2$-standard. Otherwise when $k_1 > l_2$ we write

$$f = [y, z_{j_1}]z_{l_2}z_{k_1}z_{l_3} \cdots z_{l_{n-1}} - [y, z_{j_1}]z_{l_2}z_{k_1}z_{l_3} \cdots z_{l_{n-1}}.$$
Once again by Remark 5.19 we obtain that the first summand is $T_2$-standard. Moreover, using the identities (ii) and Lemma 5.3, the second summand can be expressed as a linear combination of polynomials of the form

$$g = z_{i_1} \cdots z_{i_{n-1}}[y, z_j].$$

Therefore we consider the polynomials of the form $g$. Obviously we may assume that $i_1 < \cdots < i_{n-2}$. If furthermore $i_{n-2} < i_{n-1}$, or $i_{n-2} = n$ then the polynomial $g$ is already $T_2$-standard. So let $i_{n-2} > i_{n-1}$ and $i_{n-2} \neq n$. Then we have $j = n$ and $i_{n-2} = n - 1$. Denote $i_{n-1} = i$. Using the identity (2) of Lemma 5.10 we obtain

$$g = w(z_i z_{n-1}[y, z_n] + z_i[y, z_{n-1}]z_n - z_{n-1}[y, z_i]z_n)$$

where $w = z_{i_1} \cdots z_{i_{n-3}}$.

If $n = 3$, we are done. Else, observe that by Remark 5.19, $wz_i z_{n-1}[y, z_n] = z_1 \cdots z_{n-1}[y, z_n]$. Moreover, the third summand of the right-hand side above is already $T_2$-standard.

Thus we are left with the polynomial $h = wz_i[y, z_{n-1}]z_n$ where $i < n - 2$. Using Remark 5.19 and Lemma 5.3 we obtain

$$h = w'(z_i z_{n-2}[y, z_n] + [z_n-2, z_i][y, z_{n-1}]z_n)$$

$$= w'(z_i z_{n-2}[y, z_{n-1}]z_n - z_n[z_{n-2}, z_i][y, z_{n-1}])$$

$$= w'(z_i z_{n-2}[y, z_{n-1}]z_n - z_{n-2}z_n z_i[y, z_{n-1}] + z_i z_n z_{n-2}[y, z_{n-1}]).$$

Here $w' = z_1 \cdots \hat{z}_i \cdots z_{n-3}$.

Now we may assume that the polynomial $w'z_{n-2}z_n z_i[y, z_{n-1}]$ is already $T_2$-standard. Applying Remark 5.19 to $w'z_i z_{n-2}[y, z_{n-1}]z_n$ and $w'z_i z_n z_{n-2}[y, z_{n-1}]$ we may assume that these polynomials are also $T_2$-standard.

Let $f = z_{i_1} \cdots z_{i_r}[y, z_j]z_{k_1} \cdots z_{k_s}$ be a $T_1$-standard polynomial such that $r, s > 0$. Applying the identity (vi) we may assume that $r = n - 2$ and $s = 1$. Furthermore, as it was done above, we may also suppose that $i_1 < \cdots < i_{n-3}$. Then using the identity (iv) we have

$$f = wz_i[y, z_j]z_k$$

where $w = z_{i_1} \cdots z_{i_{n-3}}$ and $i = i_{n-2} < k$. Suppose that $f$ is not $T_2$-standard. The first case we consider is when $j = n$. By identity (2) of Lemma 5.10 we have that

$$f = w(z_n z_i[y, z_k] - z_i z_n[y, z_k] + z_n[y, z_i]z_k).$$

By identity (iv) we have $z_n[y, z_i]z_k = zk[y, z_i]z_n$ modulo $I$. Therefore, we need only show how to rewrite the polynomial $g = wz_k[y, z_i]z_n$ as a combination of $T_2$-standard polynomials, in the case when $n > 3$ and $i_{n-3} > k$. In this case we have that

$$g = w'(zk z_{i_{n-3}}[y, z_i]z_n + [z_{i_{n-3}}, z_k][y, z_i]z_n)$$

where $w' = z_{i_1} \cdots z_{i_{n-4}}$. The first summand here is already $T_2$-standard, up to applying Remark 5.19. Concerning the polynomial $w'[z_{i_{n-3}}, z_k][y, z_i]z_n$ we note that by Lemma 5.3 it is
a linear combination of \( T_1 \)-standard polynomials of the type \( z_{l_1} \cdots z_{l_{n-1}} [y, z_i] \), and this type has already been dealt with.

Now we turn to the case when \( f = wz_i[y, z_j]z_k \) with \( k = n \) and \( i_{n-3} > i \). Actually this last case was done in the course of the latter argument.

So, when \( f \) is not \( T_2 \)-standard, the only case that remains to consider is when \( i_{n-3} = n \). We have in it

\[
f = w'(z_iz_n[y, z_j]z_k + [z_n, z_i][y, z_j]z_k) = w'(z_iz_k[y, z_j]z_n - z_k[z_n, z_i][y, z_j])
\]

by applying the identities (iv) and Lemma 5.3. The result now follows from the above considered cases.

\[ \Box \]

6. Computations with generic matrices

In this section we show that the elements from the canonical forms \( S_3 \) and \( T_2 \) are linearly independent modulo the \( * \)-identities of \( A \). In order to do this we need a convenient model for the relatively free algebra in the variety of algebras with involution generated by \( A \). Let \( y_{ij}^k, z_{ij}^k \) be commuting variables and consider the matrices

\[
y_k = \begin{pmatrix}
y_{11}^k & y_{12}^k & y_{13}^k \\
0 & y_{22}^k & y_{12}^k \\
0 & 0 & y_{11}^k
\end{pmatrix}, \quad z_k = \begin{pmatrix}
z_{11}^k & z_{12}^k & 0 \\
0 & 0 & -z_{12}^k \\
0 & 0 & -z_{11}^k
\end{pmatrix}, \quad k = 1, 2, \ldots
\]

Let \( D \) be the \( F \)-subalgebra of \( UT_3(F[y_{ij}^k, z_{ij}^k]) \) generated by these matrices. Then one defines an involution \( * \) on \( D \) in the usual manner. The matrices \( y_k \) are symmetric while \( z_k \) are skew-symmetric with respect to \( * \).

**Lemma 6.1.** The algebra \( D \) is isomorphic to the relatively free algebra in the variety of algebras with involution generated by \( A \).

**Proof.** The proof repeats verbatim the one for the generic matrix algebra and so we omit it. \[ \Box \]

Now we compute the elements of the normal form \( S_3 \) on the corresponding generic matrices. All these computations are straightforward (but somewhat lengthy), therefore we give only the final results. First we observe that

\[
[z_{i_1}, z_{i_2}] = (z_{i_1}^1 z_{i_2}^2 - z_{i_1}^2 z_{i_2}^1)(e_{12} - e_{23}),
\]

\[
[y_{i_1}, z_{i_2}] = ((y_{i_1}^1 - y_{i_2}^1) y_{i_1}^2 z_{i_2}^2 - y_{i_1}^1 y_{i_2}^2 z_{i_1}^1)(e_{12} + e_{23}) - 2(y_{i_1}^1 z_{i_2}^1 + y_{i_2}^1 z_{i_1}^1)e_{13},
\]

\[
[y_{i_1}, y_{i_2}] = ((y_{i_1}^1 - y_{i_2}^1) y_{i_2}^2 y_{i_1}^2 - y_{i_2}^1 (y_{i_1}^2 - y_{i_2}^2))(e_{12} - e_{23}).
\]
Lemma 6.2. The $S_3$-standard polynomials in $\Gamma_{m,0}$ are linearly independent modulo the $*$-identities of $A$.

Proof. We compute in $D$. Suppose that $f$ is a linear combination of $S_3$-standard elements that vanishes when evaluated on the generic matrices $y_i$. Consider first the commutator $v = [y_{i_1}, y_{i_2}, \ldots, y_{i_k}]$. When $k$ is even we have in $D$ that

$$v = \left(\left(y_{i_1}^{i_1} - y_{i_2}^{i_1}\right)y_{i_2}^{i_2} - y_{i_2}^{i_1}\left(y_{i_1}^{i_2} - y_{i_2}^{i_2}\right)\right)\prod_{t=3}^{k} (y_{i_1}^{i_t} - y_{i_2}^{i_t})(e_{12} - e_{23}),$$

and when $k$ is odd,

$$v = -\left(\left(y_{i_1}^{i_1} - y_{i_2}^{i_1}\right)y_{i_2}^{i_2} - y_{i_2}^{i_1}\left(y_{i_1}^{i_2} - y_{i_2}^{i_2}\right)\right)\prod_{t=3}^{k-1} (y_{i_1}^{i_t} - y_{i_2}^{i_t})(y_{i_1}^{i_k} - y_{i_2}^{i_k})(e_{12} + e_{23} - 2y_{i_1}^{i_k}e_{13}).$$

Observe that in both cases there appear nonzero coefficients of $e_{12}$ (and of $e_{23}$ as well). If we have a product of two commutators then the only nonzero coefficient may be that of $e_{13}$. Hence we can deal with each of these cases separately. If $v$ is an $S_3$-standard commutator we can recover it uniquely by looking at the right-hand side of the corresponding equality above. Namely we determine the first two variables of $v$ since only these contribute with entries $y_{i_2}^{i_1}$ as coefficients of $e_{12}$, and then the rest of the variables must be ordered uniquely in accordance to the $S_3$-standardness. Therefore we have a one-to-one correspondence between the set of $S_3$-standard commutators and the polynomials of $F[y_{ij}^p]$ that are $(1,2)$-entry of the evaluation of the commutators in $D$. Note that these polynomial in commuting variables have all different leading monomials in a suitable ordering.

One deals with the product of two commutators in a similar way. Write $v = \alpha(e_{12} \pm e_{23}) + \beta e_{13}$ where $\alpha \neq 0$ but $\beta$ might be 0. If $w = [y_{j_1}, y_{j_2}]$ and the product $vw$ is $S_3$-standard then one gets that

$$vw = -\alpha(\left(y_{j_1}^{j_1} - y_{j_2}^{j_1}\right)y_{j_2}^{j_2} - y_{j_2}^{j_1}\left(y_{j_1}^{j_2} - y_{j_2}^{j_2}\right))e_{13}.$$ 

Now the coefficient of $e_{13}$ is a polynomial from $F[y_{ij}^p]$ that is represented as a product of irreducible polynomials. By each of the quadratic factors we can recover the pairs $(i_1, i_2)$ and $(j_1, j_2)$ and using the $S_3$-standardness we determine uniquely the commutators $v$ and $w$. Again it remains defined a one-to-one correspondence between the set of $S_3$-standard products of two commutators and a set of linear independent polynomials in commuting variables. This proves the lemma. \qed

Lemma 6.3. The $S_3$-standard polynomials in $\Gamma_{0,n}$ are linearly independent modulo the $*$-identities of $A$. 

Proof. Once again we compute in $D$. There is only one product of the form $z_1 z_2 \cdots z_n$ and it has nonzero coefficient of $e_{11}$. Now, let $v = [z_{i_1}, z_{i_2}, \ldots, z_{i_m}]$ be a commutator containing only skew symmetric variables. Then in $D$ one has

$$v = (-1)^n (z_{i_1} z_{i_2} - z_{i_2} z_{i_1}) \prod_{t=3}^n z_{i_t} (e_{12} - e_{23}).$$

Furthermore, the element $w = z_{i_1} [z_{j_1}, z_{j_2}, \ldots, z_{j_n-1}]$ has the following evaluation in $D$:

$$w = (z^j_{i_1} z^j_{i_2} - z^j_{i_2} z^j_{i_1}) \prod_{s=3}^{n-1} z_{i_1} (z_{i_1} e_{12} - z_{i_2} e_{13}).$$

Since these matrices have a nonzero entry in different positions we can deal with each of the above three cases separately. Moreover, from the above formulas we recover uniquely the variables $z_{i_1}$ and $z_{i_2}$ in the commutator $v$; the rest of the $z$’s is determined by the $S_3$-standardness. For the polynomial $w$, from the coefficient of $e_{12}$ we recover $j_1$ and $j_2$, and afterwards from the one of $e_{13}$ we determine $i_1$. Thus the proof of the lemma is complete. □

**Lemma 6.4.** The $S_3$-standard polynomials in $\Gamma_{m,n}$, $m > 1$, $n > 0$, are linearly independent modulo the $*$-identities of $A$.

Proof. The computations are rather similar to the ones in the two previous lemmas but here we have quite a few cases to consider. We recall that according to Definition 5.16, an $S_3$-standard polynomial in $\Gamma_{m,n}$ is either a single commutator $v$ or the product $vw$ of two of them or else the product $z_h v$ of a single skew-symmetric variables and a commutator. Note that the entry in the position $(2, 3)$ of the evaluation of the commutator $v$ in $D$ is not zero and the corresponding entry is zero for the other $S_3$-standard polynomials. Now we can deal separately with the case of a single commutator.

Clearly $v$ is either of the form $v = [z_{i_1}, z_{i_2}, \ldots, z_{i_n}, y_{j_1}, \ldots, y_{j_m}]$, or of the form $v = [y_{j_1}, z_{i_1}, \ldots, z_{i_n}, y_{j_2}, \ldots, y_{j_m}]$. Since $v$ is $S_3$-standard, it is uniquely determined once given its first variable. Now, computing in $D$, for the first case if $m$ is even we have

$$v = (-1)^n (z^i_{j_1} z^i_{j_2} - z^i_{j_2} z^i_{j_1}) \prod_{t=3}^n z^i_{j_t} \prod_{s=1}^m (y^j_{s_1} - y^j_{s_2}) (e_{12} - e_{23}),$$

and for $m$ odd, respectively

$$v = (-1)^{n-1} (z^i_{j_1} z^i_{j_2} - z^i_{j_2} z^i_{j_1}) \prod_{t=3}^n z^i_{j_t} \prod_{s=1}^{m-1} (y^j_{s_1} - y^j_{s_2}) ((y^j_{s_1} - y^j_{s_2}) (e_{12} + e_{23}) - 2 y^j_{s_2} e_{13}).$$

Analogously, in the second case, when $m$ is even we obtain

$$v = (-1)^n ((y^j_{s_1} - y^j_{s_2}) z^i_{j_t} - y^j_{s_2} z^i_{j_t}) \prod_{s=2}^n z^i_{j_t} \prod_{t=2}^m (y^j_{t_1} - y^j_{t_2}) (e_{12} - e_{23}),$$

and for $m$ odd, respectively

$$v = (-1)^{n-1} ((y^j_{s_1} - y^j_{s_2}) z^i_{j_t} - y^j_{s_2} z^i_{j_t}) \prod_{s=2}^n z^i_{j_t} \prod_{t=2}^{m-1} (y^j_{t_1} - y^j_{t_2}) ((y^j_{t_1} - y^j_{t_2}) (e_{12} + e_{23}) - 2 y^j_{s_2} e_{13}).$$
and if $m$ is odd we have

$$v = (-1)^{n-1}((y_{11}^j - y_{22}^j)z_{12}^j - y_{12}^j z_{11}^j) \prod_{s=2}^{n} z_{11}^s \prod_{t=2}^{m-1} (y_{11}^{j_t} - y_{22}^{j_t})((y_{11}^{j_m} - y_{22}^{j_m})(e_{12} + e_{23}) - 2y_{12}^{j_m} e_{13}).$$

In all cases the first variable of the commutator $v$ can be recovered by the polynomial that appears as coefficient of $e_{12}$. We note that in all cases the commutators $v$ are skew-symmetric or symmetric elements.

Notice that for the product of two commutators $vw$ the only nonzero entry of the evaluation in $D$ is the one in position $(1, 3)$. Since the evaluation of the product $zhv$ has a nonzero coefficient of $e_{12}$ we can consider each of these cases separately.

We start with the product $zh[x_{j_1}, x_{j_2}, \ldots, x_{j_{22}}] = z_h v$. Here $v$ stands for some among the four commutators above, or further $v$ may be a commutator in the variables $y$ only.

Let $v = \alpha(e_{12} - e_{23})$ be the first of these commutators, and let $v' = \beta(e_{12} - e_{23})$ be the third of them, $\alpha, \beta$ being nonzero. Then $zhv = zh^h v = z_h^h \alpha e_{12} - z_h^h \alpha e_{13}$, and analogously $zhv' = z_h^h \beta e_{12} - z_h^h \beta e_{13}$. Thus in both cases, dividing the coefficients of $e_{12}$ and $e_{23}$ we determine $zh$, and then proceed as above.

The case when $v$ and $v'$ are commutators respectively of the second and of the fourth type considered in this proof, is rather similar. Write

$$v = \alpha(y_{11}^{j_m} - y_{22}^{j_m})(e_{12} + e_{23}) - 2\alpha y_{12}^{j_m} e_{13}; \quad v' = \beta(y_{11}^{j_m} - y_{22}^{j_m})(e_{12} + e_{23}) - 2\beta y_{12}^{j_m} e_{13}$$

and then compute the products $zhv$ and $zhv'$

$$zhv = z_h^h \alpha(y_{11}^{j_m} - y_{22}^{j_m})e_{12} + \alpha(-2z_h^h y_{12}^{j_m} + z_h^h (y_{11}^{j_m} - y_{22}^{j_m})) e_{13};$$

$$zhv' = z_h^h \beta(y_{11}^{j_m} - y_{22}^{j_m})e_{12} + \beta(-2z_h^h y_{12}^{j_m} + z_h^h (y_{11}^{j_m} - y_{22}^{j_m})) e_{13}.$$ 

Once more we divide the coefficients of $e_{12}$ and of $e_{13}$ and thus determine $zh$, and then we continue in the known manner.

The element $zhv$ where $v$ is a commutator in the letters $y$ only is dealt in the same way by using the formulas from the proof of Lemma 6.2.

The last case to consider is the product of two commutators $vw$. Note that necessarily $w = [y_{k_1}, z_{k_2}]$. (If we have $w = [y_{k_1}, y_{k_2}]$ then according to the $S_3$-standardness all variables in $v$ must be $y$ and this was done in Lemma 6.2.)

Let $v = [y_{j_1}, y_{j_2}, \ldots, y_{j_s}]$. When $s$ is even we have $v = \alpha(e_{12} - e_{23})$ and $vw = \alpha((y_{11}^{k_1} - y_{22}^{k_1}) z_{12}^{k_2} - y_{12}^{k_1} z_{11}^{k_2}) e_{13}$, here the coefficient $\alpha$ was computed in Lemma 6.2. If $s$ is odd then $v = \beta((y_{11}^{k_1} - y_{22}^{k_1}) z_{12}^{k_2} - y_{12}^{k_1} z_{11}^{k_2}) e_{13}$. As in Lemma 6.2, by looking at the quadratic irreducible factors of the coefficient of $e_{13}$ we recover $k_1$, and then once again as in Lemma 6.2 we determine the correct order of the variables in $v$.

There is only one $S_3$-standard element of the form $[y_{j_1}, z_{i_1}, \ldots, z_{i_s}] [y_{k_1}, z_{k_2}]$. So the only remaining case is when $v = [y_{j_1}, z_{i_1}, \ldots, z_{i_k}, y_{j_2}, \ldots, y_{j_s}]$ and $w = [y_{p_1}, z_{p_2}]$. Then we use the
results about $v$ obtained at the very beginning of this proof and proceed in the same manner as above. □

Lemma 6.5. The $T_2$-standard elements in $\Gamma_{1,n}$ are linearly independent modulo the *-identities of $A$.

Proof. Let $v = [y, z_j]$, then in $D$ we have $e_{13}v = ve_{13} = 0$. We compute first the product $z_1z_2 \cdots z_k$. We have

$$z_1z_2 \cdots z_k = \prod_{s=1}^{k} z_1^s (e_{11} + (-1)^s e_{33}) + \prod_{s=1}^{k-1} z_1^s z_2^s e_{12} + (-1)^k z_1^1 \prod_{s=2}^{k} z_1^s e_{23} + \alpha e_{13}$$

for some nonzero polynomial $\alpha$.

Note that the polynomials $z_1 \cdots \hat{z}_j \cdots z_n [y, z_j]$ and $z_1 \cdots \hat{z}_i \cdots z_n z_i [y, z_j]$ have zero coefficient of $e_{23}$ in $D$ since the matrices corresponding to the variables $z$ have zero entry in position $(2, 2)$, and $[y, z_j]$ has zero entries on the main diagonal. For the same reasons $[y, z_j] z_1 \cdots \hat{z}_j \cdots z_n$ has zero coefficient for $e_{12}$. Therefore $z_1 \cdots \hat{z}_j \cdots z_{n-1} [y, z_j] z_n$ has non-zero entry only in position $(1, 3)$. It follows that we can deal with these three cases separately.

If $w = [y, z_j] z_1 \cdots \hat{z}_j \cdots z_n$ then

$$w = (-1)^{n-1} \left( (y_{11} - y_{22}) z_{12}^j z_{11}^j - y_{12} z_{11}^j \prod_{s=1}^{n} z_{11}^s e_{23} \right) + \left( (y_{11} - y_{22}) z_{12}^j - y_{12} z_{11}^j \prod_{s=2}^{n} z_{11}^s e_{12} - 2(y_{12} z_{11}^j + y_{13} z_{11}^j) \prod_{s=1}^{n} z_{11}^s e_{13} \right).$$

Here in all the products the index $s$ ranges over the indicated segment and $s \neq j$. The coefficient of $e_{23}$ determines the index $j$ since the only entry $z_{12}^j$ that appears in it is when $s = j$. In the same way we may argue for the case $j = n$.

Let $w = z_1 \cdots \hat{z}_j \cdots z_{n-1} [y, z_j] z_n$, then

$$w = - \left( \prod_{s=1}^{n-1} z_{11}^s (y_{11} - y_{22}) z_{12}^j - y_{12} z_{11}^j \right) z_{12}^n z_{11}^n - 2 \prod_{s=1}^{n-1} z_{11}^s (y_{12} z_{11}^j + y_{13} z_{11}^j) z_{11}^n$$

$$+ \prod_{s=1}^{n-2} z_{11}^s z_{12}^{n-1} (y_{11} - y_{22}) z_{12}^j - y_{12} z_{11}^j z_{11}^n e_{13}.$$ 

Here for the index $s$ we require $s \neq j$. The entries of the type $z_{12}$ are only $z_{12}^j$ and $z_{12}^n$. Hence one recovers uniquely $j$. We leave to the reader the small modifications that are to be done when $j = n - 1$.

Consider now $w = z_1 \cdots \hat{z}_j \cdots z_n [y, z_j]$ and $w' = z_1 \cdots \hat{z}_i \cdots z_k z_i [y, z_k]$. By evaluating the polynomials in $D$ we obtain:
\[ w = \prod_{s=1}^{n} z_{s1}^{i}\left((y_{11} - y_{22})z_{12}^{j} - y_{12}z_{11}^{j}\right)e_{12} \]

\[ \quad + \left( -2 \prod_{s=1}^{n} z_{s1}^{i}\left(y_{12}z_{12}^{j} + y_{13}z_{11}^{j}\right) + \prod_{s=1}^{n-1} z_{s1}^{i}\left((y_{11} - y_{22})z_{12}^{j} - y_{12}z_{11}^{j}\right) \right) e_{13}, \]

\[ w' = \prod_{s=1}^{n} z_{s1}^{i}\left((y_{11} - y_{22})z_{12}^{k} - y_{12}z_{11}^{k}\right)e_{12} \]

\[ \quad + \left( -2 \prod_{s=1}^{n} z_{s1}^{i}\left(y_{12}z_{12}^{k} + y_{13}z_{11}^{k}\right) + \prod_{s=1}^{n} z_{s1}^{i}\left((y_{11} - y_{22})z_{12}^{k} - y_{12}z_{11}^{k}\right) \right) e_{13}, \]

where in the first formula the index \( s \) does not assume value \( j \) and in the second one it does not assume the values \( i, k \). From the coefficient of \( e_{12} \) we determine \( z_{j} \) and \( z_{k} \) for the polynomials \( w \) and \( w' \) respectively. Moreover, if \( j = k \) (hence \( j \neq n \)) we may distinguish the polynomials \( w \) and \( w' \) by considering which of the variables \( z_{12} \) or \( z_{12}' \) occurs in the coefficient of \( e_{13} \) and we are done. \( \Box \)

We recall that in the case of characteristic zero any \( T(\ast) \)-ideal in the free algebra with involution is completely determined by its \( Y \)-proper multilinear components. Then, by Propositions 5.8, 5.12, 5.17, 5.20 and the lemmas of the present section we obtain the following theorem.

**Theorem 6.6.** Let \( F \) be a field of characteristic zero and \( UT_{3} = UT_{3}(F) \). The ideal \( T(UT_{3}, \ast) \) is generated as \( T(\ast) \)-ideal by the identities (i)–(vi) of Proposition 5.1. A linear basis of the space of the \( Y \)-proper multilinear polynomials \( \Gamma_{m,n}(UT_{3}, \ast) \) is given by the \( T_{2} \)-standard polynomials when \( m = 1, n > 0 \), and by the \( S_{3} \)-standard polynomials otherwise.

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**References**


